



RESEARCH ARTICLE

Algebraic relations among Goss's zeta values on elliptic curves

Nathan Green¹ and Tuan Ngo Dac²

¹Louisiana Tech University, 501 Dan Reneau Dr, 71270, USA; E-mail: ngreen@latech.edu.

²Normandie Université, Université de Caen Normandie - CNRS, Laboratoire de Mathématiques Nicolas Oresme (LMNO), UMR 6139, Caen, 14000, France; E-mail: ngodac@unicaen.fr.

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Abstract

In 2007 Chang and Yu determined all the algebraic relations among Goss's zeta values for $A = \mathbb{F}_q[\theta]$, also known as the Carlitz zeta values. Goss raised the problem of determining all algebraic relations among Goss's zeta values at positive integers for a general base ring A , but very little is known. In this paper, we develop a general method, and we determine all algebraic relations among Goss's zeta values for the base ring A which is the coordinate ring of an elliptic curve defined over \mathbb{F}_q . To our knowledge, this is the first work tackling Goss's problem when the base ring has class number strictly greater than 1.

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Introduction

0.1. Background

The study of the Riemann zeta function $\zeta(\cdot)$ and its special values $\zeta(n)$ for $n \in \mathbb{N}$ and $n \geq 2$ is a classical topic in number theory. A well-known analogy between the arithmetic of number fields and global function fields suggests that one can replace \mathbb{Z} by the coordinate ring of a curve defined over a finite field and \mathbb{Q} by its field of fractions, and study similarly defined objects. In [16], Carlitz carries out this analogy for the case where $A = \mathbb{F}_q[\theta]$ (q a power of a prime) and $K = \mathbb{F}_q(\theta)$ (which are the coordinate ring and function field of the curve \mathbb{P}^1 over \mathbb{F}_q), wherein he defines zeta values $\zeta_A(n)$ which are considered as the analogues of the Riemann zeta function. Many years after Carlitz's pioneering work, Goss showed that these values could be realized as the special values of the so-called Goss-Carlitz zeta function $\zeta_A(\cdot)$ over a suitable generalization of the complex plane. Goss's zeta functions are a special case of the L -functions he introduced in [30] for more general rings A . The special values of this type of L -function, called Goss's zeta values, have been at the heart of function field arithmetic for the last forty years. Various works have revealed the importance of these zeta values for both their

independent interest and for their applications to a wide variety of arithmetic problems, including multiple zeta values (see the excellent articles [54, 53] for an overview and also [46] for some recent results), Anderson's log-algebraicity identities (see [2, 3, 6, 36, 51]) and Taelman's units and the class formula à la Taelman (see [11, 24, 25, 26, 28, 43, 50] for recent progress and [10] for an overview).

For $A = \mathbb{F}_q[\theta]$, the transcendence of the Carlitz zeta values at positive integers $\zeta_A(n)$ ($n \geq 1$) was first proved by Jing Yu [55]. Further, all algebraic relations among these values were determined by Jing Yu [56] and by Chieh-Yu Chang and Jing Yu [23]. These results are very surprising when compared to the extremely limited knowledge we have about the transcendence of values of Riemann's zeta function at odd positive integers greater than 3 (see section 3 of [44] for a description of known classical results). Goss raised the problem of extending the work of Chang and Yu to a more general setting. For a finite class of curves, such that the coordinate ring A has class number one, several partial results about Goss's zeta values have been obtained by a similar method (see, for example, [42]). However, to our knowledge, nothing is known when the class number of A is greater than 1. One difficulty of extending these results to rings with arbitrary class number is that one must define the zeta values as sums over ideals, rather than over monic elements, which greatly complicates some of the calculations. Additionally, Anderson generating functions are more complicated in this situation (see §2.1) and require more sophisticated analysis to realize their evaluations as periods (see §3.3).

In this paper, we provide the first step towards the resolution of the above problem and develop a conceptual method to deal with the genus 1 case. The advantage of working in the genus 1 case (elliptic curves) is that we have an explicit group law on the curve which we often exploit in our arguments. However, where possible, we strive to give general arguments in our proofs which will readily generalize to curves of arbitrary genus. Our results determine all algebraic relations among Goss's zeta values attached to the base ring A which is the coordinate ring of an elliptic curve over a finite field. To do so, we reduce the study of Goss's zeta values, which are fundamentally analytic objects, to that of Anderson's zeta values, which are of arithmetic nature (see Section 5.3 for details). Then we use a generalization of Anderson-Thakur's theorem (Theorem 1.8) on elliptic curves to construct zeta t -motives attached to Anderson's zeta values. We apply the work of Hardouin [37] on Tannakian groups in positive characteristic and compute the Galois groups attached to zeta t -motives. Finally, we apply the transcendence method introduced by Papanikolas [47] (which relies heavily on [4]) to obtain our algebraic independence result.

