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ON INDESTRUCTIBLE STRONGLY GUESSING MODELS

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Abstract. In [15] we defined and proved the consistency of the principle $\mathrm{GM}^+(\omega_3, \omega_1)$ which implies that many consequences of strong forcing axioms hold simultaneously at ω_2 and ω_3. In this paper we formulate a strengthening of $\mathrm{GM}^+(\omega_3, \omega_1)$ that we call $\mathrm{SGM}^+(\omega_3, \omega_1)$. We also prove, modulo the consistency of two supercompact cardinals, that $\mathrm{SGM}^+(\omega_3, \omega_1)$ is consistent with ZFC. In addition to all the consequences of $\mathrm{GM}^+(\omega_3, \omega_1)$, the principle $\mathrm{SGM}^+(\omega_3, \omega_1)$, together with some mild cardinal arithmetic assumptions that hold in our model, implies that any forcing that adds a new subset of ω_2 either adds a real or collapses some cardinal. This gives a partial answer to a question of Abraham [1] and extends a previous result of Todorčević [16] in this direction.

1. Introduction

One of the driving themes of research in set theory in recent years has been the search for higher forcing axioms. Since most applications of strong forcing axioms such as the Proper Forcing Axiom (PFA) and Martin's Maximum (MM) can be factored through some simple, but powerful combinatorial principles, it is natural to look for higher cardinal versions of such principles. One such principle is $\mathrm{ISP}(\omega_2)$ which was formulated by Viale and Weiß [19]. They derived it from PFA and showed that it has a number of structural consequences on the set-theoretic universe. For instance, it implies the tree property at ω_2, and the failure of $\square(\lambda)$, for regular $\lambda \geq \omega_2$. Moreover, Krueger [10] showed that it implies the Singular Cardinal Hypothesis (SCH). A strengthening of $\mathrm{ISP}(\omega_2)$ was studied by Cox and Krueger [6]. They showed that this strengthening, which we refer to as $\mathrm{SGM}(\omega_2, \omega_1)$, is consistent with $2^{\aleph_0} > \aleph_2$. It is therefore a natural candidate for a generalization to higher cardinals. In [15] we introduced one such generalization, $\mathrm{GM}^+(\omega_3, \omega_1)$, and showed that if there are two supercompact cardinals then there is a generic extension $V[G]$ of the universe in which $\mathrm{GM}^+(\omega_3, \omega_1)$ holds. In addition to $\mathrm{ISP}(\omega_2)$, the principle $\mathrm{GM}^+(\omega_3, \omega_1)$ implies that the tree property also holds at ω_3, and that the approachability ideal $I[\omega_2]$ restricted to ordinals of cofinality ω_1 is the non stationary ideal restricted to this set. The consistency of the latter was previously shown by Mitchell [14] starting from a model with a greatly Mahlo cardinal. In the current paper we further strengthen the principle $\mathrm{GM}^+(\omega_3, \omega_1)$ in the spirit of Cox and Krueger [6]. We call this new principle $\mathrm{SGM}^+(\omega_3, \omega_1)$. We show its consistency by a modification of the forcing notion from [15], again starting from two supercompact cardinals. As a new application we show that $\mathrm{SGM}^+(\omega_3, \omega_1)$, together with some mild cardinal arithmetic assumptions that hold in our model, implies that any forcing that adds a new subset of

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ω_2 either adds a real or collapses some cardinal. This gives a partial answer to a question of Abraham [1] and extends a previous result of Todorčević [16] in this direction.

The paper is organized as follows. In §2 we recall some preliminaries and background facts. In §3 we state the main theorem and derive the application of $\mathrm{SGM}^+(\omega_3, \omega_1)$ to the problem of Abraham mentioned above. In §4 we review the definitions on virtual models from [15]. In §5 we describe the main forcing and prove the relevant facts about it. The pieces are then put together in §6 to derive the main theorem of this paper. We warn the reader that the current paper relies heavily on the concepts and results from [15], hence having access to that paper is necessary for the full understanding of our results.

2. Preliminaries

Special trees. Let $T = (T, \leq_T)$ be a tree of height ω_1. Recall that T is called *special* if there is a function $f : T \to \omega$, such that for every $s <_T t$ in T, we have $f(s) \neq f(t)$. If T is a tree of height ω_1, the standard forcing $\mathbb{S}(T)$ to generically specialize T consists of finite partial specializing functions, ordering by reverse inclusion. If T has no cofinal branches then $\mathbb{S}(T)$ has the countable chain condition, see [2]. We say that T is *weakly special* if there is a function $f : T \to \omega$ such that whenever $s, t, u \in T$ and $f(s) = f(t) = f(u)$, if $s <_T t$ and $s <_T u$ then t and u are comparable. Baumgartner [3, Theorem 7.3] proved that for a tree T of height ω_1 which has no branches of length ω_1, T is special iff T is weakly special. He also showed [3, Theorem 7.5] that if every tree of height and size ω_1 that has no branches of length ω_1 is special, then every tree of height and size ω_1 that has at most ω_1 branches is weakly special. We recall the following well-known fact, see [6, Proposition 4.3] for a proof.

Proposition 2.1. *If $V \subseteq W$ are models of set theory with $\omega_1^V = \omega_1^W$, $T \in V$ is a tree of height ω_1 which is weakly special in V, then every branch of T in W of length ω_1 is in V.*

2.1

Guessing models. Throughout this paper by a model M, we mean a set or a class such that $(M, \in)$ satisfies a sufficient fragment of ZFC. For a set or class M, we say that a set $x \subseteq M$ is *bounded in M* if there is $y \in M$ such that $x \subseteq y$. Let us call a transitive model R *powerful* if it includes every set bounded in R. We say that x is *guessed* in M if there is $x^* \in M$ with $x^* \cap M = x \cap M$. Suppose that γ is an M-regular cardinal. We say x is *γ-approximated* in M if for every $a \in M$ with $|a|^M < \gamma$, we have $a \cap x \in M$. We say that M is *γ-guessing in a model N* if for every $x \in N$ bounded in M, if x is γ-approximated in M, then x is guessed in M.

Definition 2.2 ([18])**.** *Let γ be a regular cardinal. A set M is said to be* γ-guessing *if M is γ-guessing in V.*

Recall from [9] that a pair (M, N) of transitive models with $\gamma \in M \subseteq N$ has the *γ-approximation property*, if M is γ-guessing in N.

Theorem 2.3 (Krueger [10])**.** *Let $M_0 \subseteq M_1 \subseteq M_2$ be transitive models. If (M_0, M_1) and (M_1, M_2) have the ω_1-approximation property, then so does (M_0, M_2).*

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Theorem 2.4 (Cox–Krueger [5]). *Suppose that R is a powerful model and $M \prec R$. Let also γ be a regular cardinal in M with $\gamma \subseteq M$. Then the following are equivalent.*

(1) M is a γ-guessing model.
(2) The pair $(\overline{M}, V)$ has the γ-approximation property.

2.4

Indestructible and special guessing models. The notion of an indestructible guessing model was introduced by Cox and Krueger [6].

Definition 2.5. *An ω_1-guessing model M of size ω_1 is* indestructible *if M remains ω_1-guessing in any outer universe with the same ω_1.*

Let us first recall that a set X of size ω_1 is *internally unbounded* if $X \cap \mathcal{P}_{\omega_1}(X)$ is cofinal in $\mathcal{P}_{\omega_1}(X)$, and is *internally club* if $X \cap \mathcal{P}_{\omega_1}(X)$ contains a closed unbounded subset of $\mathcal{P}_{\omega_1}(X)$. It is easily seen that if M is an ω_1-sized elementary submodel of a powerful model R, then M is internally unbounded (I.U.) if and only if there is an I.U. sequence for M, i.e., a $\subseteq$-increasing and $\in$-sequence $\langle M_\alpha : \alpha < \omega_1 \rangle$ of countable sets with union M, and that M is internally club (I.C.) if and only if there is an I.C. sequence for M, i.e. a $\subseteq$-continuous I.U. sequence for M.

It was shown by Krueger [10] that if R is a powerful model and $M \prec R$ with $\omega_1 \subseteq M$ is an ω_1-guessing model of size ω_1, then M is internally unbounded.

Let us fix an I.U. sequence $\langle M_\alpha : \alpha < \omega_1 \rangle$ for an I.U. ω_1-guessing model M of size ω_1. For every $\alpha < \omega_1$, let

$$T_\alpha(M) \coloneqq \{(z, f) \in M : f : z \cap M_\alpha \to 2\}.$$

Let also

$$T(M) = \bigcup_{\alpha < \omega_1} T_\alpha(M).$$

We order $T(M)$ by $(z, f) \leq (z', f')$ if and only if $z = z'$ and $f \subseteq f'$.

Notice that $T(M) \subseteq M$, and hence $|T(M)| \leq \omega_1$. Clearly $T(M)$ is a tree of height ω_1. Assume that b is an uncountable branch through $T(M)$. Then there is $z \in M$ such that for all $(z', f') \in b$, we have $z' = z$. Let

$$F = \bigcup \{f : (z, f) \in b\}.$$

Then $F : z \cap M \mapsto 2$ is a function which is ω_1-approximated in M. Since M is an ω_1-guessing model, there is $F^* \in M$ with $F \restriction (z \cap M) = F^* \restriction (z \cap M)$. Given F^*, we can recover the branch b, so $T(M)$ has at most ω_1 branches. By Proposition 2.1 we now have the following [6, Proposition 4.4].

Proposition 2.6 (Cox-Krueger [6]). *Suppose M is an I.U. ω_1-guessing model of cardinality ω_1. Then there is a tree T of size and height ω_1 with ω_1 cofinal branches such that T being weakly special implies that M is an indestructible ω_1-guessing.*

2.6

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Strong properness.

Definition 2.7 (Mitchell [13])**.** *Let $\mathbb{P}$ be a forcing notion and A a set. We say that $p \in \mathbb{P}$ is* $(A, \mathbb{P})$-strongly generic *if for every $q \leq p$ there is a condition $q \restriction A \in A$ such that any $r \in A$ with $r \leq q \restriction A$ is compatible with q.*

Definition 2.8. *Let $\mathbb{P}$ be a forcing notion, and $\mathcal{S}$ be a collection of sets. We say that $\mathbb{P}$ is* strongly proper *for $\mathcal{S}$ if for every $A \in \mathcal{S}$ and $p \in A \cap \mathbb{P}$, there exists an $(A, \mathbb{P})$-strongly generic condition $q \leq p$.*

A forcing notion $\mathbb{P}$ is called strongly proper, if for every sufficiently large regular cardinal θ there is a club $\mathcal{C}$ in $\mathcal{P}_{\omega_1}(H_\theta)$ such that $\mathbb{P}$ is strongly proper for $\mathcal{C}$.

Definition 2.9. *A forcing notion $\mathbb{P}$ has the γ-approximation property if for every V-generic filter $G \subseteq \mathbb{P}$, the pair $(V, V[G])$ has the γ-approximation property.*

The following are standard.

Lemma 2.10. *Every strongly proper forcing has the ω_1-approximation property.*

$\boxed{2.10}$

Lemma 2.11. *Suppose R is a powerful model, $M \prec R$ is an ω_1-guessing model, and that $\mathbb{P}$ is a forcing with the ω_1-approximation property. Then M remains ω_1-guessing in $V^{\mathbb{P}}$.*

Proof. It follows easily from Theorems 2.3 and 2.4. $\boxed{2.11}$

Magidor models. The following definition is motivated by Magidor's reformulation of a supercompact cardinal, see [12].

Definition 2.12 (Magidor model, [15])**.** *We say that a model M is a κ-*Magidor model *if, letting $\overline{M}$ be the transitive collapse of M and π the collapsing map, $\overline{M} = V_{\bar{\gamma}}$, for some $\bar{\gamma} < \kappa$ with $\mathrm{cof}(\bar{\gamma}) \geq \pi(\kappa)$, and $V_{\pi(\kappa)} \subseteq M$. If κ is clear from the context, we omit it.*

Proposition 2.13. *Suppose κ is supercompact and $\mu > \kappa$ with $\mathrm{cof}(\mu) \geq \kappa$. Then the set of κ-Magidor models is stationary in $\mathcal{P}_\kappa(V_\mu)$.*

Proof. See [15, Proposition 3.21]. $\boxed{2.13}$

Some lemmata.

Lemma 2.14 (Neeman [7])**.** *Suppose that M is a 0-guessing model which is sufficiently elementary in some transitive model A. Suppose that $\mathbb{P} \in M$ is a forcing. Then every $(M, \mathbb{P})$-generic condition is $(M, \mathbb{P})$-strongly generic.*

$\boxed{2.14}$

We need the following

Lemma 2.15. *Let $\mathbb{P} \in H_\theta$ be a forcing with the ω_1-approximation property. Assume that $\mathcal{P}(\mathbb{P}) \in H_\theta$. Suppose $M \prec H_\theta$ is countable and $\mathbb{P} \in M$. Let $Z \in M$ and let $f : M \cap Z \to 2$ be a function that is not guessed in M. If $p \in \mathbb{P}$ is $(M, \mathbb{P})$-generic, then*

$$p \Vdash \text{“}\check{f} \text{ is not guessed in } M[\dot{G}].\text{”}$$

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Proof. Pick a V-generic filter $G \subseteq \mathbb{P}$ containing p. Assume towards a contradiction that for some function $g : Z \to 2$ in $M[G]$, $g \restriction M \cap Z = f$. Suppose $x \in [Z]^\omega \cap M$. Then $x \subseteq M$ and hence $g \restriction x = f \restriction x \in V$ as $M[G] \cap V = M$. We have that $g \restriction x \in M$. Now by the elementarity of $M[G]$ in $H_\theta[G]$ and the definability of V in $V[G]$ (see [11, 20],) we have g is countably approximated in V. Since $\mathbb{P}$ has the ω_1-approximation property, it follows that $g \in V$ and hence $g \in M$. Therefore f is guessed in M, which is a contradiction. 2.15

Lemma 2.16. *Let $\mathbb{P}$ be a forcing. Assume $\dot{f} : Z \to 2$ is a $\mathbb{P}$-name for a function and M is a model with $\dot{f}, Z \in M$. Let $p \in \mathbb{P}$ be an $(M, \mathbb{P})$-generic condition that forces $\dot{f} \restriction M \cap Z = \check{g}$, for some function g. If g is guessed in M, then p decides $\dot{f}$.*

Proof. Suppose that $h \in M$ is a function so that $h \restriction M \cap Z = g$, we will show that $p \Vdash$ "$\dot{f} = \check{h}$". Fix $q \leq p$. We claim that there is a condition below q that forces $\dot{f} = \check{h}$. If not, the set

$$D = \{r \in \mathbb{P} : \exists \zeta \in Z \text{ such that } r \Vdash \check{h}(\zeta) \neq \dot{f}(\zeta)\}$$

that belongs to M is pre-dense below q. Since q is $(M, \mathbb{P})$-generic, there is $r \in D \cap M$ compatible with q. So there exists $\zeta \in M \cap Z$ such that $r \Vdash$ "$\dot{f}(\zeta) \neq \check{h}(\zeta) = \check{g}(\zeta)$". This is a contradiction, since r is compatible with q and q forces $\dot{f}(\zeta) = \check{g}(\zeta)$. 2.16

