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# THE SPIRIT OF CAPITALISM AND RATIONAL BUBBLES

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This study provides an infinite-horizon model of rational bubbles in a production economy. A bubble can arise when the pursuit of status is modeled explicitly, capturing the notion of "the spirit of capitalism." Using a parameterized model, I demonstrate the specific conditions for the existence of bubbles and their implications. Bubbles crowd out investment, stimulate consumption, and slow economic growth. I also discuss a stochastic bubble that bursts with an exogenous probability. I show that there could be multiple stochastic bubbly equilibria. Moreover, I suggest that taxing wealth properly can eliminate bubbles and achieve the social optimum.

**Keywords:** Bubbles, Spirit of Capitalism, Multiple Equilibria, Wealth Tax

## 1. INTRODUCTION

"The spirit of capitalism seems to be a driving force behind stock-market volatility and economic growth," Bakshi and Chen (1996) suggest, based on their analytical and empirical examinations. However, they do not discuss the phenomenon of bubbles, which are one caise of enormous volatility in the stock market. This study is motivated by exploring the linkage between "the spirit of capitalism" and bubbles and develops a theory of rational bubbles driven by the pursuit of status.

My theory is based on an infinite-horizon model of a production economy with "the spirit of capitalism." As Max Weber (1958) argued, "the spirit of capitalism" suggests that people acquire wealth not simply for implied material rewards but also for the social status gained by the accumulation of wealth. Following the method of Kurz (1968) and Zou (1994), I model "the spirit of capitalism" by placing the wealth term directly into the utility function. The bubble in my model is in the pricing of a zero-dividend asset.[1]

The model sheds light on how people's pursuit of status can lead to rational bubbles. More concretely, in an infinite-horizon general equilibrium framework, bubbles can arise provided that the ratio of the marginal utility of wealth over the

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marginal utility of consumption is positive as time goes to infinity. The condition, technically, prevents the transversality condition (TVC) in ruling out bubbles. Intuitively, when the happiness of holding one more unit of wealth is not trivial relative to the utility of consuming one more unit of goods, individuals would like to hold the asset with a bubble in its price to enjoy the resulting status, rather than sell out the asset for the purpose of consumption. However, once the ratio of marginal utilities is zero, there is no incentive for the pursuit of status. Eventually, individuals will sell out the asset with the bubble for material rewards. It is this behavior that expels bubbles.

Technically, the way to introduce rational bubbles by "the spirit of capitalism" is similar to the method used to model a credit-driven bubble, which is adopted in Kocherlakota (2009) and a series of papers by Miao and Wang (2012a, 2012b, 2013, 2014). Both methods focus on preventing the TVC in ruling out bubbles. A credit-driven bubble can exist because of the binding credit constraint. When credit is scarce, indicated by the binding credit constraint, individuals have incentives to hold the asset with a bubble as collateral for more outside credit. The mechanism guarantees the existence of bubbles.

Using a parameterized model, this study provides specific conditions for determining whether a bubble can exist. The conditions mainly depend on the extent to which individuals care about status. When the enthusiasm for status seeking is strong enough, physical capital will be overaccumulated so that a bubbleless economy is dynamically inefficient. Under this situation, bubbles driven by the pursuit of status can arise and mitigate the overaccumulation of capital because the incentive for holding an asset with a bubble is not trivial relative to the incentive for consumption in this environment. The requirement for dynamic inefficiency in a bubbleless economy is consistent with the necessary condition for the existence of bubbles in Tirole's overlapping generations (OLG) framework.

The impact of a bubble driven by status seeking is robust both in the neoclassical growth model and in the endogenous growth model. Bubbles stimulate consumption, crowd out investment, and slow economic growth. This result is similar to the findings obtained based on Tirole's OLG framework, such as those of Grossman and Yanagawa (1993). The infinite-horizon framework that I adopt, however, eliminates the concern of incomplete markets induced by the OLG framework and easily connects with a vast literature on asset pricing. However, my result is opposite to the impact of a credit-driven bubble. The reason is simple. A bubble driven by status seeking, like Tirole's bubble, absorbs funds from the capital market, whereas a credit-driven bubble helps to raise funds for investment.

In addition to deterministic rational bubbles, my model can also analyze stochastic bubbles. A stochastic bubble that bursts with an exogenous constant probability can exist only if the probability of bursting is less than some upper limit that measures the extent of dynamic inefficiency in a bubbleless economy. The more dynamically inefficient the economy is, the more possible it is for a stochastic bubble to arise. This finding is consistent with what is suggested by Weil (1987), which is based on Tirole's OLG framework. As demonstrated by my explicit solution, given capital stock, a higher probability that a bubble collapses will reduce the size of the bubble, decrease consumption, and boost economic growth. The following intuition helps us to understand this result. Facing a greater likelihood of collapse, individuals will reduce their bubbly assets and invest more in physical capital. The higher investment stimulates economic growth but sacrifices some consumption. Moreover, there are multiple stochastic bubbly equilibria. Each equilibrium is associated with a distinct probability that the bubble crashes.

This study also explores the socially optimal policy. The reason for the inefficiency of a competitive economy is that "the spirit of capitalism" induces the overaccumulation of physical capital. Two tax policies that restrict capital accumulation can make the competitive economy return to the social optimum. One tax is imposed on investment return, the other on holding wealth. The optimal tax rates are both equal to the ratio of the marginal utility of wealth to the marginal utility of consumption. The former policy focuses on returns from physical capital and still allows bubbles, whereas a tax on wealth can eliminate bubbles.

Placing wealth in the utility function appears to be similar to placing money in the utility function (MIU). Some may believe that the bubble in the pricing of zero-dividend assets is simply identical to the positive value of money in MIU models. This belief, however, is not accurate. In typical MIU models, money enters the utility function independent of the physical capital stock. In my model, however, the wealth term includes both the bubble and the physical capital, which implies that the marginal utility of the bubble depends on the physical capital. Whereas money is neutral, as argued by Sidrauski (1967), the bubble in my model has an impact on the real economy.

This paper is related to many current papers on rational bubbles. Introducing financial frictions into Tirole's OLG framework, Farhi and Tirole (2012) analyze the interaction of liquidity and bubbles, and Martin and Ventura (2012) provide a stylized model of economic growth with bubbles. Kocherlakota (2009) and Miao and Wang (2013) demonstrate that bubbles on collateral can emerge when credit is sufficiently scarce. Unlike these papers, I offer a simple model of rational bubbles driven by the pursuit of status and illustrate a robust result that is consistent with the finding made by Tirole (1985). In addition, Kamihigashi (2008) shows that bubbles can arise when "the spirit of capitalism" is modeled by a restrictive utility function.[2] Unlike that described in his paper, my model does not require any restriction on the properties of the preference function and my results are very robust.

The rest of the paper is organized as follows. Section 2 establishes an infinite-horizon model with "the spirit of capitalism" and provides the necessary condition for the existence of bubbles. Section 3 uses a parameterized function of preference to study the bubbleless equilibrium and the bubbly equilibrium, both in a neoclassical growth model and in an endogenous growth model. Section 4 analyzes a stochastic bubbly economy where the bubble bursts with an exogenous probability. Section 5 discusses socially optimal policies for eliminating bubbles. Section 6 provides concluding remarks.

## 2. THE EXISTENCE OF BUBBLES

As Blanchard and Watson (1982) demonstrated, it is the TVC that rules out rational bubbles from the infinite-horizon general equilibrium framework. This section demonstrates that introducing "the spirit of capitalism" can ensure that the TVC still holds at an equilibrium with bubbles and provides the necessary condition for the existence of bubbles. In addition, I also show that introducing a bubble by way of "the spirit of capitalism" is technically similar to the method of modeling a credit-driven bubble.

I construct the general model as follows.

Time is continuous. An infinite number of identical individuals, who live forever, are continuously and evenly distributed over the area [0,1]. Every individual can rent his or her physical capital to firms that are owned by all of the individuals and receives the lump-sum transfer of the firms' profit, $\Pi$, and a rental at a rate of $r$. In the model, the capital stock is denoted by $k$, and $r$ is equal to the real interest rate. Each individual is also able to invest in financial assets. For convenience, I suppose that there is only one kind of zero-dividend asset in this economy. Based on the standard definition, the fundamental value of the asset should be zero. Therefore, once the price of the asset, which is denoted by $q$, is positive, I say that an asset bubble exists. The total supply of the asset is normalized by 1. The amount of the financial asset held by the individual is denoted by $s$.

The representative individual's optimization question is given by

$$\max_{c} \int_{0}^{\infty} e^{-\rho t} U(c, a) dt, \quad \rho > 0,$$

subject to

$$a \equiv qs + k,$$

$$\dot{a} = rk - c + \dot{q}s + \Pi,$$

where $\rho$ is the rate of time preference and $U(c, a)$ is the utility function, which is continuous, differentiable, strictly increasing, and concave in all of its arguments. Here, $c$ is the amount of consumption, and $a$ is the amount of wealth, which is equal to the sum of the asset's value, $qs$, and physical capital, $k$. Following the method of Zou (1994) and Bakshi and Chen (1996), I place the wealth term directly into the utility function to model "the spirit of capitalism."