0.2. Statement of Results

Let us give now more precise statements of our results.

Let X be a geometrically connected smooth projective curve over a finite field \mathbb{F}_q of characteristic p , having q elements. We denote by K its function field and fix a place ∞ of K of degree $d_\infty = 1$. We denote by A the ring of elements of K which are regular outside ∞ . The ∞ -adic completion K_∞ of K is equipped with the normalized ∞ -adic valuation $v_\infty : K_\infty \rightarrow \mathbb{Z} \cup \{+\infty\}$. The completion \mathbb{C}_∞ of a fixed algebraic closure \overline{K}_∞ of K_∞ comes with a unique valuation extending v_∞ , which we also denote by v_∞ .

To define Goss's zeta values (our exposition closely follows [32, §8.2-8.7]), we let $\pi \in K_\infty^\times$ be a uniformizer so that we can identify K_∞ with $\mathbb{F}_q((\pi))$. For $x \in \overline{K}_\infty^\times$, one can write $x = \pi^{v_\infty(x)} \text{sgn}(x)\langle x \rangle$, where $\text{sgn}(x) \in \overline{\mathbb{F}}_q^\times$ and $\langle x \rangle \in (1 + \pi\mathbb{F}_q[[\pi]])$ is a 1-unit. If we denote by $\mathcal{I}(A)$ the group of fractional ideals of A , then Goss defines a group homomorphism

$$[\cdot]_A : \mathcal{I}(A) \rightarrow \overline{K}_\infty^\times$$

such that for $x \in K^\times$, we have $[xA]_A = x/\text{sgn}(x)$. Note that the definition of ideal exponentiation technically depends on the choice of uniformizer $\pi \in K_\infty^*$ and the choice of sign function (see [32, Thm. 8.2.15-16]). However, in this paper, we apply it only in the case of function fields for rank 1 sign-normalized Drinfeld modules for elliptic curves, so there is a canonical choice for each of these (see Section 1.2).

Let E/K be a finite extension, and let O_E be the integral closure of A in E . Then Goss defined a zeta function $\zeta_{O_E}(\cdot)$ (see [32, §8.6]) over a suitable generalization of the complex plane (see [32, §8.1]). We are interested in Goss's zeta values for $n \in \mathbb{N}$ given by

$$\zeta_{O_E}(n) = \sum_{d \geq 0} \sum_{\substack{\mathfrak{I} \in \mathcal{I}(O_E), \mathfrak{I} \subset O_E, \\ \deg(N_{E/K}(\mathfrak{I}))=d}} \left[\frac{O_E}{\mathfrak{I}} \right]_A^{-n} \in \overline{K}_\infty^\times$$

(details of this definition are in §5.2), where $\mathcal{I}(O_E)$ denotes the group of fractional ideals of O_E .

0.3. Carlitz zeta values (the genus 0 case).

We set our curve X to be the projective line $\mathbb{P}^1/\mathbb{F}_q$ equipped with the infinity point $\infty \in \mathbb{P}^1(\mathbb{F}_q)$. Then $A = \mathbb{F}_q[\theta]$, $K = \mathbb{F}_q(\theta)$ and $K_\infty = \mathbb{F}_q((1/\theta))$, where θ is an independent variable. Let A_+ be the set of monic polynomials in A .

Since the class number of A is 1, by the above discussion, Goss's map is given by $[xA]_A = x/\text{sgn}(x)$ for $x \in K^\times$. Then the Carlitz zeta values, which are special values of the Carlitz-Goss zeta function, are given by

$$\zeta_A(n) := \sum_{a \in A_+} \frac{1}{a^n} \in K_\infty^\times, \quad n \in \mathbb{N}.$$

Carlitz noticed that these values are intimately related to the so-called Carlitz module C that is the first example of a Drinfeld module. Then he proved two fundamental theorems about these values. In analogy with the classical Euler formulas, Carlitz's first theorem asserts that for the so-called Carlitz period $\tilde{\pi} \in \overline{K}_\infty^\times$, we have the Carlitz-Euler relations

$$\frac{\zeta_A(n)}{\tilde{\pi}^n} \in K \quad \text{for all } n \geq 1, n \equiv 0 \pmod{q-1}.$$

His second theorem states that $\zeta_A(1)$ is the logarithm of 1 of the Carlitz module C , which is the first example of log-algebraicity identities. Anderson extended this theory by giving many more log-algebraic identities [3].

