NOTE ON WEAK DIMENSION OF ALGEBRAS

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Let Λ be a K-algebra over a commutative ring K. Harada [5] has introduced the notion of weak dimension of algebras Λ (denoted by w. dim Λ) analogous to the dimension of algebras in Cartan and Eilenberg [3].

In case Λ is semiprimary algebra over a field K with radical N such that $(\Lambda/N : K) < \infty$ then dim Λ = w. dim Λ.

We prove here some results for w. dim Λ, analogous to the results proved in Eilenberg [4] for dim Λ.

Throughout this note, unless stated otherwise, we shall assume that K is a commutative ring, Λ is a K-algebra which is K-flat and Γ is a regular K-algebra. We first notice that

$$(1) \qquad H_n(\Lambda, B \otimes_\Gamma C) \cong Tor_n^{\Lambda \otimes \Gamma^*}(C, B)$$

for $({}_\Lambda B_\Gamma, {}_\Gamma C_\Lambda)$.

This follows from [5, (*) of § 1] on replacing $(\Lambda, \Gamma, \Sigma)$ by $(\Lambda, \Lambda^*, \Gamma)$.

Imposing further conditions on K, Λ, Γ we deduce, rather easily, from (1) a few results some of which we use for the proofs of our main results viz. Theorems 1 and 2.

COROLLARY 1. If Λ is a K-algebra with K regular then

$$\text{w. gl. dim } \Lambda \leqq \text{w. dim } \Lambda.$$

COROLLARY 2. If Λ is K-flat and regular then

$$\text{w. gl. dim } \Lambda = \text{w. dim } \Lambda.$$

From Corollaries 1 and 2 one obtains Theorem 1 of [5] as

COROLLARY 3. Let Λ be a K-algebra and K be regular. Then w. dim $\Lambda = 0$ if and only if Λ^e is regular.

COROLLARY 4. If Λ is K-flat and Γ is regular then

$$\text{l. w. dim}_{\Lambda \otimes \Gamma^*} \Gamma \leqq \text{w. gl. dim } \Lambda \otimes \Gamma^* \leqq \text{w. dim } \Lambda.$$

COROLLARY 5. Let Λ be K-flat and $\Gamma = \Lambda/I$, where I is a two sided ideal of Λ, be regular. Then

$$H_n(\Lambda, C) \cong \text{Tor}_n^{\Lambda \otimes \Gamma^*}(C, \Gamma).$$

PROOF. Replace B by Γ in (1).

We state (without proof) the following

LEMMA 1. [1, Lemma 6]. Let Λ be an arbitrary ring, I a nilpotent right ideal in Λ and T a (covariant or contravariant) half-exact functor defined for all right Λ-modules. If $T(A) = 0$ for each right Λ-module A such that $AI = 0$ then $T = 0$.

Now we give sufficient conditions under which the inequalities in Corollary 4 become equalities.

THEOREM 1. Let Λ be K-flat and $\Gamma = \Lambda/I$ be regular where I is a two sided nilpotent ideal in Λ. Then

$$\text{l. w. dim}_{\Lambda\otimes\Gamma^*}\Gamma = \text{w. gl. dim } \Lambda\otimes\Gamma^* = \text{w. dim } \Lambda.$$

PROOF. Assume that l.w. $\dim_{\Lambda\otimes\Gamma^*}\Gamma = n$. By Corollary 5, we have $H_{n+1}(\Lambda, C) = 0$ for all right $\Lambda\otimes\Gamma^*$-modules C. Now the exact sequence

$$\Lambda\otimes I^* \to \Lambda\otimes\Lambda^* \xrightarrow{\varphi} \Lambda\otimes\Gamma^* \to 0$$

and I nilpotent imply that kernel φ is nilpotent. Hence, by Lemma 1, we have $H_{n+1}(\Lambda, C) = 0$ for all $\Lambda\otimes\Lambda^*$-modules C, i.e. w. dim $\Lambda \leqq n$. Hence the theorem follows.

COROLLARY 6. Let Λ be a semiprimary algebra over a field K and $(\Lambda/N : K) < \infty$ where N is the radical of Λ. Then

$$\dim \Lambda = \text{l. gl. dim } \Lambda\otimes\Gamma^* = \text{l. dim}_{\Lambda\otimes\Gamma^*}\Gamma$$

where $\Gamma = \Lambda/N$.

REMARK. Theorem 1 of [4] is a particular case of this Corollary.

We now make three further assumptions (i) K regular (ii) $\Gamma\otimes\Gamma^*$ regular and (iii) Γ K-projective.