Lemma 2.17. *Assume $M \prec H_\theta$ is countable and $Z \in M$. Suppose that $z \mapsto f_z$ is a function on $[Z]^\omega$ in M so that f_z is a function with $z \subseteq \mathrm{dom}(f_z)$. Let $f : M \cap Z \to 2$ be a function that is not guessed in M. Let $B \in M$ be a cofinal subset of $[Z]^\omega$. Then there is $B^* \in M$ which is a cofinal subset of B such that for every $z \in B^*$, $f_z \nsubseteq f$.*

Proof. For every $\zeta \in Z$, let

$$A^\epsilon_\zeta = \{z \in B : f_z(\zeta) = \epsilon\}, \text{ where } \epsilon = 0, 1.$$

Note that the sequence

$$\langle A^\epsilon_\zeta : \zeta \in Z, \epsilon \in \{0, 1\}\rangle$$

belongs to M. We are done if there is some $\zeta \in Z$ such that both A^0_ζ and A^1_ζ are cofinal in B, as then one can find by elementarity such $\zeta \in M \cap Z$ and let $B^* = A^{1-f(\zeta)}_\zeta$. Therefore, let us assume that for every $\zeta \in Z$, there is a unique $\epsilon \in \{0, 1\}$ such that A^ϵ_ζ is cofinal in B. Now define h on Z by letting $h(\zeta)$ be ϵ if and only if A^ϵ_ζ is cofinal in $[Z]^\omega$. Obviously h belongs to M. But $h \restriction M \cap Z \neq f$, since f is not guessed in M. So there must exist $\zeta \in M \cap Z$ such that $h(\zeta) \neq f(\zeta)$, which in turn implies that $A^{1-f(\zeta)}_\zeta$ is a cofinal subset of B and belongs to M. Let B^* be $A^{1-f(\zeta)}_\zeta$. 2.17

Definition 2.18. *Let T be a tree of height ω_1. We say that $p \in \mathbb{S}(T)$ if p is a partial function,* $\mathrm{dom}(p)$ *is a finite subset of T, $p : \mathrm{dom}(p) \to \omega$, and if $x, y \in \mathrm{dom}(p)$ are distinct and $p(x) = p(y)$, then x and y are incomparable in T. The order on $\mathbb{S}(T)$ is the reverse inclusion.*

It is well-known that if T has no uncountable branches, then $\mathbb{S}(T)$ has the ccc. We also need the following fact which was shown by Chodounský-Zapletal [4]. We provide a proof for completeness.

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Proposition 2.19 (Chodounský–Zapletal [4])**.** *Suppose T is a tree of height ω_1 without uncountable branches. Then $\mathbb{S}(T)$ has the ω_1-approximation property.*

Proof. Suppose μ is a cardinal and $\dot{f}$ is an $\mathbb{S}(T)$-name for a function from μ to 2 which is countably approximated in V. It suffices to show that some condition in $\mathbb{S}(T)$ decides $\dot{f}$. Fix a sufficiently large regular cardinal θ and a countable $M \prec H_\theta$ containing all the relevant objects. Since $\dot{f}$ is forced to be countably approximated in V we can pick a condition $p \in \mathbb{S}(T)$ and a function $g : M \cap \mu \to 2$ such that p forces that $\dot{f} \restriction (M \cap \mu) = \check{g}$. Since $\mathbb{S}(T)$ is ccc, p is $(M, \mathbb{S}(T))$-generic. Therefore, by Lemma 2.16 we may assume that g is not guessed in M. Let $d = \mathrm{dom}(p)$, let $d_0 = d \cap M$, and $p_0 = p \restriction d_0$. Let $n = |d \setminus d_0|$. Let C be the collection of $x \in [T \cup \mu]^\omega$ such that $d_0 \subseteq x$ and $T \cap x$ is an initial segment of T under $\leq_T$. If $x \in C \cap M$, then there is a condition $p_x \leq p_0$, and a function $g_x : \mu \cap x \to 2$ such that $p_x \restriction x = p_0$, $|p_x \setminus p_0| = n$, and p_x forces that $\dot{f} \restriction (x \cap \mu)$ equals g_x. This is witnessed by the condition p. By the elementarity of M such a pair (p_x, g_x) exists, for all $x \in C$. Moreover, we can pick the assignment $x \mapsto (p_x, g_x)$ in M. Let $d_x = \mathrm{dom}(p_x)$. By Lemma 2.17 we can find $C^* \in M$, a cofinal subset of C under inclusion such that $g_x \nsubseteq g$, for all $x \in C^*$. This implies that p_x and p are incompatible, for all $x \in C^* \cap M$. Let us fix an enumeration $d \setminus d_0 = \{t(0), \ldots, t(n-1)\}$, and, for each $x \in C^*$, an enumeration $d_x \setminus d_0 = \{t_x(0), \ldots, t_x(n-1)\}$.

Claim 2.20. *Let $B \in M$ be a cofinal subset of C^*, and $i, j < n$. Then there is $B^* \in M$, a cofinal subset of B, such that for every $x \in B^* \cap M$, $t(i)$ and $t_x(j)$ are incomparable.*

Proof. For $t \in T$, let $\mathrm{pred}_T(t)$ be the set of predecessors of t in T, and let h_t be the characteristic function of $\mathrm{pred}_T(t)$ as a subset of T. Let $h = h_{t(i)} \restriction (T \cap M)$. For $x \in B$ let $h_x = h_{t_x(j)} \restriction (T \cap x)$. Suppose first that h is not guessed in M. Then by Lemma 2.17 we can find $B^* \in M$, a cofinal subset of B, such that $h_x \nsubseteq h$, for all $x \in B^*$. This implies that if $x \in B^* \cap M$ then $t(i)$ and $t_x(j)$ do not have the same predecessors in $T \cap x$ and hence are incomparable, as desired. Suppose now that h is guessed in M and let $k \in M$ be a function such that $k \restriction (T \cap M) = h$. Since $h^{-1}(1)$ is a chain and is closed downwards in T, the same is true for $k^{-1}(1)$. T has no uncountable chains, so $k^{-1}(1)$ is countable and hence we must have $k^{-1}(1) \subseteq M$. Moreover, $k^{-1}(1) = \mathrm{pred}_T(t(i)) \cap M$. By the elementarity of M there is $s \in M$ such that $k^{-1}(1) = \mathrm{pred}_T(s)$. Now, let B^* be the set of $x \in B$ such that $s \in x$. Then $B^* \in M$ is a cofinal subset of B. If $x \in B^* \cap M$, then since $t_x(j) \notin x$, it follows that $t_x(j) \notin \mathrm{pred}_T(s)$. It follows that $t_x(j)$ is not below $t(i)$. Also, $t_x(j)$ cannot be above $t(i)$ in T, since then $t_x(j)$ and all its predecessors in T are in M and $t(i)$ is not in M. It follows that B^* is as required. 2.20

Now, applying the above claim repeatedly for all pairs $i, j < n$ we can find a cofinal subset B of C^* such that, for all $x \in B \cap M$, every element of $d_x \setminus d_0$ is incomparable with every element of $d \setminus d_0$. It follows that if $x \in B \cap M$ then $p_x \cup p \in \mathbb{S}(T)$, which is a contradiction.

2.19

3. The main theorems

We recall the guessing model principle introduced by Viale and Weiß [19]. We let $\mathrm{GM}(\kappa, \gamma)$ be the statement that $\mathrm{GM}(\kappa, \gamma, H_\theta)$ holds, for all sufficiently large regular θ,

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where $\mathrm{GM}(\kappa, \gamma, H_\theta)$ states that

$$\mathfrak{G}_{\kappa,\gamma}(H_\theta) = \{M \in \mathcal{P}_\kappa(H_\theta) : M \prec H_\theta \text{ and } M \text{ is } \gamma\text{-guessing}\}$$

is stationary in $\mathcal{P}_\kappa(H_\theta)$.

Definition 3.1 (Cox–Krueger [6]). $\mathrm{SGM}(\omega_2, \omega_1)$ *states that for every sufficiently large regular cardinal* θ *the following set is stationary in* $\mathcal{P}_{\omega_2}(H_\theta)$.

$$\mathfrak{G}_{\omega_2,\omega_1}(H_\theta) = \{M \in \mathcal{P}_{\omega_2}(H_\theta) : M \prec H_\theta \textit{ and } M \textit{ is indestructibly } \omega_1\textit{-guessing}\}$$

Notice that it was proved in [6] that Suslin's Hypothesis follows from $\mathrm{SGM}(\omega_2, \omega_1)$, but not from $\mathrm{GM}(\omega_2, \omega_1)$.

Let us now recall the notion of a strongly γ-guessing model introduced and studied by the authors in [15].

Definition 3.2. *Let* $\gamma \leq \kappa$ *be regular uncountable cardinals. A model* M *of cardinality* κ^+ *is called* strongly γ-guessing *if it is the union of an* $\in$*-increasing chain* $\langle M_\xi : \xi < \kappa^+\rangle$ *of* γ*-guessing models of cardinality* κ *such that* $M_\xi = \bigcup\{M_\eta : \eta < \xi\}$*, for every* ξ *of cofinality* κ.

It was proved in Remark 2.14 of [15] that every strongly γ-guessing model is γ-guessing.

Definition 3.3. $\mathrm{GM}^+(\kappa^{++}, \gamma)$ *is the statement that* $\mathrm{GM}^+(\kappa^{++}, \gamma, H_\theta)$ *holds, for all sufficiently large regular* θ*, where* $\mathrm{GM}^+(\kappa^{++}, \gamma, H_\theta)$ *states that*

$$\mathfrak{G}^+_{\kappa^{++},\gamma}(H_\theta) = \{M \in \mathcal{P}_{\kappa^{++}}(H_\theta) : M \prec H_\theta \textit{ and } M \textit{ is strongly } \gamma\textit{-guessing}\}.$$

is stationary in $\mathcal{P}_{\kappa^{++}}(H_\theta)$.

Definition 3.4. *A model* M *of cardinality* ω_2 *is* indestructibly strongly ω_1-guessing *if it is the union of an increasing chain* $\langle M_\xi : \xi < \omega_2\rangle$ *of indestructibly* ω_1*-guessing models of cardinality* ω_1 *such that* $M_\xi = \bigcup\{M_\eta : \eta < \xi\}$*, for every* ξ *of cofinality* ω_1.

Definition 3.5. *The principle* $\mathrm{SGM}^+(\omega_3, \omega_1)$ *states that* $\mathfrak{G}^+_{\omega_3,\omega_1}(H_\theta)$ *is stationary, for all large enough* θ*, where*

$$\mathfrak{G}^+_{\omega_3,\omega_1}(H_\theta) \coloneqq \{M \in \mathcal{P}_{\omega_3}(H_\theta) : M \prec H_\theta \textit{ is indestructibly and strongly } \omega_1\textit{-guessing}\}.$$

Definition 3.6. *For a regular cardinal* κ*, we let* $\mathrm{MP}(\kappa^+)$ *denote the statement that every forcing that adds a new subset of* κ^+ *whose initial segments are in the ground model collapses some cardinal* $\leq 2^\kappa$.

Todorčević [16] showed that if every tree of size and height ω_1 with at most ω_1 cofinal branches is weakly special and $2^{\aleph_0} < \aleph_{\omega_1}$ then $\mathrm{MP}(\omega_1)$ holds. The principle was also studied by Golshani and Shelah in [8], where they showed that $\mathrm{MP}(\kappa^+)$ is consistent, for every regular cardinal κ.

Proposition 3.7 (Cox–Krueger [6]). *If* $\mathrm{SGM}(\omega_2, \omega_1)$ *and* $2^{\aleph_0} < \aleph_{\omega_1}$ *then* $\mathrm{MP}(\omega_1)$ *holds.*

3.7

We shall prove that $\mathrm{SGM}^+(\omega_3, \omega_1)$ implies $\mathrm{MP}(\omega_2)$, and since $\mathrm{SGM}(\omega_2, \omega_1)$ follows from $\mathrm{SGM}^+(\omega_3, \omega_1)$, we get the simultaneous consistency of $\mathrm{MP}(\omega_1)$ and $\mathrm{MP}(\omega_2)$.

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Theorem 3.8. *Suppose that* $V \subseteq W$ *are transitive models of* ZFC. *Assume that* $\mathrm{SGM}^+(\omega_3, \omega_1)$ *and* $2^{\omega_1} < \aleph_{\omega_2}$ *hold in* V. *Suppose that* W *has a subset of* ω_2^V *that does not belong to* V. *Then either* $\mathcal{P}^V(\omega_1) \neq \mathcal{P}^W(\omega_1)$ *or some* V*-cardinal* $\leq 2^{\omega_1}$ *is no longer a cardinal in* W.

Proof. Let $x \in W \setminus V$ be a subset of ω_2^V, and suppose that $\mathcal{P}^V(\omega_1) = \mathcal{P}^W(\omega_1)$. We will show that some cardinal $\leq 2^{\omega_1}$ is no longer a cardinal in W. By $\mathcal{P}^V(\omega_1) = \mathcal{P}^W(\omega_1)$, every initial segment of x belongs to V. Let

$$\mathfrak{X} = \{x \cap \gamma : \gamma < \omega_2\}.$$

Note that $\mathfrak{X}$ is bounded in V. Assume towards a contradiction that every V-cardinal $\leq 2^{\omega_1}$ remains a cardinal in W. Working in W, let $\mu \geq \omega_2$ be the least cardinal so that there is a set M in V of cardinality μ such that $M \cap \mathfrak{X}$ is of size ω_2. Thus $\mu \leq 2^{\omega_1}$.

Claim 3.9. $\mu = \omega_2$.

Proof. Assume towards a contradiction that $\mu > \omega_2$ and M is a witness to it, then one can work in V and write M as the union of an increasing sequence $\langle M_\xi : \xi < \mathrm{cof}^{\mathrm{V}}(\mu)\rangle$ of subsets of M in V whose size are less than μ. Since $\mu \leq 2^{\omega_1} < \aleph_{\omega_2}$ and every cardinal $\leq 2^{\omega_1}$ is a cardinal in W, we have $\mathrm{cof}^W(\mu) = \mathrm{cof}^V(\mu) \neq \omega_2$. Thus either μ is of cofinality at most ω_1, and then by the pigeonhole principle, there is $\xi < \mathrm{cof}(\mu)$ such that $|M_\xi \cap \mathfrak{X}| = \omega_2$, or μ is regular, and then there is some $\xi < \mu = \mathrm{cof}(\mu)$ such that $M \cap \mathfrak{X} \subseteq M_\xi$. In either case, we get a contradiction since $|M_\xi| < \mu$. 3.9

By the above claim, $\mu = \omega_2$. Let M be a witness for $\mu = \omega_2$, and let $\mathfrak{X}' = M \cap \mathfrak{X}$. Notice that $V \models |M| = \omega_2$. Since M is in V and V satisfies $\mathrm{SGM}^+(\omega_3, \omega_1)$, one can cover M by an indestructibly strongly ω_1-guessing model N of size ω_2. Working in W, x is countably approximated in N, since if $\gamma \in N \cap \omega_2$, then there is $\gamma' > \gamma$ in N such that $x \cap \gamma' \in \mathfrak{X}' \subseteq N$, and hence $x \cap \gamma \in N$. On the other hand, N is a guessing model in W by $\mathrm{SGM}^+(\omega_3, \omega_1)$ in V and both ω_1 and ω_2 are cardinals in W. Thus x is guessed in N, but then x must be in N since $\omega_2 \subseteq N$. Therefore, x is in V, which is a contradiction! 3.8

The following corollaries are immediate.

