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SOME NEW RESULTS ON THE LARGEST ORDER STATISTICS FROM MULTIPLE-OUTLIER GAMMA MODELS

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In this paper, we carry out stochastic comparisons of the largest order statistics arising from multiple-outlier gamma models with different both shape and scale parameters in the sense of various stochastic orderings including the likelihood ratio order, star order and dispersive order. It is proved, among others, that the weak majorization order between the scale parameter vectors along with the majorization order between the shape parameter vectors imply the likelihood ratio order between the largest order statistics. A quite general sufficient condition for the star order is presented. The new results established here strengthen and generalize some of the results known in the literature. Numerical examples and applications are also provided to explicate the theoretical results.

1. INTRODUCTION

Order statistics play a prominent role in statistical inference, reliability theory, life testing, operations research, and many other areas. The kth order statistic $X_{k:n}$ arising from the sample $X_1, \ldots, X_n$ corresponds to the lifetime of a $(n-k+1)$-out-of-n system, which is a very popular structure of redundancy in fault-tolerant systems that have been studied extensively. In particular, $X_{n:n}$ and $X_{1:n}$ correspond the lifetimes of parallel and series systems, respectively. A large number of papers have appeared on various aspects of order statistics when the observations are independent and identically distributed (i.i.d.), but for the case when observations are non-i.i.d., not too much work is available in the literature due to the complexity of the distribution theory; see, for example, David and Nagaraja [9], Balakrishnan and Rao [3,4] and Balakrishnan [2] for comprehensive discussions on this topic.

Pledger and Proschan [23] may be the first to stochastically compare the order statistics arising from independent but non-identically distributed (i.ni.d.) exponential random variables. After that, many researchers have paid attentions to this topic and established colorful results, including Proschan and Sethuraman [24], Boland, EL-Neweihi, and Proschan [6],

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Hu [11], Kochar and Rojo [15], Dykstra, Kochar, and Rojo [10], Khaledi and Kochar [13], Bon and Păltănea [8], Kochar and Xu [16], Păltănea [22], Zhao and Balakrishnan [29], Joo and Mi [12], Mao and Hu [18], Khaledi, Farsinezhad, and Kochar [14] and Kochar and Xu [17]. Gamma distribution is one of the most commonly used distributions in reliability and life testing. If X is a gamma random variable with shape parameter r and scale parameter λ, in its standard form X has the probability density function

$$f(x; r, \lambda) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}, \quad x > 0.$$

It is a quite flexible family of distributions with decreasing, constant, and increasing failure rates when $0 < r < 1$, $r = 1$ and $r > 1$, respectively. Meanwhile, gamma distribution has been widely used to describe the lifetime of components in shock model and minimal repairs. In this paper, we will focus our attentions on the stochastic comparisons of the largest order statistic arising from multiple-outlier gamma models with different both shape and scale parameters.

Let X_1 and X_2 be independent gamma variables with X_i having shape parameter r_i and scale parameter λ_i, $i = 1, 2$, and X_1^*, X_2^* be another set of independent gamma variables with X_i^* having shape parameter r_i^* and scale parameter λ_i^*, $i = 1, 2$. Suppose $r_1 \geq r_2$, $r_1^* \geq r_2^*$ and $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$. Zhao and Zhang [32] proved that

$$(r_1, r_2) \overset{\mathrm{m}}{\succeq} (r_1^*, r_2^*), (\lambda_1, \lambda_2) \overset{\mathrm{w}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{2:2} \geq_{\mathrm{lr}} X_{2:2}^*. \tag{1}$$

The pertinent definitions and notions such as stochastic orders, majorization and related orders may refer to Barlow and Proschan [5], Bon and Păltănea [7], Marshall and Olkin [19], Shaked and Shanthikumar [27] and Marshall, Olkin, and Arnold [20]. Furthermore, it was also proved that, if $r_1 = r_1^* \geq r_2^* = r_2$, $\lambda_1 \leq \lambda_2$, and $\lambda_1^* \leq \lambda_2^*$, then

$$\frac{\lambda_2}{\lambda_1} \geq \frac{\lambda_2^*}{\lambda_1^*} \Longrightarrow X_{2:2} \geq_{\star} X_{2:2}^*. \tag{2}$$

With the help of (2), they further proved, if $r_1 = r_1^* \geq r_2^* = r_2$ and $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$, then,

$$(\lambda_1, \lambda_2) \overset{\mathrm{w}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{2:2} \geq_{\mathrm{disp}} X_{2:2}^*. \tag{3}$$

These results in (1)–(3) generalize and strengthen some results established earlier in the literature for the exponential case. It should be also mentioned that Zhao [28] and Zhao and Balakrishnan [30] established some results similar to those in (1) and (3) for the special case when all the shape parameters are common.

On the other hand, Zhao and Balakrishnan [31] studied the stochastic properties on the largest order statistics arising from multiple-outlier gamma models with common shape parameter. To be specific, let $X_1, \ldots, X_n$ be independent random variables following the multiple-outlier gamma model with common shape parameter $r > 0$ and scale parameters

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{p}, \underbrace{\lambda_2, \ldots, \lambda_2}_{q}),$$

where $p, q \in \mathcal{Z}_+ = \{1, 2, \ldots\}$ and $p + q = n$. For simplicity, we will use $\lambda_1 \mathbf{1}_p$ to denote $(\underbrace{\lambda_1, \ldots, \lambda_1}_{p})$, where $\mathbf{1}_p$ is a p-dimensional vector with all values being 1. Therefore, the scale parameters vector can be written as $(\lambda_1 \mathbf{1}_p, \lambda_2 \mathbf{1}_q)$. And let $Y_1, \ldots, Y_n$ be another set of independent random variables following the multiple-outlier gamma model with common shape parameter r and scale parameters $(\lambda_1^*\mathbf{1}_p, \lambda_2^*\mathbf{1}_q)$. They proved, under the condition $\lambda_1 \le \lambda_1^* \le \lambda_2^* \le \lambda_2$ and $0 < r \le 1$, that

$$(\lambda_1\mathbf{1}_p, \lambda_2\mathbf{1}_q) \overset{\text{w}}{\succeq} (\lambda_1^*\mathbf{1}_p, \lambda_2^*\mathbf{1}_q) \Longrightarrow X_{n:n} \ge_{\text{lr}} Y_{n:n} \tag{4}$$

and

$$(\lambda_1\mathbf{1}_p, \lambda_2\mathbf{1}_q) \overset{\text{p}}{\succeq} (\lambda_1^*\mathbf{1}_p, \lambda_2^*\mathbf{1}_q) \Longrightarrow X_{n:n} \ge_{\text{hr[disp]}} Y_{n:n}. \tag{5}$$

Moreover, they further obtained that

$$\frac{\lambda_2}{\lambda_1} \ge \frac{\lambda_2^*}{\lambda_1^*} \Longrightarrow X_{n:n} \ge_{\star} Y_{n:n}. \tag{6}$$

However, all the results in (4)–(6) require that all the shape parameters are common and restricted in the interval (0,1]. Motivated by this, we will pursue the ordering properties on the largest order statistics arising from multiple-outlier gamma models having different shape parameters. Let $X_1, X_2, \ldots, X_n$ be independent random variables following the multiple-outlier gamma model with respective shape parameters and scale parameters $(r_1\mathbf{1}_p, r_2\mathbf{1}_q), (\lambda_1\mathbf{1}_p, \lambda_2\mathbf{1}_q)$, where $p, q \in \mathcal{Z}_+$ and $p + q = n$. Let $Y_1, Y_2, \ldots, Y_n$ be another set of independent random variables following the multiple-outlier gamma model with respective shape parameters and scale parameters $(r_1^*\mathbf{1}_p, r_2^*\mathbf{1}_q), (\lambda_1^*\mathbf{1}_p, \lambda_2^*\mathbf{1}_q)$. Under the conditions that $p \ge q$, $r_1 \ge r_2$, $r_1^* \ge r_2^*$ and $\lambda_1 \le \lambda_1^* \le \lambda_2^* \le \lambda_2$, it is proved that

$$(r_1, r_2) \overset{\text{m}}{\succeq} (r_1^*, r_2^*), (\lambda_1, \lambda_2) \overset{\text{w}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{n:n} \ge_{\text{lr}} Y_{n:n}. \tag{7}$$

We also prove that, if $r_1 = r_1^* \ge r_2^* = r_2$, $\lambda_1 \le \lambda_2$ and $\lambda_1^* \le \lambda_2^*$, then

$$\frac{\lambda_2}{\lambda_1} \ge \frac{\lambda_2^*}{\lambda_1^*} \Longrightarrow X_{n:n} \ge_{\star} X_{n:n}^*. \tag{8}$$

Based on (8), we establish that, if $p \ge q$, $r_1 = r_1^* \ge r_2^* = r_2$ and $\lambda_1 \le \lambda_1^* \le \lambda_2^* \le \lambda_2$, then,

$$(\lambda_1, \lambda_2) \overset{\text{w}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{n:n} \ge_{\text{disp}} X_{n:n}^*. \tag{9}$$

It can be seen that the new results in (7)–(9) generalize and strengthen all the results in (1)–(6) established earlier in the literature.

Throughout this paper, the term *increasing* is used for *monotone non-decreasing* and *decreasing* is used for *monotone non-increasing*.

2. LIKELIHOOD RATIO ORDERING

In this section, we stochastically compare the largest order statistics arising from two multiple-outlier gamma samples in terms of the likelihood ratio ordering. First, we present several lemmas that will be helpful for proving the main results. The first one turns out to be a useful tool for showing the monotonicity of a fraction whose numerator and denominator are integrals or summations.

LEMMA 2.1 (Misra and van der Meulen [21]): *Let Θ be a subset of a real line and U be a nonnegative random variable having a cumulative distribution function (c.d.f) belonging* *to a stochastically ordered family* $\mathcal{P} = \{H(\cdot|\theta), \theta \in \Theta\}$*; that is, for* $\theta_1, \theta_2 \in \Theta$, $H(\cdot|\theta_1) \leq_{\rm st} [\geq_{\rm st}] H(\cdot|\theta_2)$ *whenever* $\theta_1 < \theta_2$*. Suppose a real function* $\psi(u,\theta)$ *on* $\Re \cdot \Theta$ *is measurable in* u *for each* θ *such that* $\mathsf{E}_\theta[\psi(U,\theta)]$ *exists. Then the following hold:*

(i) $\mathsf{E}_\theta[\psi(U,\theta)]$ *is increasing in* θ *if* $\psi(u,\theta)$ *is increasing in* θ *and increasing [decreasing] in* u*;*

(ii) $\mathsf{E}_\theta[\psi(U,\theta)]$ *is decreasing in* θ *if* $\psi(u,\theta)$ *is decreasing in* θ *and decreasing [increasing] in* u*.*

The next two lemmas can be found in Zhao [28] and Zhao and Zhang [32], respectively.

LEMMA 2.2 (Zhao [28]): *For* $r > 0$ *and* $y \in \Re_+$*, the function*

$$f_{r,y}(x) = x + \frac{y^{r-1}e^{-xy}}{\int_0^y u^{r-1}e^{-xu}du}$$

is increasing in $x \in \Re_+$*.*

LEMMA 2.3 (Zhao and Zhang [32]): *For* $\lambda > 0$ *and* $t \in \Re_+$*, the function*

$$g_{\lambda,t}(r) = r - \frac{t^r e^{-\lambda t}}{\int_0^t u^{r-1}e^{-\lambda u}du}$$

is increasing in $r \in \Re_+$*.*

LEMMA 2.4: *Suppose* $0 < \lambda_1 \leq \lambda_2$*,* $0 < \lambda_1^* \leq \lambda_2^*$*,* $p \geq q \geq 1$ *(*$p, q \in \mathcal{Z}_+$*) and* $0 < r_2 \leq r_1$*. If* $(\lambda_1, \lambda_2) \stackrel{\rm m}{\succeq} (\lambda_1^*, \lambda_2^*)$*, then the function*

$$\Psi(y,t) = \frac{pe^{-\lambda_1(1-y)t} + qy^{r_1-r_2}e^{-\lambda_2(1-y)t}}{pe^{-\lambda_1^*(1-y)t} + qy^{r_1-r_2}e^{-\lambda_2^*(1-y)t}}$$

is increasing in $t \in \Re_+$*, while is decreasing in* $y \in (0,1)$*.*

PROOF: For simplicity, denote $\lambda_1(1-y) = a_1$, $\lambda_2(1-y) = a_2$, $\lambda_1^*(1-y) = a_1^*$, and $\lambda_2^*(1-y) = a_2^*$, which satisfied $(a_1, a_2) \stackrel{\rm m}{\succeq} (a_1^*, a_1^*)$. Taking the derivative of $\Psi(y,t)$ with respect to t gives rise to

$$\begin{aligned}
\Psi_t'(y,t) &\stackrel{\rm sgn}{=} \left[-a_1pe^{-a_1t} - a_2qy^{r_1-r_2}e^{-a_2t}\right]\left[pe^{-a_1^*t} + qy^{r_1-r_2}e^{-a_2^*t}\right] \\
&\quad - \left[pe^{-a_1t} + qy^{r_1-r_2}e^{-a_2t}\right]\left[-a_1^*pe^{-a_1^*t} - a_2^*qy^{r_1-r_2}e^{-a_2^*t}\right] \\
&= (a_1^* - a_1)p^2e^{-(a_1^*+a_1)t} + (a_2^* - a_2)q^2y^{2(r_1-r_2)}e^{-(a_2^*+a_2)t} \\
&\quad + (a_2^* - a_1)pqy^{r_1-r_2}e^{-(a_2^*+a_1)t} + (a_1^* - a_2)pqy^{r_1-r_2}e^{-(a_1^*+a_2)t} \\
&\geq q^2e^{-(a_2^*+a_2)t}\left[a_1^* - a_1 + a_2^* - a_2\right] + \left[a_2^* - a_1 + a_1^* - a_2\right]pqy^{r_1-r_2}e^{-(a_1^*+a_2)t} \\
&= 0,
\end{aligned}$$

where the last inequality holds due to $a_1 + a_2 = a_1^* + a_2^*$, $p \geq q$ and $y \in (0,1)$. Then, $\Psi(y,t)$ is increasing in $t \in \Re_+$.