With these assumptions we prove

THEOREM 2. Let Λ be a K-algebra and $\Gamma = \Lambda/I$, where I is a two sided nilpotent ideal in Λ, be K-projective. If K and $\Gamma\otimes\Gamma^*$ are regular then w. dim Λ = l. w. $\dim_\Lambda \Gamma = n$ where n is the smallest integer such that $\text{Tor}^\Lambda_{n+1}(\Gamma, \Gamma) = 0$. If no such integer exists, then we take $n = \infty$.

For the proof of this theorem we need the following

LEMMA 2. Let Λ be a K-algebra and $\Gamma = \Lambda/I$ be a K-projective algebra. If $\Gamma\otimes\Gamma^*$ is regular then

$$\text{Tor}_n^{\Lambda\otimes\Gamma^*}(C, A) \cong C\otimes_{\Gamma\otimes\Gamma^*}\text{Tor}_n^\Lambda(\Gamma, A)$$

for $({}_\Lambda A_\Gamma, {}_\Gamma C_\Gamma)$.

PROOF. Consider the natural isomorphism

$$C\otimes_{\Lambda\otimes\Gamma^*}A \cong C\otimes_{\Gamma\otimes\Gamma^*}(\Gamma\otimes_\Lambda A).$$

Let X be a $\Lambda \otimes \Gamma^*$-projective resolution of A. Since Γ is K-projective, it follows that X is a Λ-projective resolution of A. Since $\Gamma \otimes \Gamma^*$ is regular, $\otimes_{\Gamma \otimes \Gamma^*}$ is an exact functor and therefore replacing A by X and passing to homology, we get

$$\mathrm{Tor}_{n+1}^{\Lambda \otimes \Gamma^*}(C, A) \cong C \otimes_{\Gamma \otimes \Gamma^*} \mathrm{Tor}_{n+1}^{\Lambda}(\Gamma, A).$$

Corollary 8. Let Λ and Γ be as in Lemma 2. If further Λ is K-flat and Γ is regular then $H_{n+1}(\Lambda, C) \cong C \otimes_{\Gamma \otimes \Gamma^*} \mathrm{Tor}_{n+1}^{\Lambda}(\Gamma, \Gamma)$.

Proof. Take $A = \Gamma$ in Lemma 2.

Proof of the Theorem 2. We know that $n \leqq \mathrm{l.\,w.\,dim}_\Lambda \Gamma$. Since K is regular, by Corollary 1, we have $\mathrm{l.\,w.\,dim}_\Lambda \Gamma \leqq \mathrm{w.\,dim}\, \Gamma$. By Corollary 3, K and $\Gamma \otimes \Gamma^*$ are regular give $\mathrm{w.\,dim}\, \Gamma = 0$ and therefore by Corollary 2, Γ is regular. Thus by Corollary 8, we have $H_{n+1}(\Lambda, C) = 0$ for all two sided Γ-modules C. The exact sequence

$$I \otimes \Lambda^* + \Lambda \otimes I^* \to \Lambda \otimes \Lambda^* \xrightarrow{\varphi} \Gamma \otimes \Gamma^* \to 0$$

and I nilpotent imply that kernel of φ is nilpotent. Then by Lemma 1, we have $H_{n+1}(\Lambda, C) = 0$ for all two sided Λ-modules C, i.e. $\mathrm{w.\,dim}\, \Lambda \leqq n$. Hence the result.

Corollary 9. Let Λ be a semiprimary algebra over a field K. Let $\Gamma = \Lambda/N$ be of finite degree over K and $\Gamma \otimes \Gamma^*$ be semisimple. Then

$$\dim \Lambda = \mathrm{l.\,dim}_\Lambda \Gamma = \mathrm{gl.\,dim}\, \Lambda.$$

Remark. Theorem 2 of [4] is a particular case of this Corollary.

References

[1] Auslander, M., 'On dimension of modules and algebras III', *Nagoya Math.* J., 9 (1955), 67—77.

[2] Auslander, M., 'On dimension of modules and algebras VI', *Nagoya Math.* J. 11 (1957), 61—65.

[3] Cartan, H. and Eilenberg, S., *Homological algebra* (Princeton University Press), 1956.

[4] Eilenberg, S., 'Algebras of cohomologically finite dimension', *Commentrii Math. Helv.* 28 (1954), 310—319.

[5] Harada, M., 'The weak dimension of algebras and its applications', *J. Inst. Poly. Osaka City Univ.*, Vol. 9, No. 2 (1958), 47—58.

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