Corollary 3.10. *Suppose that* $V \subseteq W$ *are transitive models of* ZFC. *Assume that* $\mathrm{SGM}^+(\omega_3, \omega_1)$, $2^{\omega_0} < \aleph_{\omega_1}$ *and* $2^{\omega_1} < \aleph_{\omega_2}$ *hold in* V. *Suppose that* W *has a new subset of* ω_2^V. *Then either* W *contains a real which is not in* V *or some cardinal* $\leq 2^{\omega_1}$ *in* V *is collapsed in* W.

3.10

Corollary 3.11. *Assume* $2^{\aleph_0} < \aleph_{\omega_1}$ *and* $2^{\aleph_1} < \aleph_{\omega_2}$ *hold. Then* $\mathrm{SGM}^+(\omega_3, \omega_1)$ *implies both* $\mathrm{MP}(\omega_1)$ *and* $\mathrm{MP}(\omega_2)$.

3.11

Our main theorem reads as follows.

Theorem 3.12 (Main Theorem)**.** *Assume there are two supercompact cardinals. There is a generic extension* $V[G]$ *of* V *in which* $\mathrm{SGM}^+(\omega_3, \omega_1)$ *and* $2^{\aleph_0} = 2^{\aleph_1} = 2^{\aleph_2} = \aleph_3$ *hold.*

3.12

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4. An overview of virtual models

We review the theory of virtual models from the authors' paper [15].

The general setting. We consider the language $\mathcal{L}$ obtained by adding a single constant symbol c and a predicate U to the standard language of set theory. Let us say that an $\mathcal{L}$-structure $\mathcal{A} = (A, \in, \kappa, U)$ is *suitable* if A is a transitive set and $\mathcal{A} \models$ "ZFC" in the expanded language, where $\kappa = c^{\mathcal{A}}$ is an uncountable regular in $\mathcal{A}$. For a suitable structure $\mathcal{A}$ and an ordinal $\alpha \in A \setminus \kappa$, let $\mathcal{A}_\alpha = (A \cap V_\alpha, \in, \kappa, U \cap V_\alpha)$. Finally, we let

$$E_{\mathcal{A}} = \{\alpha \in \mathrm{ORD}^A : \mathcal{A}_\alpha \prec \mathcal{A}\}.$$

Note that E_A is a closed subset of $\mathrm{ORD}^{\mathcal{A}}$. It is not definable in $\mathcal{A}$, but $E_{\mathcal{A}} \cap \alpha$ is uniformly definable in $\mathcal{A}$ with parameter α, for each $\alpha \in E_{\mathcal{A}}$. For $\alpha \in E_{\mathcal{A}}$, we let $\mathrm{next}_{\mathcal{A}}(\alpha)$ be the least ordinal in $E_{\mathcal{A}}$ above α, if such an ordinal exists. Otherwise, we leave $\mathrm{next}_{\mathcal{A}}(\alpha)$ undefined. We shall often abuse notation and refer to the structure $\mathcal{A}$ by A.

Lemma 4.1. *Suppose M is an elementary submodel of a suitable structure A, and that $\alpha \in E_A$. If $(M \cap \mathrm{ORD}^A) \setminus \alpha \neq \varnothing$, then $\min(M \cap \mathrm{ORD}^A \setminus \alpha) \in E_A$.*

Proof. See [17, Lemma 1.1] or [15, Lemma 3.1]. 4.1

If M is a submodel of a suitable structure A and X is a subset of A, let the operation Hull be defined by

$$\mathrm{Hull}(M, X) = \{f(\bar{x}) : f \in M, \bar{x} \in X^{<\omega}, f \text{ is a function, and } \bar{x} \in \mathrm{dom}(f)\}.$$

For $\alpha \in E_A$, we let $\mathscr{A}_\alpha$ denote the set of all transitive $\mathcal{L}$-structures $\mathcal{H} \in A$ which are elementary extensions of $\mathcal{A}_\alpha$ and have the same cardinality as A_α. Note that if $\mathcal{H} \in \mathscr{A}_\alpha$ and $\alpha \in \mathcal{H}$, then $E_{\mathcal{H}} \cap \alpha = E_A \cap \alpha$. For $\mathcal{H} \in \mathscr{A}_\alpha$, we will refer to A_α as the *standard part* of $\mathcal{H}$. Note that if $\mathcal{H}$ has nonstandard elements, then $\alpha \in E_{\mathcal{H}}$.

Lemma 4.2. *Suppose M is an elementary submodel of a suitable structure A, and that $X \subseteq A$. Let $\delta = \sup(M \cap \mathrm{ORD}^A)$, and suppose that $X \cap A_\delta$ is nonempty. Then $\mathrm{Hull}(M, X)$ is the least elementary submodel of A containing M and $X \cap A_\delta$ as subsets.*

Proof. See [17, Lemma 1.2] or [15, Lemma 3.3]. 4.2

Virtual models in V_λ. Assume that $\kappa < \lambda$ are supercompact and inaccessible cardinals, respectively. Consider the $\mathcal{L}$-structure

$$V_\lambda \coloneqq (V_\lambda, \in, \kappa, U).$$

Let $E = E_{V_\lambda}$ and $\mathrm{next}(\alpha) = \mathrm{next}_{V_\lambda}(\alpha)$.

Definition 4.3. *Suppose $\alpha \in E$. We let $\mathscr{V}_\alpha$ denote the collection of all substructures M of V_λ of size less than κ so that if we let $A = \mathrm{Hull}(M, V_\alpha)$, then $A \in \mathscr{A}_\alpha$ and M is an elementary submodel of A. The members of $\mathscr{V}_\alpha$ are called α-*models.

We write $\mathscr{V}_{<\alpha}$ for $\bigcup\{\mathscr{V}_\gamma : \gamma \in E \cap \alpha\}$ and $\mathscr{V}$ for $\mathscr{V}_{<\lambda}$. The collections $\mathscr{V}_{\leq\alpha}$ and $\mathscr{V}_{\geq\alpha}$ are defined in the obvious way. For $M \in \mathscr{V}$, $\eta(M)$ denotes the unique ordinal α such that $M \in \mathscr{V}_\alpha$. Note that if $M \in \mathscr{V}_\alpha$, then $\sup(M \cap \mathrm{ORD}) \geq \alpha$. In general, M is not elementary in V_λ, in fact, this only happens if $M \subseteq V_\alpha$. In this case, we will say that M

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is a *standard α-model*. We refer to the members of $\mathscr{V}$ as *virtual models*. We also refer to members of $\mathscr{V}^A$, for some suitable structure A with $A \subseteq V_\lambda$, as *general virtual models*.

Suppose $M, N \in \mathscr{V}$ and $\alpha \in E$. We say that an isomorphism $\sigma : M \to N$ is an *α-isomorphism* if it has an extension to an isomorphism $\bar{\sigma} : \mathrm{Hull}(M, V_\alpha) \to \mathrm{Hull}(N, V_\alpha)$. We say that M and N are *α-isomorphic* and write $M \cong_\alpha N$ if there is an α-isomorphism between them. Note that σ and $\bar{\sigma}$, if they exist, are unique.

It is clear that $\cong_\alpha$ is an equivalence relation, for every $\alpha \in E$. If $\alpha < \beta$ are in E, then for every β-model M there is a canonical representative of the $\cong_\alpha$-equivalence class of M which is an α-model.

Definition 4.4. *Suppose α and β are members of E and M is a β-model. Let $\overline{\mathrm{Hull}(M, V_\alpha)}$ be the transitive collapse of $\mathrm{Hull}(M, V_\alpha)$, and let π be the collapsing map. We define $M \restriction \alpha$ to be $\pi[M]$, i.e., the image of M under the collapsing map of $\mathrm{Hull}(M, V_\alpha)$.*

Note that if $A \in \mathscr{A}_\alpha$ then $\mathscr{V}_\alpha^A \subseteq \mathscr{V}_\alpha$. Therefore, if $A, B \in \mathscr{A}_\alpha$, $M \in \mathscr{V}^A$, and $N \in \mathscr{V}^B$, we can still write $M \cong_\alpha N$ if $M \restriction \alpha = N \restriction \alpha$. It is straightforward to check that if $\alpha \leq \beta$ are in E, and $M \in \mathscr{V}$. Then $(M \restriction \beta) \restriction \alpha = M \restriction \alpha$. For each $\alpha \in E$, the virtual version of the membership relation is defined as follows.

Suppose $M, N \in \mathscr{V}$ and $\alpha \in E$. We write $M \in_\alpha N$ if there is $M' \in N$ with $M' \in \mathscr{V}^N$ such that $M' \cong_\alpha M$. If this is the case, we say that M is α-in N.

Lemma 4.5. *Let $\alpha \leq \beta$ be in E. Suppose $M, N \in \mathscr{V}_{\geq\beta}$ and $M \in_\beta N$. Then $M \restriction \alpha \in_\alpha N \restriction \alpha$.*

Proof. See [15, Proposition 3.15]. 4.5

Definition 4.6. *For $\alpha \in E$, we let $\mathscr{C}_\alpha$ denote the collection of countable models in $\mathscr{V}_\alpha$, and we let $\mathscr{U}_\alpha$ be the collection of all $M \in \mathscr{V}_\alpha$ that are κ-Magidor models.*

The collections $\mathscr{C}_{<\alpha}$, $\mathscr{C}_{\leq\alpha}$, $\mathscr{C}_{\geq\alpha}$, $\mathscr{U}_{<\alpha}$, $\mathscr{U}_{\leq\alpha}$, and $\mathscr{U}_{\geq\alpha}$ are defined similarly. We write $\mathscr{C}$ for $\mathscr{C}_{<\lambda}$, and $\mathscr{U}$ for $\mathscr{U}_{<\lambda}$, $\mathscr{C}_{\mathrm{st}}$ for the collection of standard models in $\mathscr{C}$, and $\mathscr{U}_{\mathrm{st}}$ for the standard models in $\mathscr{U}$. Note that both classes $\mathscr{C}$ and $\mathscr{U}$ are closed under projections.

Lemma 4.7. *$\mathscr{C}_{\mathrm{st}}$ contains a club in $\mathcal{P}_{\omega_1}(V_\lambda)$, and $\mathscr{U}_{\mathrm{st}}$ is stationary in $\mathcal{P}_\kappa(V_\lambda)$.*

Proof. See [15, Propositions 3.19 and 3.24]. 4.7

Definition 4.8. *Let $M \in \mathscr{V}$. We say that M is* active *at $\alpha \in E$ if $\eta(M) \geq \alpha$ and $\mathrm{Hull}(M, V_{\kappa_M}) \cap E \cap \alpha$ is unbounded in $E \cap \alpha$, where $\kappa_M = \sup(M \cap \kappa)$. We say that M is* strongly active *at α if $\eta(M) \geq \alpha$ and $M \cap E \cap \alpha$ is unbounded in $E \cap \alpha$.*

One can easily see that a Magidor model is active at α if and only if it is strongly active at α.

Notation 4.9. *For a model $M \in \mathscr{V}$, let $a(M) = \{\alpha \in E : M$ is active at $\alpha\}$ and $\alpha(M) = \max(a(M))$.*

Note that $a(M)$ is a closed subset of E of size at most $|\mathrm{Hull}(M, V_{\kappa_M})|$. We now recall the definition of a *meet* from [15] which is the virtual version of intersection defined only for two models of different types. Suppose $N \in \mathscr{U}$ and $M \in \mathscr{C}$. Let $\overline{N}$ be the transitive collapse of N, and let π be the collapsing map. Note that if $\overline{N} \in M$, then $\overline{N} \cap M$

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is a countable elementary submodel of $\overline{N}$. Then $\overline{N} \cap M \in \overline{N}$ since $\overline{N}$ is closed under countable sequence. Note that $\pi^{-1}(\overline{N} \cap M) = \pi^{-1}[\overline{N} \cap M]$, and this model is elementary in N.

Definition 4.10. *Suppose $N \in \mathscr{U}$ and $M \in \mathscr{C}$. Let $\alpha = \max(a(N) \cap a(M))$. We will define $N \wedge M$ if $N \in_\alpha M$. Let $\overline{N}$ be the transitive collapse of N, and let π be the collapsing map. Set*

$$\eta = \sup(\sup(\pi^{-1}[\overline{N} \cap M] \cap \mathrm{ORD}) \cap E \cap (\alpha + 1)).$$

We define the meet of N and M to be $N \wedge M = \pi^{-1}[\overline{N} \cap M] \restriction \eta$.

Remark 4.11. It was proved in [15] that $N \wedge M \in \mathscr{C}_\eta$ is an η-model, and, furthermore, if $N \wedge M$ is active at γ, then $(N \wedge M) \restriction \gamma = N \restriction \gamma \wedge M \restriction \gamma$. More intersection-like properties of $\wedge$ are found on [15, Pages 14–16].

Notation 4.12. *Let $\alpha \in E$ and let $\mathcal{M}$ be a set of virtual models. We let*

$$\mathcal{M} \restriction \alpha = \{M \restriction \alpha : M \in \mathcal{M}\} \textit{ and } \mathcal{M}^\alpha = \{M \restriction \alpha : \ M \in \mathcal{M} \textit{ is active at } \alpha\}.$$

Definition 4.13. *Let $\alpha \in E$ and let $\mathcal{M}$ be a subset of $\mathscr{U} \cup \mathscr{C}$. We say $\mathcal{M}$ is an α-*chain *if for all distinct $M, N \in \mathcal{M}$, either $M \in_\alpha N$ or $N \in_\alpha M$, or there is a $P \in \mathcal{M}$ such that either $M \in_\alpha P \in_\alpha N$ or $N \in_\alpha P \in_\alpha M$.*

Lemma 4.14. *Suppose $\alpha \in E$ and $\mathcal{M}$ is a finite subset of $\mathscr{U} \cup \mathscr{C}$. Then $\mathcal{M}$ is an α-chain if and only if there is an enumeration $\{M_i : i < n\}$ of $\mathcal{M}$ such that $M_0 \in_\alpha \cdots \in_\alpha M_{n-1}$.*

Proof. See [15, Proposition 3.41]. 4.14

5. Virtual models as side conditions

We recall the definition of our pure side conditions forcing with decorations introduced in [15]. We give its basic properties and refer the reader to the above paper for the proofs. Recall that $\kappa < \lambda$ are supercomapct and inaccessible, respectively.

Definition 5.1 (pure side conditions). *Suppose $\alpha \in E$. We say that $p = \mathcal{M}_p$ belongs to $\mathbb{M}^\kappa_\alpha$ if:*

(1) $\mathcal{M}_p \subseteq \mathscr{C}_{\leq\alpha} \cup \mathscr{U}_{\leq\alpha}$ is finite and closed under meets, and
(2) $\mathcal{M}^\delta_p$ is a δ-chain, for all $\delta \in E \cap (\alpha + 1)$.

We let $\mathcal{M}_q \leq \mathcal{M}_p$ if for all $M \in \mathcal{M}_p$ there is $N \in \mathcal{M}_q$ such that $N \restriction \eta(M) = M$. Finally, let $\mathbb{M}^\kappa_\lambda = \bigcup\{\mathbb{M}^\kappa_\alpha : \alpha \in E\}$ with the same order.