We next prove that, for each fixed $t \in \Re_+$, the function $\Psi(y,t)$ is decreasing in $y \in (0,1)$. Notice that

$$\begin{aligned}
\Psi'_y(y,t) \overset{\text{sgn}}{=} & \Big[p\lambda_1 t e^{-\lambda_1(1-y)t} + q(r_1-r_2)y^{r_1-r_2-1}e^{-\lambda_2(1-y)t} + q\lambda_2 t y^{r_1-r_2}e^{-\lambda_2(1-y)t}\Big] \\
& \times \Big[pe^{-\lambda_1^*(1-y)t} + qy^{r_1-r_2}e^{-\lambda_2^*(1-y)t}\Big] \\
& - \Big[p\lambda_1^* t e^{-\lambda_1^*(1-y)t} + q(r_1-r_2)y^{r_1-r_2-1}e^{-\lambda_2^*(1-y)t} + q\lambda_2^* t y^{r_1-r_2}e^{-\lambda_2^*(1-y)t}\Big] \\
& \times \Big[pe^{-\lambda_1(1-y)t} + qy^{r_1-r_2}e^{-\lambda_2(1-y)t}\Big] \\
= & \, p^2(\lambda_1-\lambda_1^*)te^{-(\lambda_1+\lambda_1^*)(1-y)t} + q^2(\lambda_2-\lambda_2^*)ty^{2(r_1-r_2)}e^{-(\lambda_2+\lambda_2^*)(1-y)t} \\
& + pqe^{-(\lambda_2+\lambda_1^*)(1-y)t}y^{r_1-r_2-1}[yt(\lambda_2-\lambda_1^*)+(r_1-r_2)] \\
& + pqe^{-(\lambda_1+\lambda_2^*)(1-y)t}y^{r_1-r_2-1}[yt(\lambda_1-\lambda_2^*)-(r_1-r_2)] \\
\le & \, p^2e^{-(\lambda_1+\lambda_1^*)(1-y)t}[(\lambda_1-\lambda_1^*)+(\lambda_2-\lambda_2^*)]t \\
& + pqe^{-(\lambda_1+\lambda_2^*)(1-y)t}y^{r_1-r_2-1}[yt(\lambda_1+\lambda_2-\lambda_1^*-\lambda_2^*)] \\
= & \, 0,
\end{aligned}$$

where the last inequality holds due to $p \ge q \ge 0$, $y \in (0,1)$ and $\lambda_1+\lambda_2 = \lambda_1^*+\lambda_2^*$. So, $\Psi(y,t)$ is decreasing in $y \in (0,1)$, and the proof is completed. ■

LEMMA 2.5: *Suppose* $0 < r_2 \le r_1$, $0 < r_2^* \le r_1^*$, $p \ge q \ge 1$ *(*$p, q \in \mathcal{Z}_+$*) and* $b \ge 0$*. If*

$$(r_1, r_2) \overset{\text{m}}{\succeq} (r_1^*, r_2^*),$$

then the function

$$\nu(y,t) = \frac{py^{r_2}e^{b(1-y)t} + qy^{r_1}}{py^{r_2^*}e^{b(1-y)t} + qy^{r_1^*}}$$

is increasing in $t \in \Re_+$*, while is decreasing in* $y \in (0,1)$*.*

PROOF: It is known from the assumption that $r_2 \le r_2^* \le r_1^* \le r_1$ and $r_1+r_2 = r_1^*+r_2^*$. Taking the derivative of $\nu(y,t)$ with respect to t yields that

$$\begin{aligned}
\nu'_t(y,t) \overset{\text{sgn}}{=} & \Big[b(1-y)py^{r_2}e^{b(1-y)t}\Big]\Big[py^{r_2^*}e^{b(1-y)t} + qy^{r_1^*}\Big] \\
& - \Big[b(1-y)py^{r_2^*}e^{b(1-y)t}\Big]\Big[py^{r_2}e^{b(1-y)t} + qy^{r_1}\Big] \\
\overset{\text{sgn}}{=} & \, y^{r_1^*+r_2} - y^{r_2^*+r_1} \\
\overset{\text{sgn}}{=} & \, r_2^* + r_1 - r_1^* - r_2 \\
\ge & \, 0,
\end{aligned}$$

which implies that $\nu(y,t)$ is increasing in $t \in \Re_+$.

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Taking the derivative of $\nu(y,t)$ with respect to y, we have

$$\begin{aligned}
\nu_y'(y,t) &\overset{\text{sgn}}{=} \left[pr_2 y^{r_2-1}e^{b(1-y)t} - pbty^{r_2}e^{b(1-y)t} + qr_1 y^{r_1-1}\right] \times \left[py^{r_2^*}e^{b(1-y)t} + qy^{r_1^*}\right] \\
&\quad - \left[pr_2^* y^{r_2^*-1}e^{b(1-y)t} - pbty^{r_2^*}e^{b(1-y)t} + qr_1^* y^{r_1^*-1}\right] \times \left[py^{r_2}e^{b(1-y)t} + qy^{r_1}\right] \\
&= p^2(r_2 - r_2^*)y^{r_2+r_2^*-1}e^{2b(1-y)t} + pq(r_2 - r_1^* - bty)y^{r_2+r_1^*-1}e^{b(1-y)t} \\
&\quad pq(r_1 - r_2^* + bty)y^{r_1+r_2^*-1}e^{b(1-y)t} + q^2(r_1 - r_1^*)y^{r_1+r_1^*-1} \\
&\le p^2(r_1 + r_2 - r_1^* - r_2^*)y^{r_2+r_2^*-1} + pq(r_1 + r_2 - r_1^* - r_2^*)y^{r_2+r_1^*-1}e^{b(1-y)t} \\
&= 0,
\end{aligned}$$

which implies that $\nu(y,t)$ is decreasing in $y \in (0,1)$. Hence, the desired result follows. ■

We are now ready to present the main results of this section.

THEOREM 2.6: *Let $X_1, X_2, \ldots, X_n$ be independent random variables following the multiple-outlier gamma model with respective shape and scale parameters $(r_1\mathbf{1}_p, r_2\mathbf{1}_q)$, $(\lambda_1\mathbf{1}_p, \lambda\mathbf{1}_q)$, where $p, q \in \mathcal{Z}_+$ and $p+q=n$. Let $Y_1, Y_2, \ldots, Y_n$ be another set of independent random variables following the multiple-outlier gamma model with respective shape and scale parameters $(r_1\mathbf{1}_p, r_2\mathbf{1}_q)$, $(\lambda_2\mathbf{1}_p, \lambda\mathbf{1}_q)$. Suppose $r_1 \ge r_2$ and $\lambda \ge \lambda_2 \ge \lambda_1$, then*

$$X_{n:n} \ge_{\text{lr}} Y_{n:n}.$$

PROOF: To obtain the desired results, it is enough to show that $X_{n:n} \ge_{\text{rh}} Y_{n:n}$ and the ratio of their reversed hazard rate functions (i.e., $\phi(t) = r_{X_{n:n}}/r_{Y_{n:n}}$) is increasing in $t \in \Re_+$. Notice that $r_{X_{n:n}} = pr_{X_1} + qr_{X_n}$ and $r_{Y_{n:n}} = pr_{Y_1} + qr_{Y_n}$. Due to the facts that $r_{X_n} = r_{Y_n}$ and $r_{X_1} \ge r_{Y_1}$, it follows that $r_{X_{n:n}} \ge r_{Y_{n:n}}$. To simplify the notations, we define

$$h_{r,\lambda}(t) = \frac{t^{r-1}e^{-\lambda t}}{\int_0^t u^{r-1}e^{-\lambda u}du}$$

and

$$k_{r,\lambda}(t) = \frac{e^{-\lambda t}}{\int_0^t u^{r-1}e^{-\lambda u}du}.$$

In what follows, we need to show that the function

$$\begin{aligned}
\phi(t) &= \frac{ph_{r_1,\lambda_1}(t) + qh_{r_2,\lambda}(t)}{ph_{r_1,\lambda_2}(t) + qh_{r_2,\lambda}(t)} \\
&= \frac{pt^{r_1-r_2}k_{r_1,\lambda_1}(t) + qk_{r_2,\lambda}(t)}{pt^{r_1-r_2}k_{r_1,\lambda_2}(t) + qk_{r_2,\lambda}(t)}
\end{aligned}$$

is increasing in $t \in \Re_+$. Taking the derivative of $\phi(t)$ with respect to t gives rise to

$$\begin{aligned}
&\phi'(t)\left(pt^{r_1-r_2}k_{r_1,\lambda_2}(t)+qk_{r_2,\lambda}(t)\right)^2\\
&\quad=\left[-p\left(\lambda_1-\frac{r_1-r_2}{t}+h_{r_1,\lambda_1}(t)\right)t^{r_1-r_2}k_{r_1,\lambda_1}(t)-q\left(\lambda+h_{r_2,\lambda}(t)\right)k_{r_2,\lambda}(t)\right]\\
&\qquad\times\left(pt^{r_1-r_2}k_{r_1,\lambda_2}(t)+qk_{r_2,\lambda}(t)\right)\\
&\qquad-\left[-p\left(\lambda_2-\frac{r_1-r_2}{t}+h_{r_1,\lambda_2}(t)\right)t^{r_1-r_2}k_{r_1,\lambda_2}(t)-q\left(\lambda+h_{r_2,\lambda}(t)\right)k_{r_2,\lambda}(t)\right]\\
&\qquad\times\left(pt^{r_1-r_2}k_{r_1,\lambda_1}(t)+qk_{r_2,\lambda}(t)\right)\\
&\quad=p^2\left(\lambda_2+h_{r_1,\lambda_2}(t)-\lambda_1-h_{r_1,\lambda_1}(t)\right)t^{2(r_1-r_2)}k_{r_1,\lambda_1}(t)k_{r_1,\lambda_2}(t)\\
&\qquad+pq\left(\lambda+h_{r_2,\lambda}(t)+\frac{r_1-r_2}{t}-\lambda_1-h_{r_1,\lambda_1}(t)\right)t^{r_1-r_2}k_{r_1,\lambda_1}(t)k_{r_2,\lambda}(t)\\
&\qquad+pq\left(\lambda_2+h_{r_1,\lambda_2}(t)-\frac{r_1-r_2}{t}-\lambda-h_{r_2,\lambda}(t)\right)t^{r_1-r_2}k_{r_1,\lambda_2}(t)k_{r_2,\lambda}(t)\\
&\quad=A+B,\quad \text{say},
\end{aligned}$$

where

$$\begin{aligned}
A&=p^2\left(\lambda_2+h_{r_1,\lambda_2}(t)-\lambda_1-h_{r_1,\lambda_1}(t)\right)t^{2(r_1-r_2)}k_{r_1,\lambda_1}(t)k_{r_1,\lambda_2}(t)\\
&=p^2\left(f_{r_1,t}(\lambda_2)-f_{r_1,t}(\lambda_1)\right)t^{2(r_1-r_2)}k_{r_1,\lambda_1}(t)k_{r_1,\lambda_2}(t)
\end{aligned}$$

and

$$\begin{aligned}
B&=pq\left(\lambda+h_{r_2,\lambda}(t)+\frac{r_1-r_2}{t}-\lambda_1-h_{r_1,\lambda_1}(t)\right)t^{r_1-r_2}k_{r_1,\lambda_1}(t)k_{r_2,\lambda}(t)\\
&\quad+pq\left(\lambda_2+h_{r_1,\lambda_2}(t)-\frac{r_1-r_2}{t}-\lambda-h_{r_2,\lambda}(t)\right)t^{r_1-r_2}k_{r_1,\lambda_2}(t)k_{r_2,\lambda}(t).
\end{aligned}$$

