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SOME EXPLICIT RESULTS ON ONE KIND OF STICKY DIFFUSION

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Abstract

In this paper we derive several explicit results on one special sticky diffusion process which is constructed as a time-changed version of a diffusion with no sticky points. A theorem concerning the process-related Green operators defined on some nonnegative piecewise continuous functions is provided. Then, based on this theorem, we explore the distributional properties of the sticky diffusion. A financial application is presented where we compute the value of the European vanilla call option written on the underlying with sticky price dynamics.

Keywords: Sticky diffusion; first hitting time; Green's operator; distributional properties; option pricing

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Secondary 91G20

1. Introduction

Sticky reflecting diffusion processes were first discussed by Feller [13], where the second-derivative boundary condition for the infinitesimal generator of the process X is used to characterize the stickiness of X at the boundary. An interesting feature of this kind of process is that the set of times spent at the sticky boundary forms a Cantor set with positive Lebesgue measure (see [29]). We refer the reader to [14], [17], [33], and [34] for studies on sticky reflecting diffusions. For applications of sticky diffusions in storage and queueing models, see [25] and [36].

Generally, the sticky behavior can appear at any state of the diffusion, or in other words a sticky point may not necessarily be a boundary. For instance, in [1], the sticky Brownian motion on $\mathbb{R}$ was constructed as the strong limit of a sequence of time-changed random walks. Using different approaches, [4] and [11] showed the weak existence and uniqueness of solutions of the stochastic differential equation (SDE) system which governs the sticky Brownian motion. More recently, the general pathwise characterization for diffusions with sticky points in terms of an SDE and an occupation time formula was presented in [29], in which the authors derived some basic results on the distributional properties of the sticky Brownian motion such as the characteristic function. In fact, the explicit expressions for the Green's function and the transition density of the sticky Brownian motion were documented in [6] (Appendix 1, No. 8),

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and the distributions of some other quantities like the local time at the sticky point and the first hitting time can be found in [1]. Moreover, based on the method of random time change even much more complicated sticky diffusions can be constructed, and one may consult [16] (Section 2.10) for more details. For a discussion on the optimal stopping problem of the sticky Brownian motion, please refer, for example, to [8] and [30]. Other papers devoted to the investigation of sticky (reflecting) Brownian motions include [8], [12], [18], [27], and [35]. However, to the best of the authors' knowledge, many basic probabilistic properties for more general diffusions with sticky points are still unknown.

There are interesting practical applications of processes with sticky points. As illustrated in [4], if a corporation has a takeover offer at \$10, then it is very likely that the stock price will stay precisely at \$10 for a great deal of time but not confined to it. Here, \$10 can be viewed as the sticky point of the diffusion process which models the stock price. Also, mounting empirical evidence (see, e.g., [3], [5], and [31]) has discovered the existence of clustering phenomena in the prices of stocks or commodities like gold and oil, meaning that the price appears at certain levels much more often than other possible levels. Thus, diffusion with sticky points can nicely capture the price behavior in such situations.

Due to the theoretical significance and the potential applications of sticky processes, we explore in the current study some fundamental properties of one special kind of sticky diffusion and provide one of its financial applications. First, based on the result on the first hitting time, we obtain an important theorem concerning the Green operator acting on the nonnegative piecewise continuous function. The ordinary differential equation (ODE) satisfied by the resulting function under certain conditions is derived. Then, with the help of this theorem, we analyze the distributional properties of the studied sticky diffusion by finding explicit expressions for the Laplace transform of the probability mass at the sticky point, and the Green's function and the Laplace transform of the expected value of the process. As a financial application, we use this process to characterize the underlying asset price, and compute the value of the related European vanilla call option using the Laplace transform approach. The impact of the sticky point on the option value is also revealed. The central idea we follow to tackle all the above problems may also be applied to some other processes with sticky points (including sticky boundaries).

The rest of this paper is organized as follows. In Section 2 we give a brief description of the sticky diffusion studied in this paper. In Section 3 we present the main theorem, which is helpful in deriving the conclusions in the subsequent sections. The distributional properties are exhibited in Section 4. In Section 5, we deduce the Laplace transform associated with the value of the vanilla call option whose underlying price is driven by the proposed sticky process.

2. Preliminaries

Let $\{B_t,\ t \geq 0\}$ be a standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t,\ t \geq 0\}, \mathbb{P})$. For $S^* > 0$ consider on $\mathbb{R}_+$ a diffusion process satisfying the SDE

$$\mathrm{d}Y_t = \mu(Y_t)Y_t\mathrm{d}t + \sigma(Y_t)Y_t\mathrm{d}B_t, \tag{1}$$

where the coefficient functions admit

$$\mu(x) = \begin{cases} \mu_1, & x < S^*, \\ \mu_2, & x \geq S^*, \end{cases} \qquad \sigma(x) = \begin{cases} \sigma_1, & x < S^*, \\ \sigma_2, & x \geq S^*, \end{cases}$$

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with $\sigma_i > 0$ for $i = 1, 2$. The SDE (1) has a unique non-exploding solution by the observation that the existence and uniqueness of weak solutions hold for the log process $\{X_t = \log(Y_t),\ t \geq 0\}$,

$$\mathrm{d}X_t = \left[\mu(\mathrm{e}^{X_t}) - \frac{1}{2}\sigma^2(\mathrm{e}^{X_t})\right]\mathrm{d}t + \sigma(\mathrm{e}^{X_t})\mathrm{d}B_t,$$

according to Theorem 1.3 in [24], and Feller's test for explosions guarantees the non-explosion. When $\mu_1 = \mu_2$ and $\sigma_1 = \sigma_2$, $\{Y_t,\ t \geq 0\}$ degenerates to the classical geometric Brownian motion model prevailing in the field of financial engineering.

In this paper we focus on a new process $\{S_t,\ t \geq 0\}$ which will be introduced through the standard approach based on the random time change (see, e.g., [20], [23], [28], and [29]). First note that since $\{Y_t,\ t \geq 0\}$ is a continuous semimartingale, it has the symmetric semimartingale local time process $\{L_t^Y(a),\ t \geq 0\}$ at each $a > 0$ defined (see, e.g., page 150 in [12] or page 212 in [26]) by

$$L_t^Y(a) = |Y_t - a| - |Y_0 - a| - \int_0^t \operatorname{sgn}(Y_s - a)\,\mathrm{d}Y_s,$$

with sgn being the sign function of the form

$$\operatorname{sgn}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

Also, denoting by $\{\langle Y\rangle_t = \int_0^t \sigma^2(Y_s)Y_s^2\,\mathrm{d}s,\ t \geq 0\}$ the quadratic variation process of $\{Y_t,\ t \geq 0\}$, we have the occupation time formula (see, e.g., page 216 in [26])

$$\int_{\mathbb{R}} L_t^Y(a)g(a)\,\mathrm{d}a = \int_0^t g(Y_s)\,\mathrm{d}\langle Y\rangle_s$$

for any given bounded Borel measurable function g. Now introduce the random time

$$\eta(t) = t + \alpha L_t^Y(S^*), \qquad \alpha > 0. \tag{2}$$

Apparently, $\eta(t)$ is continuous and strictly increasing, which enables us to define its inverse function $\eta^{-1}(t)$. Then $\{S_t,\ t \geq 0\}$ is constructed as the time-changed version of $\{Y_t,\ t \geq 0\}$:

$$S_t = Y_{\eta^{-1}(t)}. \tag{3}$$

Furthermore, in light of the definition of the symmetric semimartingale local time, it is straightforward to show that $L_t^S(S^*) = L_{\eta^{-1}(t)}^Y(S^*)$, which in turn indicates that (see Lemma 2.2 in [29])

$$\eta^{-1}(t) = t - \alpha L_t^S(S^*). \tag{4}$$

On the other hand, using the definition (2) and the equality $\int_0^t \mathbf{1}_{\{Y_s=l\}}\,\mathrm{d}s = 0$ for any $l > 0$ and $t > 0$, we have

$$\begin{aligned}\int_0^t \mathbf{1}_{\{S_s=l\}}\,\mathrm{d}s &= \int_0^{\eta^{-1}(t)} \mathbf{1}_{\{Y_s=l\}}\,\mathrm{d}\eta(s)\\ &= \int_0^{\eta^{-1}(t)} \mathbf{1}_{\{Y_s=l\}}\,\mathrm{d}s + \alpha\int_0^{\eta^{-1}(t)} \mathbf{1}_{\{Y_s=l\}}\,\mathrm{d}L_s^Y(S^*)\\ &= \begin{cases}\alpha L_t^S(S^*), & l=S^*,\\ 0, & l\neq S^*.\end{cases}\end{aligned} \tag{5}$$