Suppose $\mathcal{M}_p \in \mathbb{M}^\kappa_\lambda$. Let

$$\mathcal{L}(\mathcal{M}_p) = \{M \restriction \alpha : M \in \mathcal{M}_p \text{ and } \alpha \in a(M)\}.$$

We say that $M \in \mathcal{L}(\mathcal{M}_p)$ is $\mathcal{M}_p$-*free* if every $N \in \mathcal{M}_p$ with $M \in_{\eta(M)} N$ is strongly active at $\eta(M)$. Let $\mathcal{F}(\mathcal{M}_p)$ denote the set of all $M \in \mathcal{L}(\mathcal{M}_p)$ that are $\mathcal{M}_p$-free. Note that a model $M \in \mathcal{L}(\mathcal{M}_p)$ that is not $\mathcal{M}_p$-free is not $\mathcal{M}_q$-free, for any $\mathcal{M}_q \leq \mathcal{M}_p$.

Definition 5.2 (side conditions with decorations). *Suppose $\alpha \in E \cup \{\lambda\}$. We say that a pair $p = (\mathcal{M}_p, d_p)$ belongs to $\mathbb{P}^\kappa_\alpha$ if $\mathcal{M}_p \in \mathbb{M}^\kappa_\alpha$, d_p is a finite partial function from $\mathcal{F}(\mathcal{M}_p)$ to $\mathcal{P}_\omega(V_\kappa)$ such that*

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(⁎) *if* $M \in \mathrm{dom}(d_p)$*,* $N \in \mathcal{M}_p$*, and* $M \in_{\eta(M)} N$*, then* $d_p(M) \in N$*.*

We say that $q \leq p$ *if* $\mathcal{M}_q \leq \mathcal{M}_p$ *in* $\mathbb{M}^\kappa_\alpha$*, and, for every* $M \in \mathrm{dom}(d_p)$*, there exists some* $\gamma \in E \cap (\eta(M)+1)$ *with* $M \restriction \gamma \in \mathrm{dom}(d_q)$ *such that* $d_p(M) \subseteq d_q(M \restriction \gamma)$*.*

We call d_p the decoration of p. The order on $\mathbb{P}^\kappa_\lambda$ is transitive. We will say that q is *stronger* than p if q forces that p belongs to the generic filter, in other words, any $r \leq q$ is compatible with p. We write $p \sim q$ if each of p and q is stronger than the other. We identify equivalent conditions, often without saying so. Our forcing does not have the greatest lower bound property, but if p and q have a greatest lower bound, we will denote it by $p \wedge q$. To be precise, we should refer to $p \wedge q$ as the $\sim$-equivalence class of a greatest lower bound, but we ignore this point as it should not cause any confusion. Note that if $p \in \mathbb{P}^\kappa_\alpha$ and $M \in \mathcal{M}_p$ is a δ-model that is not active at δ, we may replace M with $M \restriction \alpha(M)$ and get an equivalent condition. Thus, if $\alpha \in E$ and $\mathrm{cof}(\alpha) \geq \kappa$, then $\mathbb{P}^\kappa_\alpha$ is forcing equivalent to $\bigcup\{\mathbb{P}^\kappa_\gamma : \gamma \in E \cap \alpha\}$. Suppose $\alpha, \beta \in E$ and $\alpha \leq \beta$. For every $p \in \mathbb{P}^\kappa_\beta$, we let $\mathcal{M}_{p\restriction\alpha} = \mathcal{M}_p \restriction \alpha$ and $d_{p\restriction\alpha} = d_p \restriction \mathcal{F}(\mathcal{M}_p \restriction \alpha)$. It is easy to see that $p \restriction \alpha = (\mathcal{M}_{p\restriction\alpha}, d_{p\restriction\alpha}) \in \mathbb{P}^\kappa_\alpha$. The following is straightforward.

Lemma 5.3. *Suppose that* $\alpha, \beta \in E$ *with* $\alpha \leq \beta$*. Let* $p \in \mathbb{P}^\kappa_\beta$ *and let* $q \in \mathbb{P}^\kappa_\alpha$ *be such that* $q \leq p \restriction \alpha$*. Then there exists* $r \in \mathbb{P}^\kappa_\beta$ *such that* $r \leq p, q$*.*

Proof. We let $\mathcal{M}_r = \mathcal{M}_p \cup \mathcal{M}_q$. Note that $\mathcal{M}_r$ is closed under meets. We define d_r by letting $d_r(M) = d_q(M)$ if $M \in \mathrm{dom}(d_q)$, and $d_r(M) = d_p(M)$ if $M \in \mathrm{dom}(d_p)$ with $\eta(M) > \alpha$. It is straightforward that r is as required. 5.3

Remark 5.4. The condition r from the previous lemma is the greatest lower bound of p and q, so we will let $r := p \wedge q$.

Definition 5.5. *Assume* $p \in \mathbb{P}^\kappa_\lambda$*, and that* $M \in \mathscr{C} \cup \mathscr{U}$ *be such that* $p \in M$*. Let* $\mathcal{M}_{p^M}$ *be the closure of* $\mathcal{M}_p \cup \{M\}$ *under meets, and let* d_{p^M} *be defined on*

$$\mathrm{dom}(d_{p^M}) = \{N \restriction \delta : N \in \mathrm{dom}(d_p) \text{ and } \delta = \sup(M \cap \eta(N))\},$$

by letting $d_{p^M}(N \restriction \delta) = d_p(N)$*.*

Lemma 5.6. p^M *is the weakest condition extending* p *with* $M \in \mathcal{M}_{p^M}$*.*

Proof. [15, Lemma 4.11]. 5.6

Notation 5.7. *For virtual models* N, M*, we set* $\alpha(N, M) = \max(a(N) \cap a(M))$*.*

We are now about to give the restriction of a condition to a given model. We start with Magidor models.

Definition 5.8. *Let* $p \in \mathbb{P}^\kappa_\lambda$*. Assume that* $M \in \mathcal{L}(\mathcal{M}_p)$ *is a Magidor model. For* $N \in \mathcal{M}_p$*, we let* $N \restriction M = N \restriction \alpha(N, M)$ *if* $\kappa_N < \kappa_M$*, otherwise* $N \restriction M$ *is undefined. Let* $p \restriction M$ *be defined by*

$$\mathcal{M}_{p\restriction M} = \{N \restriction M : N \in \mathcal{M}_p\} \text{ and } d_{p\restriction M} = d_p \restriction (\mathrm{dom}(d_p) \cap M).$$

Proposition 5.9. *Suppose* $p \in \mathbb{P}^\kappa_\lambda$ *and* $M \in \mathcal{L}(\mathcal{M}_p)$ *is a Magidor model. Then* $p \restriction M \in \mathbb{P}^\kappa_\lambda \cap M$ *and* $p \leq p \restriction M$*. Moreover, if* $q \in M \cap \mathbb{P}^\kappa_\lambda$ *extends* $p \restriction M$*. Then* q *is compatible with* p *and the meet* $p \wedge q$ *exists.*

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Proof. See [15, Lemmata 4.13 and 4.14]. $\boxed{5.9}$

As a corollary, we have that the forcing $\mathbb{P}^\kappa_\lambda$ is strongly proper for $\mathscr{U}$. We now define an analogue of $p \restriction M$ for countable models $M \in \mathcal{L}(\mathcal{M}_p)$. Now suppose $p \in \mathbb{P}^\kappa_\alpha$ and $M \in \mathcal{M}_p$ is either a standard countable model or a countable model with $\alpha \in M$. Suppose $N \in \mathcal{M}_p$ and $N \in_\gamma M$, where $\gamma = \max(a(M) \cap a(N))$. If $\gamma \in M$, then $N \restriction \gamma \in M$, but it may be that $\gamma \notin M$, but then, if we let $\gamma^* = \min(M \cap \lambda \setminus \gamma)$, we have that $\gamma^* \in E \cap M$ by Lemma 4.1. Thus there is a γ^*-model $N^* \in M$ which is γ-isomorphic to $N \restriction \gamma$. Moreover, such N^* is unique[1].

Definition 5.10. *Let $p \in \mathbb{P}^\kappa_\alpha$. Assume that $M \in \mathcal{L}(\mathcal{M}_p)$ is a countable model that is either standard or contains α. Suppose that $N \in \mathcal{M}_p$, and let $\gamma = \alpha(M, N)$. If $N \in_\gamma M$, we let $\gamma^* = \min(M \cap \lambda \setminus \gamma)$. We define $N \restriction M$ to be the unique γ^*-model $N^* \in M$ such that $N^* \cong_\gamma N$. Otherwise, $N \restriction M$ is undefined. Let*

$$\mathcal{M}_{p\restriction M} = \{N \restriction M : N \in \mathcal{M}_p\} \textit{ and}$$

$$\mathrm{dom}(d_{p\restriction M}) = \{N \restriction M : N \in \mathrm{dom}(d_p) \textit{ and } N \in_{\eta(N)} M\}.$$

If $N \in \mathrm{dom}(d_p)$ and $N \in_{\eta(N)} M$, we then let $d_{p\restriction M}(N \restriction M) = d_p(N)$. Let also

$$p \restriction M \coloneqq (\mathcal{M}_{p\restriction M}, d_{p\restriction M}).$$

Remark 5.11. Suppose $N \in \mathrm{dom}(d_p)$ and let $\eta = \eta(N)$. If $N \in_\eta M$ then M is strongly active at η since N is $\mathcal{M}_p$-free. If $\eta \in M$ then we put N in $\mathrm{dom}(d_{p\restriction M})$ and keep the same decoration at N. If $\eta \notin M$ we lift N to the least level η^* of M above η, we put the resulting model N^* in $\mathrm{dom}(d_{p\restriction M})$ and copy the decoration of N to N^*. If $P \in \mathcal{M}_p$ is such that $P \restriction \eta = N$ then $(P \restriction M) \restriction \eta^* = N^*$. Moreover, we can recover N from N^* as $N^* \restriction \sup(\eta^* \cap M)$. Thus, the function $d_{p\restriction M}$ is well-defined. Note also that $p \restriction M \in M$.

Proposition 5.12. *Let $p \in \mathbb{P}^\kappa_\alpha$ and let $M \in \mathscr{C}$ such that $\alpha \in M$ and $M \restriction \alpha \in \mathcal{M}_p$. Then $p \restriction M \in \mathbb{P}^\kappa_\alpha$, and for any $q \in \mathbb{P}^\kappa_\alpha \cap M$ with $q \le p \restriction M$, p and q are compatible, and the meet $p \wedge q$ exists.*

Proof. See [15, Lemma 4.26]. $\boxed{5.12}$

Note that if $p, q \in \mathbb{P}^\kappa_\lambda$ are such that $q \le p$ and $M \in \mathcal{M}_p$ then $q \restriction M \le p \restriction M$. It follows from Lemma 5.6 and Proposition 5.12 that $\mathbb{P}^\kappa_\lambda$ is strongly proper for $\mathscr{C}_{\mathrm{st}}$, and that $\mathbb{P}^\kappa_\alpha$ is strongly proper for the collection of all models $M \in \mathscr{C}$ with $\alpha \in M$.

For a filter F in $\mathbb{P}^\kappa_\lambda$, we set $\mathcal{M}_F \coloneqq \bigcup\{\mathcal{M}_p : \ p \in F\}$, and for a V-generic filter G in $\mathbb{P}^\kappa_\lambda$, we let $G_\alpha = G \cap \mathbb{P}^\kappa_\alpha$, for all $\alpha \in E$. The following is easy.

Lemma 5.13. *For every $\delta \in E$ with $\mathrm{cof}(\delta) < \kappa$, $\mathcal{M}^\delta_G$ is a δ-chain.*

$\boxed{5.13}$

Proposition 5.14. *Let $\delta \in E$ with $\mathrm{cof}(\delta) < \kappa$. Suppose $M \in \mathcal{M}^\delta_G$ is a Magidor model that is not the least model in $\mathcal{M}^\delta_G$. Then*

$$M \cap V_\delta = \bigcup\{Q \cap V_\delta : Q \in_\delta M \textit{ and } Q \in \mathcal{M}^\delta_G\}.$$

Proof. See [15, Proposition 4.32]. $\boxed{5.14}$

[1]See the paragraph above Definition 4.21 of [15]

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Proposition 5.15. *The forcing $\mathbb{P}^\kappa_\lambda$ collapses all uncountable cardinals below κ to ω_1 and those between κ and λ to κ.*

Proof. See [15, Theorems 4.33 and 4.34]. $\boxed{5.15}$

Notice $\mathbb{P}^\kappa_\lambda$ is λ-c.c., see Proposition 5.32.

Definition 5.16. *Suppose that $G \subseteq \mathbb{P}^\kappa_\lambda$ is V-generic, and that $\alpha \in E$ is of cofinality less than κ. We let $C_\alpha(G) = \{\kappa_M : M \in \mathcal{M}^\alpha_G\}$.*

Proposition 5.17. *Let G be a V-generic filter over $\mathbb{P}^\kappa_\lambda$. Then $C_\alpha(G)$ is a club in κ, for all $\alpha \in E$ of cofinality $<\kappa$. Moreover, if $\alpha < \beta$ then $C_\beta(G) \setminus C_\alpha(G)$ is bounded in κ.*

Proof. See [15, Lemma 4.37]. $\boxed{5.17}$

The iteration. For a condition $\mathcal{M}_p \in \mathbb{M}^\kappa_\lambda$ and an ordinal $\gamma \in E$, we let

$$\mathcal{M}_p(\gamma) := \{M \in \mathcal{M}_p^{\mathrm{next}(\gamma)} : \gamma \in M\}.$$

In other words, $\mathcal{M}_p(\gamma)$ consists of those models in $\mathcal{M}_p\restriction \mathrm{next}(\gamma)$ that are strongly active at $\mathrm{next}(\gamma)$.

Let $E^+ = E \cup \{\alpha + 1 : \alpha \in E\}$. We will define by induction the poset $\mathbb{Q}^\kappa_\alpha$, for $\alpha \in E^+ \cup \{\lambda\}$. Let $\mathrm{Fn}(\omega_1, \omega)$ denote the poset of finite partial functions from ω_1 to ω, ordered under reverse inclusion. Conditions in $\mathbb{Q}^\kappa_\alpha$ will be triples of the form $p = (\mathcal{M}_p, d_p, w_p)$, where $(\mathcal{M}_p, d_p) \in \mathbb{P}^\kappa_\alpha$, and w_p is a finite function with $\mathrm{dom}(w_p) \subseteq E \cap \alpha$, such that $w_p(\gamma) \in \mathrm{Fn}(\omega_1, \omega)$, for all $\gamma \in \mathrm{dom}(w_p)$. If p is such a triple and $\gamma < \alpha$ is in E, we let $p \restriction \gamma$ denote the triple $(\mathcal{M}_p \restriction \gamma, d_p \restriction \gamma, w_p \restriction \gamma)$, where $(\mathcal{M}_p \restriction \gamma, d_p \restriction \gamma)$ is defined as in $\mathbb{P}^\kappa_\alpha$ and $w_p \restriction \gamma$ is the restriction of w_p to $\mathrm{dom}(w_p) \cap \gamma$.