The current value Hamiltonian[3] of this optimization question can be written as

$$\begin{aligned} \mathcal{H} = {} & U(c, a) + \lambda (a - qs - k) \\ & + \mu (rk - c + \dot{q}s + \Pi). \end{aligned}$$

The first-order conditions are given by the Euler equation

$$\frac{\dot{\mu}}{\mu} = \rho - \frac{U_a'}{\mu} - r, \tag{1}$$

where $\mu = U_c'$, and the nonarbitrage condition

$$\frac{\dot{q}}{q} = r, \tag{2}$$

which indicates that the growth rate of the bubble is equal to the real interest rate. The TVCs can be written as follows:

$$\lim_{t\to\infty} e^{-\rho t}\mu k = 0, \tag{3}$$

$$\lim_{t\to\infty} e^{-\rho t}\mu q s = 0. \tag{4}$$

There are an infinite number of homogeneous firms, which are continuously and evenly distributed over the area [0,1]. The representative firm wishes to maximize its current profit,

$$\Pi \equiv f(k) - \delta k - rk,$$

where $\delta > 0$ is the depreciation rate of physical capital. From the first-order condition, the rate of rental (also the real interest rate) is given by

$$r = f'(k) - \delta. \tag{5}$$

At equilibrium, the goods market clearing condition is given by

$$\dot{k} = f(k) - \delta k - c, \tag{6}$$

and the asset market clearing condition is

$$s = 1.$$

Thus, the TVC (4) can be rewritten as

$$\lim_{t\to\infty} e^{-\rho t}\mu q = 0. \tag{7}$$

If the value of $q$ is some positive number, then a bubble exists. Equation (2) implies that the bubble grows at speed $r$. To ensure that there is no deviation from the TVC (7), the growth rate of the product of $\mu$ and $q$ ultimately needs to be less than $\rho$. It is easy to find that the growth rate of $\mu q$ is equal to $\rho - U_a'/U_c'$ by combining the Euler equation (1) with equation (2). Here, I use the fact that $\mu = U_c'$. Thus, as long as

$$\lim_{t\to\infty} \frac{U_a'}{U_c'} > 0, \tag{8}$$

the product of $\mu$ and $q$ will eventually grow at a rate that is less than $\rho$, and the TVC (7) will hold. Therefore, technically, the condition (8) is necessary for the existence of bubbles.

The necessary condition (8) can be understood by considering the following intuition. Individuals, in an economy with "the spirit of capitalism," not only care about the expected consumption flows provided by their wealth, but also enjoy holding the wealth itself. When the condition (8) is satisfied, the happiness gained from holding one more unit of wealth is nontrivial relative to the happiness gained from consuming one more good, even at the end of the world. Therefore, even if a bubble cannot provide any material rewards, people still like to hold it to enjoy the increase in their own wealth.

The condition also reveals the necessity of "the spirit of capitalism" for the existence of bubbles. If there is no "spirit of capitalism" in the economy, then the marginal utility of wealth, $U_a'$, is zero. There will be a deviation from the TVC (7) if there is a bubble. Thus, the TVC rules out bubbles, which can be explained intuitively. In this case, people hold wealth only because of the expected consumption flows. They do not feel happy from holding the wealth itself. Thus, no one likes to hold an asset whose price includes a bubble forever because no material reward flow supports the bubble. Eventually, they will sell out the asset simply for material rewards. This behavior expels bubbles.

Here, I need to emphasize that any further restriction on the form of preference function is not necessary at all to guarantee the existence of bubbles. For example, my model does not need the restriction $\lim_{a\to\infty} U_a' > 0$, which is required by Kamihigashi (2008). When $\lim_{a\to\infty} U_a' = 0$, as long as the marginal utility of consumption, $U_c'$, also converges to zero, the condition (8) may still hold. In this case, it is also possible for bubbles to arise. The following sections of this paper verify this insight.

Moreover, I would like to stress that my method for introducing rational bubbles is technically similar to the method for modeling a credit-driven bubble that is advocated by Kocherlakota (2009) and Miao and Wang (2013). In their models, the binding credit constraint makes sure that the TVC is not violated when bubbles emerge. This result can be understood by considering the following line of thought. The credit constraint binds only when credit is scarce. Because bubbles can function as collateral, individuals have an incentive to hold them for outside credit. Therefore, individuals might not sell out bubbles simply for the purpose of consumption.

## 3. THE PARAMETERIZED MODEL

This section uses a parameterized model to illustrate that rational bubbles can arise in an economy with "the spirit of capitalism" provided that condition (8) is satisfied. I consider both a neoclassical growth model and an endogenous growth model and offer explicit conditions under which a bubbly equilibrium can exist. Dynamic analysis is also included.

For convenience, the parameterized utility function of the representative individual takes the form

$$\log c + \eta \log \frac{a}{\bar{a}}, \qquad (9)$$

where $\eta > 0$ measures the magnitude of "the spirit of capitalism", $\bar{a}$ denotes the average wealth level, and the ratio of the individual's own wealth to the average wealth determines the his or her wealth status. This specification is consistent with the main goal of the spirit of capitalism as well as the notion of "catching up with the Joneses" argued by Abel (1990). Moreover, the log-form utility function guarantees the uniqueness of the bubbleless steady state and dispels the concern about multiple steady states introduced by the wealth effect.[4] It is also necessary to emphasize that the utility function (9) implies that

$$\lim_{a \to \infty} U'_a = 0,$$

which indicates that Kamihigashi's restriction is violated. The specific form of the condition (8) is currently given by

$$\lim_{t \to \infty} \frac{\eta c}{a} > 0. \tag{10}$$

The following section explores the bubbleless equilibrium, the bubbly equilibrium, and their dynamics in a neoclassical growth model and an endogenous growth model.

### 3.1. Neoclassical Growth Model

In the neoclassical growth model, the production function, $f(k)$, takes the form $Ak^{\alpha}$, where $A$ is the technology level and $0 < \alpha < 1$. The real interest rate, $r$, is equal to $\alpha A k^{\alpha-1} - \delta$, which is a decreasing function of the physical capital, $k$. With the production of decreasing returns to scale, our discussion focuses on steady states in which all variables are constants. In the steady states, the TVCs are trivial.

The following proposition suggests that there exist two steady states in the economy. One is the bubbleless steady state, in which the value of the bubble, $q$, is equal to zero. The other is the bubbly steady state, in which the value of the bubble is some positive constant. I use an asterisk to denote a variable's value in the bubbleless steady state and double asterisks to denote a variable's value in the bubbly steady state.

PROPOSITION 1. *(a) There always exists a unique bubbleless steady state in which* $q^* = 0$, $k^* = [\frac{\rho+(1+\eta)\delta}{(\eta+\alpha)A}]^{1/(\alpha-1)}$, *and* $c^* = [\frac{\rho+(1+\eta)\delta}{\eta+\alpha} - \delta]k^*$.

*(b) With the parameter restriction*

$$\eta\delta(1-\alpha) > \rho\alpha, \tag{11}$$

*there exists a unique bubbly steady state in which* $k^{**} = (\frac{\delta}{\alpha A})^{1/(\alpha-1)}$, $c^{**} = \frac{\delta(1-\alpha)}{\alpha}k^{**}$, *and* $q^{**} = [\frac{\eta\delta(1-\alpha)}{\rho\alpha} - 1]k^{**} > 0$.

*(c) Under the parameter restriction (11),* $k^{**} < k^*$ *and* $c^{**} > c^*$.

The parameter restriction (11) in Proposition 1 illustrates that whether a bubble arises depends on the extent to which people care about their status. When the extent, measured by the parameter $\eta$, is large enough, a bubble can exist. The size of the bubble is also dependent on the value of $\eta$. A higher value of $\eta$ leads to a larger bubble. These results are intuitive. The enthusiasm for status seeking is due to the happiness gained by holding wealth. When the enthusiasm is strong enough, people like to hold an asset with a bubble to enjoy the resulting status but not sell out the asset. At the same time, the stronger pursuit of status stimulates greater wealth accumulation, which causes bubbles to arise.

The proposition also implies that the existence of a bubble requires the dynamic inefficiency of a bubbleless economy. The extent of this dynamic inefficiency is generally measured by the difference between the economic growth rate and the real interest rate. The difference in the bubbleless steady state is given by

$$0 - r(k^*) = \frac{(1-\alpha)\eta\delta - \alpha\rho}{\eta + \alpha}.$$

The value is positive under the restriction (11), which indicates that the bubbleless economy is dynamically inefficient. The requirement is the same as for the argument made by Tirole (1985), which is based on an OLG framework. In both papers, the dynamic inefficiency is derived from the overaccumulation of physical capital. Bubbles can absorb redundant funds and mitigate the overaccumulation of capital. However, the reasons provided for the overaccumulation of capital in the two papers are different. The OLG framework in Tirole (1985) leads to an incomplete market and makes capital function as a store of value, whereas in this study, people accumulate too much capital because of status seeking.