Combining this result with (4) yields

$$\eta^{-1}(t) = \int_0^t \mathbf{1}_{\{S_s\neq S^*\}}\,\mathrm{d}s. \tag{6}$$

Consequently,

$$\begin{aligned}S_t &= Y_0 + \int_0^{\eta^{-1}(t)} \mu(Y_s)Y_s\,\mathrm{d}s + \int_0^{\eta^{-1}(t)} \sigma(Y_s)Y_s\,\mathrm{d}B_s\\ &= Y_0 + \int_0^t \mu(Y_{\eta^{-1}(s)})Y_{\eta^{-1}(s)}\,\mathrm{d}\eta^{-1}(s) + \int_0^t \sigma(Y_{\eta^{-1}(s)})Y_{\eta^{-1}(s)}\,\mathrm{d}B_{\eta^{-1}(s)}\\ &= S_0 + \int_0^t \mu(S_s)S_s\mathbf{1}_{\{S_s\neq S^*\}}\,\mathrm{d}s + \int_0^t \sigma(S_s)S_s\,\mathrm{d}V_s,\end{aligned}$$

where for the last equality we use the expression (6) and the notation $V_t = B_{\eta^{-1}(t)}$. Then, we define, as in [29], the Brownian motion

$$W_t = B_{\eta^{-1}(t)} + \widetilde{B}_{\int_0^t \mathbf{1}_{\{S_s=S^*\}}\mathrm{d}s},$$

with $\{\widetilde{B}_t,\ t\geq 0\}$ being a Brownian motion that is independent of $\{B_t,\ t\geq 0\}$. In this way,

$$V_t = \int_0^t \mathbf{1}_{\{S_s\neq S^*\}}\,\mathrm{d}W_s,$$

and we finally have

$$S_t = S_0 + \int_0^t \mu(S_s)S_s\mathbf{1}_{\{S_s\neq S^*\}}\,\mathrm{d}s + \int_0^t \sigma(S_s)S_s\mathbf{1}_{\{S_s\neq S^*\}}\,\mathrm{d}W_s. \tag{7}$$

The above semimartingale representation will facilitate the discussion on the properties of $\{S_t,\ t\geq 0\}$ in the next section.

The formula (5) tells us that once the process $\{S_t,\ t\geq 0\}$ reaches the level S^* the amount of time spent at S^* will be positive, since the symmetric semimartingale local time increases only when $S=S^*$. By comparison, the process spends zero time at $l\neq S^*$. So S^* is called the 'sticky' point of $\{S_t,\ t\geq 0\}$. More generally, we may construct $\{S_t,\ t\geq 0\}$ with multiple sticky points in a manner similar to the above. In fact, [29] demonstrated that all one-dimensional sticky diffusions can be built by time-changing diffusions with no sticky points, thereby obtaining the pathwise descriptions of this kind of process. It is noteworthy that we can alternatively

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characterize a sticky diffusion through its infinitesimal generator and the generator's domain of definition (see, e.g., [15], [20], and [29]), so that the speed measure of the process puts positive masses at the sticky points. In our case, the infinitesimal generator of $\{S_t,\ t \geq 0\}$ has the form

$$(\mathcal{A}f)(x) = \frac{1}{2}\sigma^2(x)x^2 f''(x) + \mu(x)xf'(x), \qquad x \neq S^*, \tag{8}$$

and $(\mathcal{A}f)(S^*) = \lim_{x\to S^*} (\mathcal{A}f)(x)$, with domain of definition

$$\begin{aligned}\mathcal{D}(\mathcal{A}) = \big\{&f \in \mathcal{C}_b^2((0,\infty)\backslash\{S^*\}) \cap \mathcal{C}_b((0,\infty))\colon \\ &\mathcal{A}f \in \mathcal{C}_b((0,\infty)),\ f'(S^*+) - f'(S^*-) = 2\alpha(\mathcal{A}f)(S^*)\big\}.\end{aligned}$$

The scale density $\mathfrak{s}(x)$ and the speed measure $\mathfrak{m}(\mathrm{d}x)$ of $\{S_t,\ t \geq 0\}$ are given by

$$\mathfrak{s}(x) = \begin{cases} \Big(\dfrac{x}{S^*}\Big)^{-\frac{2\mu_1}{\sigma_1^2}}, & x < S^*, \\ \Big(\dfrac{x}{S^*}\Big)^{-\frac{2\mu_2}{\sigma_2^2}}, & x \geq S^*, \end{cases} \qquad \mathfrak{m}(\mathrm{d}x) = m(x)\mathrm{d}x + \alpha\delta_{S^*}(\mathrm{d}x),$$

where

$$m(x) = \begin{cases} \dfrac{2}{\sigma_1^2(S^*)^2}\Big(\dfrac{x}{S^*}\Big)^{\frac{2\mu_1}{\sigma_1^2}-2}, & x < S^*, \\ \dfrac{2}{\sigma_2^2(S^*)^2}\Big(\dfrac{x}{S^*}\Big)^{\frac{2\mu_2}{\sigma_2^2}-2}, & x \geq S^*, \end{cases}$$

and $\delta_{S^*}(\mathrm{d}x)$ denotes the delta measure with unit mass concentrated at S^*.

3. Main theorem

The aim of this section is to provide a theorem concerning the Green operators of $\{S_t,\ t \geq 0\}$. To achieve this, we need explicit expressions for the Laplace transform of the first hitting time for the process $\{S_t,\ t \geq 0\}$. Define the first hitting time $\tau_z^S = \inf\{t \geq 0\colon S_t = z\}$ and let $S_0 = Y_0 = x\ (>0)$. By $\mathbb{E}_x[\cdot]$ we denote the conditional expectation $\mathbb{E}[\cdot \mid S_0 = Y_0 = x]$. Then the result is exhibited in the following proposition.