Let us recall that we are working with a suitable structure $(V_\lambda, \in, \kappa, U)$. To be precise, $U : \lambda \to V_\lambda$ is a (bookkeeping) function that we regard it as a binary predicate. Thus let us assume that U is onto and for every $x \in V_\lambda$, the set $\{\alpha < \lambda : U(\alpha) = x\}$ is unbounded.

Definition 5.18. *For $\alpha \in E$, we let $\mathbb{Q}^\kappa_\alpha$ consist of triples $p = (\mathcal{M}_p, d_p, w_p)$, where*

(1) $(\mathcal{M}_p, d_p) \in \mathbb{P}^\kappa_\alpha$.
(2) w_p is a finite function with $\mathrm{dom}(w_p) \subseteq E \cap \alpha$ such that, if $\gamma \in \mathrm{dom}(w_p)$, then $U(\gamma)$ is a $\mathbb{Q}^\kappa_\gamma$-term for a relation on ω_1 such that $\dot{T}_\gamma = (\check{\omega}_1, U(\gamma))$ is forced to be a tree of height ω_1 without uncountable branches, $w_p(\gamma) \in \mathrm{Fn}(\omega_1, \omega)$, and

$$p \restriction \gamma \Vdash_{\mathbb{Q}^\kappa_\gamma} \check{w}_p(\gamma) \in \dot{\mathbb{S}}(\dot{T}_\gamma).$$

For conditions $p, q \in \mathbb{Q}^\kappa_\alpha$, we say p is stronger than q and write $p \leq q$, if and only if,

(1) $(\mathcal{M}_p, d_p) \leq (\mathcal{M}_q, d_q)$ in $\mathbb{P}^\kappa_\alpha$,
(2) $\mathrm{dom}(w_p) \supseteq \mathrm{dom}(w_q)$, and $w_q(\gamma) \subseteq w_p(\gamma)$, for every $\gamma \in \mathrm{dom}(w_q)$.

We let $\mathbb{Q}^\kappa_{\alpha+1}$ denote the set of triples p as above, but with $\mathrm{dom}(w_p) \subseteq \alpha + 1$. Let $\mathbb{Q}^\kappa_\lambda = \bigcup_{\alpha \in E} \mathbb{Q}^\kappa_\alpha$ with the same order.

Note that if $\alpha = \min(E)$ then $\mathbb{Q}^\kappa_\alpha = \mathbb{P}^\kappa_\alpha$. Also, for $\alpha \in E$, $\mathbb{Q}^\kappa_{\alpha+1}$ is isomorphic to $\mathbb{Q}^\kappa_\alpha * \dot{\mathbb{S}}(\dot{T}_\alpha)$, if $U(\alpha)$ has the right form, otherwise $\mathbb{Q}^\kappa_{\alpha+1} \cong \mathbb{Q}^\kappa_\alpha$. For $p \in \mathbb{Q}^\kappa_\lambda$ and $\alpha \in E$, we let

$$p \restriction (\alpha + 1) = (\mathcal{M}_p \restriction \alpha, d_p \restriction \alpha, w_p \restriction (\alpha + 1)).$$

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The order is transitive and whenever $p \in \mathbb{Q}^\kappa_\lambda$ and $\alpha \in E^+$, then $p \restriction \alpha \in \mathbb{Q}^\kappa_\alpha$. Moreover, if $p \leq q$ then $p \restriction \alpha \leq q \restriction \alpha$.

Proposition 5.19. *Suppose $p \in \mathbb{Q}^\kappa_\beta$, and that $\alpha \leq \beta$ are in E^+. If q is a condition in $\mathbb{Q}^\kappa_\alpha$ that extends $p \restriction \alpha$, then p is compatible with q in $\mathbb{Q}^\kappa_\beta$.*

Proof. Let $(\mathcal{M}_r, d_r)$ be $(\mathcal{M}_p, d_p) \wedge (\mathcal{M}_q, d_q)$ as defined in Lemma 5.3. Let also w_r be defined on $\mathrm{dom}(w_p) \cup \mathrm{dom}(w_q)$ by

$$w_r(\gamma) = \begin{cases} w_q(\gamma) & \text{if } \gamma < \alpha, \\ w_p(\gamma) & \text{if } \gamma \geq \alpha. \end{cases}$$

It is evident that r is a condition which extends p and q. $\boxed{5.19}$

Remark 5.20. The condition r from the previous lemma is the greatest lower bound of p and q, so we will denote it by $r \coloneqq p \wedge q$.

The following corollary is immediate.

Corollary 5.21. *For every $\alpha \leq \beta$ in $E^+ \cup \{\lambda\}$, $\mathbb{Q}^\kappa_\alpha$ is a complete suborder of $\mathbb{Q}^\kappa_\beta$.*

$\boxed{5.21}$

Proposition 5.22. *Let $p \in \mathbb{Q}^\kappa_\lambda$ and let $M \in \mathscr{C} \cup \mathscr{U}$ be such that $p \in M$. Then there is a condition $p^M \leq p$ with $M \in \mathcal{M}_{p^M}$.*

Proof. Let $\mathcal{M}_{p^M}$ and d_{p^M} be defined as in Definition 5.5. We let $p^M = (\mathcal{M}_{p^M}, d_{p^M}, w_p)$. It is clear that p^M is the required condition. $\boxed{5.22}$

We need the following lemma in several proofs.

Lemma 5.23. *Let $\alpha \in E$. Suppose $M \in \mathscr{C} \cup \mathscr{U}$ with $\eta(M) > \alpha$ and $\alpha \in M$. Let $p \in \mathbb{Q}^\kappa_\alpha$ be a condition with $M \restriction \alpha \in \mathcal{M}_p$. Let $\beta = \max(\mathrm{dom}(w_p) \cap M)$. Let $G_{\beta+1}$ be a V-generic filter on $\mathbb{Q}^\kappa_{\beta+1}$ with $p \restriction \beta + 1 \in G_{\beta+1}$. Assume that $q \in M \cap \mathbb{Q}^\kappa_\alpha$ is a condition such that:*

- *$(\mathcal{M}_q, d_q) \leq (\mathcal{M}_p, d_p) \restriction M$,*
- *$M \cap \mathrm{dom}(w_p) \subseteq \mathrm{dom}(w_q)$, and*
- *$q \restriction \beta + 1 \in G_{\beta+1}$.*

Then p and q are compatible in $\mathbb{Q}^\kappa_\alpha$.

Proof. First, by Proposition 5.12, we have that $(\mathcal{M}_q, d_q)$ and $(\mathcal{M}_p, d_p)$ are compatible in $\mathbb{P}^\kappa_\alpha$ and the meet $(\mathcal{M}_q, d_q) \wedge (\mathcal{M}_p, d_r)$ exists. Let us denote this meet by $(\mathcal{M}, d)$. Let us also fix $r \in G_{\beta+1}$ extending $p \restriction \beta + 1$ and $q \restriction \beta + 1$. Let $(\mathcal{M}_s, d_s)$ be the meet $(\mathcal{M}, d) \wedge (\mathcal{M}_r, d_r)$ as defined in Lemma 5.3. We now define w_s on $\mathrm{dom}(w_r) \cup \mathrm{dom}(w_p) \cup \mathrm{dom}(w_q)$ by letting:

$$w_s(\gamma) = \begin{cases} w_r(\gamma) & \text{if } \gamma \in \mathrm{dom}(w_r) \\ w_q(\gamma) & \text{if } \gamma \in \mathrm{dom}(w_q) \setminus \mathrm{dom}(w_r), \\ w_p(\gamma) & \text{if } \gamma \in \mathrm{dom}(w_p) \setminus (\mathrm{dom}(w_r) \cup \mathrm{dom}(w_q)). \end{cases}$$

It is easy to see that $s = (\mathcal{M}_s, d_s, w_s)$ is a condition in $\mathbb{Q}^\kappa_\alpha$ extending p and q. $\boxed{5.23}$

Proposition 5.24. *Let $\alpha \in E$. Suppose $M \in \mathscr{C} \cup \mathscr{U}$ with $\eta(M) > \alpha$ and $\alpha \in M$. Let p be a condition with $M \restriction \alpha \in \mathcal{M}_p$. Then*

(1) if $p \in \mathbb{Q}^\kappa_\alpha$, then p is $(M, \mathbb{Q}^\kappa_\alpha)$-generic,
(2) if $p \in \mathbb{Q}^\kappa_{\alpha+1}$, then p is $(M, \mathbb{Q}^\kappa_{\alpha+1})$-generic.

Proof. First note that under our assumptions both $\mathbb{Q}^\kappa_\alpha$ and $\mathbb{Q}^\kappa_{\alpha+1}$ belong to M. $\mathbb{Q}^\kappa_{\alpha+1}$ is either equal to $\mathbb{Q}^\kappa_\alpha$ or is isomorphic to $\mathbb{Q}^\kappa_\alpha * \dot{\mathbb{S}}(\dot{T}_\alpha)$. Since $\dot{\mathbb{S}}(\dot{T}_\alpha)$ is forced to be ccc, (2) follows from (1). If $\alpha = \min(E)$ then $\mathbb{Q}^\kappa_\alpha$ is isomorphic to $\mathbb{P}^\kappa_\alpha$, and by Proposition 5.12, it is then strongly proper for all models $M \in \mathscr{C} \cup \mathscr{U}$ with $\eta(M) > \alpha$.

Suppose now that α is not the least element of E and (2) holds for all $\beta < \alpha$. Let $D \in M$ be a dense subset of $\mathbb{Q}^\kappa_\alpha$. We may assume without loss of generality that $p \in D$. Let $\beta = \max(\mathrm{dom}(w_p) \cap M)$. Pick a V-generic filter $G_{\beta+1}$ on $\mathbb{Q}^\kappa_{\beta+1}$ such that $p \restriction \beta+1 \in G_{\beta+1}$. By elementarity of $M[G_{\beta+1}]$ in $\mathrm{Hull}(M, V_{\eta(M)})[G_{\beta+1}]$, there is a condition $q \in D \cap M[G_{\beta+1}]$ satisfying the following.

(1) $(\mathcal{M}_q, d_q) \leq (\mathcal{M}_p, d_p) \restriction M$,
(2) $M \cap \mathrm{dom}(w_p) \subseteq \mathrm{dom}(w_q)$, and
(3) $q \restriction \beta + 1 \in G_{\beta+1}$.

By the inductive assumption, $p \restriction \beta+1$ is $(M, \mathbb{Q}^\kappa_{\beta+1})$-generic, and hence $M[G_{\beta+1}] \cap V = M$. Therefore, we can find such q in M. By Lemma 5.23, p and q are compatible. 5.24

Proposition 5.25. *Suppose that $\lambda^* > \lambda$ is a sufficiently large regular cardinal. Let $p \in \mathbb{Q}^\kappa_\lambda$. Suppose that $M^* \prec H_{\lambda^*}$ with $\kappa, \lambda \in M^*$ is such that $M := M^* \cap V_\lambda \in \mathcal{M}_p$. Then p is $(M^*, \mathbb{Q}^\kappa_\lambda)$-generic.*

Proof. Observe that $\eta = \sup(M^* \cap \lambda)$ is in E, and that M is an η-model. The rest is as in the proof of Proposition 5.24. 5.25

Corollary 5.26. *For every $\alpha \in E^+ \cup \{\lambda\}$, $\mathbb{Q}^\kappa_\alpha$ is proper.*

5.26

Corollary 5.27. *For every $\alpha \in E^+ \cup \{\lambda\}$, $\mathbb{Q}^\kappa_\alpha$ is strongly proper for $\mathscr{U}$.*

Proof. This follows from Proposition 5.24 and Lemma 2.14. 5.27

Definition 5.28. *Let $\alpha \in E$ and let G_α be V-generic over $\mathbb{Q}^\kappa_\alpha$. In the model $V[G_\alpha]$, let $\mathscr{C}_{\mathrm{st}}[G_\alpha]$ denote the set of all $M \in \mathscr{C}_{\mathrm{st}}$ such that $\eta(M) > \alpha$, $\alpha \in M$ and $M \restriction \alpha \in \mathcal{M}^\alpha_{G_\alpha}$.*

As in [15, Lemma 5.2] we have the following.

Proposition 5.29. *$\mathscr{C}_{\mathrm{st}}[G_\alpha]$ is a stationary subset of $\mathcal{P}_{\omega_1}(V_\lambda)$ in the model $V[G_\alpha]$.*

5.29

Let $\alpha \in E^+$. Assume that G_α is a V-generic filter on $\mathbb{Q}^\kappa_\alpha$. One can form the quotient forcing $\mathbb{Q}^\kappa_\lambda / G_\alpha$. Suppose that also $M \in \mathscr{C}_{>\alpha}$, $M \restriction \alpha \in \mathcal{M}^\alpha_{G_\alpha}$, and $p \in M \cap \mathbb{Q}^\kappa_\lambda / G_\alpha$. It is then easily seen that p^M belongs to $\mathbb{Q}^\kappa_\lambda / G_\alpha$. By an argument, as in Proposition 5.24, we have the following.

Proposition 5.30. *Let $\lambda^* > \lambda$ be a sufficiently large regular cardinal. Suppose that $\alpha \in E$, and $G_\alpha \subseteq \mathbb{Q}^\kappa_\alpha$ is a V-generic filter. Let $p \in \mathbb{Q}^\kappa_\lambda / G_\alpha$. Suppose $M^* \prec H_{\lambda^*}$ contains all the relevant objects, and $M = M^* \cap V_\lambda$ belongs to $\mathcal{M}_p$. Then p is $(M^*[G_\alpha], \mathbb{Q}^\kappa_\lambda / G_\alpha)$-generic.*

5.30

Suppose $\alpha \in E^+$ and let G_α be V-generic over $\mathbb{Q}^\kappa_\alpha$. In $V[G_\alpha]$ fix a large regular cardinal λ^* and let $\mathscr{S}$ be the collection of all countable models of the form $M[G_\alpha]$, where $M \prec H^V_{\lambda^*}$ contains the relevant objects, and $M \cap V_\lambda \in \mathscr{C}_{\rm st}[G_\alpha]$. Now, by Propositions 5.29 and 5.30 we have the following.

Corollary 5.31. *$\mathscr{S}$ is stationary in $\mathcal{P}_{\omega_1}([H_{\lambda^*}[G_\alpha])$ and $\mathbb{Q}^\kappa_\lambda/G_\alpha$ is $\mathscr{S}$-proper in $V[G_\lambda]$.*

5.31

Proposition 5.32. *$\mathbb{Q}^\kappa_\lambda$ satisfies the λ-c.c.*

Proof. Assume that $A \subseteq \mathbb{Q}^\kappa_\lambda$ is of size λ. For every $p \in \mathbb{P}^\kappa_\lambda$, let $a(p) = \bigcup\{a(M) : M \in \mathcal{M}_p\}$, which is a closed subset of E of size $<\kappa$. By a standard Δ-system argument, we can find a subset B of A of size λ so that there are a and d subsets of E such that $a(p) \cap a(q) = a$ and $\mathrm{dom}(w_p) \cap \mathrm{dom}(w_q) = d$, for all distinct $p, q \in B$. Note that a is closed, and if we let $\gamma = \max(a)$, then $\gamma \in E$. Since B has size λ, by a simple counting argument, we can assume there is $\mathcal{M} \in \mathbb{M}^\kappa_\gamma$ such that $\mathcal{M}_p \restriction \gamma = \mathcal{M}$ and that $w_p \restriction d = w_q \restriction d$, for all $p \in B$. Now, pick distinct $p, q \in B$, and define $\mathcal{M}_r = \mathcal{M}_p \cup \mathcal{M}_q$, $d_r = d_p \cup d_q$, and also $w_r = w_p \cup w_q$. Let $r = (\mathcal{M}_r, d_r, w_r)$. It is straightforward to check that $r \in \mathbb{Q}^\kappa_\lambda$ and $r \leq p, q$. 5.32

Putting everything together, we have the following.