Comparing the bubbleless steady state and the bubbly steady state, I can easily determine a bubble's impact on the real economy. Bubbles, as one type of wealth, are a substitute for physical capital in the process of status seeking. Thus, they will crowd out investment. As a result of the bubbles' wealth effect, individuals also consume more. The implications of bubbles are also consistent with those noted by Tirole (1985).

Next, I consider the stability of the steady states and the local dynamics of economic equilibrium. The following proposition summarizes the analysis of the stability of the bubbly steady state and the bubbleless steady state.

PROPOSITION 2. *Under the restriction (11), both the bubbly steady state and the bubbleless steady state are local saddle points. Moreover, the stable manifold of the bubbly steady state is one-dimensional, whereas that of the bubbleless steady state is two-dimensional.*

Proposition 2 also reveals the local dynamics in the neighborhood of the steady states. Around the bubbly steady state, given an initial value of the state variable, physical capital $k_0$, there is a unique pair of initial consumption and initial bubble values, $\{c_0, q_0\}$, which makes sure the initial economy eventually converges to the bubbly steady state. However, in the neighborhood of the bubbleless steady state, given the initial value of capital $k_0$, there are multiple pairs $\{c_0, q_0\}$, each of which guarantees that the initial economy finally converges to the bubbleless steady state. In this sense, there exist a series of asymptotically bubbleless equilibria whose initial values of bubbles are positive. The result is consistent with the finding reported by Tirole (1985). Miao and Wang (2013) also suggest that similar local dynamics occur in a model of credit-driven bubbles.

## 3.2. Endogenous Growth Model

Following Romer (1986) and Xie (1991), I assume that the production function in the endogenous growth model has some positive externality. The specific form of $f(k)$ is given by

$$Ak^{\alpha}\bar{k}^{1-\alpha}, \tag{12}$$

where $\bar{k}$ is the average capital stock and $0 < \alpha < 1$. At equilibrium, $k = \bar{k}$, and the real interest rate is equal to $\alpha A - \delta$, which is a constant. To keep the following discussions interesting, I assume that $\alpha A - \delta > 0$ in this paper. In addition, the form of the production function that I adopt here is the same as that in Grossman and Yanagawa (1993), who discuss bubbles in an OLG framework. The setup makes it easy to compare the results of each study.

In the endogenous growth model, my discussions focus on the balanced growth path (BGP) on which all variables grow at certain constant speeds. Now, it is necessary to pay more attention to the condition (8) that prevents the TVC (7) from ruling out bubbles. From the real resource constraint, I can obtain that

$$\frac{\dot{c}}{c} \le \frac{\dot{k}}{k} \le \frac{\dot{a}}{a}. \tag{13}$$

To ensure that the condition (10) is satisfied, it is required that

$$\lim_{t\to\infty} \frac{\dot{c}}{c} \ge \lim_{t\to\infty} \frac{\dot{a}}{a}.$$

Therefore, on the BGP with bubbles, the growth rate of consumption cannot be less than the growth rate of bubbles or the growth rate of physical capital; i.e., $\frac{\dot{c}}{c} \ge \frac{\dot{q}}{q}$ and $\frac{\dot{c}}{c} \ge \frac{\dot{k}}{k}$. Otherwise, bubbles will be ruled out by the TVC (7). Together with the constraint (13), I obtain that

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} \ge \frac{\dot{q}}{q}$$

on the BGP with bubbles. This point is clearly detailed in Appendix C.

The following proposition provides three possible BGPs in the endogenous growth model. One is the bubbleless BGP, on which the value of the bubble is zero. The second is the quasi-bubbleless BGP, on which the value of the bubble

**Table 1.** Comparison between the bubbly BGP and the bubbleless BGP

| | Bubbly BGP | vs. | Bubbleless BGP |
|---|---|---|---|
| Consumption | $(1-\alpha)Ak$ | $>$ | $\frac{(1-\alpha)A+\rho}{\eta+1}k$ |
| Bubble | $\left[\frac{\eta(1-\alpha)A}{\rho}-1\right]k$ | $>$ | 0 |
| Growth rate | $\alpha A-\delta$ | $<$ | $A-\delta-\frac{(1-\alpha)A+\rho}{\eta+1}$ |
| Saving rate | $\alpha$ | $<$ | $\left[A-\frac{(1-\alpha)A+\rho}{\eta+1}\right]/A$ |

is positive but eventually trivial relative to the values of real economic variables. The third is the bubbly BGP.

PROPOSITION 3. *(a) There always exists a bubbleless BGP on which* $q=0$, $c=\frac{(1-\alpha)A+\rho}{\eta+1}k$, *and* $\frac{\dot{c}}{c}=\frac{\dot{k}}{k}=A-\delta-\frac{(1-\alpha)A+\rho}{\eta+1}$.

*(b) With the parameter restriction*

$$\eta(1-\alpha)A>\rho, \tag{14}$$

*there exists a quasi-bubbleless BGP on which the value of the bubble eventually is trivial relative to the values of real economic variables. The quasi-bubbleless BGP can be described by* $\frac{\dot{q}}{q}=r=\alpha A-\delta$, $c=\frac{(1-\alpha)A+\rho}{\eta+1}k$, *and* $\frac{\dot{c}}{c}=\frac{\dot{k}}{k}=A-\delta-\frac{(1-\alpha)A+\rho}{\eta+1}>r$.

*(c) Given the restriction (14), there exists a bubbly balanced growth path on which* $c=(1-\alpha)Ak$, $q=[\frac{\eta(1-\alpha)A}{\rho}-1]k$, *and* $\frac{\dot{c}}{c}=\frac{\dot{k}}{k}=\frac{\dot{q}}{q}=r=\alpha A-\delta$.

Proposition 3 suggests effects of the economic environment on bubbles in the endogenous growth model similar to those in the neoclassical growth model. The condition for the existence of bubbles and the size of a bubble both depend on the extent to which people care about their status. The dynamic inefficiency of the bubbleless economy is also necessary for the existence of bubbles because the parameter restriction (14) implies that the value of $[\eta(1-\alpha)A-\rho]/(\eta+1)$, which measures the dynamic inefficiency of the bubbleless economy, is positive.

Moreover, the implications of bubbles in the endogenous growth model are also the same as for those in the neoclassical growth model. I can clarify this point by comparing the bubbly BGP and the bubbleless BGP. The comparison is illustrated in Table 1. Given the parameter restriction (14) and the same capital level, consumption in the bubbly economy is higher than that in the bubbleless economy. Together with the fact that the investment is equal to $Ak-c$, investment in the bubbly economy must be lower than that in the bubbleless economy. Therefore, bubbles stimulate consumption and crowd out investment, which also explains why the bubbleless economy has a higher growth rate than the bubbly economy. This result is consistent with the findings reported by Grossman and Yanagawa (1993), whose bubbles exist in Tirole's OLG framework.

Now, I consider the dynamics of the endogenous growth model. For convenience in using a phase diagram, I define that

$$\tilde{c} \equiv ce^{-rt},$$
$$\tilde{k} \equiv ke^{-rt},$$
$$\tilde{q} \equiv qe^{-rt}.$$

Here, $r \equiv \alpha A - \delta$ is the real interest rate. The newly defined variable $\tilde{x}$ can be understood as $x$'s value discounted by the real interest rate. The economy can be described by the following system of equations:

$$\tilde{q} \equiv q_0,$$
$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{\eta\tilde{c}}{q_0 + \tilde{k}} - \rho,$$
$$\dot{\tilde{k}} = (1-\alpha)A\tilde{k} - \tilde{c}.$$

On the bubbly BGP, $\tilde{c}$ and $\tilde{k}$ both eventually converge to non-negative constants, which depend on the initial bubble value, $q_0$. However, on the bubbleless BGP and the quasi-bubbleless BGP, both of the discounted variables will diverge.

When the initial value of the bubble, $q_0$, is equal to zero, the dynamics of the bubbleless economy is the same as that of the classical endogenous growth models. Given the initial physical capital, $k_0$, an appropriate value of initial consumption, $c_0$, ensures that the economy stays on the bubbleless BGP forever.