Proposition 3.1. *Set*

$$\gamma_1^{(i)} = \frac{-(\mu_i - \frac{1}{2}\sigma_i^2) - \sqrt{(\mu_i - \frac{1}{2}\sigma_i^2)^2 + 2\theta\sigma_i^2}}{\sigma_i^2},$$

$$\gamma_2^{(i)} = \frac{-(\mu_i - \frac{1}{2}\sigma_i^2) + \sqrt{(\mu_i - \frac{1}{2}\sigma_i^2)^2 + 2\theta\sigma_i^2}}{\sigma_i^2}$$

for $i = 1, 2$. *If* $0 < L \leq x \leq U$, *then*

$$\mathbb{E}_x[e^{-\theta\tau_U^S}] = \frac{I_\theta(x)}{I_\theta(U)}, \qquad \mathbb{E}_x[e^{-\theta\tau_L^S}] = \frac{D_\theta(x)}{D_\theta(L)},$$

where

$$I_\theta(x)=\begin{cases}x^{\gamma_2^{(1)}}, & 0<x<S^*,\\ a_1x^{\gamma_1^{(2)}}+a_2x^{\gamma_2^{(2)}}, & x\ge S^*,\end{cases} \tag{9}$$

$$D_\theta(x)=\begin{cases}a_3x^{\gamma_1^{(1)}}+a_4x^{\gamma_2^{(1)}}, & 0<x<S^*,\\ x^{\gamma_1^{(2)}}, & x\ge S^*,\end{cases}$$

and the coefficients are given by

$$a_1=\frac{2\alpha\theta S^*+\gamma_2^{(1)}-\gamma_2^{(2)}}{\gamma_1^{(2)}-\gamma_2^{(2)}}(S^*)^{\gamma_2^{(1)}-\gamma_1^{(2)}},\qquad a_2=\frac{-2\alpha\theta S^*+\gamma_1^{(2)}-\gamma_2^{(1)}}{\gamma_1^{(2)}-\gamma_2^{(2)}}(S^*)^{\gamma_2^{(1)}-\gamma_2^{(2)}},$$

$$a_3=\frac{2\alpha\theta S^*+\gamma_2^{(1)}-\gamma_1^{(2)}}{\gamma_2^{(1)}-\gamma_1^{(1)}}(S^*)^{\gamma_1^{(2)}-\gamma_1^{(1)}},\qquad a_4=\frac{-2\alpha\theta S^*+\gamma_1^{(2)}-\gamma_1^{(1)}}{\gamma_2^{(1)}-\gamma_1^{(1)}}(S^*)^{\gamma_1^{(2)}-\gamma_2^{(1)}}.$$

Proof. Write $\tau_z^Y=\inf\{t\ge 0\colon Y_t=z\}$ as the first time for the process $\{Y_t,\ t\ge 0\}$ to reach z. Noting that $\tau_z^S=\eta(\tau_z^Y)$, we have

$$\mathbb{E}_x[\mathrm{e}^{-\theta\tau_U^S}]=\mathbb{E}_x[\mathrm{e}^{-\theta\tau_U^Y-\alpha\theta L^Y_{\tau_U^Y}(S^*)}] \tag{10}$$

and

$$\mathbb{E}_x[\mathrm{e}^{-\theta\tau_L^S}]=\mathbb{E}_x[\mathrm{e}^{-\theta\tau_L^Y-\alpha\theta L^Y_{\tau_L^Y}(S^*)}]. \tag{11}$$

To save space, we only provide the proof for the up-hitting case (10); the down-hitting case (11) can be addressed by similar arguments.

First, one can easily verify that the function I_θ given by (9) is a solution of the ODE

$$(\mathcal{A}f)(x)=\theta f(x),\qquad x\in(0,\infty)\backslash\{S^*\},$$

where the operator $\mathcal{A}$ is defined in (8). Also, I_θ belongs to $\mathcal{C}([0,\infty))\cap\mathcal{C}^2((0,\infty)\backslash\{S^*\})$ (noting that $\gamma_2^{(1)}>0$) and satisfies

$$\frac{1}{2}I'_\theta(S^*+)-\frac{1}{2}I'_\theta(S^*-)-\alpha\theta I_\theta(S^*)=0. \tag{12}$$

Now, since I_θ can be written as the difference of two convex functions (see Problem 3.6.24 in [22]), the generalized Itô formula (i.e., formula (4.3) on page 150 in [12]) can be applied:

$$\begin{aligned}I_\theta(Y_t)=I_\theta(x)+\int_0^t(\mathcal{A}I_\theta)(Y_s)\mathrm{d}s+\int_0^t\sigma(Y_s)Y_sI'_\theta(Y_s)\mathrm{d}B_s\\ +\frac{1}{2}[I'_\theta(S^*+)-I'_\theta(S^*-)]L_t^Y(S^*),\end{aligned}$$

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where $(\mathcal{A}I_\theta)(S^*) = \lim_{x\to S^*}(\mathcal{A}I_\theta)(x) = \theta I_\theta(S^*)$, and I'_θ should be understood as the symmetric derivative of I_θ with $I'_\theta(S^*) = [I'_\theta(S^*+) + I'_\theta(S^*-)]/2$. Hence,

$$
\begin{aligned}
\mathrm{e}^{-\theta t-\alpha\theta L_t^Y(S^*)} I_\theta(Y_t) = I_\theta(x) &+ \int_0^t \mathrm{e}^{-\theta s-\alpha\theta L_s^Y(S^*)} I_\theta(Y_s)\,\mathrm{d}[-\theta s - \alpha\theta L_s^Y(S^*)] \\
&+ \int_0^t \mathrm{e}^{-\theta s-\alpha\theta L_s^Y(S^*)}\,\mathrm{d}I_\theta(Y_s) \\
&= I_\theta(x) + [\tfrac{1}{2} I'_\theta(S^*+) - \tfrac{1}{2} I'_\theta(S^*-) - \alpha\theta I_\theta(S^*)] \\
&\quad \times \int_0^t \mathrm{e}^{-\theta s-\alpha\theta L_s^Y(S^*)}\,\mathrm{d}L_s^Y(S^*) \\
&\quad + \int_0^t \mathrm{e}^{-\theta s-\alpha\theta L_s^Y(S^*)} \sigma(Y_s) Y_s I'_\theta(Y_s)\,\mathrm{d}B_s \\
&= I_\theta(x) + \int_0^t \mathrm{e}^{-\theta s-\alpha\theta L_s^Y(S^*)} \sigma(Y_s) Y_s I'_\theta(Y_s)\,\mathrm{d}B_s,
\end{aligned}
\tag{13}
$$

where for the last equality we use (12). Next we show that the Laplace transform of τ_U^S can be expressed in terms of $I_\theta(x)$. In fact, it follows from (13) that

$$\{Z_{t\wedge\tau_U^Y} = \mathrm{e}^{-\theta(t\wedge\tau_U^Y)-\alpha\theta L^Y_{t\wedge\tau_U^Y}(S^*)} I_\theta(Y_{t\wedge\tau_U^Y}),\ t \geq 0\}$$

is a local martingale. Also, since $Y_0 = x \leq U$, we have $0 < Y_{t\wedge\tau_U^Y} \leq U$. Therefore, $\sup_{t\geq 0,\omega\in\Omega} |Z_{t\wedge\tau_U^Y}(\omega)|$ is bounded above, implying that $\{Z_{t\wedge\tau_U^Y},\ t\geq 0\}$ is actually a martingale. Then the dominated convergence theorem leads to the desired result via

$$
\begin{aligned}
I_\theta(x) &= \lim_{t\to\infty} \mathbb{E}_x[\mathrm{e}^{-\theta(t\wedge\tau_U^Y)-\alpha\theta L^Y_{t\wedge\tau_U^Y}(S^*)} I_\theta(Y_{t\wedge\tau_U^Y})] \\
&= I_\theta(U)\mathbb{E}_x[\mathrm{e}^{-\theta\tau_U^Y-\alpha\theta L^Y_{\tau_U^Y}(S^*)} \mathbf{1}_{\{\tau_U^Y<\infty\}}] \\
&= I_\theta(U)\mathbb{E}_x[\mathrm{e}^{-\theta\tau_U^S}].
\end{aligned}
$$

□

In the study of diffusion processes or their applications, it is usually of vital importance to consider the Green operators defined on some function space $\mathcal{D}$,

$$G_\theta : f \in \mathcal{D} \to \mathbb{E}_\bullet\bigg[\int_0^\infty \mathrm{e}^{-\theta t} f(X_t)\,\mathrm{d}t\bigg], \quad \theta > 0,$$

for some process $\{X_t,\ t\geq 0\}$. In the theorem below, the ODE satisfied by the function $G_\theta f$ associated with $\{S_t,\ t\geq 0\}$ under certain restrictive conditions is provided. One way to obtain the results in subsequent sections is to utilize this theorem.