From Lemma 2.2, we have $A \geq 0$. We next show that $B \geq 0$. Since the function $k_{r,\lambda}(t)$ is decreasing in λ, it follows that

$$k_{r_1,\lambda_1}(t)k_{r_2,\lambda}(t) \geq k_{r_1,\lambda_2}(t)k_{r_2,\lambda}(t). \tag{10}$$

On the other hand, upon applying Lemmas 2.2 and 2.3, we have

$$f_{r_1,t}(\lambda)=\lambda+\frac{t^{r_1-1}e^{-\lambda t}}{\int_0^t u^{r_1-1}e^{-\lambda u}du} \geq \lambda_1+\frac{t^{r_1-1}e^{-\lambda_1 t}}{\int_0^t u^{r_1-1}e^{-\lambda_1 u}du}=f_{r_1,t}(\lambda_1) \tag{11}$$

and

$$g_{\lambda,t}(r_1)=r_1-\frac{t^{r_1}e^{-\lambda t}}{\int_0^t u^{r_1-1}e^{-\lambda u}du} \geq r_2-\frac{t^{r_2}e^{-\lambda t}}{\int_0^t u^{r_2-1}e^{-\lambda u}du}=g_{\lambda,t}(r_2). \tag{12}$$

Combining the inequality (11) with (12), we have

$$\begin{aligned}
\lambda+h_{r_2,\lambda}(t)+\frac{r_1-r_2}{t}-\lambda_1-h_{r_1,\lambda_1}(t)&=\left[f_{r_1,t}(\lambda)-f_{r_1,t}(\lambda_1)\right]+\frac{1}{t}\left[g_{\lambda,t}(r_1)-g_{\lambda,t}(r_2)\right]\\
&\geq 0.
\end{aligned}$$

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Then, it follows from (10) and (11) that

$$\begin{aligned} B &\geq pq\left(\lambda + h_{r_2,\lambda}(t) + \frac{r_1 - r_2}{t} - \lambda_1 - h_{r_1,\lambda_1}(t)\right) t^{r_1-r_2} k_{r_1,\lambda_2}(t) k_{r_2,\lambda}(t) \\ &\quad + pq\left(\lambda_2 + h_{r_1,\lambda_2}(t) - \frac{r_1 - r_2}{t} - \lambda - h_{r_2,\lambda}(t)\right) t^{r_1-r_2} k_{r_1,\lambda_2}(t) k_{r_2,\lambda}(t) \\ &\overset{\text{sgn}}{=} \lambda_2 + h_{r_1,\lambda_2}(t) - \lambda_1 - h_{r_1,\lambda_1}(t) \\ &= f_{r_1,t}(\lambda_2) - f_{r_1,t}(\lambda_1) \\ &\geq 0. \end{aligned}$$

Therefore, $\phi'(t) \geq 0$, which means that $\phi(t)$ is increasing in $t \in \Re_+$. Hence, the proof is completed. ■

Remark 2.7: The result in Theorem 2.6 has been obtained in Zhao and Zhang [32] for the special case $p = q = 1$, and in Zhao and Balakrishnan [31] for the special case $r_1 = r_2$.

Remark 2.8: The result in Theorem 2.6 still holds when replacing the condition $\lambda \geq \lambda_2 \geq \lambda_1$ with $\lambda_2 \geq \lambda \geq \lambda_1$. Here we omit the proof for simplicity because it is quite similar to that of Theorem 2.6.

The following result, however, presents a more general version.

THEOREM 2.9: *Under the assumptions of Theorem 2.6, if $r_1 \geq r_2$ and $\lambda_1 \leq \min\{\lambda_2, \lambda\}$, then*

$$X_{n:n} \geq_{\text{lr}} Y_{n:n}.$$

Remark 2.10: Suppose that the condition $\lambda \geq \lambda_2 \geq \lambda_1$ in Theorem 2.6 is replaced by $\lambda_2 \geq \lambda_1 \geq \lambda$ with other assumptions unchanged, the reversed hazard order still holds, that is, $X_{n:n} \geq_{\text{rh}} Y_{n:n}$. Here we omit the proof because it is quite similar to that of Theorem 2.6. The likelihood ratio order, however, cannot be established. A counterexample is given to support this assertion in the following.

Example 2.11: Set $p = 3$, $q = 2$, $r_1 = 3.5$, $r_2 = 1.5$, $\lambda_1 = 2.1$, $\lambda_2 = 3.8$ and $\lambda = 1.5$ in Theorem 2.6, we have $\lambda_2 \geq \lambda_1 \geq \lambda$. Figure 1 plots the ratio of the density functions $f_{X_{n:n}}(t)$ and $f_{Y_{n:n}}(t)$. It can be observed that the ratio $f_{X_{n:n}}(t)/f_{Y_{n:n}}(t)$ is neither increasing nor decreasing in $t \in (0, +\infty)$.

THEOREM 2.12: *Let $X_1, X_2, \ldots, X_n$ be independent random variables following the multiple-outlier gamma model with respective shape parameters and scale parameters $(r_1\mathbf{1}_p, r_2\mathbf{1}_q)$ and $(\lambda_1\mathbf{1}_p, \lambda_2\mathbf{1}_q)$, where $p, q \in \mathcal{Z}_+$ and $p + q = n$. Let $Y_1, Y_2, \ldots, Y_n$ be another set of independent random variables following the multiple-outlier gamma model with respective shape parameters and scale parameters $(r_1\mathbf{1}_p, r_2\mathbf{1}_q)$ and $(\lambda_1^*\mathbf{1}_p, \lambda_2^*\mathbf{1}_q)$. Suppose $p \geq q$, $r_1 \geq r_2$, $\lambda_1 \leq \lambda_2$ and $\lambda_1^* \leq \lambda_2^*$. We then have*

$$(\lambda_1, \lambda_2) \overset{\text{m}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{n:n} \geq_{\text{lr}} Y_{n:n}.$$

PROOF: Denote $f_{X_{n:n}}(t)[f_{Y_{n:n}}(t)]$ the density function of $X_{n:n}[Y_{n:n}]$. It suffices to prove that $\Delta(t) = f_{X_{n:n}}(t)/f_{Y_{n:n}}(t)$ is increasing in $t \in \Re_+$. Let $r_{X_1}[r_{Y_1}]$ and $r_{X_n}[r_{Y_n}]$ be the

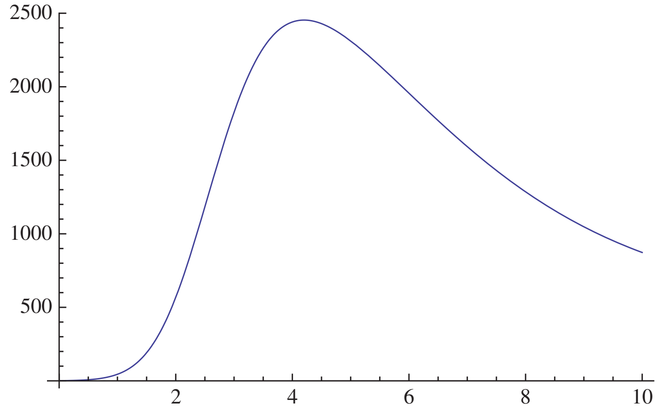

FIGURE 1. Plot of $f_{X_{5:5}}(t)/f_{Y_{5:5}}(t)$ when $p=3$, $q=2$, $r_1=3.5$, $r_2=1.5$, $\lambda_1=2.1$, $\lambda_2=3.8$ and $\lambda=1.5$.

reversed hazard rate functions of $X_1[Y_1]$ and $X_n[Y_n]$. We first show $X_{n:n} \geq_{\mathrm{rh}} Y_{n:n}$, that is, $r_{X_{n:n}} \geq r_{Y_{n:n}}$, where $r_{X_{n:n}}$ and $r_{Y_{n:n}}$ are the reversed hazard rate functions of $X_{n:n}$ and $Y_{n:n}$, respectively. Thus, we need to show $pr_{X_1}+qr_{X_n} \geq pr_{Y_1}+qr_{Y_n}$. From Zhao and Zhang [32], we have $X_{2:2} \geq_{\mathrm{lr}} Y_{2:2}$, which implies $X_{2:2} \geq_{\mathrm{rh}} Y_{2:2}$, that is, $r_{X_{2:2}} \geq r_{Y_{2:2}}$. Based on the definition of reversed hazard rate function, we have $r_{X_1}+r_{X_n} \geq r_{Y_1}+r_{Y_n}$. Since $p \geq q$, $r_{X_1} \geq r_{Y_1}$ and $r_{X_n} \leq r_{Y_n}$, it follows that $pr_{X_1}+qr_{X_n} \geq pr_{Y_1}+qr_{Y_n}$, that is, $X_{n:n} \geq_{\mathrm{rh}} Y_{n:n}$.

It is enough to prove that

$$\begin{aligned}\Delta(t) &= \frac{f_{X_{n:n}}(t)}{f_{Y_{n:n}}(t)}\\ &= \frac{pF_1^{p-1}(t)F_2^q(t)f_1(t)+qF_1^p(t)F_2^{q-1}(t)f_2(t)}{pF_1^{*p-1}(t)F_2^{*q}(t)f^*{}_1(t)+qF_1^{*p}(t)F_2^{*q-1}(t)f_2^*(t)}\\ &= \frac{F_1^{p-1}(t)F_2^{q-1}(t)[pF_2(t)f_1(t)+qF_1(t)f_2(t)]}{F_1^{*p-1}(t)F_2^{*q-1}(t)[pF_2^*(t)f_1^*(t)+qF_1^*(t)f_2^*(t)]}\end{aligned}$$

is increasing in $t \in \Re_+$. Since $X_{n-2:n-2} \geq_{\mathrm{rh}} Y_{n-2:n-2}$, it holds that

$$\frac{F_1^{p-1}(t)F_2^{q-1}(t)}{F_1^{*p-1}(t)F_2^{*q-1}(t)}$$

is increasing in $t \in \Re_+$. Hence, it suffices to show that

$$\begin{aligned}\delta(t) &= \frac{pF_2(t)f_1(t)+qF_1(t)f_2(t)}{pF_2^*(t)f_1^*(t)+qF_1^*(t)f_2^*(t)}\\ &= \frac{p\frac{\lambda_1^{r_1}}{\Gamma(r_1)}t^{r_1-1}e^{-\lambda_1 t}\int_0^t \frac{\lambda_2^{r_2}}{\Gamma(r_2)}u^{r_2-1}e^{-\lambda_2 u}du+q\frac{\lambda_2^{r_2}}{\Gamma(r_2)}t^{r_2-1}e^{-\lambda_2 t}\int_0^t \frac{\lambda_1^{r_1}}{\Gamma(r_1)}u^{r_1-1}e^{-\lambda_1 u}du}{p\frac{\lambda_1^{*r_1}}{\Gamma(r_1)}t^{r_1-1}e^{-\lambda_1^* t}\int_0^t \frac{\lambda_2^{*r_2}}{\Gamma(r_2)}u^{r_2-1}e^{-\lambda_2^* u}du+q\frac{\lambda_2^{*r_2}}{\Gamma(r_2)}t^{r_2-1}e^{-\lambda_2^* t}\int_0^t \frac{\lambda_1^{*r_1}}{\Gamma(r_1)}u^{r_1-1}e^{-\lambda_1^* u}du}\\ &\propto \frac{pt^{r_1-1}e^{-\lambda_1 t}\int_0^t u^{r_2-1}e^{-\lambda_2 u}du+qt^{r_2-1}e^{-\lambda_2 t}\int_0^t u^{r_1-1}e^{-\lambda_1 u}du}{pt^{r_1-1}e^{-\lambda_1^* t}\int_0^t u^{r_2-1}e^{-\lambda_2^* u}du+qt^{r_2-1}e^{-\lambda_2^* t}\int_0^t u^{r_1-1}e^{-\lambda_1^* u}du}\end{aligned}$$