Next, I consider the case in which the initial bubble value is positive, i.e., $q_0 > 0$. The dynamic analysis is based on a phase diagram, which is presented in Figure 1. In the figure, the locus of $\frac{\dot{\tilde{c}}}{\tilde{c}} = 0$ and the locus of $\dot{\tilde{k}} = 0$ are both straight lines. Given the restriction (14), the slope of the locus of $\frac{\dot{\tilde{c}}}{\tilde{c}} = 0$ is less than that of the locus of $\dot{\tilde{k}} = 0$. The two loci have an intersection point whose horizontal ordinate is denoted by $\tilde{k}_0^*(q_0)$. The intersection point represents the bubbly balanced growth path. The value of $\tilde{k}_0^*(q_0)$ is given by $\rho q_0/[\eta(1-\alpha)A - \rho]$. In the space above the locus of $\frac{\dot{\tilde{c}}}{\tilde{c}} = 0$ and below the locus of $\dot{\tilde{k}} = 0$, there exists at least one trajectory converging to the quasi-bubbleless BGP.

Using the phase diagram, it is easy to conclude the following about the economic dynamics. Given $q_0 > 0$ and $\tilde{k}_0 \equiv k_0$, if the initial value of capital, $k_0$, is equal to $\tilde{k}_0^*(q_0)$, then an appropriate value of initial consumption, $c_0 = \tilde{c}_0$, can ensure that the initial economy will be located at the intersection point of the two loci. In this case, the economy will stay on the bubbly balanced growth path forever. If $k_0 > \tilde{k}_0^*(q_0)$, then an appropriate value of initial consumption can guarantee that the initial economy is on the trajectory converging to the quasi-bubbleless balanced growth path, whereas if $k_0 < \tilde{k}_0^*(q_0)$, no economic equilibrium exists.

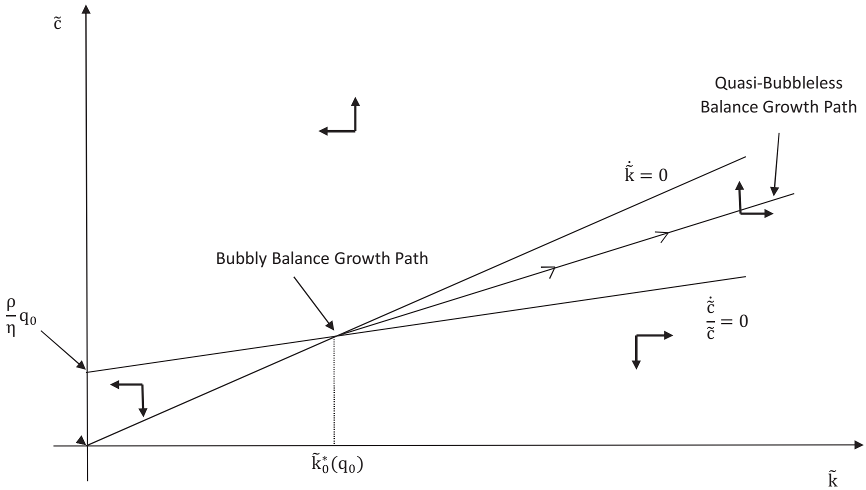


**FIGURE 1.** Dynamics of the endogenous growth model.

These findings can also be expressed from the perspective of the initial bubble value. The inverse function of $\tilde{k}_0^*(q_0)$ implies that a threshold bubble value, $q_0^*(k_0)$, is equal to $[\eta(1-\alpha)A/\rho-1]k_0$. Here, I use the fact that $\tilde{k}_0 \equiv k_0$. With the restriction (14), for any initial capital stock, $k_0 > 0$, there is a unique corresponding threshold value, $q_0^*(k_0) > 0$. If the initial bubble value $q_0$ is equal to $q_0^*(k_0)$, the locus of $\frac{\dot{\tilde{c}}}{\tilde{c}} = 0$ and the locus of $\dot{\tilde{k}} = 0$ intersect at the point whose horizontal ordinate is equal to $k_0$. An appropriate initial consumption, $c_0$, can ensure that the economy is located at the intersection point. If the initial value of the bubble $q_0$ is positive but lower than the threshold value of $q_0^*(k_0)$, then the horizontal ordinate of the intersection point is less than $k_0$. Thus, an appropriate initial value of consumption will ensure that the initial economy is on the trajectory converging to the quasi-bubbleless BGP. If the initial value of the bubble $q_0$ is higher than the threshold value of $q_0^*(k_0)$, then the horizontal ordinate of the intersection point is greater than $k_0$. There is no economic equilibrium in this case.

The following proposition summarizes our analysis regarding the economic dynamics.

PROPOSITION 4. *(a) If the initial value of the bubble, $q_0$, is equal to zero, the economy stays on the bubbleless BGP.*

*(b) Under the parameter restriction (14), given any initial capital stock, $k_0 > 0$, there is a unique threshold bubble value, $q_0^*(k_0) = [\eta(1-\alpha)A/\rho - 1]k_0$. If the initial bubble value, $q_0$, is positive but lower than $q_0^*(k_0)$, then the economy converges to the quasi-bubbleless BGP; if the initial bubble value, $q_0$, is equal to $q_0^*(k_0)$, then the economy stays on the bubbly BGP forever; if the initial bubble value, $q_0$, is higher than $q_0^*(k_0)$, then no economic equilibrium exists.*

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## 4. STOCHASTIC BUBBLES

This section discusses a stochastic bubbly economy, in which bubbles might burst with an exogenous constant probability.

The utility function and the production function are given by (9) and (12), respectively. Suppose that the bubble still exists at the current moment; i.e., $q > 0$. The process of the bubble can be described as follows:

$$dq = \begin{cases} \varphi q dt, & \text{with probability } 1 - \varepsilon dt \\ -q, & \text{with probability } \varepsilon dt, \end{cases} \tag{15}$$

where $\varphi > 0$ and $\varepsilon > 0$. The process means that the bubble will grow at the rate of $\varphi$ with probability $1 - \varepsilon dt$, and will burst with probability $\varepsilon dt$. If the bubble bursts, i.e., $q = 0$, then the price of the asset will always be zero, as indicated by the process outlined in the preceding. This implies that the bubble cannot be reborn.

The process of the asset volume held by the representative individual, $s$, is given by

$$\dot{s} = \iota, \tag{16}$$

where $\iota$ is a choice variable that measures the increment of the asset held by the representative individual given the asset price. Thus, the budget constraint can be written as follows:

$$\dot{k} = Ak^{\alpha}\bar{k}^{1-\alpha} - \delta k - c - q\iota. \tag{17}$$

Following the method of Rebelo and Xie (1999), I also conjecture the process of average capital $\bar{k}$ to be as follows:

$$\dot{\bar{k}} = \varphi \bar{k}. \tag{18}$$

Via the solving process presented in Appendix D, I verify that this law of motion holds.

The optimal question of the representative individual can be rewritten by solving the following at each time $t \in [0, \infty)$:

$$V(q, \bar{k}, k, s) \equiv \max_{c,\iota} E_t \int_t^{\infty} e^{-\rho(x-t)} U(c(x), q(x)s(x) + k(x))dx,$$

subject to the constraints (15), (16), (17), and (18). An explicit solution of the stochastic bubbly economy is presented in the following proposition.

PROPOSITION 5. *(a) When* $0 < \varepsilon < \bar{\varepsilon} \equiv \frac{\eta(1-\alpha)A-\rho}{\eta+1}$*, a stochastic bubble can exist. The stochastic bubbly economy is described by*

$$c = \frac{(1-\alpha)A(\rho + \varepsilon)}{\varepsilon\eta + \rho + \varepsilon} k, \tag{19}$$

$$q = \left[ \frac{\eta(1-\alpha)A}{\varepsilon\eta + \rho + \varepsilon} - 1 \right] k, \tag{20}$$

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = \frac{\dot{q}}{q} = A - \delta - \frac{(1-\alpha)A(\rho+\varepsilon)}{\varepsilon\eta + \rho + \varepsilon}. \tag{21}$$

*(b) If the stochastic bubble bursts, the economy immediately jumps to the bubbleless BGP described in part (a) of Proposition 3.*

This proposition shows that a stochastic bubble can exist only if the value of $\varepsilon$, which measures the probability that the bubble bursts, is less than an upper limit, $\bar{\varepsilon}$. To guarantee that the value of $\bar{\varepsilon}$ is positive, the parameters restriction (14) should be satisfied. Because the value of $\bar{\varepsilon}$ is just equal to the difference between the growth rate of the bubbleless economy and the real interest rate, a positive $\bar{\varepsilon}$ implies that the bubbleless economy is dynamically inefficient.

The preceding analysis implies that the existence of a stochastic bubble also requires dynamic inefficiency of the bubbleless economy. This result is consistent with the argument of Weil (1987), which is based on Tirole's OLG framework. Moreover, my explicit solution of the stochastic bubbly economy allows us to analyze other economic issues more intuitively.