Theorem 3.1. *Let $f:[0,\infty)\to[0,\infty)$ be a piecewise continuous function with the discontinuity set D, which does not include the point S^*. Assume that v is a function of the class $\mathcal{C}([0,\infty)) \cap \mathcal{C}^1([0,\infty)\backslash\{S^*\}) \cap \mathcal{C}^2((0,\infty)\backslash(D\cup\{S^*\}))$ with finite left and right derivatives of orders one and two at S^*, and satisfies*

$$|v(x)| \leq H_1 + H_2 x, \quad x \in [\delta, \infty), \tag{14}$$

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for some constants $\delta > 0$, $H_i \geq 0$, $i = 1, 2$. *In addition, for some nonnegative constant* Q, *if* v *satisfies*

$$\begin{cases} \frac{1}{2}\sigma^2(x)x^2 v''(x) + \mu(x)xv'(x) + f(x) = \theta v(x), \ x \in (0, \infty)\backslash(D \cup \{S^*\}), \ \theta > Q, & (15) \\ v'(S^* +) - v'(S^* -) = 2\alpha[\theta v(S^*) - f(S^*)], & (16) \end{cases}$$

then it must be nonnegative and admit the following stochastic representation:

$$v(x) = \mathbb{E}_x\left[\int_0^\infty e^{-\theta t} f(S_t)\, dt\right], \quad x \in (0, \infty), \tag{17}$$

for any $\theta > \max\{Q, \mu_2\}$. *In particular, when* $H_2 = 0$ *(17) holds for all* $\theta > Q$.

Proof. Recalling the definition of the operator $\mathcal{A}$ in (8), we conclude from the ODE (15) as well as the continuities at S^* of f and v that $\mathcal{A}v$ is continuous at S^*. Then, on the basis of the semimartingale representation (7), an application of the generalized Itô formula (i.e., formula (4.3) on page 150 in [12]) to the process $\{v(S_t),\ t \geq 0\}$ produces

$$\begin{aligned} v(S_t) &= v(x) + \int_0^t (\mathcal{A}v)(S_s)\mathbf{1}_{\{S_s \neq S^*\}}\mathrm{d}s + \int_0^t \sigma(S_s)S_s v'(S_s)\mathbf{1}_{\{S_s \neq S^*\}}\mathrm{d}W_s \\ &\quad + \frac{1}{2}[v'(S^* +) - v'(S^* -)]L_t^S(S^*) \\ &= v(x) + \int_0^t (\mathcal{A}v)(S_s)\mathrm{d}s + \int_0^t \sigma(S_s)S_s v'(S_s)\mathbf{1}_{\{S_s \neq S^*\}}\mathrm{d}W_s \\ &\quad + \Big[\frac{1}{2}v'(S^* +) - \frac{1}{2}v'(S^* -) - \alpha(\mathcal{A}v)(S^*)\Big]L_t^S(S^*) \\ &= v(x) + \int_0^t (\mathcal{A}v)(S_s)\mathrm{d}s + \int_0^t \sigma(S_s)S_s v'(S_s)\mathbf{1}_{\{S_s \neq S^*\}}\mathrm{d}W_s, \end{aligned}$$

where the second equality comes from (5), and the third equality comes from (16) and the fact that

$$(\mathcal{A}v)(S^*) = \theta v(S^*) - f(S^*).$$

Thus, given (15), it follows immediately that

$$\begin{aligned} \mathrm{e}^{-\theta t}v(S_t) &= v(x) + \int_0^t \mathrm{e}^{-\theta s}[(\mathcal{A}v)(S_s) - \theta v(S_s)]\mathrm{d}s \\ &\quad + \int_0^t \mathrm{e}^{-\theta s}\sigma(S_s)S_s v'(S_s)\mathbf{1}_{\{S_s \neq S^*\}}\mathrm{d}W_s \\ &= v(x) - \int_0^t \mathrm{e}^{-\theta s}f(S_s)\mathrm{d}s + \int_0^t \mathrm{e}^{-\theta s}\sigma(S_s)S_s v'(S_s)\mathbf{1}_{\{S_s \neq S^*\}}\mathrm{d}W_s. \end{aligned} \tag{18}$$

What is more, noting that for any $n > x$ there exists a constant $V > 0$, which may depend on n, such that

$$\mathbb{E}_x\left[\int_0^\infty \mathbf{1}_{\{S_s \neq S^*, s \leq \tau_n^S\}}\mathrm{e}^{-2\theta s}\sigma^2(S_s)S_s^2(v'(S_s))^2\,\mathrm{d}s\right] \leq Vn^2,$$

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it follows that the process $\{\int_0^{t\wedge\tau_n^S} \mathrm{e}^{-\theta s}\sigma(S_s)S_s v'(S_s)\mathbf{1}_{\{S_s\neq S^*\}}\,\mathrm{d}W_s,\ t\geq 0\}$ is a martingale. Therefore, (18) results in

$$v(x)=\mathbb{E}_x\bigg[\int_0^{t\wedge\tau_n^S}\mathrm{e}^{-\theta s}f(S_s)\mathrm{d}s\bigg]+\mathbb{E}_x[\mathrm{e}^{-\theta(t\wedge\tau_n^S)}v(S_{t\wedge\tau_n^S})].$$

Thanks to the nonnegativeness of f, it holds that

$$\lim_{t\to\infty}\mathbb{E}_x\bigg[\int_0^{t\wedge\tau_n^S}\mathrm{e}^{-\theta s}f(S_s)\mathrm{d}s\bigg]=\mathbb{E}_x\bigg[\int_0^{\tau_n^S}\mathrm{e}^{-\theta s}f(S_s)\mathrm{d}s\bigg]$$

by the monotone convergence theorem. Also, since $0<S_{t\wedge\tau_n^S}\leq n$ for $n>x$ and the function v is continuous in $[0, n]$, we have by the dominated convergence theorem that

$$\lim_{t\to\infty}\mathbb{E}_x[\mathrm{e}^{-\theta(t\wedge\tau_n^S)}v(S_{t\wedge\tau_n^S})]=\mathbb{E}_x[\mathrm{e}^{-\theta\tau_n^S}v(S_{\tau_n^S})].$$

Hence,

$$v(x)=\mathbb{E}_x\bigg[\int_0^{\tau_n^S}\mathrm{e}^{-\theta s}f(S_s)\mathrm{d}s\bigg]+\mathbb{E}_x[\mathrm{e}^{-\theta\tau_n^S}v(S_{\tau_n^S})].$$

Then, again from the monotone convergence theorem, we deduce that the first term on the right-hand side converges to $\mathbb{E}_x[\int_0^\infty \mathrm{e}^{-\theta s}f(S_s)\mathrm{d}s]$ as $n\to\infty$. Now, if the second term converges to 0 we can obtain (17). This is indeed the case because by virtue of Proposition 3.1 and the assumption (14), for any $n\geq\max\{\delta, S^*\}$ and $\theta>\max\{Q, \mu_2\}$,

$$\begin{aligned}|\mathbb{E}_x[\mathrm{e}^{-\theta\tau_n^S}v(S_{\tau_n^S})]| &\leq |v(n)|\mathbb{E}_x[\mathrm{e}^{-\theta\tau_n^S}]\\ &\leq (H_1+H_2n)\frac{I_\theta(x)}{a_1n^{\gamma_1^{(2)}}+a_2n^{\gamma_2^{(2)}}}\\ &\to 0\end{aligned}$$

as $n\to\infty$, thanks to

$$\gamma_2^{(2)}-1=\frac{-(\mu_2+\frac{1}{2}\sigma_2^2)+\sqrt{(\mu_2+\frac{1}{2}\sigma_2^2)^2+2\sigma_2^2(\theta-\mu_2)}}{\sigma_2^2}>0.$$

In particular, when $H_2=0$ only $\theta>Q$ is required. This completes the proof. □

We state that the growth rate (14) is satisfied by the respective v for all the examples in the subsequent sections.