Many years after the work of Carlitz, Anderson and Thakur [5] developed an explicit theory of tensor powers of the Carlitz module $C^{\otimes n}$ ($n \in \mathbb{N}$) and expressed $\zeta_A(n)$ as the last coordinate of the logarithm of a special algebraic point of $C^{\otimes n}$. Using this result, Yu proved that $\zeta_A(n)$ is transcendental in [55] and that the only \overline{K} -linear relations among the Carlitz zeta values and powers of the Carlitz period are the above Carlitz-Euler relations in [56].

For algebraic relations among Carlitz zeta values, we have the trivial relations coming from working in characteristic p , which state that for $m, n \in \mathbb{N}$,

$$\zeta_A(p^m n) = (\zeta_A(n))^{p^m}.$$

Extending the previous works of Yu, Chang and Yu [23] proved that the Carlitz-Euler relations and the Frobenius relations give rise to all algebraic relations among the Carlitz zeta values. To prove this result, Chang and Yu use the connection between Anderson $\mathbb{F}_q[\theta]$ -modules and t -motives as well as the powerful criterion for transcendence introduced by Anderson-Brownawell-Papanikolas in [4] and the criterion for algebraic independence developed by Papanikolas in [47]. This latter criterion, which we will also use in our present paper, states roughly that the dimension of the motivic Galois group of a t -motive is equal to the transcendence degree of its attached period matrix.

0.4. Goss’s zeta values on elliptic curves (the genus 1 case).

In a series of papers [34, 33, 36], Papanikolas and the first author carried out an extensive study to move from the projective line $\mathbb{P}^1/\mathbb{F}_q$ (the genus 0 case) to elliptic curves over \mathbb{F}_q (the genus 1 case).

We work with an elliptic curve X defined over \mathbb{F}_q with defining equation given in (1.1) equipped with a rational point $\infty \in X(\mathbb{F}_q)$. Then $A = \mathbb{F}_q[\theta, \eta]$ is the coordinate ring of X , where θ and η satisfy (1.1). We denote by $K = \mathbb{F}_q(\theta, \eta)$ its fraction field and by $H \subset K_\infty$ the Hilbert class field of A .

The class number $\text{Cl}(A)$ of A equals the number of rational points $X(\mathbb{F}_q)$ on the elliptic curve X , which also equals the degree of extension $[H : K]$; that is,

$$\text{Cl}(A) = |X(\mathbb{F}_q)| = [H : K].$$

For a prime ideal \mathfrak{p} of A of degree 1 corresponding to an \mathbb{F}_q -rational point on X , we let \mathfrak{p}^{-1} be the inverse fractional ideal of \mathfrak{p} and consider the sum

$$\zeta_A(\mathfrak{p}, n) = \sum_{\substack{a \in \mathfrak{p}^{-1}, \\ \text{sgn}(a)=1}} \frac{1}{a^n}, \quad n \in \mathbb{N}.$$

The sums $\zeta_A(\mathfrak{p}, n)$ where \mathfrak{p} runs through the set \mathcal{P} of prime ideals of A of degree 1 are the elementary blocks in the study of Goss’s zeta values on elliptic curves. When the extension E/K is trivial (i.e., $E = K$), the Goss zeta value $\zeta_A(n)$ can be expressed as a \overline{K} -linear combination of $\zeta_A(\mathfrak{p}, n)$. When $E = H$, the zeta value $\zeta_{O_H}(n)$ (which is a regulator in the sense of Taelman [6, 50]) can be written as a product of \overline{K} -linear combinations of $\zeta_A(\mathfrak{p}, n)$. This is done explicitly in §5.3–5.4.

Contrary to the $\mathbb{F}_q[\theta]$ -case, one main issue present in the higher genus case is that Goss’s zeta function is fundamentally analytic in nature; it has no explicit dependence on the arithmetic of Drinfeld modules. To overcome this problem, Anderson introduced the so-called Anderson zeta values $\zeta_\rho(b_i, n)$ (see (1.21) for a precise definition) indexed by a K -basis $\{b_i\}_{i=1}^m \in O_H$ of H . These zeta values are also \overline{K} -linear combinations of $\zeta_A(\mathfrak{p}, n)$, and thus they contain the same information as Goss’s zeta values. The crucial point is that Anderson’s zeta values are of arithmetic nature and intimately related to a canonical rank 1 sign normalized Drinfeld A -module ρ (see §1.3 for a summary).