By comparing the explicit solutions of the deterministic bubbleless economy, the deterministic bubbly economy,[5] and the stochastic bubbly economy, it is easy to determine the impact of the uncertainty resulting from the stochastic bubble. Given the fact that

$$\frac{(1-\alpha)A + \rho}{\eta + 1} < \frac{(1-\alpha)A(\rho+\varepsilon)}{\varepsilon\eta + \rho + \varepsilon} < (1-\alpha)A,$$

it is clear that consumption in the stochastic economy is greater than that in the bubbleless economy but less than that in the deterministic bubbly economy. Because the aggregate investment is equal to $Ak - c$, investment in the stochastic economy is less than that in the deterministic bubbleless economy but greater than that in the deterministic bubbly economy. This comparison also suggests that the growth rate in the stochastic economy is higher than that in the bubbly economy without uncertainty and lower than that in the deterministic bubbleless economy. Thus, stochastic bubbles also stimulate consumption, crowd out investment, and slow economic growth. This result is similar to what I obtain in the deterministic case, except that the impact of stochastic bubbles is weaker.

The explicit example also directly illustrates how the probability that the bubble bursts, measured by $\varepsilon$, affects the real economy. The fact that $\partial(c/k)/\partial\varepsilon < 0$ implies a negative relationship between consumption and the probability of bursting. Because the growth rate of the real economy is equal to $A - \delta - c/k$, there is a positive relationship between the probability measured by $\varepsilon$ and the economic growth rate. A lower probability of bursting leads to higher consumption and a lower growth rate, whereas the results are just the opposite with a higher probability.

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The size of a stochastic bubble is also affected by the probability that the bubble collapses. A higher probability would reduce the size of the bubble, and a lower probability allows a larger bubble size. As the probability that the bubble crashes, measured by $\varepsilon$, converges to zero, the stochastic bubbly economy approaches the deterministic bubbly economy, whereas as the value of $\varepsilon$ converges to its upper limit, $\bar{\varepsilon}$, the stochastic bubbly economy converges to the deterministic bubbleless economy.

These relationships are consistent with our intuition. When a bubble has a higher probability of bursting, the expected wealth decreases. By the wealth effect, consumption will also decrease. Investors will adjust their portfolios and put more weight on physical capital. Thus, the value of financial assets will be even lower. Because the fundamental value of the financial asset will always be zero, the size of the bubble would be reduced. At the same time, higher investment will stimulate economic growth.

Furthermore, how the economic environment affects the upper limit of the probability that a bubble collapses, measured by $\bar{\varepsilon}$, provides a hint regarding under what circumstances this type of stochastic bubble is most likely to arise. From Proposition 5, it is easy to find that higher values of $\eta$ will raise the upper limit, $\bar{\varepsilon}$. A higher upper limit on the probability of bursting indicates a higher probability of the existence of stochastic bubbles. Thus, this type of stochastic bubble more likely occurs in an economy in which people more care about their status, which is consistent with our results for the deterministic case. The value of $\bar{\varepsilon}$ also measures the size of the dynamic inefficiency of the bubbleless economy. Therefore, stochastic bubbles more likely emerge in an economy with greater dynamic inefficiency. This result is also consistent with the finding reported by Weil (1987).

In addition, the explicit example of a stochastic bubble also verifies the existence of a series of stochastic bubbly equilibria. Equation (20) implies that

$$\varepsilon = \frac{\eta(1-\alpha)A/(q/k+1)-\rho}{\eta+1}.$$

To satisfy the restriction $0 \le \varepsilon < \bar{\varepsilon}$, it is necessary that $0 < q \le [\eta(1-\alpha)A/\rho - 1]k$. Because the growth rate of the bubble is equal to the growth rate of the real economy at this stochastic bubbly equilibrium, the preceding inequality will hold at any time. Thus, the following proposition regarding stochastic bubbly equilibrium can be obtained naturally.

PROPOSITION 6. *In an economy with the "spirit of capitalism," the preference function and production function are given by (9) and (12), respectively. Given the parameter restriction (14), for any positive initial value of bubble,* $q_0$, *which is not greater than the threshold value of* $[\eta(1-\alpha)A/\rho - 1]k_0$, *there exists a stochastic* *bubbly equilibrium, at which the process of the bubble is given by*

$$dq = \begin{cases} \varphi q dt, & \text{with probability } 1 - \varepsilon dt \\ -q, & \text{with probability } \varepsilon dt, \end{cases}$$

*and the real economy grows at the rate* $\varphi$*, which is equal to* $A - \delta - \frac{(1-\alpha)A(\rho+\varepsilon)}{\varepsilon\eta+\rho+\varepsilon}$. *The value of* $\varepsilon$ *is given by*

$$\frac{\eta(1-\alpha)A/(q_0/k_0+1)-\rho}{\eta+1}.$$

## 5. DISCUSSIONS OF OPTIMAL POLICY

In this section, I first explore the socially optimal solution for an economy with the "spirit of capitalism." I then discuss two policies that can help the competitive economy achieve the social optimum.

### 5.1. Social Optimum

The parameterized utility function of an individual is given by (9). Because the total number of individuals is normalized to 1, the aggregate wealth, $a$, has to be equal to the average wealth, $\bar{a}$. This implies that the wealth term in the objective function of the central planner should be zero. Thus, the optimization question of the central planner can be given by

$$\max_c \int_0^\infty e^{-\rho t} \log c dt,$$

subject to the real resource constraint

$$\dot{k} = f(k) - \delta k - c. \tag{22}$$

Here, I consider a general form of the production function.

The social optimum of the economy can be described by the previously mentioned real resource constraint (22), the Euler equation

$$-\frac{\dot{c}}{c} = \rho - [f'(k) - \delta], \tag{23}$$

and the TVC

$$\lim_{t\to\infty} e^{-\rho t}\frac{k}{c} = 0.$$

It is clear that no bubble exists at the social optimum. Because the marginal utility of wealth is always zero, the central planner has no incentive to pursue wealth for any other purpose than future consumption flows. Therefore, there is also no overaccumulation of physical capital at the social optimum.

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By comparing the social optimum with the competitive economy, I can draw the following conclusions. It is the status seeking of individuals that distorts the competitive economy. In the bubbleless economy, physical capital is accumulated excessively. The emergence of a bubble relieves the situation but cannot help the economy reach the social optimum. This is because bubbles cannot totally take the place of physical capital in the process of status seeking.

## 5.2. Discussions on Policies

Given the preceding analysis, it is natural to ask whether a policy for restricting capital accumulation can push the competitive economy to the social optimum. To answer the question, I consider the following two tax policies.

*Investment income tax.* Suppose the tax rate on investment returns is denoted by $\tau$. The total income of the government from the tax is paid back to the households by the lump-sum transfer $T$. Thus, the budget constraint of the representative individual can be written as

$$\dot{a} = (r - \tau)k - c + \dot{q}s + \Pi + T.$$

From the first-order conditions of the representative individual's optimal question, I obtain the nonarbitrage condition

$$\frac{\dot{q}}{q} = r - \tau \tag{24}$$

and the Euler equation

$$\frac{\dot{\mu}}{\mu} = \rho - \eta\frac{c}{a} - (r - \tau). \tag{25}$$

To allow the economy to achieve the social optimum, the Euler equation (25) should be consistent with the equation (23). Thus, the efficient tax rate on investment return, $\tau^{\mathrm{E}}$, should be equal to $\eta c/a$, which is the ratio of the marginal utility of wealth to the marginal utility of consumption, $U_a'/U_c'$. Intuitively, the efficient tax rate is just equal to the additional investment return rate from the happiness of gaining wealth. When the government sets the tax rate to $\tau^{\mathrm{E}}$, the investment income tax will completely offset the incentive for capital accumulation from status seeking. Therefore, the efficient tax rate can eliminate the capital overaccumulation.

However, under the preceding efficient tax rate, a bubble might still exist because

$$\frac{\dot{\mu}}{\mu} + \frac{\dot{q}}{q} = \rho - \tau^{\mathrm{E}} < \rho,$$

which implies that the TVC (7) is satisfied. To illustrate this point, I consider the following example. Assume the production function is given by $Ak^{\alpha}$. In the steady state, I obtain that $\alpha Ak^{\alpha-1} - \delta = \eta c/a = \rho$. It is easy to check that the capital,

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$k$, and the consumption, $c$, are both at an economically efficient level. However, with the parameter restriction (11), I can obtain that

$$\frac{a}{k} = \frac{\eta}{\rho}\frac{\rho + (1-\alpha)\delta}{\alpha} > \frac{\eta}{\rho}\frac{(1-\alpha)\delta}{\alpha} > 1.$$

This result implies that

$$q > 0.$$

Therefore, the efficient tax rate on investment returns, $\tau^{\mathrm{E}}$, cannot eliminate bubbles.

*Wealth tax.* In reality, however, bubbles may burst at any time. The probability that a bubble bursts is difficult to investigate. The existence of bubbles implies some kind of uncertainty, which distorts the economy. Thus, the desired policies should not only push the economy to the social optimum but also expel bubbles. Here, I demonstrate that a tax on wealth is such a policy.