4. Distributional properties

In this section we show how to derive some distributional properties associated with $\{S_t,\ t\geq 0\}$ by using Theorem 3.1. To start with, we prove that the distribution of $\{S_t,\ t\geq 0\}$ may contain a point mass at the sticky point S^* by presenting the Laplace transform of $\mathbb{P}_x(S_t=S^*)$ with respect to t. Similarly, the probability mass can also appear at the absorbing boundary (if it exists) of a diffusion process (see [23]).

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Proposition 4.1.

$$\int_0^\infty e^{-\theta t}\mathbb{P}_x(S_t=S^*)dt=\begin{cases} q_1x^{\gamma_2^{(1)}}, & 0<x<S^*,\\ q_2x^{\gamma_1^{(2)}}, & x\ge S^*,\end{cases}$$

where

$$q_1=\frac{2\alpha(S^*)^{1-\gamma_2^{(1)}}}{\gamma_2^{(1)}-\gamma_1^{(2)}+2\alpha\theta S^*},\qquad q_2=\frac{2\alpha(S^*)^{1-\gamma_1^{(2)}}}{\gamma_2^{(1)}-\gamma_1^{(2)}+2\alpha\theta S^*}.$$

Proof. Note that $\mathbb{P}_x(S_t=S^*)=\mathbb{E}_x(\mathbf{1}_{\{S_t=S^*\}})$ and we cannot apply Theorem 3.1 directly because the function

$$\zeta(x)\triangleq\begin{cases}1, & x=S^*,\\ 0, & x\neq S^*\end{cases}$$

is discontinuous at S^*. Instead, we turn to computing, for a sufficiently small ϵ, the expectation $\mathbb{E}_x[\int_0^\infty \mathrm{e}^{-\theta t}\mathbf{1}_{\{S^*-\epsilon<S_t<S^*+\epsilon\}}\,\mathrm{d}t]$; then the result follows by letting $\epsilon\to 0$. To be specific, denote

$$v_\epsilon(x)=\begin{cases} h_1x^{\gamma_2^{(1)}}, & 0<x<S^*-\epsilon,\\ \dfrac{1}{\theta}+h_2x^{\gamma_1^{(1)}}+h_3x^{\gamma_2^{(1)}}, & S^*-\epsilon\le x<S^*,\\ \dfrac{1}{\theta}+h_4x^{\gamma_1^{(2)}}+h_5x^{\gamma_2^{(2)}}, & S^*\le x<S^*+\epsilon,\\ h_6x^{\gamma_1^{(2)}}, & x\ge S^*+\epsilon,\end{cases}$$

which is a solution of the ODE (15) on $[0,\infty)\backslash\{S^*\pm\epsilon, S^*\}$ with

$$f(x)=\begin{cases}1, & x\in(S^*-\epsilon,S^*+\epsilon),\\ 0, & x\in[0,\infty)\backslash(S^*-\epsilon,S^*+\epsilon).\end{cases}$$

By assuming the continuity of v_ϵ at the points $S^*\pm\epsilon$ and S^*, the continuity of v'_ϵ at the points $S^*\pm\epsilon$, and the condition (16), we can determine the coefficients h_i, $i=1,2,\ldots,6$. In particular, we have

$$\begin{aligned}h_1={}&\frac{(S^*)^{-\gamma_2^{(1)}}(S^*-\epsilon)^{-\gamma_1^{(1)}-\gamma_2^{(1)}}(S^*+\epsilon)^{-\gamma_2^{(2)}}}{\theta(\gamma_2^{(1)}-\gamma_1^{(1)})(\gamma_2^{(1)}-\gamma_1^{(2)}+2\alpha\theta S^*)}[-\gamma_1^{(2)}(\gamma_1^{(1)}-\gamma_2^{(1)})(S^*)^{\gamma_2^{(2)}}(S^*-\epsilon)^{\gamma_1^{(1)}+\gamma_2^{(1)}}\\ &-\gamma_1^{(1)}(\gamma_2^{(1)}-\gamma_1^{(2)})(S^*)^{\gamma_2^{(1)}}(S^*-\epsilon)^{\gamma_1^{(1)}}(S^*+\epsilon)^{\gamma_2^{(2)}}+\gamma_2^{(1)}(\gamma_1^{(1)}-\gamma_1^{(2)})(S^*)^{\gamma_1^{(1)}}(S^*-\epsilon)^{\gamma_2^{(1)}}\\ &\times(S^*+\epsilon)^{\gamma_2^{(2)}}-2\alpha\theta\gamma_1^{(1)}(S^*)^{1+\gamma_2^{(1)}}(S^*-\epsilon)^{\gamma_1^{(1)}}(S^*+\epsilon)^{\gamma_2^{(2)}}+2\alpha\theta\gamma_2^{(1)}(S^*)^{1+\gamma_1^{(1)}}(S^*-\epsilon)^{\gamma_2^{(1)}}\\ &\times(S^*+\epsilon)^{\gamma_2^{(2)}}],\\ h_6={}&\frac{(S^*)^{-\gamma_1^{(2)}}(S^*-\epsilon)^{-\gamma_1^{(1)}}(S^*+\epsilon)^{-\gamma_1^{(2)}-\gamma_2^{(2)}}}{\theta(\gamma_1^{(2)}-\gamma_2^{(2)})(\gamma_1^{(2)}-\gamma_2^{(1)}-2\alpha\theta S^*)}[\gamma_2^{(1)}(\gamma_1^{(2)}-\gamma_2^{(2)})(S^*)^{\gamma_1^{(1)}}(S^*+\epsilon)^{\gamma_1^{(2)}+\gamma_2^{(2)}}\\ &+\gamma_1^{(2)}(\gamma_2^{(2)}-\gamma_2^{(1)})(S^*)^{\gamma_2^{(2)}}(S^*-\epsilon)^{\gamma_1^{(1)}}(S^*+\epsilon)^{\gamma_1^{(2)}}-\gamma_2^{(2)}(\gamma_1^{(2)}-\gamma_2^{(1)})(S^*)^{\gamma_1^{(2)}}(S^*-\epsilon)^{\gamma_1^{(1)}}\\ &\times(S^*+\epsilon)^{\gamma_2^{(2)}}-2\alpha\theta\gamma_1^{(2)}(S^*)^{1+\gamma_2^{(2)}}(S^*-\epsilon)^{\gamma_1^{(1)}}(S^*+\epsilon)^{\gamma_1^{(2)}}+2\alpha\theta\gamma_2^{(2)}(S^*)^{1+\gamma_1^{(2)}}(S^*-\epsilon)^{\gamma_1^{(1)}}\\ &\times(S^*+\epsilon)^{\gamma_2^{(2)}}].\end{aligned}$$

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It can be readily checked that v_ϵ satisfies (14) where $\delta = S^* + \epsilon$, $H_1 = h_6(S^* + \epsilon)^{\gamma_1^{(2)}}$, and $H_2 = 0$. Consequently,

$$v_\epsilon(x) = \mathbb{E}_x\bigg[\int_0^\infty \mathrm{e}^{-\theta t}\mathbf{1}_{\{S^*-\epsilon<S_t<S^*+\epsilon\}}\,\mathrm{d}t\bigg]$$

by Theorem 3.1, and thus q_1 and q_2 are the limit values of h_1 and h_6 as $\epsilon \to 0$, respectively. □

Note that when $\alpha = 0$ both q_1 and q_2 become zero, suggesting that $\mathbb{P}_x(S_t = S^*) = 0$.