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$$
\begin{aligned}
&= \frac{\int_0^1 py^{r_2-1}e^{-(\lambda_1+\lambda_2 y)t} + qy^{r_1-1}e^{-(\lambda_2+\lambda_1 y)t}dy}{\int_0^1 py^{r_2-1}e^{-(\lambda_1^*+\lambda_2^* y)t} + qy^{r_1-1}e^{-(\lambda_2^*+\lambda_1^* y)t}dy} \\
&= \mathsf{E}_t\Psi(Y,t)
\end{aligned}
$$

is increasing in $t \in \Re_+$, where

$$
\Psi(y,t) = \frac{py^{r_2-1}e^{-(\lambda_1+\lambda_2 y)t} + qy^{r_1-1}e^{-(\lambda_2+\lambda_1 y)t}}{py^{r_2-1}e^{-(\lambda_1^*+\lambda_2^* y)t} + qy^{r_1-1}e^{-(\lambda_2^*+\lambda_1^* y)t}}, \quad y \in (0,1),
$$

and the distribution function of the random variable Y belongs to the family $\mathcal{P} = \{H(\cdot|t), t \in R^+\}$ with density function

$$
h(y|t) = c(t)[py^{r_2-1}e^{-(\lambda_1^*+\lambda_2^* y)t} + qy^{r_1-1}e^{-(\lambda_2^*+\lambda_1^* y)t}]
$$

and a normalizing constant $c(t)$ such that $\int_0^1 h(y|t)dy = 1$. Observe that

$$
\Psi(y,t) = \frac{pe^{-\lambda_1(1-y)t} + qy^{r_1-r_2}e^{-\lambda_2(1-y)t}}{pe^{-\lambda_1^*(1-y)t} + qy^{r_1-r_2}e^{-\lambda_2^*(1-y)t}},
$$

is increasing in t, while is decreasing in $y \in (0,1)$ due to Lemma 2.4. Denote by $a = r_1 - r_2 \geq 0$, for $t_2 \geq t_1 \geq 0$, we can show

$$
\begin{aligned}
\omega(y) &= \frac{h(y|t_2)}{h(y|t_1)} \\
&\propto \frac{py^{r_2-1}e^{-(\lambda_1^*+\lambda_2^* y)t_2} + qy^{r_1-1}e^{-(\lambda_2^*+\lambda_1^* y)t_2}}{py^{r_2-1}e^{-(\lambda_1^*+\lambda_2^* y)t_1} + qy^{r_1-1}e^{-(\lambda_2^*+\lambda_1^* y)t_1}} \\
&\propto \frac{pe^{\lambda_2^* t_2(1-y)} + qy^a e^{\lambda_1^* t_2(1-y)}}{pe^{\lambda_2^* t_1(1-y)} + qy^a e^{\lambda_1^* t_1(1-y)}}
\end{aligned}
$$

is decreasing in $y \in (0,1)$ by observing

$$
\begin{aligned}
\omega'(y) &\overset{\text{sgn}}{=} \left[p\lambda_2^* t_1 e^{\lambda_2^* t_1(1-y)} + q\left(\lambda_1^* t_1 - \frac{a}{y}\right)y^a e^{\lambda_1^* t_1(1-y)}\right]\left[pe^{\lambda_2^* t_2(1-y)} + qy^a e^{\lambda_1^* t_2(1-y)}\right] \\
&\quad - \left[p\lambda_2^* t_2 e^{\lambda_2^* t_2(1-y)} + q\left(\lambda_1^* t_2 - \frac{a}{y}\right)y^a e^{\lambda_1^* t_2(1-y)}\right]\left[pe^{\lambda_2^* t_1(1-y)} + qy^a e^{\lambda_1^* t_1(1-y)}\right] \\
&= p^2\lambda_2^*(t_1-t_2)e^{\lambda_2^*(1-y)(t_1+t_2)} + pq\left(\lambda_2^* t_1 - \lambda_1^* t_2 + \frac{a}{y}\right)y^a e^{(\lambda_2^* t_1+\lambda_1^* t_2)(1-y)} \\
&\quad + pq\left(\lambda_1^* t_1 - \lambda_2^* t_2 - \frac{a}{y}\right)y^a e^{(\lambda_1^* t_1+\lambda_2^* t_2)(1-y)} + q^2\lambda_1^*(t_1-t_2)y^{2a}e^{\lambda_1^*(t_1+t_2)(1-y)} \\
&\leq p^2\lambda_2^*(t_1-t_2)e^{\lambda_2^*(1-y)(t_1+t_2)} + pq\left(\lambda_2^* t_1 - \lambda_1^* t_2 + \frac{a}{y}\right)y^a e^{(\lambda_2^* t_1+\lambda_1^* t_2)(1-y)} \\
&\quad + pq\left(\lambda_1^* t_1 - \lambda_2^* t_2 - \frac{a}{y}\right)y^a e^{(\lambda_1^* t_2+\lambda_2^* t_1)(1-y)} + q^2\lambda_1^*(t_1-t_2)y^{2a}e^{\lambda_1^*(t_1+t_2)(1-y)} \\
&= p^2\lambda_2^*(t_1-t_2)e^{\lambda_2^*(1-y)(t_1+t_2)} + q^2\lambda_1^*(t_1-t_2)y^{2a}e^{\lambda_1^*(t_1+t_2)(1-y)} \\
&\quad + pq(\lambda_2^*+\lambda_1^*)(t_1-t_2)y^a e^{(\lambda_2^* t_1+\lambda_1^* t_2)(1-y)} \\
&\leq 0.
\end{aligned}
$$

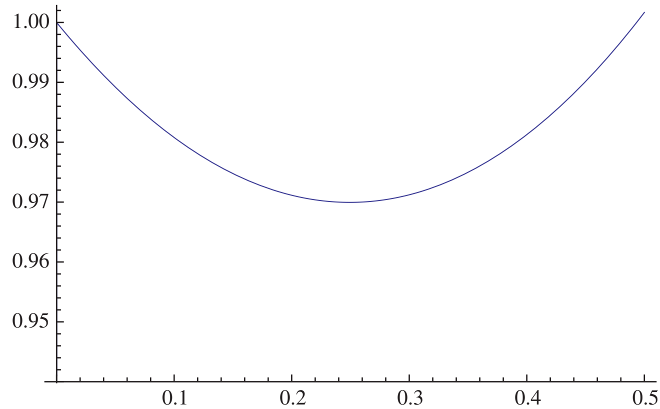

FIGURE **2.** Plot of $f_{X_{5:5}}(t)/f_{Y_{5:5}}(t)$ when $p = 2$, $q = 3$, $r_1 = 2$, $r_2 = 1.2$, $\lambda_1 = 0.8$, $\lambda_2 = 3.2$, $\lambda_1^* = 1.5$ and $\lambda_2 = 2.5$.

We then have $H(\cdot|t_1) \geq_{\rm lr} H(\cdot|t_2)$, which implies that $H(\cdot|t_1) \geq_{\rm st} H(\cdot|t_2)$ whenever $t_2 \geq t_1 \geq 0$. Upon using Lemma 2.1, we conclude that $\mathsf{E}_t\Psi(Y,t)$ is increasing in $t \in (0,\infty)$. Hence, the proof is completed. ■

One may wonder whether the result in Theorem 2.12 holds when $p < q$, we will show that the answer is negative by adopting the following example.

Example 2.13: Set $p = 2$, $q = 3$, $r_1 = 2$, $r_2 = 1.2$, $\lambda_1 = 0.8$, $\lambda_2 = 3.2$, $\lambda_1^* = 1.5$ and $\lambda_2^* = 2.5$ in Theorem 2.12, we have $(0.8, 3.2) \overset{\rm m}{\succeq} (1.5, 2.5)$. Figure 2 plots the ratio of density functions $f_{X_{n:n}}(t)$ and $f_{Y_{n:n}}(t)$. It can be observed that the function $f_{X_{n:n}}(t)/f_{Y_{n:n}}(t)$ is neither increasing nor decreasing in $t \in (0, +\infty)$.

Combining the result in Theorem 2.6 with that in Theorem 2.12, we have the following theorem.

THEOREM 2.14: *Under the assumptions of Theorem 2.12, if $p \geq q$, $r_1 \geq r_2$ and $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$, we then have*

$$(\lambda_1, \lambda_2) \overset{\rm w}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{n:n} \geq_{\rm lr} Y_{n:n}.$$

PROOF: It is known from the assumption that $\lambda_1 + \lambda_2 \leq \lambda_1^* + \lambda_2^*$. The result follows from Theorem 2.12 for the case when $\lambda_1 + \lambda_2 = \lambda_1^* + \lambda_2^*$. In what follows, we only need to consider the case when $\lambda_1 + \lambda_2 < \lambda_1^* + \lambda_2^*$. In this case, there exists some λ satisfying $\lambda_1 < \lambda \leq \lambda_1^*$ and $\lambda + \lambda_2 = \lambda_1^* + \lambda_2^*$. Let $Z_{n:n}$ denote the lifetime of the parallel system consisting of n independent gamma variables $Z_1, Z_2, \ldots, Z_n$, where $Z_1, \ldots, Z_p$ have common shape parameter r_1 and common scale parameter λ and $Z_{p+1}, \ldots, Z_n$ have common shape parameter r_2 and common scale parameter λ_2. Apparently, $(\lambda, \lambda_2) \overset{\rm m}{\succeq} (\lambda_1^*, \lambda_2^*)$. Upon using Theorem 2.12, it holds that $Z_{n:n} \geq_{\rm lr} Y_{n:n}$. On the other hand, we have $X_{n:n} \geq_{\rm lr} Z_{n:n}$ for $\lambda_1 < \lambda \leq \lambda_2$ from Theorem 2.6. Then, the desired result can be obtained immediately. ■

The following corollary is a direct consequence of Theorem 2.14.

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Corollary 2.15: *Let $X_1, X_2, \ldots, X_n$ be independent random variables following the multiple-outlier gamma model with respective shape parameters and scale parameters $(r_1\boldsymbol{1}_p, r_2\boldsymbol{1}_q)$, and $(\lambda_1\boldsymbol{1}_p, \lambda_2\boldsymbol{1}_q)$, where $p, q \in \mathcal{Z}_+$ and $p+q=n$. Let $Y_1, Y_2, \ldots, Y_n$ be another set of independent random variables following the multiple-outlier gamma model with respective shape parameters $(r_1\boldsymbol{1}_p, r_2\boldsymbol{1}_q)$ and common scale parameter λ. Suppose $p \geq q$, $r_1 \geq r_2$ and $\lambda \leq \max\{\lambda_1, \lambda_2\}$. Then,*

$$\lambda \geq \frac{\lambda_1+\lambda_2}{2} \Longrightarrow X_{n:n} \geq_{\rm lr} Y_{n:n}.$$

We next turn to discussing the case of comparing the largest order statistics arising from multiple-outlier gamma models in terms of the relationship between shape parameter vectors.