Corollary 5.33. *$\mathbb{Q}^\kappa_\lambda$ preserves ω_1, κ and λ, and forces that $\kappa = \omega_2^{V[G_\lambda]}$ and $\lambda = \omega_3^{V[G_\lambda]}$.*

Proof. The preservation of ω_1 and κ is guaranteed by Corollaries 5.26 and 5.27, respectively. It is easily seen that $\mathbb{P}^\kappa_\lambda$ is a complete suborder of $\mathbb{Q}^\kappa_\lambda$, and hence Propositions 5.15 and 5.32 imply that in generic extensions by $\mathbb{Q}^\kappa_\lambda$, $\kappa = \omega_2$ and $\lambda = \omega_3$. 5.33

Proposition 5.34. *For every $\alpha \in E^+ \cup \{\lambda\}$, $\mathbb{Q}^\kappa_\alpha$ has the ω_1-approximation property.*

Proof. We proceed by induction. Suppose $\alpha \in E$ and we have established that $\mathbb{Q}^\kappa_\alpha$ has the ω_1-approximation property. Recall that $\mathbb{Q}^\kappa_{\alpha+1}$ is either $\mathbb{Q}^\kappa_\alpha$ or is isomorphic to $\mathbb{Q}^\kappa_\alpha * \dot{\mathbb{S}}(\dot{T}_\alpha)$, for some $\dot{T}_\alpha$ which is a name for a tree of size and height ω_1 without uncountable branches. Then by Proposition 2.19 $\dot{\mathbb{S}}(\dot{T}_\alpha)$ is forced to have the ω_1-approximation property over $V[G_\alpha]$, where G_α is a generic over $\mathbb{Q}^\kappa_\alpha$. By Theorem 2.3 we conclude that $\mathbb{Q}^\kappa_{\alpha+1}$ has the ω_1-approximation property. Recall also that $\mathbb{Q}^\kappa_{\min(E)}$ is strongly proper for $\mathscr{C}$, by Proposition 5.12, and hence has the ω_1-approximation property by Lemma 2.10.

Suppose now that α is not the least element of E and that the conclusion holds for every ordinal in $E^+ \cap \alpha$. Let $\dot{f}$ be a $\mathbb{Q}^\kappa_\alpha$-name for a function $\mu \to 2$ which is forced by a condition p to be countably approximated in V. We may assume that μ is a cardinal in V. Suppose $\lambda^* > \mu, \lambda$ is a sufficiently large regular cardinal. Pick a countable $M^* \prec H_{\lambda^*}$ containing the relevant objects. Thus $M = M^* \cap V_\lambda$ is a standard virtual model. Let $q \leq p^M$ be a condition which decides $\dot{f} \restriction M^* \cap \mu$ and forces it to be equal to some function $g : M^* \cap \mu \to 2$ which is in V. By Proposition 5.25 q is $(M^*, \mathbb{Q}^\kappa_\alpha)$-generic. By Lemma 2.16, it suffices to show that g is guessed in M^*. Assume towards a contradiction that g is not guessed in M^*. Let $\gamma = \max(\mathrm{dom}(w_q) \cap M)$, and pick some V-generic filter $G_{\gamma+1}$ on $\mathbb{Q}^\kappa_{\gamma+1}$ such that $(q \restriction \gamma, w_q(\gamma)) \in G_{\gamma+1}$. Working in $V[G_{\gamma+1}]$, we show that in $M^*[G_{\gamma+1}]$ there is an assignment $x \mapsto (q_x, g_x)$ on $[\mu]^\omega \cap V$ with the following properties.

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(1) $q_x \in \mathbb{Q}^\kappa_\alpha$,
(2) $(\mathcal{M}_{q_x}, d_{q_x}) \leq (\mathcal{M}_q, d_q) \restriction M$,
(3) $q_x \restriction \gamma + 1 \in G_{\gamma+1}$,
(4) $g_x : x \to 2$, $g_x \in V$,
(5) $q_x \Vdash \dot{f} \restriction x = \check{g}_x$.

If $x \in [\mu]^\omega \cap M^*$ then $x \mapsto (q, g \restriction x)$ satisfies all the above properties. By elementarity of $M^*[G_{\gamma+1}]$ in $H_{\lambda^*}[G_{\gamma+1}]$, such an assignment exists for all $x \in [\mu]^\omega \cap V$. Moreover, by elementarity again there is such an assignment in $M^*[G_{\gamma+1}]$. Now, by the inductive assumption and Lemma 2.15, g is not guessed in $M^*[G_{\gamma+1}]$. Since $\mathbb{Q}^\kappa_\alpha$ is proper, $[\mu]^\omega \cap V$ is a cofinal in $[\mu]^\omega$. By Lemma 2.17 there is $B \in M^*[G_{\gamma+1}]$, a cofinal subset of $[\mu]^\omega \cap V$, such that for every $x \in B \cap M^*[G_{\gamma+1}]$, $g_x \nsubseteq g$, and hence q_x is incompatible with q. On the other hand, conditions (1)-(3) above enable us to use Lemma 5.23 to make sure that q_x and q are compatible. Thus, for every such $x \in B \cap M^*[G_{\gamma+1}]$, q_x and q are compatible, and hence we get a contradiction. $\boxed{5.34}$

One can use Proposition 5.30 and the proof of Proposition 5.34 to prove the following.

Proposition 5.35. *Suppose $\alpha, \beta \in E^+ \cup \{\lambda\}$ and $\alpha < \beta$. Suppose that G_α is a V-generic filter over $\mathbb{Q}^\kappa_\alpha$. Then $\mathbb{Q}^\kappa_\beta / G_\alpha$ has the ω_1-approximation property over $V[G_\alpha]$.*

$\boxed{5.35}$

The following corollary is immediate.

Corollary 5.36. *Suppose $\alpha, \beta \in E^+ \cup \{\lambda\}$ and $\alpha < \beta$. Let G_β be a V-generic filter over $\mathbb{Q}^\kappa_\beta$, and let $G_\alpha = G_\beta \cap \mathbb{Q}^\kappa_\alpha$. Then the pairs $(V, V[G_\alpha])$ and $(V[G_\alpha], V[G_\beta])$ have the ω_1-approximation property.*

$\boxed{5.36}$

Quotients by Magidor models. The first lemma of this subsection states that $\mathbb{Q}^\kappa_\lambda$ is strongly proper for $\mathscr{U}$ in a canonical way.

Definition 5.37. *Let $p \in \mathbb{Q}^\kappa_\alpha$. Assume $N \in \mathcal{L}(\mathcal{M}_p)$ is a Magidor model. We let*

$$p \restriction N = (\mathcal{M}_{p \restriction N}, d_{p \restriction N}, w_p \restriction N).$$

Lemma 5.38. *Let $p \in \mathbb{Q}^\kappa_\lambda$, and let $N \in \mathcal{L}(\mathcal{M}_p)$ be a Magidor model. Assume that $\alpha \leq \eta(N)$ is in E. Then $p \restriction \alpha \restriction N = (p \restriction N) \restriction \alpha$.*

Proof. It is enough to show that if $M \in \mathcal{M}_p$, then $(M \restriction N) \restriction \alpha = M \restriction \alpha \restriction N$. This is of course clear from the definition of $M \restriction N$. $\boxed{5.38}$

Lemma 5.39. *Let $p \in \mathbb{Q}^\kappa_\alpha$, and assume that $N \in \mathcal{L}(\mathcal{M}_p)$ is a Magidor model. Then*

(1) $p \restriction N \in \mathbb{Q}^\kappa_\alpha \cap N$.
(2) Every condition $q \in N$ extending $p \restriction N$ is compatible with p.

Proof. We prove both items simultaneously by induction on α. Note that both are true for the minimum of E. Thus let us assume that $\alpha > \min(E)$. Note that $p \restriction N$ belongs to N. So all we have to show is that $p \restriction N$ is a condition in $\mathbb{Q}^\kappa_\alpha$ that satisfies the second item.

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One needs first to show that for every $\gamma \in \mathrm{dom}(p) \cap N$, $(p \restriction N) \restriction \gamma$, which is a condition by the inductive hypothesis and Lemma 5.38, forces in $\mathbb{Q}^{\kappa}_{\gamma}$ that "$\check{w}_p(\gamma) \in \dot{\mathbb{S}}(\dot{T}_\gamma)$". Assume towards a contradiction that this is not the case, so there is $q \in \mathbb{Q}^{\kappa}_{\gamma}$ with $q \leq (p \restriction N) \restriction \gamma$ which forces $\check{w}_p(\gamma)$ is not in $\dot{\mathbb{S}}(\dot{T}_\gamma)$". Since both $p \restriction N \restriction \gamma$ and $w_p(\gamma)$ belong to N, we can find such q in N by elementarity. Therefore, q is not compatible with $p \restriction \gamma$. This contradicts the second item above applied to $p \restriction \gamma, N \restriction \gamma$, and q. Hence the first item follows.

To see the second item, we will define a common extension of p, q, say r by letting $(\mathcal{M}_r, d_r) = (\mathcal{M}_p, d_p) \wedge (\mathcal{M}_q, d_q)$, $\mathrm{dom}(w_r) = \mathrm{dom}(w_p) \cup \mathrm{dom}(w_q)$, and that

$$w_r(\gamma) = \begin{cases} w_q(\gamma) & \text{if } \gamma \in N, \\ w_p(\gamma) & \text{if } \gamma \notin N. \end{cases}$$

It follows from Proposition 5.9 that r is a common extension of p and q. 5.39

Remark 5.40. As before, we let the above r be the meet of p and q, which we denote by $p \wedge q$.

Suppose that $N \in \mathscr{U}$ and $\alpha \in E$. We set

$$\mathbb{Q}^{\kappa}_{\alpha} \restriction N := \{p \in \mathbb{Q}^{\kappa}_{\alpha} : N \restriction \alpha \in \mathcal{M}_p\} \quad \text{and} \quad \mathbb{Q}^{\kappa}_{\alpha,N} := \mathbb{Q}^{\kappa}_{\alpha} \cap N.$$

By Lemma 5.39, the condition $\mathbf{1}_N := (\{N \restriction \alpha\}, \varnothing, \varnothing)$ is $(N, \mathbb{Q}^{\kappa}_{\alpha})$-strongly generic, and therefore the mapping

$$\begin{aligned} \mathbb{Q}^{\kappa}_{\alpha,N} &\longrightarrow \mathbb{Q}^{\kappa}_{\alpha} \restriction N \\ p &\longmapsto p^{N \restriction \alpha} \end{aligned}$$

is a complete embedding. Moreover, if p is a condition in $\mathbb{Q}^{\kappa}_{\alpha} \restriction N$, then $p \restriction N \in \mathbb{Q}^{\kappa}_{\alpha,N}$ is such that if $q \in \mathbb{Q}^{\kappa}_{\alpha,N}$ extends $p \restriction N$, then p and q are compatible, and indeed the meet $p \wedge q$ exists.

For a V-generic filter $G_{\alpha,N} \subseteq \mathbb{Q}^{\kappa}_{\alpha} \cap N$ we can form the following quotient forcing

$$\mathbb{R}^{N}_{\alpha} := \mathbb{Q}^{\kappa}_{\alpha} \restriction N / G_{\alpha,N}.$$

Definition 5.41. *Let $G_{\alpha,N}$ be a V-generic filter over $\mathbb{Q}^{\kappa}_{\alpha} \cap N$. Let $\mathscr{C}_{\mathrm{st}}[G_{\alpha,N}]$ denote the set of all models $M \in \mathscr{C}_{\mathrm{st}}$ such that $\alpha, N \in M$ and $(N \wedge M) \restriction \alpha \in \mathcal{M}_{G_{\alpha,N}}$.*

Note that $\mathscr{C}_{\mathrm{st}}[G_{\alpha,N}]$ is stationary in $\mathcal{P}_{\omega_1}(V_\lambda)$ in the model $V[G_{\alpha,N}]$, but we do not need this and therefore avoid a proof.

Lemma 5.42 (Factorization Lemma)**.** *Suppose $N \in \mathscr{U}$, and that $\alpha \leq \beta \leq \eta(N)$ are in E. Let $G_{\alpha,N} \subseteq \mathbb{Q}^{\kappa}_{\alpha} \cap N$ be a V-generic filter. The mapping*

$$\begin{aligned} \rho : \mathbb{Q}^{\kappa}_{\beta} \restriction N / G_{\alpha,N} &\longrightarrow \mathbb{R}^{N}_{\alpha} \times (\mathbb{Q}^{\kappa}_{\beta} \cap N)/G_{\alpha,N} \\ p &\longmapsto (p \restriction \alpha, p \restriction N) \end{aligned}$$

is a projection in $V[G_{\alpha,N}]$.

Proof. It is easy to observe that ρ is a well-defined, order-preserving mapping that respects the maximal conditions. Assume that the arbitrary elements

$$p \in \mathbb{Q}^{\kappa}_{\beta} \restriction N/G_{\alpha,N} \text{ and } (r, s) \in \mathbb{Q}^{\kappa}_{\alpha} \restriction N/G_{\alpha,N} \times (\mathbb{Q}^{\kappa}_{\beta} \cap N)/G_{\alpha,N} \text{ with}$$

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$$(r,s) \leq (p \restriction \alpha, p \restriction N)$$

are given. Our goal is to find a condition $q \in \mathbb{Q}^{\kappa}_{\beta} \restriction N/G_{\alpha,N}$ with $q \leq p$ such that $\rho(q) \leq (r,s)$.