In [36], Papanikolas and the first author developed an explicit theory of the above Drinfeld A -module ρ . They gave a new proof of Anderson’s celebrated log-algebraicity theorem on elliptic curves and proved that $\zeta_\rho(b_i, 1)$ can be realized as the logarithm of ρ evaluated at a prescribed algebraic point. In [34, 33], the first author introduced the tensor powers $\rho^{\otimes n}$ for $n \in \mathbb{N}$ and proved basic properties of Anderson modules $\rho^{\otimes n}$. Then he obtained a generalization of Anderson-Thakur’s theorem for small values $n < q$. By a completely different approach based on the notion of Stark units and Pellarin’s L -series, Anglès, Tavares Ribeiro and the second author [8] proved a generalization of Anderson-Thakur’s theorem for all $n \in \mathbb{N}$. It states that for any $n \in \mathbb{N}$, Anderson’s zeta values $\zeta_\rho(b_i, n)$ can be written as the last coordinate of the logarithm of $\rho^{\otimes n}$ evaluated at an algebraic point. ¹

In this paper, using the aforementioned works, we generalize the work of Chang and Yu [23] for the Carlitz zeta values and determine all algebraic relations among Anderson’s zeta values on elliptic curves.

Theorem A (Theorem 4.3). *Let $m \in \mathbb{N}$ and $\{b_1, \dots, b_h\}$ be a K -basis of H with $b_i \in B$. We consider the following set:*

$$\mathcal{A} = \{\pi_\rho\} \cup \{\zeta_\rho(b_i, n) : 1 \leq i \leq h, 1 \leq n \leq m \text{ such that } q - 1 \nmid n \text{ and } p \nmid n\},$$

where π_ρ is a generator of the period lattice attached to ρ . Then the elements of \mathcal{A} are algebraically independent over \overline{K} . We also classify all algebraic relations between such zeta values, and thus these algebraic independence results are the best possible in this setting.

¹In fact, this theorem holds for any general base ring A ; see [8].

As an application, we also determine all algebraic relations among Goss’s zeta values defined for function fields of elliptic curves (see also Theorem 5.3).

Theorem B (Corollary 5.4). *Let $m \in \mathbb{N}$ and L be an extension of K such that $L \subset H$. We consider the following set:*

$$\mathcal{G}_L = \{\pi_\rho\} \cup \{\zeta_{O_L}(n) : 1 \leq n \leq m \text{ such that } q - 1 \nmid n \text{ and } p \nmid n\}.$$

Then the elements of \mathcal{G}_L are algebraically independent over \overline{K} . We also classify all algebraic relations between such zeta values, and thus these algebraic independence results are the best possible in this setting.

Remark 0.1. We note that the conditions $q - 1 \nmid n$ and $p \nmid n$ are necessary, else one gets known relations, such as

$$\frac{\zeta_{O_L}(q - 1)}{\pi_\rho^{q-1}} \in \overline{K}, \quad \zeta_{O_L}(pn) = \zeta_{O_L}(n)^p.$$

We also prove algebraic independence of periods and logarithms of tensor powers of Drinfeld modules.

Theorem C (Theorem 3.13). *Suppose that $\mathbf{u}_1, \dots, \mathbf{u}_m \in \text{Mat}_{n \times 1}(\mathbb{C}_\infty)$ such that $\text{Exp}_\rho^{\otimes n}(\mathbf{u}_i) = \mathbf{v}_i \in \text{Mat}_{n \times 1}(\overline{K})$ and denote the j th entry of \mathbf{u}_i as $\mathbf{u}_{i,j}$. If $\pi_\rho^n, \mathbf{u}_{1,n}, \dots, \mathbf{u}_{m,n}$ are linearly independent over K , then they are algebraically independent over \overline{K} .*

Let us sketch our proof and highlight the advances beyond [23].