Suppose the tax rate on wealth is denoted by $\theta$. The total income of the government from the tax is paid back to the households by the lump-sum transfer $T$. The budget constraint of households is given by

$$\dot{a} = rk - c + \dot{q}s + \Pi - \theta a + T.$$

By solving the optimal question of households, I obtain the nonarbitrage condition

$$\frac{\dot{q}}{q} = r$$

and the Euler equation

$$\frac{\dot{\mu}}{\mu} = \rho - \eta\frac{c}{a} - r + \theta.$$

Clearly, to allow the economy to achieve the social optimum, the efficient tax rate on wealth, $\theta^{\mathrm{E}}$, should also be the ratio of the marginal utility of wealth to the marginal utility of consumption, $\eta c/a$. With the efficient tax rate, $\theta^{\mathrm{E}}$, I can obtain that

$$\frac{\dot{\mu}}{\mu} + \frac{\dot{q}}{q} = \rho.$$

This implies that the TVC (7) will rule out bubbles.

The following intuitions help us to understand this result. The efficient tax rate, $\theta^{\mathrm{E}}$, is just equal to the additional return rate from the happiness of wealth gaining. Given the tax is on wealth, the efficient tax rate will completely offset the incentive for wealth accumulation from status seeking. Thus, it is trivial that the efficient wealth tax rate can eliminate capital overaccumulation. Moreover, people would sell out a bubble sooner or later, because it cannot provide any material reward. This behavior expels bubbles.

## 6. CONCLUSION

This study focuses on rational bubbles driven by the pursuit of status in an infinite-horizon model. In an economy with the "spirit of capitalism," as long as the marginal utility of holding wealth is eventually nontrivial relative to the marginal utility of consumption, rational bubbles can emerge. The analysis of economic dynamics yields results similar to those reported by Tirole (1985). However, my infinite-horizon model eliminates the concern over an incomplete market generated by the structure of OLG. Because an infinite-horizon framework is the common basis for a vast literature on asset pricing and macroeconomics, many economic issues regarding bubbles can be discussed based on my framework.

In addition, this study also discusses an economy in which a bubble might burst with an exogenous probability. This study provides a simple theoretical foundation for discussing the economic implications of the collapse of a bubble. As an interesting direction for further study, issues regarding financial crisis can be explored by introducing a banking sector into this framework.

### NOTES

1. Please refer to Zhou (2013) for a discussion on a rational bubble in the pricing of a positive-dividend asset.
2. Specifically, Kamihigashi's bubble requires that the marginal utility of wealth not decline to zero as wealth goes to infinity. The implications of bubbles in his paper depend on the production function.
3. More details in Kamien and Schwartz (1991).
4. Kurz (1968) demonstrates that it is possible for multiple steady states to arise when the wealth term enters the utility function directly.
5. The explicit solutions of the deterministic bubbleless economy and the deterministic bubbly economy are provided in part (a) and part (c) of Proposition 3, respectively.

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# APPENDIX A. PROOF OF PROPOSITION 1

The neoclassical growth economy can be described by the following system of equations:

$$\dot{k} = Ak^{\alpha} - \delta k - c, \quad \textbf{(A.1)}$$

$$\dot{q} = (\alpha A k^{\alpha-1} - \delta)q, \quad \textbf{(A.2)}$$

$$\dot{c} = c\left[\eta \frac{c}{k+q} + \alpha A k^{\alpha-1} - \delta - \rho\right]. \quad \textbf{(A.3)}$$

In the bubbleless steady state, the bubble value is equal to zero; i.e., $q^* = 0$. From equation (A.1) and equation (A.3), I can obtain the two equations in the steady state as follows:

$$Ak^{\alpha} - \delta k = c,$$

$$\delta + \rho = \eta \frac{c}{k} + \alpha A k^{\alpha-1}.$$

It is easy to solve this system of equations. The unique solution is given by $k^* = [\frac{\rho+(1+\eta)\delta}{(\eta+\alpha)A}]^{1/(\alpha-1)}$ and $c^* = [\frac{\rho+(1+\eta)\delta}{(\eta+\alpha)A}]^{1/(\alpha-1)}[\frac{\rho+(1+\eta)\delta}{\eta+\alpha} - \delta]$. In addition, $0 < \alpha < 1$ ensures that $c^* > 0$.

In the bubbly steady state, the bubble value is some positive constant. From equation (A.2), I can obtain that $k^{**} = (\frac{\delta}{\alpha A})^{1/(\alpha-1)}$. Using equation (A.1), I obtain that

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$c^{**} = (1-\alpha)(\frac{\delta}{\alpha})^{\alpha/(\alpha-1)} A^{1/(1-\alpha)}$. By equation (A.3), I obtain that $q^{**} = [\frac{\eta\delta(1-\alpha)}{\rho\alpha} - 1](\frac{\delta}{\alpha A})^{1/(\alpha-1)}$. To guarantee that $q^{**} > 0$, I need a parameter restriction given by $\eta\delta(1-\alpha) > \rho\alpha$.

The restriction implies that $\alpha A k^{*\alpha-1} - \delta < 0$, whereas the real interest rate int the bubbly steady state, $\alpha A k^{**\alpha-1} - \delta$, is equal to zero. Given the decreasing return of the production function $Ak^{\alpha}$, I know that $k^{**} < k^{*}$. In any steady state, $c = Ak^{\alpha} - \delta k$. Because $Ak^{\alpha} - \delta k$ is a strictly concave function, it is easy to observe that $c^{**} > c^{*}$. ■

# APPENDIX B. PROOF OF PROPOSITION 2

This proof mainly follows the method of Miao and Wang (2013).

## B.1. AROUND THE BUBBLY STEADY STATE

To verify the stability of the bubbly steady state, first, I have to log-linearize the dynamic system of (A.1), (A.2), and (A.3) around the steady state $\{k^{**}, q^{**}, c^{**}\}$.

I define

$$\hat{x} \equiv \log \frac{x}{x^{**}},$$

which implies that

$$\frac{\dot{x}}{x} = \dot{\hat{x}}.$$

The log-linearized system is given by

$$\begin{pmatrix} \dot{\hat{k}} \\ \dot{\hat{q}} \\ \dot{\hat{c}} \end{pmatrix} = \Psi \begin{pmatrix} \hat{k} \\ \hat{q} \\ \hat{c} \end{pmatrix},$$

where

$$\Psi \equiv \begin{pmatrix} (\alpha-1)A(k^{**})^{\alpha-1} + \frac{c^{**}}{k^{**}} & 0 & -\frac{c^{**}}{k^{**}} \\ \alpha(\alpha-1)A(k^{**})^{\alpha-1} & 0 & 0 \\ \alpha(\alpha-1)A(k^{**})^{\alpha-1} - \frac{\eta c^{**}k^{**}}{(k^{**}+q^{**})^2} & -\frac{\eta c^{**}q^{**}}{(k^{**}+q^{**})^2} & \frac{\eta c^{**}}{k^{**}+q^{**}} \end{pmatrix}.$$

Here, $\Psi_{1,1} = 0$, $\Psi_{1,3} < 0$, $\Psi_{2,1} < 0$, $\Psi_{3,1} < 0$, $\Psi_{3,2} < 0$.

The characteristic equation of the matrix $\Psi$ is given by

$$F(\lambda) = |\Psi - \lambda I| = -\lambda^3 + \rho\lambda^2 + \Psi_{1,3}\Psi_{3,1}\lambda + \Psi_{1,3}\Psi_{2,1}\Psi_{3,2}.$$

The equation can be rewritten as

$$\begin{aligned} F(\lambda) &= -(\lambda-\psi_1)(\lambda-\psi_2)(\lambda-\psi_3) \\ &= -\lambda^3 + (\psi_1+\psi_2+\psi_3)\lambda^2 \\ &\quad - (\psi_1\psi_2 + \psi_1\psi_3 + \psi_2\psi_3)\lambda + \psi_1\psi_2\psi_3. \end{aligned}$$

Thus, I determine that the characteristic roots $\psi_1$, $\psi_2$, and $\psi_3$ must satisfy the following conditions:

$$\psi_1 + \psi_2 + \psi_3 = \rho > 0, \tag{B.1}$$

$$\psi_1\psi_2 + \psi_1\psi_3 + \psi_2\psi_3 = -\Psi_{1,3}\Psi_{3,1} < 0, \tag{B.2}$$

$$\psi_1\psi_2\psi_3 = \Psi_{1,3}\Psi_{2,1}\Psi_{3,2} < 0. \tag{B.3}$$

Because $F(0) < 0$, $F(-\infty) = +\infty$, matrix $\Psi$ has at least one negative characteristic root, which I denote by $\psi_1$. I discuss the other two characteristic roots, $\psi_2$ and $\psi_3$, by considering two cases.

