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Introduction: computability of the physical

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Albert Einstein encapsulated a commonly held view within the scientific community when he wrote in his book *Out of My Later Years* (Einstein 1950, page 54)

> 'When we say that we understand a group of natural phenomena, we mean that we have found a constructive theory which embraces them.'

This represents a dual challenge to the scientist: on the one hand, to explain the real world in a very basic, and if possible, mathematical, way; but on the other, to characterise the extent to which this is even possible. Recent years have seen the mathematics of computability play an increasingly vital role in pushing forward basic science and in illuminating its limitations within a creative coming together of researchers from different disciplines. This special issue of *Mathematical Structures in Computer Science* is based on the special session 'Computability of the Physical' at the International Conference *Computability in Europe 2010*, held at Ponta Delgada, Portugal, in June 2010, and it, together with the individual papers it contains, forms what we believe to be a special contribution to this exciting and developing process.

Yuri Manin, whose mathematical fame derives from his results in algebraic geometry and diophantine geometry, also has a strong interest in computability theory, which dates back at least to the early 1970s: indeed, his monograph *A Course in Mathematical Logic*[§] dedicates two (out of five) parts to computability theory and its applications. In particular, he included a more general presentation of Kolmogorov's complexity (the approach in Calude and Stay (2006) is close to Manin's formalism) and connections to physics.

Yuri Manin's paper *Renormalisation and Computation II: Time Cut–Off and the Halting Problem* in this special issue is the second part of a programme designed to find bridges between quantum field theory and computability theory. In the first part, Manin (2011), it was 'argued that both philosophy and technique of the perturbative renormalisation in quantum field theory could be meaningfully transplanted to the theory of computation', and several instances illustrating this general idea have been discussed. The paper included here addresses some of the issues raised in Manin (2011) in three contexts: a categorification of the algorithmic computations; an interpretation of the halting problem in terms of time cut–off and anytime algorithms from artificial intelligence (which

[§] The current edition is Manin (2010), but the first Russian edition dates from 1974 and the first English edition was published by Springer in 1977.

generalises a result in Calude and Stay (2008)); and a Hopf algebra renormalisation of the halting problem.

The analogy between algorithmic entropy and entropy as defined in statistical mechanics has been discussed in many papers, such as Szilard (1929), Solomonoff (1964), Kolmogorov (1965), Fredkin and Toffoli (1982), Levin and Zvonkin (1970), Chaitin (1975) and Bennett *et al.* (1999). More recently, Calude and Stay (2006) introduced the zeta function associated with a special type of Turing machine, the so-called tuatara machine, and proved that Tadaki's generalisation (Tadaki 2002) of Chaitin's halting probability is a zeta function. Tadaki's important paper Tadaki (2002), together with Calude *et al.* (2006), which followed it, form the basis for a theory of partial randomness. Zeta functions $Z(T)$ appear as partition functions and in expectation values in statistical systems, and the parameter T corresponds to an inverse temperature. This suggested an algorithmic approach to statistical mechanics.

A statistical mechanical interpretation of algorithmic information theory (AIT, for short) was developed by Kohtaro Tadaki in Tadaki (2008; 2009), where thermodynamic quantities, such as the partition function $Z(T)$, free energy $F(T)$, energy $E(T)$ and statistical mechanical entropy $S(T)$ were defined in terms of AIT. In this interpretation, the temperature T equals the partial randomness of the values of all these thermodynamic quantities, where the notion of partial randomness is a stronger representation of the compression rate using program-size complexity. A similar phenomenon holds true for each of the thermodynamic quantities above: the computability of its value at temperature T gives a sufficient condition for $T \in (0, 1)$ to be a fixed point on partial randomness.

Kohtaro Tadaki's paper *A statistical mechanical interpretation of algorithmic information theory III: composite systems and fixed points* in this special issue develops his statistical mechanical interpretation of AIT further, and studies its formal correspondence to classical statistical mechanics. The thermodynamic quantities in AIT are defined in terms of the halting set of an optimal prefix-free machine, which is a universal decoding algorithm used to define the notion of program-size complexity. Tadaki introduces the notion of the composition of prefix-free machines, which corresponds to the notion of composition of systems in normal statistical mechanics, and shows that there are infinitely many optimal prefix-free machines that give completely different sufficient conditions for each of the above thermodynamic quantities.

A different route to an algorithmic version of statistical mechanics is taken by John Baez and Michael Stay in their paper *Algorithmic thermodynamics*. If we fix a universal prefix-free Turing machine and consider the set X of programs that halt for this machine, the Baez–Stay interpretation regards X as a set of 'microstates' and treats any function on X as an 'observable'.

Given a collection of observables, this framework can be used to study the Gibbs ensemble, which maximises entropy subject to constraints on the expected values of these observables. This is illustrated in the paper by taking the log runtime, length and output of a program as observables analogous to the energy E, volume V and number of molecules N in a container of gas. The conjugate variables of these observables are used to define quantities analogous to the temperature, pressure and chemical potential: the 'algorithmic temperature' T, 'algorithmic pressure' P and 'algorithmic potential' μ. An analogue of the fundamental thermodynamic relation $dE = TdS - PdV + \mu dN$ is then derived, and used to study thermodynamic cycles analogous to those for heat engines.

The investigation of the values of T, P and μ for which the partition function converges leads to a very interesting phenomenon: at some points on the boundary of this domain of convergence, the partition function becomes uncomputable because at these points the partition function itself has non-trivial algorithmic entropy.