- Since we want to apply the transcendence method of Papanikolas [47] (see Section 1.6 for a summary), we will consider the $\mathbb{F}_q[t]$ -modules induced by tensor powers of Drinfeld modules, still denoted by $\rho^{\otimes n}$ (see Section 1).
- In Section 2, we construct t -motives attached to $\rho^{\otimes n}$ and give a description of their motivic Galois groups. We use this description later in the paper when applying Hardouin’s work [37] (see Section 1.7 for a summary) to extensions of these motives to calculate the dimension of their motivic Galois groups.
- In Sections 3.1 and 3.2, we construct t -motives attached to logarithms of $\rho^{\otimes n}$. Our construction uses Anderson’s generating functions as in [18] instead of polylogarithms used by Chang and Yu. This allows us to bypass the convergence issues of polylogarithms present in [23].
- In Sections 3.3 and 3.4, we present two different ways to compute periods: either by direct calculations or by using a more conceptual method due to Anderson (see [38], Section 2.5).
- In Section 3.6, we compute explicitly the Galois groups of t -motives attached to logarithms and derive an application about algebraic independence of logarithms (see Theorem 3.13). Our calculations are completely different from all aforementioned works and based on a more robust method devised by Hardouin [37] (compare our methods with [23, 18] which do not use Hardouin’s work at all, or [19] which uses her work only minimally).
- In Section 4, we use a generalization of Anderson-Thakur’s theorem on elliptic curves (see Theorem 1.8) to construct a single t -motive which is simultaneously attached to all of the Anderson zeta values we consider. Using results from Section 3, we apply the strategy of Chang-Yu to determine all algebraic relations among Anderson’s zeta values (see Theorems 4.2 and 4.3).
- In Section 5, we derive all algebraic relations among Goss’s zeta values from those among Anderson’s zeta values (see Theorem 5.3 and Corollary 5.4).

To summarize, we have solved completely the problem of determining all algebraic relations among Goss’s zeta values for function fields of elliptic curves. Although in this project we work on elliptic curves and occasionally make use of their group law, we have strived to use a general approach which

relies on such explicit calculations as little as possible. The authors are currently working to extend these ideas to curves of higher genus, where such a group law no longer exists. Some of the main difficulties to be overcome in this case are developing a theory of Anderson generating functions and proving the simplicity of the tensor powers of Drinfeld modules.

1. Background

Traditionally, proofs in transcendental number theory tend to be quite eclectic; they pull from numerous disparate areas of mathematics. Such is the case in this paper. To ease the burden on the reader, we collect here a review of the various theories on which the proofs of our main theorems rely. This review is not intended to be exhaustive, and we refer the reader to various sources listed in each section. After laying out the general notation (Section 1.1), we give a review of Anderson \mathbf{A} -modules (Section 1.2), tensor powers of sign-normalized rank 1 Drinfeld-Hayes modules (Section 1.3), Anderson-Thakur’s theorem on zeta values and logarithms (Section 1.4), linear independence of Anderson’s zeta values (Section 1.5), Papanikolas’s theory on Tannakian categories and motivic Galois groups (Section 1.6) and Hardouin’s theory on computing motivic Galois groups via the unipotent radical (Section 1.7).

1.1. Notation

1.1.1. Elliptic curves

We keep the notation of [34, 33, 36] and work on elliptic curves. Throughout this paper, let \mathbb{F}_q be a finite field of characteristic p , having q elements. Let X be an elliptic curve defined over \mathbb{F}_q given by

$$y^2 + c_1ty + c_3y = t^3 + c_2t^2 + c_4t + c_6, \quad c_i \in \mathbb{F}_q. \tag{1.1}$$

It is equipped with the rational point $\infty \in X(\mathbb{F}_q)$ at infinity, which we designate as the neutral element for the group law on X . We set $\mathbf{A} = \mathbb{F}_q[t, y]$ the affine coordinate ring of X , which is the set of functions on X regular outside ∞ and $\mathbf{K} = \mathbb{F}_q(t, y)$, its fraction field. We also fix other variables θ, η so that $A = \mathbb{F}_q[\theta, \eta]$ and $K = \mathbb{F}_q(\theta, \eta)$ are isomorphic to \mathbf{A} and \mathbf{K} . We denote the canonical isomorphism $\iota : \mathbf{K} \rightarrow K$ such that $\iota(t) = \theta$ and $\iota(y) = \eta$. Let $\lambda = \frac{dt}{2y+c_1t+c_3}$ be the invariant differential on X .