Theorem 2.16: *Let $X_1, X_2, \ldots, X_n$ be independent random variables following the multiple-outlier gamma model with respective shape parameters and scale parameters $(r_1\boldsymbol{1}_p, r\boldsymbol{1}_q)$ and $(\lambda_1\boldsymbol{1}_p, \lambda_2\boldsymbol{1}_q)$, where $p, q \in \mathcal{Z}_+$ and $p+q=n$. Let $Y_1, Y_2, \ldots, Y_n$ be another set of independent random variables following the multiple-outlier gamma model with respective shape parameters and scale parameters $(r_2\boldsymbol{1}_p, r\boldsymbol{1}_q)$ and $(\lambda_1\boldsymbol{1}_p, \lambda_2\boldsymbol{1}_q)$. Suppose that $p \geq q$, $r_1 \geq r_2 \geq r$ and $\lambda_2 \geq \lambda_1$. We then have $X_{n:n} \geq_{\rm lr} Y_{n:n}$.*

Proof: Denote by $f_i[g_i]$ and $F_i[G_i]$ the density and distribution functions of $X_i[Y_i]$, respectively, and denote by $\tilde{r}_{X_{n:n}}$ and $\tilde{r}_{Y_{n:n}}$ the reversed hazard rate functions of $X_{n:n}$ and $Y_{n:n}$, respectively. To reach the desired result, we will show that $X_{n:n} \geq_{\rm rh} Y_{n:n}$ and the ratio of their reversed hazard rate function (i.e., $\phi(t) = \tilde{r}_{X_{n:n}}(t)/\tilde{r}_{Y_{n:n}}(t)$) is increasing in $t \in \Re_+$ according to Theorem 1.C.4(b) of Shaked and Shanthikumar [27]. Notice that

$$\tilde{r}_{X_{n:n}}(t) = p\tilde{r}_{X_1}(t) + q\tilde{r}_{X_n}(t) \quad \text{and} \quad \tilde{r}_{Y_{n:n}}(t) = p\tilde{r}_{Y_1}(t) + q\tilde{r}_{Y_n}(t).$$

Since $\tilde{r}_{X_n}(t) = \tilde{r}_{Y_n}(t)$ and $r_1 \geq r_2$ implies that $X_1 \geq_{\rm rh} Y_1$ (i.e., $\tilde{r}_{X_1}(t) \geq \tilde{r}_{Y_1}(t)$), so it holds that $\tilde{r}_{X_{n:n}}(t) \geq \tilde{r}_{Y_{n:n}}(t)$. We now need to show that the function

$$\psi(t) = \left(pt^{r_1-r}k_{r_1,\lambda_1}(t) + qk_{r,\lambda_2}(t)\right)\left(pt^{r_2-r}k_{r_2,\lambda_1}(t) + qk_{r,\lambda_2}(t)\right)^{-1}$$

is increasing in $t \in \Re_+$. Taking the derivative of $\psi(t)$ with respect to t gives rise to

$$\begin{aligned}
&\psi'(t)\left(pt^{r_2-r}k_{r_2,\lambda_1}(t) + qk_{r,\lambda_2}(t)\right)^2 \\
&\quad = \left[-p\left(\lambda_1 - \frac{r_1-r}{t} + h_{r_1,\lambda_1}(t)\right)t^{r_1-r}k_{r_1,\lambda_1}(t) - q\left(\lambda_2 + h_{r,\lambda_2}(t)\right)k_{r,\lambda_2}(t)\right] \\
&\qquad \times \left(pt^{r_2-r}k_{r_2,\lambda_1}(t) + qk_{r,\lambda_2}(t)\right) \\
&\qquad - \left[-p\left(\lambda_1 - \frac{r_2-r}{t} + h_{r_2,\lambda_1}(t)\right)t^{r_2-r}k_{r_2,\lambda_1}(t) - q\left(\lambda_2 + h_{r,\lambda_2}(t)\right)k_{r,\lambda_2}(t)\right] \\
&\qquad \times \left(pt^{r_1-r}k_{r_1,\lambda_1}(t) + qk_{r,\lambda_2}(t)\right) \\
&\quad = p^2\left[\left(r_1 - th_{r_1,\lambda_1}(t) - (r_2 - th_{r_2,\lambda_1}(t))\right) \times t^{r_1+r_2-2r-1}k_{r_1,\lambda_1}(t)k_{r_2,\lambda_1}(t)\right] \\
&\qquad + pq\left[\left(\lambda_2 + h_{r,\lambda_2}(t) + \frac{r_1-r}{t} - \lambda_1 - h_{r_1,\lambda_1}(t)\right) \times t^{r_1-r}k_{r_1,\lambda_1}(t)k_{r,\lambda_2}(t)\right] \\
&\qquad - pq\left[\left(\lambda_2 + h_{r,\lambda_2}(t) + \frac{r_2-r}{t} - \lambda_1 - h_{r_2,\lambda_1}(t)\right) \times t^{r_2-r}k_{r_2,\lambda_1}(t)k_{r,\lambda_2}(t)\right] \\
&\quad = W + V, \quad \text{say,}
\end{aligned}$$

where

$$\begin{aligned} W &= p^2\left[(r_1 - th_{r_1,\lambda_1}(t) - (r_2 - th_{r_2,\lambda_1}(t))) \times t^{r_1+r_2-2r-1}k_{r_1,\lambda_1}(t)k_{r_2,\lambda_1}(t)\right] \\ &= p^2\left[(g_{\lambda_1,t}(r_1) - g_{\lambda_1,t}(r_2)) \times t^{r_1+r_2-2r-1}k_{r_1,\lambda_1}(t)k_{r_2,\lambda_1}(t)\right] \end{aligned}$$

and

$$\begin{aligned} V = pq&\left[\left(\lambda_2 + h_{r,\lambda_2}(t) + \frac{r_1 - r}{t} - \lambda_1 - h_{r_1,\lambda_1}(t)\right) \times t^{r_1-r}k_{r_1,\lambda_1}(t)k_{r,\lambda_2}(t)\right] \\ &- pq\left[\left(\lambda_2 + h_{r,\lambda_2}(t) + \frac{r_2 - r}{t} - \lambda_1 - h_{r_2,\lambda_1}(t)\right) \times t^{r_2-r}k_{r_2,\lambda_1}(t)k_{r,\lambda_2}(t)\right]. \end{aligned}$$

From Lemma 2.3, we have $W \geq 0$. We will show $V \geq 0$ in the following. Since

$$\frac{t^r}{\int_0^t u^r e^{-\lambda u}du}$$

is increasing in $r \in (0, +\infty)$, we then have that

$$\frac{t^{r_1-r}e^{-(\lambda_1+\lambda_2)t}}{\int_0^t u^{r_1-1}e^{-\lambda_1 u}du \int_0^t u^{r-1}e^{-\lambda_2 u}du} \geq \frac{t^{r_2-r}e^{-(\lambda_1+\lambda_2)t}}{\int_0^t u^{r_2-1}e^{-\lambda_1 u}du \int_0^t u^{r-1}e^{-\lambda_2 u}du},$$

that is,

$$t^{r_1-r}k_{r_1,\lambda_1}(t)k_{r,\lambda_2}(t) \geq t^{r_2-r}k_{r_2,\lambda_1}(t)k_{r,\lambda_2}(t). \tag{13}$$

On the other hand, upon applying Lemmas 2.2 and 2.3, respectively, we have the following two inequalities:

$$f_{r_1,t}(\lambda_2) = \lambda_2 + \frac{t^{r_1-1}e^{-\lambda_2 t}}{\int_0^t u^{r_1-1}e^{-\lambda_2 u}du} \geq \lambda_1 + \frac{t^{r_1-1}e^{-\lambda_1 t}}{\int_0^t u^{r_1-1}e^{-\lambda_1 u}du} = f_{r_1,t}(\lambda_1) \tag{14}$$

and

$$g_{\lambda_2,t}(r_1) = r_1 - \frac{t^{r_1}e^{-\lambda_2 t}}{\int_0^t u^{r_1-1}e^{-\lambda_2 u}du} \geq r - \frac{t^{r}e^{-\lambda_2 t}}{\int_0^t u^{r-1}e^{-\lambda_2 u}du} = g_{\lambda_2,t}(r). \tag{15}$$

Using the inequalities (14) and (15), we have

$$\begin{aligned} \lambda_2 + h_{r,\lambda_2}(t) + \frac{r_1 - r}{t} - \lambda_1 - h_{r_1,\lambda_1}(t) &= [f_{r_1,t}(\lambda_2) - f_{r_1,t}(\lambda_1)] + \frac{1}{t}[g_{\lambda_2,t}(r_1) - g_{\lambda_2,t}(r)] \\ &\geq 0. \end{aligned}$$

Then,

$$\begin{aligned} V \geq pq&\left[\left(\lambda_2 + h_{r,\lambda_2}(t) + \frac{r_1 - r}{t} - \lambda_1 - h_{r_1,\lambda_1}(t)\right) \times t^{r_2-r}k_{r_2,\lambda_1}(t)k_{r,\lambda_2}(t)\right] \\ &- pq\left[\left(\lambda_2 + h_{r,\lambda_2}(t) + \frac{r_2 - r}{t} - \lambda_1 - h_{r_2,\lambda_1}(t)\right) \times t^{r_2-r}k_{r_2,\lambda_1}(t)k_{r,\lambda_2}(t)\right] \\ &\overset{\text{sgn}}{=} r_1 - th_{r_1,\lambda_1}(t) - (r_2 - th_{r_2,\lambda_1}(t)) \\ &= g_{\lambda_1,t}(r_1) - g_{\lambda_1,t}(r_2) \\ &\geq 0, \end{aligned}$$

which means that $\psi'(t) \geq 0$, that is, $\psi(t)$ is increasing in $t \in \Re_+$. Hence, the theorem follows. ■

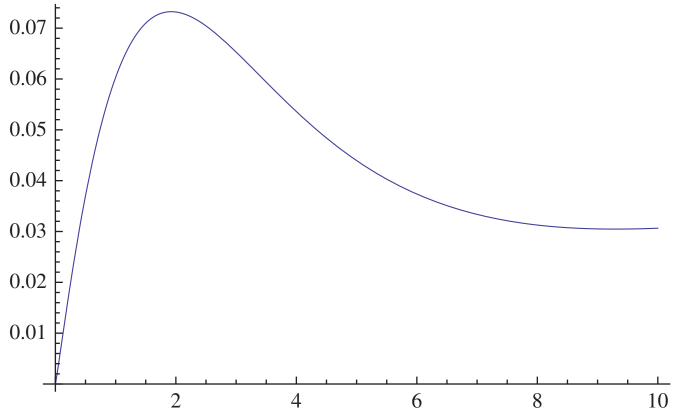

FIGURE **3.** Plot of $f_{X_{5:5}}(t)/f_{Y_{5:5}}(t)$ when $p = 3$, $q = 2$, $r_1 = 1.9$, $r_2 = 0.8$, $r = 3.1$, $\lambda_1 = 0.9$ and $\lambda_2 = 1.4$.

Remark 2.17: The result in Theorem 2.16 does not hold when replacing the condition $r \le r_2 \le r_1$ with $r_2 \le r_1 \le r$. We present a counterexample to clarify this. The result, however, holds for the reversed hazard rate order, we omit the proof details for simplicity.

Example 2.18: Set $p = 3$, $q = 2$, $r_1 = 1.9$, $r_2 = 0.8$, $r = 3.1$, $\lambda_1 = 0.9$ and $\lambda_2 = 1.4$ in Theorem 2.16, we have $r_2 \le r_1 \le r$. Figure 3 plots the ratio of density functions $f_{X_{n:n}}(t)$ and $f_{Y_{n:n}}(t)$. It can be observed that the function $f_{X_{n:n}}(t)/f_{Y_{n:n}}(t)$ is neither increasing nor decreasing in $t \in (0, +\infty)$.

THEOREM 2.19: *Let $X_1, X_2, \ldots, X_n$ be independent random variables following the multiple-outlier gamma model with respective shape parameters and scale parameters $(r_1\boldsymbol{1}_p, r_2\boldsymbol{1}_q)$ and $(\lambda_1\boldsymbol{1}_p, \lambda_2\boldsymbol{1}_q)$, where $p, q \in \mathcal{Z}_+$ and $p + q = n$. Let $Y_1, Y_2, \ldots, Y_n$ be another set of independent random variables following the multiple-outlier gamma model with respective shape parameters and scale parameters $(r_1^*\boldsymbol{1}_p, r_2^*\boldsymbol{1}_q)$ and $(\lambda_1\boldsymbol{1}_p, \lambda_2\boldsymbol{1}_q)$. Suppose that $p \ge q$, $r_1 \ge r_2$, $r_1^* \ge r_2^*$ and $\lambda_2 \ge \lambda_1$. We then have*

$$(r_1, r_2) \overset{\mathrm{m}}{\succeq} (r_1^*, r_2^*) \Longrightarrow X_{n:n} \ge_{\mathrm{lr}} Y_{n:n}.$$

PROOF: Denote $f_{X_{n:n}}(t)[f_{Y_{n:n}}(t)]$ the density function of $X_{n:n}[Y_{n:n}]$. It suffices to prove that $\eta(t) = f_{X_{n:n}}(t)/f_{Y_{n:n}}(t)$ is increasing in $t \in \Re_+$. Using an argument quite similar to that of Theorem 2.12, we can show $X_{n:n} \ge_{\mathrm{rh}} Y_{n:n}$.