(i) $\psi_2$ and $\psi_3$ are both real numbers. From equation (B.3), I can obtain that either $\psi_2 > 0$ and $\psi_3 > 0$, or $\psi_2 < 0$ and $\psi_3 < 0$. Given equation (B.1) and $\psi_1 < 0$, it is impossible that $\psi_2 < 0$ and $\psi_3 < 0$. Therefore, $\psi_2 > 0$ and $\psi_3 > 0$.

(ii) $\psi_2$ and $\psi_3$ are one pair of complex numbers. I denote them by $\psi_2 \equiv a + bi$ and $\psi_3 \equiv a - bi$. It is easy to determine that $\psi_1 + \psi_2 + \psi_3 = 2a + \psi_1$. Given equation (B.1) and $\psi_1 < 0$, the value of $a$ must be positive.

Based on this discussion, it is clear that matrix $\Psi$ has only one negative eigenvalue, which indicates that the bubbly steady state is a local saddlepoint because the dynamic system has only one state variable, $k$.

## B.2. AROUND THE BUBBLELESS STEADY STATE

First, I linearize $q$ and loglinearize $k$ and $c$ around the bubbleless steady state $\{k^*, 0, c^*\}$. The dynamic system of (A.1), (A.2), and (A.3) can be rewritten as

$$\begin{pmatrix} \dot{\hat{k}} \\ \dot{q} \\ \dot{\hat{c}} \end{pmatrix} = \Phi \begin{pmatrix} \hat{k} \\ q \\ \hat{c} \end{pmatrix},$$

where

$$\Phi \equiv \begin{pmatrix} (\alpha-1)A(k^*)^{\alpha-1} + \frac{c^*}{k^*} & 0 & -\frac{c^*}{k^*} \\ 0 & \alpha A(k^*)^{\alpha-1} - \delta & 0 \\ \alpha(\alpha-1)A(k^*)^{\alpha-1} - \eta\frac{c^*}{k^*} & -\eta\frac{c^*}{(k^*)^2} & \eta\frac{c^*}{k^*} \end{pmatrix}.$$

Given the parameter restriction (11), it is easy to determine that $\Phi_{1,1} = \alpha A(k^*)^{\alpha-1} - \delta < 0$, $\Phi_{1,3} < 0$, $\Phi_{2,2} < 0$, $\Phi_{3,1} < 0$, $\Phi_{3,2} < 0$, and $\Phi_{3,3} > 0$.

The characteristic equation of the matrix $\Phi$ is given by

$$\begin{aligned} F(\lambda) &= |\Phi - \lambda I| \\ &= -\lambda^3 + (\Phi_{1,1} + \Phi_{2,2} + \Phi_{3,3})\lambda^2 \\ &\quad - (\Phi_{2,2}\Phi_{3,3} + \Phi_{1,1}\Phi_{3,3} + \Phi_{1,1}\Phi_{2,2})\lambda \\ &\quad + \Phi_{1,1}\Phi_{2,2}\Phi_{3,3} - \Phi_{1,3}\Phi_{2,2}\Phi_{3,1}, \end{aligned}$$

where

$$\Phi_{2,2}\Phi_{3,3} + \Phi_{1,1}\Phi_{3,3} + \Phi_{1,1}\Phi_{2,2} < 0,$$

$$\Phi_{1,1}\Phi_{2,2}\Phi_{3,3} - \Phi_{1,3}\Phi_{2,2}\Phi_{3,1} > 0.$$

The equation can be rewritten as

$$\begin{aligned} F(\lambda) &= -(\lambda - \phi_1)(\lambda - \phi_2)(\lambda - \phi_3) \\ &= -\lambda^3 + (\phi_1 + \phi_2 + \phi_3)\lambda^2 \\ &\quad - (\phi_1\phi_2 + \phi_1\phi_3 + \phi_2\phi_3)\lambda + \phi_1\phi_2\phi_3, \end{aligned}$$

where $\phi_1$, $\phi_2$, and $\phi_3$ are characteristic roots. It is easy to determine that

$$\phi_1\phi_2 + \phi_1\phi_3 + \phi_2\phi_3 < 0,$$

$$\phi_1\phi_2\phi_3 > 0.$$

Because $F(0) > 0$, $F(+\infty) = -\infty$, the matrix $\Phi$ has at least one positive real characteristic root, which is denoted by $\phi_1$. Given $\phi_1\phi_2\phi_3 > 0$, I can obtain that $\phi_2\phi_3 > 0$. Together with $\phi_1\phi_2 + \phi_1\phi_3 + \phi_2\phi_3 < 0$, it is clear that $\phi_2 + \phi_3 < 0$. Therefore, the bubbleless steady state is a local saddlepoint with a two-dimensional stable manifold. ■

# APPENDIX C. PROOF OF PROPOSITION 3

The endogenous growth economy can be described by the following system of equations:

$$\dot{q} = (\alpha A - \delta)q, \tag{C.1}$$

$$-\frac{\dot{c}}{c} = \rho - \frac{\eta c}{q + k} - (\alpha A - \delta), \tag{C.2}$$

$$\dot{k} = (A - \delta)k - c, \tag{C.3}$$

together with TVCs (3) and (7).

## C.1. BUBBLELESS BALANCED GROWTH PATH

On the bubbleless BGP, the value of the bubble is zero, i.e., $q = 0$. Based on the real resource constraint, I obtain that $\frac{\dot{c}}{c} \leq \frac{\dot{k}}{k}$.

If the growth rate of consumption is less than the growth rate of capital, i.e., $\frac{\dot{c}}{c} < \frac{\dot{k}}{k}$, then $\frac{c}{k} \to 0$. From equation (C.2) and equation (C.3), I can obtain that $\frac{\dot{c}}{c} \to \alpha A - \delta - \rho$ and $\frac{\dot{k}}{k} \to A - \delta$. Thus, $\frac{\dot{k}}{k} - \frac{\dot{c}}{c} \to (1 - \alpha)A + \rho > \rho$, which indicates the TVC (3) is violated. Therefore, it must be that $\frac{\dot{c}}{c} = \frac{\dot{k}}{k}$ on the bubbleless BGP.

From equation (C.2) and equation (C.3), it is easy to obtain that $\frac{c}{k} = \frac{(1-\alpha)A+\rho}{\eta+1}$ and $\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = A - \delta - \frac{(1-\alpha)A+\rho}{\eta+1}$.

## C.2. QUASI-BUBBLELESS BALANCED GROWTH PATH

On the quasi-bubbleless BGP, the value of the bubble is not zero at all. However, the growth rate of the bubble is less than the growth rate of the real economy. Last, the value of the bubble will be trivial.

Based on the real resource constraint, I obtain that $\frac{\dot{c}}{c} \leq \frac{\dot{k}}{k}$. This finding implies that at least the growth rate of capital is higher than the growth rate of the bubble; i.e., $\frac{\dot{k}}{k} > \frac{\dot{q}}{q} = \alpha A - \delta$. If the growth rate of consumption is lower than the growth rate of capital, i.e., $\frac{\dot{c}}{c} < \frac{\dot{k}}{k}$, then the term $\frac{\eta c}{q+k}$ will converge to zero. Thus, the TVC (7) is violated. Therefore, on the quasi-bubbleless balanced growth path, $\frac{\dot{c}}{c} = \frac{\dot{k}}{k} > \frac{\dot{q}}{q} = \alpha A - \delta$.

Suppose the term $\frac{\eta c}{q+k}$ eventually converges to a positive constant, $\theta$. I can obtain that $\frac{c}{k} \rightarrow \frac{\theta}{\eta}$. Combining equation (C.2) with equation (C.3), I obtain that $\theta = \frac{\eta[(1-\alpha)A+\rho]}{\eta+1}$ and $\frac{\dot{c}}{c} = \frac{\dot{k}}{k} \rightarrow A - \delta - \frac{(1-\alpha)A+\rho}{\eta+1}$. Given the restriction (14), it is easy to determine that $A - \delta - \frac{(1-\alpha)A+\rho}{\eta+1} > \alpha A - \delta$. Thus, there exists a quasi-bubbleless BGP.

### C.3. BUBBLY BALANCED GROWTH PATH

On the bubbly BGP the growth rate of the bubble should not be less than the growth rate of the real economy. Otherwise, the value of the bubble would be trivial relative to that of the real economy, which implies that $\frac{\dot{q}}{q} \geq \frac{\dot{k}}{k}$. By the real resource constraint, I can obtain that $\frac{\dot{c}}{c} \leq \frac{\dot{k}}{k}$. However, the growth rate of consumption should not be less than the growth rate of capital. Otherwise, by equation (C.3), the growth rate of capital eventually converges to $A - \delta$, which is higher than the growth rate of the bubble, $\alpha A - \delta$. Thus, on the bubbly BGP, consumption and physical capital have the same growth rate, which is not greater than the growth rate of the bubble; i.e., $\frac{\dot{c}}{c} = \frac{\dot{k}}{k} \leq \frac{\dot{q}}{q}$.