The ∞ -adic completion K_∞ of K is equipped with the normalized ∞ -adic valuation $v_\infty : K_\infty \rightarrow \mathbb{Z} \cup \{+\infty\}$ and has residue field \mathbb{F}_q . We set $\deg := -v_\infty$ so that $\deg \theta = 2$ and $\deg \eta = 3$. The completion \mathbb{C}_∞ of a fixed algebraic closure \overline{K}_∞ of K_∞ comes with a unique valuation extending v_∞ , which we also denote by v_∞ . We define the Frobenius $\tau : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ as the \mathbb{F}_q -algebra homomorphism which sends x to x^q . Similarly, we can define \mathbf{K}_∞ equipped with v_∞ and \deg .

We set $\Xi = (\theta, \eta)$ which is a K -rational point of the elliptic curve X . We define a sign function $\mathbf{sgn} : \mathbf{A} \setminus \{0\} \rightarrow \mathbb{F}_q^\times$ as follows. For any $a \in \mathbf{A} \setminus \{0\}$, there is a unique way to write

$$a = \sum_{i \geq 0} a_i t^i + \sum_{i \geq 0} b_i t^i y, \quad a_i, b_i \in \mathbb{F}_q.$$

Recall that $\deg t = 2$ and $\deg y = 3$. The sign of a is defined to be the coefficient of the term of highest degree. It is easy to see that it extends to a group homomorphism

$$\mathbf{sgn} : \mathbf{K}_\infty^\times \rightarrow \mathbb{F}_q^\times.$$

Similarly, we can define the sign function

$$\mathbf{sgn} : K_\infty^\times \rightarrow \mathbb{F}_q^\times.$$

For any field extension L/\mathbb{F}_q , the coordinate ring of X over L is $L[t, y] = L \otimes_{\mathbb{F}_q} \mathbf{A}$. We extend the sign function to such rings $L[t, y] \setminus \{0\}$ by using the same notion of leading term, namely, by writing

$$a = \sum_{i \geq 0} a_i t^i + \sum_{i \geq 0} b_i t^i y, \quad a_i, b_i \in L,$$

and then defining $\widetilde{\text{sgn}} : L[t, y] \setminus \{0\} \rightarrow L^\times$ to be the coefficient of the leading term. This extends naturally to $L(t, y)^\times$ by taking quotients.

1.1.2. Tate algebras

We denote by \mathbb{T} the Tate algebra in the variable t with coefficients in \mathbb{C}_∞ ,

$$\mathbb{T} = \left\{ \sum_{i=0}^{\infty} b_i t^i \in \mathbb{C}_\infty[[t]] \mid |b_i|_\infty \rightarrow 0 \right\}, \tag{1.2}$$

where $|\cdot|_\infty$ is the norm on \mathbb{C}_∞ given by $|g|_\infty = q^{\deg(g)}$. The Gauss norm on \mathbb{T} is given by

$$\|f\| := \max_i \{|b_i|_\infty\}$$

for $f = \sum_{i \geq 0} b_i t^i \in \mathbb{T}$. Let \mathbb{L} be its fraction field.

Further, we define the Tate algebra \mathbb{T}_θ as the space of power series in t with coefficients in \mathbb{C}_∞ on the disc of radius $|\theta|_\infty$,

$$\mathbb{T}_\theta = \left\{ \sum_{i=0}^{\infty} b_i t^i \in \mathbb{C}_\infty[[t]] \mid q^i |b_i|_\infty \rightarrow 0 \right\}. \tag{1.3}$$

Similarly, we define the norm $\|\cdot\|_\theta$ on \mathbb{T}_θ : if $f = \sum_{i=0}^{\infty} b_i t^i \in \mathbb{T}_\theta$, then

$$\|f\|_\theta = \max_i \{q^i |b_i|_\infty\}.$$

We note that $\mathbb{T}_\theta \subset \mathbb{T}$.

1.2. Anderson \mathbf{A} -modules on elliptic curves

We briefly review the basic theory of Anderson \mathbf{A} -modules and dual \mathbf{A} -motives and the relation between them. This material follows closely to [34, §3-4], and the reader is directed there for proofs.

For R an \mathbb{F}_p -algebra, we let $R[\tau]$ denote the (non-commutative) skew-polynomial ring with coefficients in R , subject to the relation for $r \in R$,

$$\tau r = r^q \tau.$$

We similarly define $R[\sigma]$, but subject to the restriction that R must be an algebraically closed field and subject to the relation

$$\sigma r = r^{1/q} \sigma.$$

We define the i th Frobenius twisting automorphism of $\mathbb{C}_\infty[t, y]$ by

$$g \mapsto g^{(i)} : \sum_{i,j} c_{ij} t^i y^j \mapsto \sum_{i,j} c_{ij}^{q^i} t^i y^j.$$

We extend twisting to matrices $\text{Mat}_{i \times j}(\mathbb{C}_\infty[t, y])$ by twisting coordinatewise. We also define Frobenius twisting on points $P \in X(\mathbb{C}_\infty)$, also denoted by $P^{(i)}$, by raising each coordinate to the q^i power. We extend this to divisors on X in the natural way.