Now set $p(t; x, y)$ to be the transition density of $\{S_t,\ t \geq 0\}$ in the sense that

$$\mathbb{P}_x(S_t \in \mathrm{d}y) = p(t; x, y)\,\mathrm{d}y + \mathbb{P}_x(S_t = y)\delta_{S^*}(\mathrm{d}y). \tag{19}$$

Here we work with the transition density with respect to the Lebesgue measure (this setting was also adopted, for example, in [2], [9], and [10]), while [6] and [20] defined it with respect to the speed measure. For the relationship between these two definitions, see the remark at the end of this section. Since it is difficult to get the explicit form of $p(t; x, y)$, we turn to the characterization of the Green's function defined as the Laplace transform of $p(t; x, y)$:

$$G_\theta(x, y) = \int_0^\infty \mathrm{e}^{-\theta t} p(t; x, y)\,\mathrm{d}t. \tag{20}$$

Proposition 4.2.

1. *For* $y > S^*$,

$$G_\theta(x, y) = \begin{cases} p_1(y)x^{\gamma_2^{(1)}}, & 0 < x < S^*, \\ p_2(y)x^{\gamma_1^{(2)}} + p_3(y)x^{\gamma_2^{(2)}}, & S^* \leq x \leq y, \\ p_4(y)x^{\gamma_1^{(2)}}, & x > y, \end{cases}$$

where

$$p_1(y) = \frac{\gamma_1^{(2)}\gamma_2^{(2)} y^{-\gamma_2^{(2)}-1}(S^*)^{\gamma_2^{(2)}-\gamma_2^{(1)}}}{\theta(\gamma_1^{(2)} - \gamma_2^{(1)} - 2\alpha\theta S^*)},$$

$$p_2(y) = \frac{\gamma_1^{(2)}\gamma_2^{(2)} y^{-\gamma_2^{(2)}-1}(S^*)^{\gamma_2^{(2)}-\gamma_1^{(2)}}(\gamma_2^{(1)} - \gamma_2^{(2)} + 2\alpha\theta S^*)}{\theta(\gamma_1^{(2)} - \gamma_2^{(2)})(\gamma_1^{(2)} - \gamma_2^{(1)} - 2\alpha\theta S^*)},$$

$$p_3(y) = \frac{\gamma_1^{(2)}\gamma_2^{(2)} y^{-\gamma_2^{(2)}-1}}{\theta(\gamma_1^{(2)} - \gamma_2^{(2)})},$$

$$p_4(y) = \frac{\gamma_1^{(2)}\gamma_2^{(2)} y^{-\gamma_1^{(2)}-\gamma_2^{(2)}-1}(S^*)^{-\gamma_1^{(2)}}}{\theta(\gamma_1^{(2)} - \gamma_2^{(2)})(\gamma_1^{(2)} - \gamma_2^{(1)} - 2\alpha\theta S^*)}[y^{\gamma_2^{(2)}}(S^*)^{\gamma_1^{(2)}}(\gamma_1^{(2)} - \gamma_2^{(1)} - 2\alpha\theta S^*) + y^{\gamma_1^{(2)}}(S^*)^{\gamma_2^{(2)}}(\gamma_2^{(1)} - \gamma_2^{(2)} + 2\alpha\theta S^*)];$$

2. *for* $0 < y < S^*$,

$$G_\theta(x, y) = \begin{cases} p_5(y)x^{\gamma_2^{(1)}}, & 0 < x \leq y, \\ p_6(y)x^{\gamma_1^{(1)}} + p_7(y)x^{\gamma_2^{(1)}}, & y < x < S^*, \\ p_8(y)x^{\gamma_1^{(2)}}, & x \geq S^*, \end{cases}$$

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where

$$p_5(y) = \frac{\gamma_1^{(1)}\gamma_2^{(1)} y^{-\gamma_1^{(1)}-\gamma_2^{(1)}-1}(S^*)^{-\gamma_2^{(1)}}}{\theta(\gamma_1^{(1)}-\gamma_2^{(1)})(\gamma_2^{(1)}-\gamma_1^{(2)}+2\alpha\theta S^*)}[y^{\gamma_1^{(1)}}(S^*)^{\gamma_2^{(1)}}(\gamma_2^{(1)}-\gamma_1^{(2)}+2\alpha\theta S^*)$$
$$+ y^{\gamma_2^{(1)}}(S^*)^{\gamma_1^{(1)}}(\gamma_1^{(2)}-\gamma_1^{(1)}-2\alpha\theta S^*)],$$
$$p_6(y) = \frac{\gamma_1^{(1)}\gamma_2^{(1)} y^{-\gamma_1^{(1)}-1}}{\theta(\gamma_1^{(1)}-\gamma_2^{(1)})},$$
$$p_7(y) = \frac{\gamma_1^{(1)}\gamma_2^{(1)} y^{-\gamma_1^{(1)}-1}(S^*)^{\gamma_1^{(1)}-\gamma_2^{(1)}}(\gamma_1^{(2)}-\gamma_1^{(1)}-2\alpha\theta S^*)}{\theta(\gamma_1^{(1)}-\gamma_2^{(1)})(\gamma_2^{(1)}-\gamma_1^{(2)}+2\alpha\theta S^*)},$$
$$p_8(y) = \frac{\gamma_1^{(1)}\gamma_2^{(1)} y^{-\gamma_1^{(1)}-1}(S^*)^{\gamma_1^{(1)}-\gamma_1^{(2)}}}{\theta(\gamma_1^{(2)}-\gamma_2^{(1)}-2\alpha\theta S^*)}.$$

Proof. Consider the Laplace transform $u(x, y) = \int_0^\infty \mathrm{e}^{-\theta t}\mathbb{P}_x(S_t \le y)\,\mathrm{d}t$, for $x, y > 0$. Fubini's theorem implies that

$$u(x, y) = \int_0^y \int_0^\infty \mathrm{e}^{-\theta t} p(t; x, z)\,\mathrm{d}t\,\mathrm{d}z + \mathbf{1}_{\{y \ge S^*\}} \int_0^\infty \mathrm{e}^{-\theta t}\mathbb{P}_x(S_t = S^*)\,\mathrm{d}t,$$

which signifies that

$$G_\theta(x, y) = \frac{\partial u}{\partial y}(x, y) \tag{21}$$

when $y \ne S^*$. Now, similar to the discussion in the proof of Proposition 4.1, we can establish an expression for $u(x, y)$ by the method of undetermined coefficients – solving the ODE (15) accompanied by (16) with

$$f(x) = \begin{cases} 1, & x \in [0, y], \\ 0, & x \in (y, \infty), \end{cases}$$

and using the conditions that, for the variable x, $u(x, y)$ is continuous at y and S^*, and that $\partial u(x, y)/\partial x$ is continuous at y. Finally, the desired result can be obtained in closed form by virtue of the relation (21). □

Next, we offer the Laplace transform of the expected value of S_t as another application of Theorem 3.1.