Since $r \restriction N, s \restriction \alpha \in G_{\alpha,N}$, we can fix a common extension $\bar{q} \in G_{\alpha,N}$ of them. Now by applying Lemma 5.39 to $\bar{q}, N, r$, we have

$$\bar{q} \wedge r \leq \bar{q} \leq s \restriction \alpha,$$

and by applying Proposition 5.19 to $\bar{q}, s, \alpha$, we have

$$\bar{q} \wedge s \leq \bar{q} \leq r \restriction N.$$

By reverse applications of Lemma 5.39 and Proposition 5.19 to the above inequalities, the meets $(\bar{q} \wedge r) \wedge s$ and $(\bar{q} \wedge s) \wedge r$ exist. A simple calculation shows that these conditions are equal, so let us call them q^*. The canonicity of our projections guarantees that

$$q^* \restriction \alpha = (\bar{q} \wedge r) \leq r \quad \text{and} \quad q^* \restriction N = (\bar{q} \wedge s) \leq s.$$

Moreover,

$$q^* \restriction \alpha \restriction N = (q^* \restriction N) \restriction \alpha = \bar{q} \in G_{\alpha,N}.$$

Consequently,

$$q^* \in \mathbb{Q}^{\kappa}_{\beta}/G_{\alpha,N} \quad \text{and} \quad (q^* \restriction \alpha, q^* \restriction N) \leq (r,s).$$

It suffices to show that p and q^* are compatible, since the mappings $\bullet \mapsto \bullet \restriction \alpha$ and $\bullet \mapsto \bullet \restriction N$ are order-preserving. We define a common extension of them, say q. Let $\mathcal{M}_q = \mathcal{M}_p \cup \mathcal{M}_{q^*}$. For every $\delta \in E$,

$$\mathcal{M}^{\delta}_q = \begin{cases} \mathcal{M}^{\delta}_{q^*} & \text{if } \delta \leq \alpha, \qquad (\text{since } \mathcal{M}^{\delta}_p \subseteq \mathcal{M}^{\delta}_r \subseteq \mathcal{M}^{\delta}_{q^*}) \\ \\ \mathcal{M}^{\delta}_p & \text{if } \delta > \alpha. \qquad (\text{since } \mathcal{M}^{\delta}_{q^*} \subseteq \mathcal{M}^{\delta}_s \subseteq \mathcal{M}^{\delta}_p) \end{cases}$$

In either case, $\mathcal{M}^{\delta}_q$ is an δ-chain. Thus $\mathcal{M}_q$ is a condition in $\mathbb{M}^{\kappa}_{\alpha}$. We now define d_q on $\operatorname{dom}(d_p) \cup \operatorname{dom}(d_{q^*})$ by letting

$$d_q(P) = \begin{cases} d_p(P) & \text{if } \eta(P) > \alpha \text{ and } P \notin N, \\ d_{q^*}(P) & \text{otherwise.} \end{cases}$$

The function is well-defined as $\rho(q^*) \leq \rho(p)$. A proof similar to [15, Lemma 4.14] would show that every model in $\operatorname{dom}(d_q)$ is $\mathcal{M}_q$-free. We sketch the proof. Thus we assume that $P \in \mathcal{L}(\mathcal{M}_q)$, $Q \in \mathcal{M}_q$, and that $P \in_{\eta(P)} Q$. We have to show that Q is strongly active at $\eta(P)$.

- **Case 1.** $\eta(N) > \alpha$ and $P \notin N$: Then P does not belong to $\mathcal{L}(\mathcal{M}_{q^*})$, hence $Q \notin \mathcal{M}_{q^*}$, in which case, since p is a condition, Q must be strongly active at $\eta(P)$.
- **Case 2.** $\eta(N) > \alpha$ and $P \in N$: Then P is in $\mathcal{L}(\mathcal{M}_{q^*})$. We may assume that $Q \notin \mathcal{M}_{q^*}$. In particular, $Q \in \mathcal{M}_p$. Now both N and Q are active at $\eta(P)$. Therefore, we must have $P \in N \in_{\eta(P)} Q$. We can also assume that Q is countable, as otherwise, it is strongly active at $\eta(P)$. Now, we have $P \in_{\eta(P)} (N \wedge Q) \restriction N$, where the latter model belongs to $\mathcal{M}_{q^*}$. Thus $(N \wedge Q) \restriction N$ is strongly active at $\eta(P)$, and hence Q.

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- **Case 3.** $\eta(P) \leq \alpha$: In this case P is in $\mathcal{L}(\mathcal{M}_{q^*\restriction\alpha})$, then $Q \restriction \alpha \in \mathcal{M}_{q^*\restriction\alpha}$. Now $Q \restriction \alpha$, and consequently Q is strongly active at $\eta(P)$.

Using the above argument, it is easy to show that the decorative function d_q fulfils ($\circledast$) in Definition 5.2.

It remains to define w_q on $\mathrm{dom}(w_p) \cup \mathrm{dom}(w_{q^*})$. For each $\gamma \in \mathrm{dom}(w_q)$, we let

$$w_q(\gamma) = \begin{cases} w_p(\gamma) & \text{if } \gamma > \alpha \text{ and } \gamma \notin N, \\ w_{q^*}(\gamma) & \text{otherwise.} \end{cases}$$

We observe that w_q is well-defined thanks to the definition of q^*. Thus q is a condition in $\mathbb{Q}^\kappa_\beta \restriction N$, which is easily shown to be a common extension of p and q. By an easy calculation, we have

$$q \restriction \alpha \restriction N = \bar{q} \in G_{\alpha,N},$$

and therefore $q \in \mathbb{Q}^\kappa_\beta \restriction N/G_{\alpha,N}$, as required! $\boxed{5.42}$

Lemma 5.43. *Suppose $N \in \mathscr{U}$ and that $\alpha \leq \beta \leq \eta(N)$ are in E. Let $G_{\beta,N}$ be a V-generic filter over $\mathbb{Q}^\kappa_{\beta,N}$. Then in $V[G_{\beta,N}]$, $\mathbb{R}^N_\alpha$ is a complete suborder of $\mathbb{R}^N_\beta$.*

Proof. It is clear that $\mathbb{R}^N_\alpha \subseteq \mathbb{R}^N_\beta$, and furthermore, if $p \in \mathbb{R}^N_\beta$, then $p \restriction \alpha \in \mathbb{R}^N_\alpha$. Now fix $p \in \mathbb{R}^N_\beta$ and suppose that $q \in \mathbb{R}^N_\alpha$ extends $p \restriction \alpha$. Let $r = p \wedge q$ as obtained in $\mathbb{Q}^\kappa_\beta$. Therefore, we have $r \in \mathbb{Q}^\kappa_\beta \restriction N$. An easy calculation shows that

$$r\restriction N = p \restriction N \wedge q \restriction N \in G_{\beta,N},$$

where we observe that $p \restriction N \leq (q \restriction N) \restriction \alpha$, and hence the meet $p \restriction N \wedge q \restriction N$ exists by Proposition 5.19. $\boxed{5.43}$

Lemma 5.44. *Assume N is a Magidor model, $M \in \mathscr{C}_{\mathrm{st}}[G_{\alpha,N}]$, and that $p \in \mathbb{R}^N_\alpha \cap M$. Then $p^{M\restriction\alpha}$ is an $(M[G_{\alpha,N}], \mathbb{R}^N_\alpha)$-generic condition.*

Proof. Let us first show that $p^{M\restriction\alpha}$ is an $\mathbb{R}^N_\alpha$-condition. Note that by the proof of [15, Lemma 5.7], we have

$$(\mathcal{M}_p, d_p)^{M\restriction\alpha} \restriction N = \big((\mathcal{M}_p, d_p) \restriction N\big)^{N\wedge M\restriction\alpha}.$$

On the other hand, $N \wedge M \restriction \alpha$ is in $\mathcal{M}_{G_{\alpha,N}}$, which in turn implies that

$$\big((\mathcal{M}_{p\restriction N}, d_{p\restriction N})^{N\wedge M\restriction\alpha}, \varnothing\big) \in G_{\alpha,N}.$$

It follows that

$$p^{M\restriction\alpha} \restriction N = \big((\mathcal{M}_{p\restriction N}, d_{p\restriction N})^{N\wedge M\restriction\alpha}, w_p \restriction N\big) \in G_{\alpha,N}.$$

This shows that $p^{M\restriction\alpha}$ is a condition in $\mathbb{R}^N_\alpha$, and that $p^{M\restriction\alpha}$ is $(M, \mathbb{Q}^\kappa_\alpha \restriction N)$-generic. Therefore, $p^{M\restriction\alpha} \restriction N$ is $(M, \mathbb{Q}^\kappa_\alpha \cap N)$-generic, and that

$$p^{M\restriction\alpha} \restriction N \Vdash \text{“}\check{p} \text{ is } (M[\dot{G}_{\alpha,N}], \dot{\mathbb{R}}^N_\alpha)\text{-generic”}.$$

Since $p^{M\restriction\alpha} \restriction N$ belongs to $G_{\alpha,N}$, we have, in $V[G_{\alpha,N}]$, p is $(M[G_{\alpha,N}], \mathbb{R}^N_\alpha)$-generic. $\boxed{5.44}$

We now state our key lemma.

Lemma 5.45. *Suppose $N \in \mathscr{U}$ and that $\gamma \in a(N)$. Let $G_{\gamma,N}$ be a V-generic filter over $\mathbb{Q}^\kappa_\gamma \cap N$. Then in $V[G_{\gamma,N}]$, $\mathbb{R}^N_\gamma$ has the ω_1-approximation property.*

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Proof. We will prove by induction on $\beta \leq \gamma$ that $\mathbb{R}^N_\beta$ has the ω_1-approximation property over $V[G_{\gamma,N}]$, where in the definition of $\mathbb{R}^N_\beta$, we use $G_{\gamma,N} \cap \mathbb{Q}^\kappa_\beta$ as the generic filter on $\mathbb{Q}^\kappa_\beta \cap N$. Thus we fix β and assume that the lemma holds for every ordinal in $E \cap \beta$.

Suppose that $\dot{f}$ is an $\mathbb{R}^N_\beta$-term forced by a condition p to be a function on an ordinal μ that is countably approximated in $V[G_{\gamma,N}]$. We look for an exetnsion of p that decides $\dot{f}$. We first choose a countable model M elementary in V_δ with $\delta > \lambda$ such that $E, \beta, \gamma, N, p, \mu$ and $\dot{f}$ belong to M, and that

$$\big(N \wedge (M \cap V_\lambda)\big) \restriction \gamma \in \mathcal{M}_{G_{\gamma,N}}.$$

Let also

$$M_\gamma \coloneqq (M \cap V_\lambda) \restriction \gamma \quad \text{and} \quad M_\beta \coloneqq M_\gamma \restriction \beta.$$

Using Lemma 5.44, we can extend p to a condition p^{M_β} in $\mathbb{R}^N_\beta$ so that $M_\beta \in \mathcal{M}_{p^{M_\beta}}$. Since p forces that $\dot{f}$ is ω_1-approximated in $V[G_{\gamma,N}]$, there is a condition $q \in \mathbb{R}^N_\beta$ with $q \leq p^{M_\beta}$ which decides the values of $\dot{f} \restriction M \cap \mu$. Therefore, there exists a fucntion $g : M \cap \mu \to 2$ such that

$$q \Vdash \text{“}\check{g} = \dot{f} \restriction M \cap \mu\text{”}.$$

Lemmas 5.43 and 5.44 ensures that p^{M_β} is a$(M_\gamma[G_{\gamma,N}], \mathbb{R}^N_\beta)$-generic condition, and thus $(M[G_{\gamma,N}], \mathbb{R}^N_\beta)$-generic. On the other hand, by Lemma 2.16, we are done if g is guessed in $M[G_{\gamma,N}]$. Therefore, we may assume towards a contradiction that g is not guessed in $M[G_{\gamma,N}]$, and we let

$$\alpha \coloneqq \max(\mathrm{dom}(w_q) \cap M).$$

Assume that G^N_α is a $V[G_{\gamma,N}]$-generic filter on $\mathbb{R}^N_\alpha$ containing $q \restriction \alpha$, where again we use

$$G_{\alpha,N} = G_{\gamma,N} \cap \mathbb{Q}^\kappa_\alpha$$

to form the quotient forcing $\mathbb{R}^N_\alpha$. Notice that Lemmas 5.42 and 5.44 imply that $q \restriction \alpha$ is $(M[G_{\gamma,N}], \mathbb{R}^N_\alpha)$-generic, and on the other hand by the inductive hypothesis, $\mathbb{R}^N_\alpha$ has the ω_1-approximation property over $V[G_{\gamma,N}]$. Consequently, the function g is not guessed in $M[G_{\gamma,N}][G^N_\alpha]$ by Lemma 2.17. Since G^N_α is a $V[G_{\gamma,N}]$-generic filter on $\mathbb{R}^N_\alpha$, the factorization lemma (Lemma 5.42) over $V[G_{\alpha,N}]$ yields the following equality:

$$V[G_{\alpha,N}][G_{\gamma,N}][G^N_\alpha] = V[G_{\alpha,N}][G^N_\alpha][G_{\gamma,N}].$$

Note that G^N_α is a V-generic filter on $\mathbb{Q}^\kappa_\alpha \restriction N$ and $G_{\gamma,N}$ is a V-generic filter on $\mathbb{Q}^\kappa_\gamma \cap N$, and hence we have

$$V[G_{\gamma,N}][G^N_\alpha] = V[G^N_\alpha][G_{\gamma,N}].$$

Claim 5.46. *The pair $(V[G^N_\alpha], V[G^N_\alpha][G_{\gamma,N}])$ has the ω_1-approximation property.*

Proof. Let G^N_γ be a $V[G^N_\alpha][G_{\gamma,N}]$-generic filter on $\mathbb{R}^N_\gamma$. Notice that both G^N_α and G^N_γ are also V-generic filters on $\mathbb{Q}^\kappa_\gamma$ and $\mathbb{Q}^\kappa_\alpha$, respectively. Now, Corollary 5.36 implies that the pair

$$(V[G^N_\alpha], V[G^N_\gamma])$$

has the ω_1-approximation property, and so does the pair $(V[G^N_\alpha], V[G^N_\alpha][G_{\gamma,N}])$. 5.46

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Note that if $\alpha \in N$, then $\dot{\mathbb{S}}(\dot{T}_\alpha)$ is already interpreted by $G_{\alpha,N}$, and hence by $G_{\gamma,N}$ and moreover there is a generic filter on $\mathrm{val}_{G_{\gamma,N}}(\dot{\mathbb{S}}(\dot{T}_\alpha))$ in $V[G_{\gamma,N}]$. Thus in this case, the conditions in $\dot{\mathbb{S}}(\dot{T}_\alpha)$ cannot prevent us from amalgamating conditions in $\mathbb{R}^N_\beta$ (i.e., if p' and q' are conditions in $\mathbb{R}^N_\beta$ with $\alpha \in \mathrm{dom}(w_{p'}) \cap \mathrm{dom}(w_{q'})$, we do know that $w_{p'}^{G_{\gamma,N}}(\alpha)$ and $w_{q'}^{G_{\gamma,N}}(\alpha)$ are compatible.)

For the sake of simplicity, we make the following convention.