To ensure that the condition (10) is not violated, the term $\frac{\eta c}{q+k}$ should eventually converge to a positive constant. Thus, eventually, $\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = \frac{\dot{q}}{q} = \alpha A - \delta$. From equations (C.2) and (C.3), respectively, I obtain that $\frac{\eta c}{q+k} = \rho$ and $\frac{c}{k} = (1-\alpha)A$. ■

# APPENDIX D. PROOF OF PROPOSITION 5

Following the method of Rebelo and Xie (1999), I can obtain an explicit solution for the stochastic bubbly economy.

The Hamilton–Jacobi—Bellman equation is given by

$$\begin{aligned} 0 = \max_{c,\iota}\{&U(c, qs+k) + V_1(q,\bar{k},k,s)\varphi q + V_2(q,\bar{k},k,s)\varphi\bar{k} \\ &+ V_3(q,\bar{k},k,s)(Ak^{\alpha}\bar{k}^{1-\alpha} - \delta k - c - q\iota) \\ &+ V_4(q,\bar{k},k,s)\iota - (\varepsilon+\rho)V(q,\bar{k},k,s) + \varepsilon V(0,\bar{k},k,s)\}. \end{aligned}$$

I guess the form of the value function to be

$$V(q,\bar{k},k,s) \equiv \chi + h\log(qs+k) + b\log\bar{k} + \psi\log(q+\bar{k}).$$

It is easy to obtain that $V_1(q,\bar{k},k,s) = \frac{hs}{qs+k} + \frac{\psi}{q+\bar{k}}$, $V_2(q,\bar{k},k,s) = \frac{b}{\bar{k}} + \frac{\psi}{q+\bar{k}}$, $V_3(q,\bar{k},k,s) = \frac{h}{qs+k}$, and $V_4(q,\bar{k},k,s) = \frac{hq}{qs+k}$. Thus, the above

Hamilton–Jacobi–Bellman equation can be rewritten as

$$\begin{aligned} 0 = \max_{c,\iota}\{&\log c + \eta\log(qs+k) - \eta\log(q+\bar{k}) \\ &+ \frac{h\varphi qs}{qs+k} + b\varphi + \psi\varphi + \frac{h}{qs+k}(Ak^{\alpha}\bar{k}^{1-\alpha} - \delta k - c) \\ &- (\varepsilon+\rho)[\chi + h\log(qs+k) + b\log\bar{k} + \psi\log(q+\bar{k})] \\ &+ \varepsilon[\chi + h\log k + (b+\psi)\log\bar{k}]\}. \end{aligned} \tag{D.1}$$

The optimal condition for consumption is given by

$$c = \frac{qs+k}{h}. \tag{D.2}$$

The partial derivatives of equation (D.1) with respect to $\bar{k}$, $s$, $q$, and $k$ should all be zero. That is,

$$\frac{h(1-\alpha)}{qs+k}Ak^{\alpha}\bar{k}^{-\alpha} + \frac{\varepsilon\psi}{\bar{k}} = \frac{\eta}{q+\bar{k}} + \frac{\rho b}{\bar{k}} + \frac{\psi(\varepsilon+\rho)}{q+\bar{k}}, \tag{D.3}$$

$$\eta(qs+k) + h\varphi k = h(\varepsilon+\rho)(qs+k) + h(Ak^{\alpha}\bar{k}^{1-\alpha} - \delta k - c), \tag{D.4}$$

$$\begin{aligned} &\eta s(qs+k) + hs\varphi k - \frac{[\eta+\psi(\varepsilon+\rho)](qs+k)^2}{q+\bar{k}} \\ &\quad = hs(\varepsilon+\rho)(qs+k) + hs(Ak^{\alpha}\bar{k}^{1-\alpha} - \delta k - c), \end{aligned} \tag{D.5}$$

$$\begin{aligned} &\eta(qs+k) + h(\alpha Ak^{\alpha-1}\bar{k}^{1-\alpha} - \delta)(qs+k) + \frac{\varepsilon h}{k}(qs+k)^2 \\ &\quad = h(\varepsilon+\rho)(qs+k) + h\varphi qs + h(Ak^{\alpha}\bar{k}^{1-\alpha} - \delta k - c). \end{aligned} \tag{D.6}$$

From equation (D.4) and equation (D.5), I obtain that $\psi = -\frac{\eta}{\varepsilon+\rho}$. Thus, equation (D.3) can be rewritten as

$$h(1-\alpha)Ak^{\alpha}\bar{k}^{1-\alpha} = \left(\rho b + \frac{\varepsilon\eta}{\varepsilon+\rho}\right)(qs+k). \tag{D.7}$$

Equation (D.6) minus equation (D.4) is

$$\varepsilon(qs+k) = [\varphi - (\alpha Ak^{\alpha-1}\bar{k}^{1-\alpha} - \delta)]k. \tag{D.8}$$

Together with equation (D.7), I obtain that

$$k^{\alpha-1}\bar{k}^{1-\alpha} = \frac{(\varphi+\delta)(\rho b + \frac{\varepsilon\eta}{\varepsilon+\rho})}{A[\varepsilon h(1-\alpha) + (\rho b + \frac{\varepsilon\eta}{\varepsilon+\rho})\alpha]}.$$

Thus, the ratio of $k$ to $\bar{k}$ is a constant. At equilibrium, $k = \bar{k}$. Therefore, I know that

$$(\varphi+\delta)\left(\rho b + \frac{\varepsilon\eta}{\varepsilon+\rho}\right) = A\left[\varepsilon h(1-\alpha) + \left(\rho b + \frac{\varepsilon\eta}{\varepsilon+\rho}\right)\alpha\right] \tag{D.9}$$

and $k = \bar{k}$.

Substituting equation (D.2) into equation (D.4), I can obtain that

$$h\varphi k - h(Ak^{\alpha}\bar{k}^{1-\alpha} - \delta k) = [h(\varepsilon + \rho) - 1 - \eta](qs + k).$$

By substituting equation (D.8) into this equation, I obtain that

$$(\varphi + \delta)(\rho h - 1 - \eta) = A\{\alpha[h(\varepsilon + \rho) - 1 - \eta] - \varepsilon h\}. \qquad \textbf{(D.10)}$$

From equation (D.7), I know that

$$\frac{(1-\alpha)A}{\rho b + \frac{\varepsilon\eta}{\varepsilon+\rho}} k = \frac{qs + k}{h} = c. \qquad \textbf{(D.11)}$$

Because $k = \bar{k}$ and based on equation (D.2), the budget constraint can be rewritten as

$$\dot{k} = Ak - \delta k - \frac{(1-\alpha)A}{\rho b + \frac{\varepsilon\eta}{\varepsilon+\rho}} k - q\iota.$$

Because $\iota \equiv 0$ at equilibrium, I can obtain that

$$\dot{\bar{k}} = \left[ A - \delta - \frac{(1-\alpha)A}{\rho b + \frac{\varepsilon\eta}{\varepsilon+\rho}} \right] \bar{k}.$$

Thus,

$$\varphi = A - \delta - \frac{(1-\alpha)A}{\rho b + \frac{\varepsilon\eta}{\varepsilon+\rho}}. \qquad \textbf{(D.12)}$$

By solving the system of equations consisting of (D.9), (D.10), and (D.12), I can obtain that $b = 1/\rho$, $h = \frac{\eta}{\varepsilon+\rho}$, and $\varphi = A - \delta - \frac{(1-\alpha)A(\varepsilon+\rho)}{\varepsilon+\rho+\varepsilon\eta}$.

Substituting the results into equation (D.1), I obtain that

$$\begin{aligned} 0 = \max_{c,\iota} \Bigg\{ &\log \frac{(1-\alpha)A(\varepsilon+\rho)}{(\varepsilon+\rho)+\varepsilon\eta} + \log k - \log k \\ &+ \eta \log\left[\frac{\eta(1-\alpha)A}{(\varepsilon+\rho)+\varepsilon\eta} k\right] - \eta \log\left[\frac{\eta(1-\alpha)A}{(\varepsilon+\rho)+\varepsilon\eta} k\right] \\ &+ (b + \psi + h)\varphi - \rho a + \varepsilon\psi \log k + \varepsilon h \log k \Bigg\}. \end{aligned}$$

It is easy to verify that the sum of the coefficients of the $\log k$ term is zero. When $\chi$ takes the value

$$\frac{1}{\rho} \log \frac{(1-\alpha)A(\varepsilon+\rho)}{\varepsilon+\rho+\varepsilon\eta} + \frac{1}{\rho^2}\left[A - \delta - \frac{(1-\alpha)A(\varepsilon+\rho)}{\varepsilon+\rho+\varepsilon\eta}\right],$$

the sum of the constant terms is also zero. ■