Observe that

$$\begin{aligned}\eta(t) &= \frac{f_{X_{n:n}}(t)}{f_{Y_{n:n}}(t)}\\ &= \frac{F_1^{p-1}(t)F_2^{q-1}(t)[pF_2(t)f_1(t) + qF_1(t)f_2(t)]}{F_1^{*p-1}(t)F_2^{*q-1}(t)[pF_2^*(t)f_1^*(t) + qF_1^*(t)f_2^*(t)]}\end{aligned}$$

and

$$\frac{F_1^{p-1}(t)F_2^{q-1}(t)}{F_1^{*p-1}(t)F_2^{*q-1}(t)}$$

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is increasing in $t \in \Re_+$ due to the fact that $X_{n:n} \geq_{\text{rh}} Y_{n:n}$, and hence we only need to show that

$$\begin{aligned}\delta(t) &= \frac{pF_2(t)f_1(t) + qF_1(t)f_2(t)}{pF_2^*(t)f_1^*(t) + qF_1^*(t)f_2^*(t)}\\
&= \frac{p\frac{\lambda_1^{r_1}}{\Gamma(r_1)}t^{r_1-1}e^{-\lambda_1 t}\int_0^t \frac{\lambda_2^{r_2}}{\Gamma(r_2)}u^{r_2-1}e^{-\lambda_2 u}du + q\frac{\lambda_2^{r_2}}{\Gamma(r_2)}t^{r_2-1}e^{-\lambda_2 t}\int_0^t \frac{\lambda_1^{r_1}}{\Gamma(r_1)}u^{r_1-1}e^{-\lambda_1 u}du}{p\frac{\lambda_1{}^{r_1^*}}{\Gamma(r_1^*)}t^{r_1^*-1}e^{-\lambda_1 t}\int_0^t \frac{\lambda_2{}^{r_2^*}}{\Gamma(r_2^*)}u^{r_2^*-1}e^{-\lambda_2 u}du + q\frac{\lambda_2{}^{r_2^*}}{\Gamma(r_2^*)}t^{r_2^*-1}e^{-\lambda_2 t}\int_0^t \frac{\lambda_1{}^{r_1^*}}{\Gamma(r_1^*)}u^{r_1^*-1}e^{-\lambda_1 u}du}\\
&= \frac{\frac{\lambda_1^{r_1}\lambda_2^{r_2}t^{r_1+r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\int_0^1 py^{r_2-1}e^{-(\lambda_1+\lambda_2 y)t} + qy^{r_1-1}e^{-(\lambda_2+\lambda_1 y)t}dy}{\frac{\lambda_1{}^{r_1^*}\lambda_2{}^{r_2^*}t^{r_1^*+r_2^*-1}}{\Gamma(r_1^*)\Gamma(r_2^*)}\int_0^1 py^{r_2^*-1}e^{-(\lambda_1+\lambda_2 y)t} + qy^{r_1^*-1}e^{-(\lambda_2+\lambda_1 y)t}dy}\\
&\propto \frac{\int_0^1 py^{r_2-1}e^{-(\lambda_1+\lambda_2 y)t} + qy^{r_1-1}e^{-(\lambda_2+\lambda_1 y)t}dy}{\int_0^1 py^{r_2^*-1}e^{-(\lambda_1+\lambda_2 y)t} + qy^{r_1^*-1}e^{-(\lambda_2+\lambda_1 y)t}dy}\\
&= \mathsf{E}_t\nu(Y_1, t)\end{aligned}$$

is increasing in $t \in \Re_+$, where

$$\nu(y,t) = \frac{py^{r_2-1}e^{-(\lambda_1+\lambda_2 y)t} + qy^{r_1-1}e^{-(\lambda_2+\lambda_1 y)t}}{py^{r_2^*-1}e^{-(\lambda_1+\lambda_2 y)t} + qy^{r_1^*-1}e^{-(\lambda_2+\lambda_1 y)t}}, \quad y \in (0,1).$$

Here, the distribution function of the random variable Y_1 belongs to the family $\mathcal{P}_1 = \{H_1(\cdot|t), t \in R^+\}$ with density function

$$h_1(y|t) = c_1(t)\left[py^{r_2^*-1}e^{-(\lambda_1+\lambda_2 y)t} + qy^{r_1^*-1}e^{-(\lambda_2+\lambda_1 y)t}\right]$$

and a normalizing constant $c_1(t)$ such that $\int_0^1 h_1(y|t)dy = 1$. Observe that

$$\nu(y,t) = \frac{py^{r_2}e^{(\lambda_2-\lambda_1)(1-y)t} + qy^{r_1}}{py^{r_2^*}e^{(\lambda_2-\lambda_1)(1-y)t} + qy^{r_1^*}}$$

is increasing in t, while is decreasing in $y \in (0,1)$ due to Lemma 2.5. For $t_2 \geq t_1 \geq 0$ and $a = r_1^* - r_2^* \geq 0$, we have

$$\begin{aligned}\omega_1(y) &= \frac{h_1(y|t_2)}{h_1(y|t_1)}\\
&= \frac{py^{r_2^*-1}e^{-(\lambda_1+\lambda_2 y)t_2} + qy^{r_1^*-1}e^{-(\lambda_2+\lambda_1 y)t_2}}{py^{r_2^*-1}e^{-(\lambda_1+\lambda_2 y)t_1} + qy^{r_1^*-1}e^{-(\lambda_2+\lambda_1 y)t_1}}\\
&= \frac{pe^{-(\lambda_1+\lambda_2 y)t_2} + qy^a e^{-(\lambda_2+\lambda_1 y)t_2}}{pe^{-(\lambda_1+\lambda_2 y)t_1} + qy^a e^{-(\lambda_2+\lambda_1 y)t_1}}\\
&\propto \frac{pe^{\lambda_2 t_2(1-y)} + qy^a e^{\lambda_1 t_2(1-y)}}{pe^{\lambda_2 t_1(1-y)} + qy^a e^{\lambda_1 t_1(1-y)}}\end{aligned}$$

is decreasing in $y \in (0,1)$ by checking that

$$\begin{aligned}
\omega_1'(y) &\overset{\text{sgn}}{=} \left[p\lambda_2 t_1 e^{\lambda_2 t_1(1-y)} + q\left(\lambda_1 t_1 - \frac{a}{y}\right) y^a e^{\lambda_1 t_1(1-y)}\right]\left[pe^{\lambda_2 t_2(1-y)} + qy^a e^{\lambda_1 t_2(1-y)}\right] \\
&\quad - \left[p\lambda_2 t_2 e^{\lambda_2 t_2(1-y)} + q\left(\lambda_1 t_2 - \frac{a}{y}\right) y^a e^{\lambda_1 t_2(1-y)}\right]\left[pe^{\lambda_2 t_1(1-y)} + qy^a e^{\lambda_1 t_1(1-y)}\right] \\
&= p^2\lambda_2 (t_1 - t_2) e^{\lambda_2(1-y)(t_1+t_2)} + pq\left(\lambda_2 t_1 - \lambda_1 t_2 + \frac{a}{y}\right) y^a e^{(\lambda_2 t_1 + \lambda_1 t_2)(1-y)} \\
&\quad + pq\left(\lambda_1 t_1 - \lambda_2 t_2 - \frac{a}{y}\right) y^a e^{(\lambda_1 t_1 + \lambda_2 t_2)(1-y)} + q^2(\lambda_1 t_1 - \lambda_1 t_2) y^{2a} e^{(\lambda_1 t_1 + \lambda_1 t_2)(1-y)} \\
&\le p^2\lambda_2 (t_1 - t_2) e^{\lambda_2(1-y)(t_1+t_2)} + pq\left(\lambda_2 t_1 - \lambda_1 t_2 + \frac{a}{y}\right) y^a e^{(\lambda_2 t_1 + \lambda_1 t_2)(1-y)} \\
&\quad + pq\left(\lambda_1 t_1 - \lambda_2 t_2 - \frac{a}{y}\right) y^a e^{(\lambda_1 t_2 + \lambda_2 t_1)(1-y)} + q^2\lambda_1 (t_1 - t_2) y^{2a} e^{(\lambda_1 t_1 + \lambda_1 t_2)(1-y)} \\
&= p^2\lambda_2 (t_1 - t_2) e^{\lambda_2(1-y)(t_1+t_2)} + q^2\lambda_1 (t_1 - t_2) y^{2a} e^{(\lambda_1 t_1 + \lambda_1 t_2)(1-y)} \\
&\quad + pq(\lambda_2 + \lambda_1)(t_1 - t_2) y^a e^{(\lambda_2 t_1 + \lambda_1 t_2)(1-y)} \\
&\le 0.
\end{aligned}$$

Thus, we have $H_1(\cdot|t_1) \geq_{\text{lr}} H_1(\cdot|t_2)$, which implies that $H_1(\cdot|t_1) \geq_{\text{st}} H_1(\cdot|t_2)$ whenever $t_2 \geq t_1 \geq 0$. Upon using Lemma 2.1, we conclude that $\mathsf{E}_t \nu(Y_1, t)$ is increasing in $t \in (0,\infty)$. Hence, the proof is completed. ■

A counterexample is presented here to illustrate that the result does not hold for the case $p < q$ in Theorem 2.19.

Example 2.20: Set $p = 2,\ q = 3,\ r_1 = 2.5,\ r_2 = 0.5,\ r_1^* = 1.8,\ r_2^* = 1.2,\ \lambda_1 = 1$ and $\lambda_2 = 2$ in Theorem 2.19, we have $(2.5, 0.5) \overset{\text{m}}{\succeq} (1.8, 1.2)$. Figure 4 plots the ratio of density functions $f_{X_{n:n}}(t)$ and $f_{Y_{n:n}}(t)$. It can be observed that the function $f_{X_{n:n}}(t)/f_{Y_{n:n}}(t)$ is neither increasing nor decreasing in $t \in (0, +\infty)$.

Combining Theorem 2.14 with Theorem 2.19, we can reach the following general result.

THEOREM 2.21: *Let $X_1, X_2, \ldots, X_n$ be independent random variables following the multiple-outlier gamma model with respective shape parameters and scale parameters $(r_1 \boldsymbol{1}_p, r_2 \boldsymbol{1}_q)$ and $(\lambda_1 \boldsymbol{1}_p, \lambda_2 \boldsymbol{1}_q)$, where $p, q \in \mathcal{Z}_+$ and $p + q = n$. Let $Y_1, Y_2, \ldots, Y_n$ be another set of independent random variables following the multiple-outlier gamma model with respective shape parameters and scale parameters $(r_1^* \boldsymbol{1}_p, r_2^* \boldsymbol{1}_q)$ and $(\lambda_1^* \boldsymbol{1}_p, \lambda_2^* \boldsymbol{1}_q)$. Suppose that $p \geq q$, $r_1 \geq r_2$, $r_1^* \geq r_2^*$ and $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$. We then have*

$$(r_1, r_2) \overset{\text{m}}{\succeq} (r_1^*, r_2^*), (\lambda_1, \lambda_2) \overset{\text{w}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{n:n} \geq_{\text{lr}} Y_{n:n}.$$

PROOF: Let $Z_{n:n}$ be the lifetime of a parallel system consisting of n independent gamma components $Z_1, Z_2, \ldots, Z_n$, where $Z_1, \ldots, Z_p$ have common shape parameter r_1 and common scale parameter λ_1^* and $Z_{p+1}, \ldots, Z_n$ have common shape parameter r_2 and common scale parameter λ_2^*. Upon applying Theorem 2.14, we have $X_{n:n} \geq_{\text{lr}} Z_{n:n}$. On the other hand, we have that $Z_{n:n} \geq_{\text{lr}} Y_{n:n}$ from Theorem 2.19. Hence, the desired result follows. ■

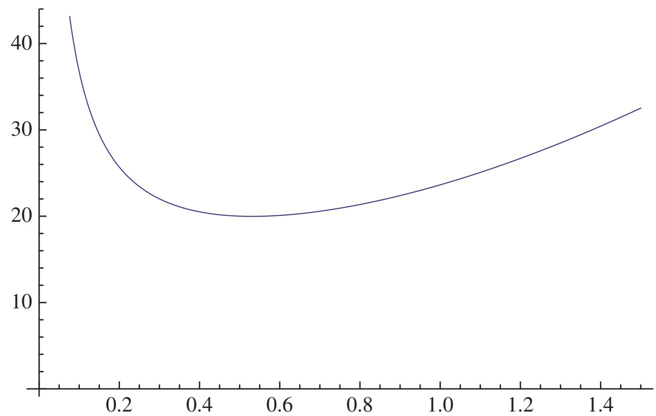

FIGURE **4.** Plot of $f_{X_{5:5}}(t)/f_{Y_{5:5}}(t)$ when $p=2$, $q=3$, $r_1=2.5$, $r_2=0.5$, $r_1^*=1.8$, $r_2^*=1.2$, $\lambda_1=1$ and $\lambda_2=2$.