The paper by Gabriel Istrate, Madhav Marathe and S. S. Ravi on *Adversarial Scheduling in Discrete Models of Social Dynamics* makes an interesting contribution to an established research tradition modelling interacting (rational) agents as nodes of a graph – see, for instance, Kittock (1994). The paper fits well with the theme of this special issue, with its stress on the physical, even though the authors emphasis is on social dynamics. The authors describe their approach as forming 'part of a foundational basis for a generative approach to social science', along the lines proposed by, say, Epstein (2007), which was, in turn, a continuation of the tradition of Axtell and Epstein (1996). Pointing to a particularly innovative aspect of the paper, the referee commented that 'I admire and endorse the four-step "principles" the authors advocate to study the role of activation order on the properties of social dynamics'.

In his 2008 book on *The Trouble with Physics*, Lee Smolin declared that 'causality itself is fundamental' in an attempt to rectify foundational problems in the standard model of particle physics. Rafael Sorkin is a leading figure in the development of a general causal framework (see, for example, Dowker (2005)) for approaching such key foundational issues. His paper *Toward a 'Fundamental Theorem of Quantal Measure Theory'* in this issue is of broad interest for the foundations of quantum mechanics as part of the programme begun by Dirac and Feynman to ground quantum mechanics firmly in space–time. In particular, it takes some crucial steps towards making the path integral mathematically rigorous. Moreover, the results are particularly important for the the causal set approach to quantum gravity, for which the path integral framework is essential.

In the quest to unify physics, the path integral is, at first sight, a promising basis for quantum theory since it places quantum mechanics in the same 'category' of theory as classical systems such as Brownian motion. Indeed, basing quantum theory on the path integral means that one uses the same language for both classical and quantum theories, *viz*. the language of events in space–time, and there is no need for the problematic shift of attention between Hilbert space and space–time that occurs in the standard textbook approach. However, the promise of that first sight has not been fully realised, in part because the path integral in quantum mechanics is not yet a rigorously defined integral over path space – in contrast to Brownian motion where we have the Wiener measure. The starting point is there: the quantal measure is defined on certain 'elementary' events (indeed, exactly the same elementary events as for Brownian motion), but then the difficulties set in. In particular, there is no analogue for quantal measures of the Kolmogorov–Caratheodory extension theorem, which, essentially, gives the Wiener measure defined on the sigma algebra generated by the elementary events.

Sorkin's paper examines the question of an extension theorem for quantal measures. It focusses on discrete theories, which sidesteps some of the trouble, but not the problems associated with infinite extent in time. It is proposed that the usual condition for existence of a unique extension (that convergence of the measure be independent of how limits are taken) be weakened so that convergence is only required when the limit is taken in a certain physically meaningful way. The proposal is to use the fact that the structure of the set of elementary events reflects the nature of the system as a physical process in time to define the measure of certain limiting events for which one wants to know the quantal measure. This situation is familiar to physicists, who often use physics to construct a cut-off prescription for (mathematically) non-convergent quantities, which are then defined through a limiting process that makes physical sense. The proposal is illustrated by studying two example systems: a 2 state unitary system and a causal set theory (a proposal for quantum gravity).

The paper by Edwin Beggs, Félix Costa and John Tucker on *The Impact of Models of a Physical Oracle on Computational Power* approaches basic questions of causality in physics from a more computability-theoretic perspective. The physics itself is Newtonian, the sophistication arises from an innovative investigation of the computational content of the physics, with interactivity modelled using Turing's oracle model of computation. The paper makes substantial progress towards an understanding of what can be computed by machines that use physical systems as oracles. Earlier work by these authors analysed 'scatter machines', consisting of Turing machines with the addition of an ideal sharp vertex under a Newtonian dynamics hypothesis. They have also worked on 'collider machines', in the form of Turing machines with the addition of possible colliding masses.

A central observation is that the computational power of the systems considered is $P/poly$ in the case of scatter machines and $P/\log^*$ for collider machines. The point is that both depend appropriately on how much can be extracted from the initial condition of the system, where this is not assumed to be necessarily computable. The initial condition can be viewed as an advice function. The difference between the $P/poly$ and $P/\log^*$ results is at the heart of the current work, which is centered on a consideration of the ideal sharp vertex. Turning to the scatter machines, and considering the vertex to be any suitable smooth curve, the authors observe that the model under consideration recognises exactly $P/\log^*$. The key factor is the more accurate account of the time needed by an experiment to distinguish two real values in a scatter machine when the sharp vertex is not perfectly 'sharp'.

The final paper we will discuss is probably the most wide-ranging and philosophical, and addresses in a very direct way the theme of this special issue. Giuseppe Longo's abstract for his paper on *Incomputability in Physics and Biology* sets out a central observation underlying the argument he develops:

> 'We note that unpredictability coincides with physical randomness in both classical and quantum frames. And today, in Physics and Biology, an understanding of randomness turns out to be a key component of the intelligibility of Nature.'

One idea he discusses is the relationship between incompleteness results in mathematical logic and unpredictability in physical systems of diverse characters. A number of the relationships are substantiated by mathematical analysis in previously published papers, as summarised in Longo (2012).

This unpredictability is related to the degree of chaos in the physical system, viewed '*à la* Poincaré' , rather than in the more classical way as an embedded unpredictability based on the potentiality of the physical system for simulating a universal Turing machine. It is difficult to say new things on such a well-worn topic, but this paper is very original, and definitely bucks the trend.

The aim of comparing algorithmic randomness with unpredictability in Nature is a recurrent theme of this special issue, and connects with important questions in many areas of science and the humanities. The various interesting and innovative papers included here represent a valuable contribution to our developing understanding of the issues involved.

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