Proposition 4.3.

$$\int_0^\infty e^{-\theta t}\mathbb{E}_x[S_t]\,dt = \begin{cases} -\dfrac{x}{\mu_1-\theta} + b_1 x^{\gamma_2^{(1)}}, & 0 < x < S^*, \\ -\dfrac{x}{\mu_2-\theta} + b_2 x^{\gamma_1^{(2)}}, & x \ge S^*, \end{cases}$$

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where

$$b_1 = \frac{(S^*)^{1-\gamma_2^{(1)}}}{(\theta-\mu_1)(\theta-\mu_2)(\gamma_2^{(1)}-\gamma_1^{(2)}+2\alpha\theta S^*)}[\mu_2(-\gamma_1^{(2)}+1+2\alpha\mu_1 S^*) + \mu_1(\gamma_1^{(2)}-1-2\alpha\theta S^*)],$$

$$b_2 = \frac{(S^*)^{1-\gamma_1^{(2)}}}{(\theta-\mu_1)(\theta-\mu_2)(\gamma_2^{(1)}-\gamma_1^{(2)}+2\alpha\theta S^*)}[\mu_2(-\gamma_2^{(1)}+1-2\alpha\theta S^*) + \mu_1(\gamma_2^{(1)}-1+2\alpha\mu_2 S^*)].$$

Proof. Denote $v(x)=\int_0^\infty \mathrm{e}^{-\theta t}\mathbb{E}_x[S_t]\,\mathrm{d}t$ and let $f(x)=x$ in Theorem 3.1. The result follows by solving the ODE (15) under the auxiliary conditions including (16) and the continuity of v at S^*. □

To close this section, we remark that the above distributional properties can also be deduced from general diffusion theory (see, e.g., [6] and [20]). Denote by $\hat{p}(t;x,y)$ the transition density of $\{S_t,\ t\geq 0\}$ with respect to the speed measure and $\hat{G}_\theta(x,y)$ the corresponding Green's function. That is to say,

$$\mathbb{P}_x(S_t\in \mathrm{d}y)=\hat{p}(t;x,y)\mathfrak{m}(\mathrm{d}y), \qquad \hat{G}_\theta(x,y)=\int_0^\infty \mathrm{e}^{-\theta t}\hat{p}(t;x,y)\,\mathrm{d}t.$$

Recalling the definitions of $p(t;x,y)$ and $G_\theta(x,y)$ – see (19) and (20) – we have, for $t>0$ and $x>0$,

$$\hat{p}(t;x,y)=\begin{cases}\dfrac{1}{m(y)}p(t;x,y), & y>0,\ y\neq S^*,\\[2mm] \dfrac{1}{\alpha}\mathbb{P}_x(S_t=S^*), & y=S^*;\end{cases}$$

$$\hat{G}_\theta(x,y)=\begin{cases}\dfrac{1}{m(y)}G_\theta(x,y), & y>0,\ y\neq S^*,\\[2mm] \dfrac{1}{\alpha}\displaystyle\int_0^\infty \mathrm{e}^{-\theta t}\mathbb{P}_x(S_t=S^*)\,\mathrm{d}t, & y=S^*.\end{cases}$$

On the other hand, it is not hard to verify that I_θ and D_θ given in Proposition 3.1 are increasing and decreasing functions, respectively. Then, from [6] (page 19, No. 11), the Wronskian defined by

$$w_\theta = I_\theta^+(x)D_\theta(x)-I_\theta(x)D_\theta^+(x)=I_\theta^-(x)D_\theta(x)-I_\theta(x)D_\theta^-(x)$$

is a constant independent of x. Here we use the notations

$$f^+(x)=\lim_{\epsilon\downarrow 0}\frac{f(x+\epsilon)-f(x)}{\mathcal{S}(x+\epsilon)-\mathcal{S}(x)}, \qquad f^-(x)=\lim_{\epsilon\downarrow 0}\frac{f(x)-f(x-\epsilon)}{\mathcal{S}(x)-\mathcal{S}(x-\epsilon)}$$

for a given function f, where $\mathcal{S}(x) = \int^x \mathfrak{s}(y)\, \mathrm{d}y$ represents the scale function of $\{S_t,\ t \geq 0\}$. The Green's function $\hat{G}_\theta(x, y)$ can now be obtained from the relation

$$\hat{G}_\theta(x, y) = \frac{1}{w_\theta} I_\theta(x \wedge y) D_\theta(x \vee y).$$

The results in Propositions 4.1 and 4.2 thus follow. Additionally, the function v in (17) can also be expressed as

$$v(x) = \int_0^\infty f(y) \hat{G}_\theta(x, y) \mathfrak{m}(\mathrm{d}y), \tag{22}$$

based on which we may calculate directly the Laplace transform in Proposition 4.3 by letting $f(x) = x$. Similarly, the forthcoming Proposition 5.1 corresponds to the case when $f(x) = (x - K)_+ = \max(x - K, 0)$.

5. A financial application

In much of the literature on option pricing where the underlying prices are described by ordinary diffusions, there is a commonly used but hidden assumption that no price clustering phenomenon exists. However, if a certain price level, say S^*, appears obviously more often than other prices over a long time horizon, then it is reasonable to regard S^* as a sticky point of the price process. In this section we take into account this factor by modeling the underlying price by the sticky diffusion $\{S_t,\ t \geq 0\}$ defined in Section 2. The use of such a process has two advantages. First, it incorporates the price clustering effect through the sticky point. Secondly, the discontinuities of the coefficient functions in the semimartingale representation (7) for $\{S_t,\ t \geq 0\}$ coincide with the empirical observation that the underlying price may show notable differences in the conditional moments of returns and the conditional variance after S^* is crossed; see [21].

A European vanilla call option is a contract that gives the buyer of that option the right, but not the obligation, to buy an underlying asset at a pre-specified price (formally called the strike price) on the expiration date. For now we concentrate on the valuation of this option written on the commodity whose price dynamics obeys $\{S_t,\ t \geq 0\}$ under the pricing (or risk-neutral) measure. Here we do not need the discounted commodity price $\{\mathrm{e}^{-rt} S_t,\ t \geq 0\}$ to be a martingale under the pricing measure, as implied, for example, by [19] (Section 33.4) and [7] (page 108, paragraph 5). In classical dynamic hedging (see, e.g., [19], Section 14.6), the instantaneously riskless portfolio consisting of the underlying and the option must have the same rate of return as the risk-free interest rate, which indicates that the discounted underlying price should be a martingale in a risk-neutral world. However, because commodities like agricultural products are not traded directly on exchange, the above dynamic hedging method no longer works. In other words, compared to exchange-traded securities like stocks, the lack of liquidity and short-selling mechanism for commodities leads to the result that the position in the commodity cannot be adjusted frequently. This suggests that the discounted underlying commodity price need not be a martingale under the pricing measure. In practice, the parameters in the underlying dynamics are calibrated to the observed futures term structure. Therefore, the risk-neutral commodity price is allowed to follow $\{S_t,\ t \geq 0\}$ given by (3).

The option value is given by

$$C(x, T) = \mathrm{e}^{-rT} \mathbb{E}_x[(S_T - K)_+],$$

where r is the constant risk-free interest rate, K is the strike price, and T is the expiration date. We expect to get the Laplace transform

$$\Lambda(x) = \int_0^\infty \mathrm{e}^{-\theta t}\mathbb{E}_x[(S_t - K)_+]\,\mathrm{d}t, \tag{22}$$

and thus

$$C(x, T) = \mathrm{e}^{-rT}\mathcal{L}^{-1}\{\Lambda(x)\}(T),$$

with $\mathcal{L}^{-1}$ representing the inverse Laplace operator. A somewhat tedious calculation leads to the following expression for Λ.

Proposition 5.1.