Convention 5.47. *We let $\mathbb{Q}$ be the interpretation of $\dot{\mathbb{S}}(\dot{T}_\alpha)$ under G^N_α if α is not in N, and let it be the trivial forcing otherwise. We also let, for every $p' \in \mathbb{R}^N_\beta$, $z_{p'} = w_{p'}^{G^N_\alpha}(\alpha)$ if $\alpha \notin N$, and let it be a canonical $\mathbb{R}^N_\alpha$-name in N for the maximal condition of the trivial forcing otherwise.*

Claim 5.48. *The forcing $\mathbb{Q}$ has the ω_1-approximation property over $V[G_{\gamma,N}][G^N_\alpha]$, and z_q is $(M[G_{\gamma,N}][G^N_\alpha], \mathbb{Q})$-generic.*

Proof. We may assume that $\alpha \notin N$ and that $\mathbb{Q}$ is nontrivial; thus $\mathbb{Q}$ is the specializing forcing of a tree $T \in V[G^N_\alpha]$ of size and height ω_1 without any cofinal branches. By Claim 5.46, the tree T is still of height ω_1 and has no cofinal branches in $V[G^N_\alpha][G_{\gamma,N}]$. On the other hand, $V[G^N_\alpha][G_{\gamma,N}] = V[G_{\gamma,N}][G^N_\alpha]$. Thus in $V[G_{\gamma,N}][G^N_\alpha]$, $\mathbb{Q}$ is a c.c.c. forcing with the ω_1-approximation property. Therefore, z_q is $(M[G_{\gamma,N}][G^N_\alpha], \mathbb{Q})$-generic. 5.48

Returning to the main body of the proof, let H be a $V[G_{\gamma,N}][G^N_\alpha]$-generic filter on $\mathbb{Q}$ containing z_q. We now work in the model $V[G_{\gamma,N}][G^N_\alpha][H]$ for the rest of the proof. By Claim 5.48 and Lemma 2.15 and the fact that we have assumed earlier that g is not guessed in $M[G_{\gamma,N}][G^N_\alpha]$, we have that g is not guessed in

$$M^* := M[G_{\gamma,N}][G^N_\alpha][H].$$

Note that for every $x \in M^* \cap [\mu]^\omega$, $x \mapsto (q, g)$ witnesses the conditions below, and so a mapping $x \mapsto (q_x, g_x)$ on $[\mu]^\omega$ exists in M^* such that:

(1) $q_x \in \mathbb{Q}^\kappa_\beta \restriction N$,
(2) $(\mathcal{M}_{q_x}, d_{q_x}) \leq (\mathcal{M}_p, d_p) \restriction M$ in $\mathbb{P}^\kappa_\beta$,
(3) $M \cap \mathrm{dom}(w_q) \subseteq \mathrm{dom}(w_{q_x})$,
(4) $q_x \restriction \alpha \in G^N_\alpha$,
(5) $z_{q_x} \in H$
(6) $q_x \restriction N \in G_{\beta,N}$, and
(7) g_x is a function with $x \subseteq \mathrm{dom}(g_x)$ such that $q_x \Vdash$ "$g_x \restriction x = \dot{f} \restriction x$".

Since g is not guessed in M^*, by Lemma 2.17, there is a set $B \in M^*$ cofinal in $[\mu]^\omega$ such that for every $x \in B \cap M^*$, $g_x \nsubseteq g$. Therefore, the conditions q_x and q are incompatible in $\mathbb{R}^\beta_N$, for every $x \in B \cap M^*$.

We now draw a contradiction using the following claim.

Claim 5.49. *For every $x \in B \cap M^*$, q_x and q are compatible.*

Proof. Fix $x \in B \cap M^*$. Thus $q_x \in M^*$, and hence $q_x \in M$ by Lemmas 5.42 and 5.44 and Claim 5.48. The conditions (1)-(5) above allow us to use Lemma 5.23 to find a common extension of q_x and q in $\mathbb{Q}^\kappa_\beta \restriction N$, say $\bar{q}_0$. However, we need to show that q_x and

q are compatible in $\mathbb{R}^N_\beta$. Let $r \in G^N_\alpha$ be a common extension of $q_x \upharpoonright \alpha$ and $q \upharpoonright \alpha$, and let also $s \in G_{\beta,N}$ be a common extension of $q_x \upharpoonright N$ and $q \upharpoonright N$. Set

$$d = \{\xi \in \mathrm{dom}(w_{\bar{q}_0}) : \alpha \leq \xi \notin N\},$$

and let

$$\bar{q}_1 := (\mathcal{M}_{\bar{q}_0}, d_{\bar{q}_0}, w_{\bar{q}_0} \upharpoonright d).$$

Notice that $\bar{q}_1 \in \mathbb{Q}^\kappa_\beta \upharpoonright N$, and that

$$(r, s) \leq (\bar{q}_1 \upharpoonright \alpha, \bar{q}_1 \upharpoonright N).$$

We now apply Lemma 5.42 to (r, s) and $\bar{q}_1$ to find a condition $\bar{q} \in \mathbb{Q}^\kappa_\beta \upharpoonright N/G_{\alpha,N}$ with $\bar{q} \leq r, s, \bar{q}_1$ such that $\bar{q} \upharpoonright N \in G_{\beta,N}$. Note that $\bar{q} \leq q_x, q$ in $\mathbb{Q}^\kappa_\beta \upharpoonright N$ but $\bar{q}$ is in $\mathbb{R}^N_\beta$, and hence q and q_x are compatible in $\mathbb{R}^N_\beta$, as required. $\boxed{5.49}$

$\boxed{5.45}$

6. The proof of theorem 3.12

Suppose $\kappa < \lambda$ are supercompact cardinals. Consider the structure $V_\lambda = (V_\lambda, \in, \kappa, U)$, where U is a suitable bookkeeping function enumerating all $\mathbb{Q}^\kappa_\lambda$-terms in V_λ which are forced to be trees of size and height ω_1 without cofinal branches. Let $G_\lambda \subseteq \mathbb{Q}^\kappa_\lambda$ be a V-generic filter. Since $\mathbb{Q}^\kappa_\lambda$ is λ-c.c, every tree of size and height ω_1 that has no cofinal branches is special in $V[G_\lambda]$. Recall that by Baumgartner's [3, Theorem 7.5], if every tree of height and size ω_1 that has no branches of length ω_1 is special, then every tree of height and size ω_1 that has at most ω_1 cofinal branches is weakly special. Thus, by Proposition 2.6, to show that $\mathrm{SGM}^+(\omega_3, \omega_1)$ holds in $V[G_\lambda]$, it suffices to show that $\mathrm{GM}^+(\omega_3, \omega_1)$ holds in $V[G_\lambda]$. Indeed, we show that the ω_1-guessing models of size ω_1 witnessing $\mathrm{GM}^+(\omega_3, \omega_1)$ are internally club.

Lemma 6.1. *Let $\alpha \in E$. Suppose that $N \in \mathcal{M}_{G_\lambda}$ is a Magidor model with $\alpha \in N$. Then $N[G_\alpha]$ is an ω_1-guessing model in $V[G_\lambda]$.*

Proof. Note that the projection $N \mapsto N \upharpoonright \alpha$ is an isomorphism and is the identity on $\mathbb{Q}^\kappa_\alpha \cap N$. Thus, $N[G_\alpha]$ and $(N \upharpoonright \alpha)[G_\alpha]$ are also isomorphic. Therefore, by replacing N with $N \upharpoonright \alpha$ we may assume that it is an α-model. Let $\overline{N}$ be the transitive collapse of N, and let π be the collapse map. For convenience, let us write $\bar{\kappa}$ for κ_N. Then $\overline{N} = V_{\bar{\gamma}}$, for some $\bar{\gamma}$ with $\mathrm{cof}(\bar{\gamma}) \geq \bar{\kappa}$ and $\pi(\kappa) = \bar{\kappa}$. Let $\bar{\alpha} = \pi(\alpha)$. Since $\alpha \in N$ and N is an α-model we have

$$\mathbb{Q}_N = \mathbb{Q}^\kappa_\lambda \cap N = \mathbb{Q}^\kappa_\alpha \cap N.$$

Let $\mathbb{Q}^{\bar{\kappa}}_{\bar{\alpha}} = \pi[\mathbb{Q}^\kappa_\alpha \cap N]$, and let $p \in G_\alpha$ be such that $N \in \mathcal{M}_p$. By Corollary 5.27 p is $(N, \mathbb{Q}^\kappa_\alpha)$-strongly generic. Consequently, $G_{\alpha,N} = G_\alpha \cap N$ is V-generic over $\mathbb{Q}^\kappa_\alpha \cap N$. It follows that

$$G^{\bar{\kappa}}_{\bar{\alpha}} = \pi[G_{\alpha,N}]$$

is V-generic over $\mathbb{Q}^{\bar{\kappa}}_{\bar{\alpha}}$. Note that

$$\overline{N[G_{\alpha,N}]} = V_{\bar{\gamma}}[G^{\bar{\kappa}}_{\bar{\alpha}}] = V_{\bar{\gamma}}^{V[G_{\alpha,N}]},$$

which belongs to $V[G_{\alpha,N}]$. It is clear that the pair $(\overline{N[G_{\alpha,N}]}, V[G_{\alpha,N}])$ has the ω_1-approximation property. On the other hand, by Lemma 5.45, the quotient forcing $\mathbb{R}^N_\alpha$ has

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the ω_1-approximation property, and thus, by Proposition 5.35, the pair $(V[G_{\alpha,N}], V[G_\lambda])$ has the ω_1-approximation property. By Theorem 2.3, the pair

$$(\overline{N[G_{\alpha,N}]}, V[G_\lambda])$$

has the ω_1-approximation property. Now Theorem 2.4 implies that $N[G_{\alpha,N}]$ is an ω_1-guessing model in $V[G_\lambda]$. $\boxed{6.1}$

One can show with a similar proof that if $\mu > \lambda$ and $N \prec V_\mu$ is a κ-Magidor model containing all the relevant parameters. Then $N[G_\lambda]$ is an ω_1-guessing model in $V[G_\lambda]$. We now prove that for all $\mu > \lambda$ the set of strongly ω_1-guessing models is stationary in $\mathcal{P}_{\omega_3}(V_\mu[G_\lambda])$.

Lemma 6.2. *Suppose that $\mu > \lambda$ and $N \prec V_\mu$ is a λ-Magidor model containing all the relevant parameters. Then $N[G_\lambda]$ is a strongly ω_1-guessing model.*

Proof. Since N is a λ-Magidor model, its transitive collapse $\overline{N}$ equals $V_{\bar\gamma}$, for some $\bar\gamma < \lambda$. Let $\bar\lambda = N \cap \lambda$, which is in E. Note that $\mathrm{cof}(\bar\lambda) \geq \kappa$, and hence the transitive collapse $\overline{N[G_\lambda]}$ of $N[G_\lambda]$ equals $V_{\bar\gamma}[G_{\bar\lambda}]$. On the other hand, the pair $(V[G_{\bar\lambda}], V[G_\lambda])$ has the ω_1-approximation property by Corollary 5.36. Therefore, $V_{\bar\gamma}[G_{\bar\lambda}]$ and hence also $N[G_\lambda]$ remains an ω_1-guessing model in $V[G_\lambda]$. To see that $V_{\bar\gamma}[G_{\bar\lambda}]$ is a strongly ω_1-guessing model, fix some $\delta \in E$ with $\delta > \bar\gamma$ and $\mathrm{cof}(\delta) < \kappa$. Note that if $M \in \mathcal{M}^\delta_{G_\lambda}$ is a Magidor model with $\bar\lambda \in M$, then by Lemma 6.1 $M[G_{\bar\lambda}]$ is an ω_1-guessing model. Moreover, if $M \in \mathcal{M}^\delta_{G_\lambda}$ is a limit of such Magidor models then by Proposition 5.14,

$$M \cap V_\delta = \bigcup \{Q \cap V_\delta : Q \in_\delta M \text{ and } Q \in \mathcal{M}^\delta_{G_\lambda}\}.$$

Hence if we let $\mathcal{G}$ be the collection of the models $(M \cap V_{\bar\gamma})[G_{\bar\lambda}]$, for Magidor models $M \in \mathcal{M}^\delta_{G_\lambda}$ with $\bar\lambda \in M$, then $\mathcal{G}$ is an $\in$-increasing chain of length ω_2 which is continuous at ω_1-limits and whose union is $V_{\bar\gamma}[G_{\bar\lambda}]$. Note that every model of size ω_1 in this chain is a continuous union of an I.C. sequence. Therefore $N[G_\lambda]$ is a strongly ω_1-guessing model in $V[G_\lambda]$, as required. $\boxed{6.2}$

The following corollary is immediate and concludes the proof of Theorem 3.12.

Corollary 6.3. *The principle* $\mathrm{SGM}^+(\omega_3, \omega_1)$ *holds in* $V[G_\lambda]$.

$\boxed{6.3}$

Remark 6.4. The decoration component in the forcing conditions plays no role in our main theorem. However, without additional effort, one can easily show that the ω_1-guessing models of size ω_1 witnessing the truth of $\mathrm{SGM}^+(\omega_3, \omega_1)$ in the final model are internally club.

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References

[1] U. Abraham. *The isomorphism types of Aronszajn trees and the forcing without the generalized continuum hypothesis*. Hebrew University of Jerusalem, 1979. Thesis (Ph.D.)–Jerusalem.

[2] J. Baumgartner, J. Malitz, and W. Reinhardt. Embedding trees in the rationals. *Proc. Nat. Acad. Sci. U.S.A.*, 67:1748–1753, 1970.

[3] J. E. Baumgartner. Iterated forcing. In *Surveys in set theory*, volume 87 of *London Math. Soc. Lecture Note Ser.*, pages 1–59. Cambridge Univ. Press, Cambridge, 1983.

[4] D. Chodounský and J. Zapletal. Why Y-c.c. *Ann. Pure Appl. Logic*, 166(11):1123–1149, 2015.

[5] S. Cox and J. Krueger. Quotients of strongly proper forcings and guessing models. *The Journal of Symbolic Logic*, 81(1):264–283, 2016.

[6] S. Cox and J. Krueger. Indestructible guessing models and the continuum. *Fund. Math.*, 239(3):221–258, 2017.

[7] T. Gilton and I. Neeman. Side conditions and iteration theorems. *Appalachian Set Theory*, 2016.

[8] M. Golshani and S. Shelah. Specializing trees and answer to a question of Williams. *J. Math. Log.*, 21(1):Paper No. 2050023, 20, 2021.

[9] J. D. Hamkins. Gap forcing. *Israel Journal of Mathematics*, 125(1):237–252, Dec 2001.

[10] J. Krueger. Guessing models imply the singular cardinal hypothesis. *Proc. Amer. Math. Soc.*, 147(12):5427–5434, 2019.

[11] R. Laver. Certain very large cardinals are not created in small forcing extensions. *Ann. Pure Appl. Logic*, 149(1-3):1–6, 2007.

[12] M. Magidor. On the role of supercompact and extendible cardinals in logic. *Israel Journal of Mathematics*, 10(2):147–157, Jun 1971.

[13] W. J. Mitchell. Adding closed unbounded subsets of ω_2 with finite forcing. *Notre Dame J. Formal Logic*, 46(3):357–371, 07 2005.

[14] W. J. Mitchell. $I[\omega_2]$ can be the nonstationary ideal on $\mathrm{Cof}(\omega_1)$. *Transactions of the American Mathematical Society*, 361(2):561–601, 2009.

[15] R. Mohammadpour and B. Veličković. Guessing models and the approachability ideal. *J. Math. Log.*, 21(2):Paper No. 2150003, 35, 2021.

[16] S. Todorčević. Some combinatorial properties of trees. *Bull. London Math. Soc.*, 14(3):213–217, 1982.

[17] B. Velickovic. Iteration of semiproper forcing revisited, 2014.

[18] M. Viale. Guessing models and generalized Laver diamond. *Ann. Pure Appl. Logic*, 163(11):1660–1678, 2012.

[19] M. Viale and C. Weiß. On the consistency strength of the proper forcing axiom. *Adv. Math.*, 228(5):2672–2687, 2011.

[20] W. H. Woodin. The continuum hypothesis, the generic-multiverse of sets, and the Ω conjecture. In *Set theory, arithmetic, and foundations of mathematics: theorems, philosophies*, volume 36 of *Lect. Notes Log.*, pages 13–42. Assoc. Symbol. Logic, La Jolla, CA, 2011.

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