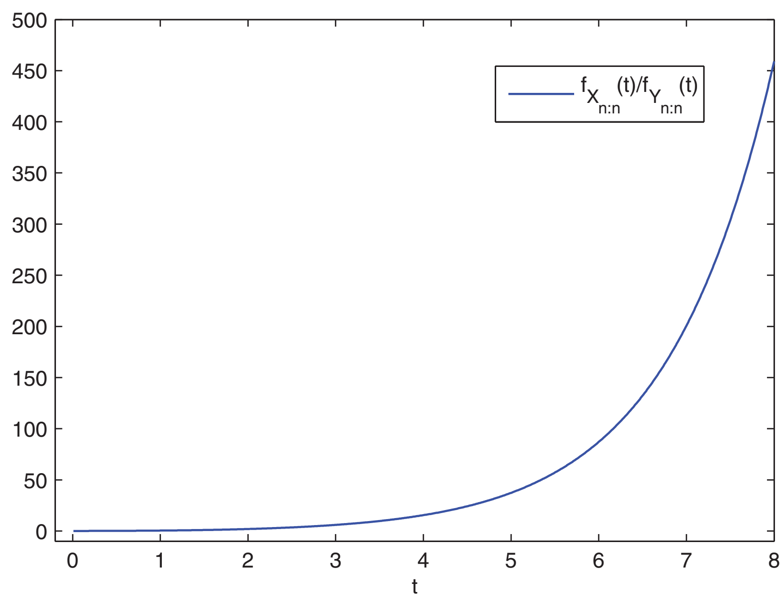

FIGURE **5.** Plot of $f_{X_{5:5}}(t)/f_{Y_{5:5}}(t)$ when $p=3$, $q=2$, $r_1=1.6$, $r_2=0.4$, $r_1^*=1.4$, $r_2^*=0.6$, $\lambda_1=1.2$, $\lambda_2=4.0$, $\lambda_1^*=2.0$ and $\lambda_2=3.8$.

Remark 2.22: The result of Theorem 2.21 generalizes that of Theorem 3.11 in Zhao and Zhang [32] from $p=q=1$ to the general case $p \geq q \geq 1$.

Finally, we present an example to illustrate the validity of the result of Theorem 2.21.

Example 2.23: Set $p=3$, $q=2$, $r_1=1.6$, $r_2=0.4$, $r_1^*=1.4$, $r_2^*=0.6$, $\lambda_1=1.2$, $\lambda_2=4.0$, $\lambda_1^*=2.0$ and $\lambda_2^*=3.8$ in Theorem 2.21, we have $(1.6,0.4) \overset{\text{m}}{\succeq} (1.4,0.6)$ and $(1.2,4.0) \overset{\text{w}}{\succeq} (2.0,3.8)$. Figure 5 plots the ratio of density functions $f_{X_{n:n}}(t)$ and $f_{Y_{n:n}}(t)$. It can be observed that the function $f_{X_{n:n}}(t)/f_{Y_{n:n}}(t)$ is increasing in $t \in \Re_+$, which is in accordance with the result of Theorem 2.21.

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3. STAR ORDERING

The following useful lemma, originally due to Saunders and Moran [25, p. 429], will be helpful for proving the main results.

LEMMA 3.1: *Let* $\{F_\lambda | \lambda \in \Re_+\}$ *be a class of distribution functions, such that* F_λ *is supported on some interval* $(a, b) \subseteq (0, \infty)$ *and has density* f_λ *which does not vanish on any subinterval of* (a, b)*. Then,*

$$F_\lambda \leq_\star F_{\lambda^*}, \quad \lambda \leq \lambda^*$$

if and only if

$$\frac{F'_\lambda(x)}{x f_\lambda(x)} \text{ is decreasing in } x,$$

where F'_λ *is the derivative of* F_λ *with respect to* λ.

THEOREM 3.2: *Under the assumptions of Theorem 2.12, if* $r_1 \geq r_2$, $\lambda_2 \geq \lambda_1$ *and* $\lambda_2^* \geq \lambda_1^*$, *then,*

$$\frac{\lambda_2}{\lambda_1} \geq \frac{\lambda_2^*}{\lambda_1^*} \Longrightarrow X_{n:n} \geq_\star Y_{n:n}.$$

PROOF: Here we will adopt the proof idea of Theorem 4.2 in Zhao and Zhang [32].

Case 1: $\lambda_1 + \lambda_2 = \lambda_1^* + \lambda_2^*$.

Without loss of generality, assume $\lambda_1 + \lambda_2 = \lambda_1^* + \lambda_2^* = 1$. In this case, we have

$$(\lambda_1, \lambda_2) \overset{\text{m}}{\succeq} (\lambda_1^*, \lambda_2^*).$$

Denote $\lambda = \lambda_2$ and $\lambda^* = \lambda_2^*$, and it is known that $\lambda \geq \lambda^*$ and $\lambda \in [1/2, 1)$. It suffices to show that

$$\frac{F'_\lambda(t)}{t f_\lambda(t)}$$

is decreasing in $t \in \Re_+$ for $\lambda \in [1/2, 1)$, where the distribution function of $X_{n:n}$ can be written as

$$F_\lambda(t) = \left[\int_0^t \frac{(1-\lambda)^{r_1}}{\Gamma(r_1)} u^{r_1 - 1} e^{-(1-\lambda)u} du\right]^p \left[\int_0^t \frac{\lambda^{r_2}}{\Gamma(r_2)} u^{r_2 - 1} e^{-\lambda u} du\right]^q,$$

with its density function as

$$\begin{aligned} f_\lambda(t) = p &\left[\int_0^t \frac{(1-\lambda)^{r_1}}{\Gamma(r_1)} u^{r_1 - 1} e^{-(1-\lambda)u} du\right]^{p-1} \left[\int_0^t \frac{\lambda^{r_2}}{\Gamma(r_2)} u^{r_2 - 1} e^{-\lambda u} du\right]^q \\ &\times \frac{(1-\lambda)^{r_1}}{\Gamma(r_1)} t^{r_1 - 1} e^{-(1-\lambda)t} + q \left[\int_0^t \frac{(1-\lambda)^{r_1}}{\Gamma(r_1)} u^{r_1 - 1} e^{-(1-\lambda)u} du\right]^p \\ &\times \left[\int_0^t \frac{\lambda^{r_2}}{\Gamma(r_2)} u^{r_2 - 1} e^{-\lambda u} du\right]^{q-1} \frac{\lambda^{r_2}}{\Gamma(r_2)} t^{r_2 - 1} e^{-\lambda t}. \end{aligned}$$

Taking the derivative to $F_\lambda(t)$ with respect to λ yields that

$$\begin{aligned}F'_\lambda(t) &= p\left[\int_0^t \frac{(1-\lambda)^{r_1}}{\Gamma(r_1)}u^{r_1-1}e^{-(1-\lambda)u}du\right]^{p-1}\left[\int_0^t \frac{\lambda^{r_2}}{\Gamma(r_2)}u^{r_2-1}e^{-\lambda u}du\right]^{q}\\ &\times\left[-r_1\int_0^t \frac{(1-\lambda)^{r_1-1}}{\Gamma(r_1)}u^{r_1-1}e^{-(1-\lambda)u}du+\int_0^t \frac{(1-\lambda)^{r_1}}{\Gamma(r_1)}u^{r_1}e^{-(1-\lambda)u}du\right]\\ &+q\left[\int_0^t \frac{(1-\lambda)^{r_1}}{\Gamma(r_1)}u^{r_1-1}e^{-(1-\lambda)u}du\right]^{p}\left[\int_0^t \frac{\lambda^{r_2}}{\Gamma(r_2)}u^{r_2-1}e^{-\lambda u}du\right]^{q-1}\\ &\times\left[r_2\int_0^t \frac{\lambda^{r_2-1}}{\Gamma(r_2)}u^{r_2-1}e^{-\lambda u}du-\int_0^t \frac{\lambda^{r_2}}{\Gamma(r_2)}u^{r_2}e^{-\lambda u}du\right].\end{aligned}$$

Upon using integration by parts, one has

$$r\int_0^t \frac{\lambda^{r-1}}{\Gamma(r)}u^{r-1}e^{-\lambda u}du=\frac{\lambda^{r-1}}{\Gamma(r)}t^{r}e^{-\lambda t}+\int_0^t \frac{\lambda^{r}}{\Gamma(r)}u^{r}e^{-\lambda u}du,$$

and thus $F'_\lambda(t)$ can be reduced to

$$\begin{aligned}F'_\lambda(t) &= p\left[\int_0^t \frac{(1-\lambda)^{r_1}}{\Gamma(r_1)}u^{r_1-1}e^{-(1-\lambda)u}du\right]^{p-1}\left[\int_0^t \frac{\lambda^{r_2}}{\Gamma(r_2)}u^{r_2-1}e^{-\lambda u}du\right]^{q}\\ &\times\left[-\frac{(1-\lambda)^{r_1-1}}{\Gamma(r_1)}t^{r_1}e^{-(1-\lambda)t}\right]+q\left[\int_0^t \frac{(1-\lambda)^{r_1}}{\Gamma(r_1)}u^{r_1-1}e^{-(1-\lambda)u}du\right]^{p}\\ &\times\left[\int_0^t \frac{\lambda^{r_2}}{\Gamma(r_2)}u^{r_2-1}e^{-\lambda u}du\right]^{q-1}\left[\frac{\lambda^{r_2-1}}{\Gamma(r_2)}t^{r_2}e^{-\lambda t}\right].\end{aligned}$$

We can compute that

$$\begin{aligned}\frac{F'_\lambda(t)}{tf_\lambda(t)} &= \frac{-p\lambda t^{r_1}e^{-(1-\lambda)t}\int_0^t u^{r_2-1}e^{-\lambda u}du+q(1-\lambda)t^{r_2}e^{-\lambda t}\int_0^t u^{r_1-1}e^{-(1-\lambda)u}du}{p\lambda(1-\lambda)t^{r_1}e^{-(1-\lambda)t}\int_0^t u^{r_2-1}e^{-\lambda u}du+q\lambda(1-\lambda)t^{r_2}e^{-\lambda t}\int_0^t u^{r_1-1}e^{-(1-\lambda)u}du}\\ &\propto \frac{-p\lambda e^{(2\lambda-1)t}\int_0^1 y^{r_2-1}e^{-\lambda yt}dy+q(1-\lambda)\int_0^1 y^{r_1-1}e^{-(1-\lambda)yt}dy}{pe^{(2\lambda-1)t}\int_0^1 y^{r_2-1}e^{-\lambda yt}dy+q\int_0^1 y^{r_1-1}e^{-(1-\lambda)yt}dy}\\ &= -\lambda+\frac{\int_0^1 qy^{r_1-1}e^{-(1-\lambda)yt}dy}{\int_0^1[qy^{r_1-1}e^{-(1-\lambda)yt}+py^{r_2-1}e^{(2\lambda-1-\lambda y)t}]dy}.\end{aligned}$$

Now, it is enough to show the function

$$\Lambda(t)=\frac{\displaystyle\int_0^1 qy^{r_1-1}e^{-(1-\lambda)yt}dy}{\int_0^1[qy^{r_1-1}e^{-(1-\lambda)yt}+py^{r_2-1}e^{(2\lambda-1-\lambda y)t}]dy}=\mathsf{E}_t k(Y_2,t)$$

is decreasing in $t\in\Re_+$ for $\lambda\in[1/2,1)$, where

$$k(Y_2,t)=\frac{qy^{r_1-1}e^{-(1-\lambda)yt}}{qy^{r_1-1}e^{-(1-\lambda)yt}+py^{r_2-1}e^{(2\lambda-1-\lambda y)t}},\quad y\in(0,1).$$