Definition 1.1. 1) An n -dimensional Anderson \mathbf{A} -module is an \mathbb{F}_q -algebra homomorphism $E : \mathbf{A} \rightarrow \text{Mat}_n(\overline{K_\infty})[\tau]$, such that for each $a \in \mathbf{A}$,

$$E_a = d[a] + A_1\tau + \dots, \quad A_i \in \text{Mat}_n(\overline{K_\infty}),$$

where $d[a] = \iota(a)\text{Id}_n + N$ for some nilpotent matrix $N \in \text{Mat}_n(\overline{K_\infty})$ (depending on a).

2) A Drinfeld module is a nontrivial one-dimensional Anderson \mathbf{A} -module $\rho : \mathbf{A} \rightarrow \overline{K_\infty}[\tau]$.

We note that the map $a \mapsto d[a]$ is a ring homomorphism.

Let E be an \mathbf{A} -Anderson module of dimension n . We introduce the exponential and logarithm functions attached to E , denoted Exp_E and Log_E , respectively. The exponential function is the unique function on \mathbb{C}_∞^n such that for all $a \in \mathbf{A}$ and $\mathbf{z} \in \mathbb{C}_\infty^n$,

$$\text{Exp}_E(d[a]\mathbf{z}) = E_a(\text{Exp}_E(\mathbf{z})) \tag{1.4}$$

and $\text{Exp}_E(\mathbf{0}) = \text{Id}_n$.

The function Log_E is then defined as the formal power series inverse of Exp_E . We note that as functions on \mathbb{C}_∞^n , the function Exp_E is everywhere convergent, whereas Log_E has a finite domain of convergence.

We briefly set out some notation regarding points and divisors on the elliptic curve X . We will denote addition of points using the group law of X by adding the points without parenthesis. For example, for $R_1, R_2 \in X$,

$$R_1 + R_2 \in X,$$

and we will denote formal sums of divisors involving points on X using the points inside parenthesis. For example, for $g \in K(t, y)$,

$$\text{div}(g) = (R_1) - (R_2).$$

Further, multiplication on the curve X will be denoted with square brackets. For example,

$$[2]R_1 \in X,$$

whereas formal multiplication of points in a divisor will be denoted with simply a number where possible, or by an expression inside parenthesis; for example, for $h \in K(t, y)$,

$$\text{div}(h) = 3(R_1) - (n + 2)(R_2).$$

1.3. Tensor powers of Drinfeld-Hayes modules on elliptic curves

We now construct the canonical rank 1 sign-normalized Drinfeld module associated to the ring \mathbf{A} to which we will attach zeta values (see [36] for a detailed account). For curves of general genus, we refer the interested reader to Hayes’s work [39, 40] (see also [3, 6, 52] or [32, §7]) for more details on sign-normalized rank one Drinfeld modules.

For the sign function defined in §1.1, a rank 1 sign-normalized Drinfeld module is a Drinfeld module $\rho : \mathbf{A} \rightarrow \overline{K_\infty}[\tau]$ such that for $a \in \mathbf{A}$, we have

$$\rho_a = \iota(a) + a_1\tau + \dots + \mathbf{sgn}(a)\tau^{\deg(a)}.$$

By Drinfeld’s seminal work [27], there exists a unique effective divisor V on X such that the divisor $V^{(1)} - V + (\Xi) - (\infty)$ is principal. In our setting of elliptic curves, the situation is much more concrete.

Recall that $H \subset K_\infty$ is the Hilbert class field of A . Then the Drinfeld divisor is the unique point $V \in X(H)$ whose coordinates have positive degree that verifies the equation on X

$$V - V^{(1)} = \Xi.$$

We remind the reader that a divisor $\sum n_P P$ on X is principal if and only if $\sum n_P P = \infty$ on X and $\sum n_P = 0$ (see [49, Cor. III.3.5]). Thus, in our situation, we conclude that the divisor $(\Xi) + (V^{(1)}) - (V) - (\infty)$ is principal, and we denote the function with that divisor $f \in H(t, y)$ (recall H is the Hilbert class field of A), normalized so that $\widehat{\text{sgn}}(f) = 1$. We call this the shtuka function associated to X ; that is,