1. *For* $K \geq S^*$,

$$\Lambda(x) = \begin{cases} c_1 x^{\gamma_2^{(1)}}, & 0 < x < S^*, \\ c_2 x^{\gamma_1^{(2)}} + c_3 x^{\gamma_2^{(2)}}, & S^* \leq x \leq K, \\ -\dfrac{K}{\theta} - \dfrac{x}{\mu_2 - \theta} + c_4 x^{\gamma_1^{(2)}}, & x > K, \end{cases}$$

where

$$c_1 = \frac{(\mu_2\gamma_1^{(2)} - \theta)K^{1-\gamma_2^{(2)}}(S^*)^{\gamma_2^{(2)}-\gamma_2^{(1)}}}{\theta(\theta - \mu_2)(\gamma_1^{(2)} - \gamma_2^{(1)} - 2\alpha\theta S^*)},$$

$$c_2 = \frac{(\mu_2\gamma_1^{(2)} - \theta)K^{1-\gamma_2^{(2)}}(S^*)^{\gamma_2^{(2)}-\gamma_1^{(2)}}(\gamma_2^{(2)} - \gamma_2^{(1)} - 2\alpha\theta S^*)}{\theta(\theta - \mu_2)(\gamma_2^{(2)} - \gamma_1^{(2)})(\gamma_1^{(2)} - \gamma_2^{(1)} - 2\alpha\theta S^*)},$$

$$c_3 = \frac{(\theta - \mu_2\gamma_1^{(2)})K^{1-\gamma_2^{(2)}}}{\theta(\theta - \mu_2)(\gamma_2^{(2)} - \gamma_1^{(2)})},$$

$$c_4 = \frac{K^{1-\gamma_1^{(2)}-\gamma_2^{(2)}}(S^*)^{-\gamma_1^{(2)}}}{\theta(\theta - \mu_2)(\gamma_1^{(2)} - \gamma_2^{(2)})(\gamma_1^{(2)} - \gamma_2^{(1)} - 2\alpha\theta S^*)}[(\theta - \mu_2\gamma_1^{(2)})K^{\gamma_1^{(2)}}(S^*)^{\gamma_2^{(2)}}$$
$$\times (\gamma_2^{(2)} - \gamma_2^{(1)} - 2\alpha\theta S^*) - (\theta - \mu_2\gamma_2^{(2)})K^{\gamma_2^{(2)}}(S^*)^{\gamma_1^{(2)}}(\gamma_1^{(2)} - \gamma_2^{(1)} - 2\alpha\theta S^*)];$$

2. *for* $K < S^*$,

$$\Lambda(x) = \begin{cases} c_5 x^{\gamma_2^{(1)}}, & 0 < x < K, \\ -\dfrac{K}{\theta} - \dfrac{x}{\mu_1 - \theta} + c_6 x^{\gamma_1^{(1)}} + c_7 x^{\gamma_2^{(1)}}, & K \leq x < S^*, \\ -\dfrac{K}{\theta} - \dfrac{x}{\mu_2 - \theta} + c_8 x^{\gamma_1^{(2)}}, & x \geq S^*, \end{cases}$$

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(a) *The case* $S^* = 110$ (b) *The case* $S^* = 90$

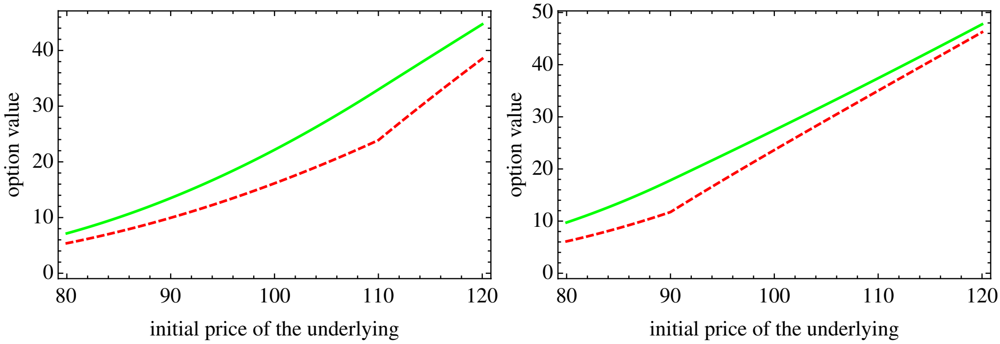

FIGURE 1: Call option values as functions of the initial underlying price $S_0 \in [80, 120]$. The solid and dashed lines correspond to the cases when $\alpha = 0$ and 0.02, respectively. The common parameters are given by $r = 0.05$, $T = 1$, $\mu_1 = 0.1$, $\mu_2 = 0.2$, $\sigma_1 = 0.3$, $\sigma_1 = 0.4$, and $K = 100$.

where

$$\begin{aligned}
c_5 &= \frac{(S^*)^{-\gamma_2^{(1)}}}{\theta(\theta-\mu_1)(\theta-\mu_2)(\gamma_1^{(1)}-\gamma_2^{(1)})(\gamma_2^{(1)}-\gamma_1^{(2)}+2\alpha\theta S^*)}\{-(\theta-\mu_1\gamma_1^{(1)})(\theta-\mu_2) \\
&\quad \times K^{1-\gamma_2^{(1)}}(S^*)^{\gamma_2^{(1)}}(\gamma_2^{(1)}-\gamma_1^{(2)}+2\alpha\theta S^*)+(\theta-\mu_1\gamma_2^{(1)})(\theta-\mu_2)K^{1-\gamma_1^{(1)}}(S^*)^{\gamma_1^{(1)}} \\
&\quad \times (\gamma_1^{(1)}-\gamma_1^{(2)}+2\alpha\theta S^*)-\theta(\gamma_1^{(1)}-\gamma_2^{(1)})S^*[\mu_2(\gamma_1^{(2)}-1)-\mu_1(\gamma_1^{(2)}-1 \\
&\quad +2\alpha S^*(\mu_2-\theta))]\}, \\
c_6 &= \frac{(\mu_1\gamma_2^{(1)}-\theta)K^{1-\gamma_1^{(1)}}}{\theta(\theta-\mu_1)(\gamma_1^{(1)}-\gamma_2^{(1)})}, \\
c_7 &= \frac{K^{-\gamma_1^{(1)}}(S^*)^{-\gamma_2^{(1)}}}{\theta(\theta-\mu_1)(\theta-\mu_2)(\gamma_1^{(1)}-\gamma_2^{(1)})(\gamma_2^{(1)}-\gamma_1^{(2)}+2\alpha\theta S^*)}\{(\theta-\mu_1\gamma_2^{(1)})(\theta-\mu_2) \\
&\quad \times K(S^*)^{\gamma_1^{(1)}}(\gamma_1^{(1)}-\gamma_1^{(2)}+2\alpha\theta S^*)-\theta(\gamma_1^{(1)}-\gamma_2^{(1)})K^{\gamma_1^{(1)}}S^*[\mu_2(\gamma_1^{(2)}-1) \\
&\quad -\mu_1(\gamma_1^{(2)}-1+2\alpha S^*(\mu_2-\theta))]\}, \\
c_8 &= \frac{K^{-\gamma_1^{(1)}}(S^*)^{-\gamma_1^{(2)}}}{\theta(\theta-\mu_1)(\theta-\mu_2)(\gamma_1^{(1)}-\gamma_2^{(1)})(\gamma_2^{(1)}-\gamma_1^{(2)}+2\alpha\theta S^*)}\{(\theta-\mu_1\gamma_2^{(1)})(\theta-\mu_2) \\
&\quad \times K(S^*)^{\gamma_1^{(1)}}-\theta K^{\gamma_1^{(1)}}S^*[\mu_2(\gamma_2^{(1)}-1+2\alpha\theta S^*)-\mu_1(\gamma_2^{(1)}-1+2\alpha\mu_2 S^*)]\}.
\end{aligned}$$

Proof. The proof is very similar to those of the propositions in the previous section (or we can use (22) with $f(x)$ equal to $(x-K)_+$) and is thus omitted. □

For illustration, we plot in Figure 1 the values of call options against the initial underlying price S_0. The Gaver–Wynn–Rho algorithm is adopted to numerically invert the Laplace transform (22); for more details on the algorithm, see [32]. The most salient feature of the

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figure is that, when a sticky point exists, the option value is remarkably lower than in the case with no sticky points. The reason for this is that the emergence of price clustering reduces the uncertainty of the underlying price, or in other words, drags down the underlying price volatility. It is also evident that no matter whether the stickiness effect exists ($\alpha = 0.02$) or not ($\alpha = 0$), the option value is always an increasing function of S_0. This accords with our intuition that the growth of S_0 makes option holders more likely to exercise their options.

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