Here, the distribution function of the random variable Y_2 belongs to the family $\mathcal{P}_2 = \{H_2(\cdot|T), t \in \Re_+\}$ with densities

$$h_2(y|t) = c_2(t)[qy^{r_1-1}e^{-(1-\lambda)yt} + py^{r_2-1}e^{(2\lambda-1-\lambda y)t}]$$

and a normalizing constant $c_2(t)$ such that $\int_0^1 h_2(y|t)dy = 1$. It is easy to see that

$$k(y,t) = \frac{q}{q + py^{r_2-r_1}e^{(2\lambda-1)(1-y)t}}$$

is decreasing in $t \in \Re_+$ while is increasing in $y \in (0,1)$. On the other hand, for $t_2 \geq t_1 \geq 0$ and $a = r_1 - r_2 \geq 0$, the function

$$\begin{aligned}\iota(y) &= \frac{h_2(y|t_2)}{h_2(y|t_1)} \\ &= \frac{qy^{r_1-1}e^{-(1-\lambda)t_2 y} + py^{r_2-1}e^{(2\lambda-1-\lambda y)t_2}}{qy^{r_1-1}e^{-(1-\lambda)t_1 y} + py^{r_2-1}e^{(2\lambda-1-\lambda y)t_1}} \\ &= \frac{qy^a e^{-(1-\lambda)t_2 y} + pe^{(2\lambda-1-\lambda y)t_2}}{qy^a e^{-(1-\lambda)t_1 y} + pe^{(2\lambda-1-\lambda y)t_1}}\end{aligned}$$

is decreasing in $y \in (0,1)$ by checking that

$$\begin{aligned}\iota'(y) &\overset{\text{sgn}}{=} \left[q(\frac{a}{y} - (1-\lambda)t_2)y^a e^{-(1-\lambda)t_2 y} - p\lambda t_2 e^{(2\lambda-1-\lambda y)t_2}\right] \\ &\quad \times \left[qy^a e^{-(1-\lambda)t_1 y} + pe^{(2\lambda-1-\lambda y)t_1}\right] \\ &\quad - \left[q(\frac{a}{y} - (1-\lambda)t_1)y^a e^{-(1-\lambda)t_1 y} - p\lambda t_1 e^{(2\lambda-1-\lambda y)t_1}\right] \\ &\quad \times \left[qy^a e^{-(1-\lambda)t_2 y} + pe^{(2\lambda-1-\lambda y)t_2}\right] \\ &= q^2(1-\lambda)(t_1 - t_2)y^{2a}e^{-(1-\lambda)(t_1+t_2)y} \\ &\quad + pq\left[(1-\lambda)t_1 - \lambda t_2 - \frac{a}{y}\right] y^a e^{-(1-\lambda)t_1 y}e^{(2\lambda-1-\lambda y)t_2} \\ &\quad + pq\left[\frac{a}{y} - (1-\lambda)t_2 + \lambda t_1\right] y^a e^{-(1-\lambda)t_2 y}e^{(2\lambda-1-\lambda y)t_1} + \lambda(t_1 - t_2)e^{(2\lambda-1-\lambda y)(t_1+t_2)} \\ &\leq q^2(1-\lambda)(t_1 - t_2)y^{2a}e^{-(1-\lambda)(t_1+t_2)y} \\ &\quad + pq\left[(1-\lambda)t_1 - \lambda t_2 - \frac{a}{y}\right] y^a e^{-(1-\lambda)t_2 y}e^{(2\lambda-1-\lambda y)t_1} \\ &\quad + pq\left[\frac{a}{y} - (1-\lambda)t_2 + \lambda t_1\right] y^a e^{-(1-\lambda)t_2 y}e^{(2\lambda-1-\lambda y)t_1} + p^2\lambda(t_1 - t_2)e^{(2\lambda-1-\lambda y)(t_1+t_2)} \\ &= q^2(1-\lambda)(t_1 - t_2)y^{2a}e^{-(1-\lambda)(t_1+t_2)y} + pq(t_1 - t_2)y^a e^{-(1-\lambda)t_2 y}e^{(2\lambda-1-\lambda y)t_1} \\ &\quad + p^2\lambda(t_1 - t_2)e^{(2\lambda-1-\lambda y)(t_1+t_2)} \\ &\leq 0,\end{aligned} \tag{16}$$

where the inequality in (16) holds due to

$$(1-\lambda)t_1 - \lambda t_2 - \frac{a}{y} \leq (1-2\lambda)t_2 - \frac{a}{y} \leq 0$$

and

$$-(1-\lambda)t_1y + (2\lambda - 1 - \lambda y)t_2 \geq -(1-\lambda)t_2y + (2\lambda - 1 - \lambda y)t_1.$$

Hence, we have $H_2(\cdot|t_1) \geq_{\rm lr} H_2(\cdot|t_2)$, which in turn implies that $H_2(\cdot|t_2) \geq_{\rm st} H_2(\cdot|t_2)$ whenever $t_2 \geq t_1$. Using Lemma 2.1 once again, we conclude that $\mathsf{E}_t k(Y_2|t)$ is decreasing in $t \in (0,\infty)$ for $\lambda \in [1/2, 1)$.

Case 2: $\lambda_1 + \lambda_2 \neq \lambda_1^* + \lambda_2^*$.

Assume that $\lambda_1 + \lambda_2 = c(\lambda_1^* + \lambda_2^*)$, where c is a scalar. We then have

$$(\lambda_1, \lambda_2) \overset{\rm m}{\succeq} (c\lambda_1^*, c\lambda_2^*).$$

Let $Y_{n:n}$ be the lifetime of a parallel system with n independent gamma components having respective shape parameters r_1 and r_2 and respective scale parameters $c\lambda_1^*$ and $c\lambda_2^*$. From the discussion of Case 1, we have

$$X_{n:n} \geq_\star Y_{n:n}.$$

On the other hand, since the star order is scale invariant, it follows that

$$X_{n:n} \geq_\star X_{n:n}^*.$$

■

Actually, the condition in Theorem 3.2 is quite general and includes many special cases (see Kochar and Xu [17]). Furthermore, due to the fact that the star order implies the Lorenz order, we have the following result which will be great interest in economics.

Corollary 3.3: *Under the assumptions of Theorem 2.12, if $r_1 \geq r_2$, $\lambda_2 \geq \lambda_1$ and $\lambda_2^* \geq \lambda_1^*$, then,*

$$\frac{\lambda_2}{\lambda_1} \geq \frac{\lambda_2^*}{\lambda_1^*} \Longrightarrow X_{n:n} \geq_{\rm Lorenz} Y_{n:n}.$$

Finally, we present a result for the dispersive order.

Theorem 3.4: *Under the assumptions of Theorem 2.12, if $p \geq q$, $r_1 \geq r_2$ and $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$, we then have*

$$(\lambda_1, \lambda_2) \overset{\rm w}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{n:n} \geq_{\rm disp} Y_{n:n}.$$

Proof: In light of Theorem 2.14, it follows that

$$(\lambda_1, \lambda_2) \overset{\rm w}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{n:n} \geq_{\rm lr} Y_{n:n} \Longrightarrow X_{n:n} \geq_{\rm st} Y_{n:n}.$$

On the other hand, we have from Theorem 3.2 that,

$$X_{n:n} \geq_\star Y_{n:n}.$$

Also, it is known from Ahmed et al. [1] that, for two continuous random X and Y, if $X \geq_\star Y$, then $X \geq_{\rm st} Y \Longrightarrow X \geq_{\rm disp} Y$. Hence, the theorem follows. ■

4. APPLICATION

Suppose that there exists two parallel systems each consisting of 10 components which have exponential lifetimes. For the first parallel system (denoted by I), we consider the case that these ten components have hazard rates vector $(0.5 \times \mathbf{1}_7, 2.6 \times \mathbf{1}_3)$. For the other parallel system (denoted by II), the components are assumed to have hazard rates vector $(1.2 \times \mathbf{1}_7, 2.1 \times \mathbf{1}_3)$. In order to enhance the reliability, we are allowed to take minimal repairs (see Shaked and Shanthikumar [26]) when the components fail. For the system I, each of components $X_1, \ldots, X_7$ is allocated eight minimal repairs while each of components X_8, X_9, X_{10} is allocated three minimal repairs. For the system II, each of components $Y_1, \ldots, Y_7$ is allocated six minimal repairs while each of components Y_8, Y_9, Y_{10} is allocated five minimal repairs. Now, a factory needs such a parallel system and the reliability engineer need to take into consideration which system should be chosen. At a first glance, it is difficult to make a decision. Denote by $X_i(r_i)$ the lifetime of component X_i with k_i minimal repairs. It is known that, by Gamma–Poisson relationship,

$$\begin{aligned} F_{X_i(k_i)}(t) &= \mathsf{P}(X_i(k_i) \le t) \\ &= \sum_{j=k+1}^{\infty} \frac{e^{-\lambda_i t}(\lambda_i t)^j}{j!} \\ &= \int_0^t \frac{\lambda_i^{k_i+1}}{\Gamma(k_i+1)} x^{k_i} e^{-\lambda_i x} dt. \end{aligned}$$

Thus, X_i with k_i minimal repairs has a gamma distribution with scale parameter λ_i and shape parameter $k_i + 1$, that is, $\Gamma(k_1 + 1, \lambda_i)$. So the lifetime of system I can be expressed as the maximum of ten gamma random variables with shape and scale parameters

$$(9 \times \mathbf{1}_7, 4 \times \mathbf{1}_3),\ (0.5 \times \mathbf{1}_7, 2.6 \times \mathbf{1}_3).$$

Similarly, the lifetime of system II can be expressed as the maximum of ten gamma random variables with shape and scale parameters

$$(7 \times \mathbf{1}_7, 6 \times \mathbf{1}_3),\ (1.2 \times \mathbf{1}_7, 2.1 \times \mathbf{1}_3).$$

Observe that $p = 7$, $q = 3$,

$$(9, 4) \overset{\text{m}}{\succeq} (7, 6) \text{ and } (0.5, 2.6) \overset{\text{w}}{\succeq} (1.2, 2.1),$$

which satisfy the conditions of Theorem 2.21. Hence, we can conclude that the lifetime of system I is superior to that of system II in terms of the likelihood ratio order, which states that the factory should choose system I in order to make the system more reliable.

5. DISCUSSION

Let $X_1, X_2, \ldots, X_n$ be independent random variables following the multiple-outlier gamma model with respective shape parameters and scale parameters $(r_1\mathbf{1}_p, r_2\mathbf{1}_q)$, $(\lambda_1\mathbf{1}_p, \lambda_2\mathbf{1}_q)$, where $p \ge 1$ and $p + q = n$. Let $Y_1, Y_2, \ldots, Y_n$ be another set of independent random variables following the multiple-outlier gamma model with respective shape parameters and scale parameters $(r_1^*\mathbf{1}_p, r_2^*\mathbf{1}_q)$, $(\lambda_1^*\mathbf{1}_p, \lambda_2^*\mathbf{1}_q)$. Suppose that $r_1 \geq r_2$, $r_1^* \geq r_2^*$ and $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$. If $p \geq q$, we then have

$$(r_1, r_2) \overset{\mathrm{m}}{\succeq} (r_1^*, r_2^*), (\lambda_1, \lambda_2) \overset{\mathrm{w}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{n:n} \geq_{\mathrm{lr}} Y_{n:n}. \quad \mathbf{(17)}$$

Especially, if $r_1 = r_1^* \geq r_2 = r_2^*$, it is shown that

$$\frac{\lambda_2}{\lambda_1} \geq \frac{\lambda_2^*}{\lambda_1^*} \Longrightarrow X_{n:n} \geq_\star Y_{n:n}.$$

Besides, under the condition $r_1 = r_1^* \geq r_2 = r_2^*$, $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$ and $p \geq q$, we also prove that

$$(\lambda_1, \lambda_2) \overset{\mathrm{w}}{\succeq} (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{n:n} \geq_{\mathrm{disp}} Y_{n:n}.$$

The results established here have generalized and extended those for the case when $p = 1$ and $q = 1$ and the case when the shape parameter r is common in the literature. We partially answer the problem for the case when $r > 1$ posed by Zhao and Balakrishnan [31]. Note that $(r_1, r_2) \overset{\mathrm{m}}{\succeq} (r_1^*, r_2^*)$, $p \geq q$ and $r_2 \leq r_2^* \leq r_1^* \leq r_1$ does not imply

$$(r_1\mathbf{1}_p, r_2\mathbf{1}_q) \overset{\mathrm{m}}{\succeq} (r_1^*\mathbf{1}_p, r_2^*\mathbf{1}_q).$$

However, the condition $(\lambda_1, \lambda_2) \overset{\mathrm{w}}{\succeq} (\lambda_1^*, \lambda_2^*)$ implies that

$$(\lambda_1\mathbf{1}_p, \lambda_2\mathbf{1}_q) \overset{\mathrm{w}}{\succeq} (\lambda_1^*\mathbf{1}_p, \lambda_2^*\mathbf{1}_q),$$

under the assumptions that $p \geq q$ and $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$. Thus, it will be very interesting to check whether the results in (17) still holds under the conditions

$$(r_1\mathbf{1}_p, r_2\mathbf{1}_q) \overset{\mathrm{m}}{\succeq} (r_1^*\mathbf{1}_p, r_2^*\mathbf{1}_q)$$

and

$$(\lambda_1\mathbf{1}_p, \lambda_2\mathbf{1}_q) \overset{\mathrm{w}}{\succeq} (\lambda_1^*\mathbf{1}_p, \lambda_2^*\mathbf{1}_q).$$

Similarly, it would be also of interest to check that if $r_1 \geq r_2$, $r_1^* \geq r_2^*$, $\lambda_1 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_2$,

$$(r_1\mathbf{1}_p, r_2\mathbf{1}_q) \overset{\mathrm{m}}{\succeq} (r_1^*\mathbf{1}_p, r_2^*\mathbf{1}_q)$$

and

$$(\lambda_1\mathbf{1}_p, \lambda_2\mathbf{1}_q) \overset{\mathrm{p}}{\succeq} (\lambda_1^*\mathbf{1}_p, \lambda_2^*\mathbf{1}_q),$$

whether it holds

$$X_{n:n} \geq_{\mathrm{hr}} Y_{n:n}?$$

We are currently working on these problems and hope to report these findings in a future paper.

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