$$\text{div}(f) = (\Xi) + (V^{(1)}) - (V) - (\infty). \tag{1.5}$$

We will denote the denominator and numerator of the shtuka function as

$$f := \frac{\nu(t, y)}{\delta(t)} := \frac{y - \eta - m(t - \theta)}{t - \alpha}, \tag{1.6}$$

where $m \in H$ is the slope on X (in the sense of [49, III.2.3]) between the collinear points $V^{(1)}, -V$ and Ξ . From [36, (19)-(20)], we get $\text{deg}(m) = q$, and

$$\text{div}(\nu) = (V^{(1)}) + (-V) + (\Xi) - 3(\infty), \quad \text{div}(\delta) = (V) + (-V) - 2(\infty). \tag{1.7}$$

Definition 1.2. 1) An abelian \mathbf{A} -motive is a $\overline{K}[t, y, \tau]$ -module M which is a finitely generated projective $\overline{K}[t, y]$ -module and free finitely generated $\overline{K}[\tau]$ -module such that for $\ell \gg 0$, we have

$$(t - \theta)^\ell (M/\tau M) = \{0\}, \quad (y - \eta)^\ell (M/\tau M) = \{0\}.$$

2) An \mathbf{A} -finite dual \mathbf{A} -motive is a $\overline{K}[t, y, \sigma]$ -module N which is a finitely generated projective $\overline{K}[t, y]$ -module and free finitely generated $\overline{K}[\sigma]$ -module such that for $\ell \gg 0$, we have

$$(t - \theta)^\ell (N/\sigma N) = \{0\}, \quad (y - \eta)^\ell (N/\sigma N) = \{0\}.$$

Note that our definitions here are in line with [15, §1.5.4], rather than the more general definition given in [38, Def. 2.4.1].

We then let $U = \text{Spec } \overline{K}[t, y]$ (i.e., the affine curve $(\overline{K} \times_{\mathbb{F}_q} X) \setminus \{\infty\}$). For a divisor D on the curve X , we let $\mathcal{L}(D)$ be the \overline{K} -vector space of rational functions g on X with $\text{div}(g) \geq -D$. The (geometric) \mathbf{A} -motive associated to X is given by

$$M_1 = \Gamma(U, \mathcal{O}_X(V)) = \bigcup_{i \geq 0} \mathcal{L}((V) + i(\infty)).$$

We make M_1 into a left $\overline{K}[t, y, \tau]$ -module by letting τ act by

$$\tau g = f g^{(1)}, \quad g \in M_1,$$

and letting $\overline{K}[t, y]$ act by left multiplication.

The (geometric) dual \mathbf{A} -motive associated to X is given by

$$N_1 = \Gamma(U, \mathcal{O}_X(-(V^{(1)}))) = \bigcup_{i \geq 1} \mathcal{L}(-(V^{(1)}) + i(\infty)) \subseteq \overline{K}[t, y]. \tag{1.8}$$

We make N_1 into a left $\overline{K}[t, y, \sigma]$ -module by letting σ act by

$$\sigma g = fg^{(-1)}, \quad g \in N_1,$$

and letting $\overline{K}[t, y]$ act by left multiplication.

We find that M_1 and N_1 are projective $\overline{K}[t, y]$ -module of rank 1, that M_1 is as a free $\overline{K}[\tau]$ -module of rank 1 and that N_1 is as a free $\overline{K}[\sigma]$ -module of rank 1 (see [36, §3] for proofs of these facts). A quick check shows that M_1 (resp. N_1) is indeed an abelian \mathbf{A} -motive (resp. \mathbf{A} -finite dual \mathbf{A} -motive).

We now follow as in [34, §3] and form the n th tensor power of M_1 and of N_1 and denote these as

$$M_n = M_1^{\otimes n} = M_1 \otimes_{\overline{K}[t,y]} \cdots \otimes_{\overline{K}[t,y]} M_1,$$

$$N_n = N_1^{\otimes n} = N_1 \otimes_{\overline{K}[t,y]} \cdots \otimes_{\overline{K}[t,y]} N_1,$$

with τ and σ action on $a \in M_n$ and $b \in N_n$ given respectively by

$$\tau a = f^n b^{(1)}, \quad \sigma b = f^n b^{(-1)}.$$

Observe that

$$M_n = \Gamma(U, \mathcal{O}_X(nV)), \quad N_n \cong \Gamma(U, \mathcal{O}_X(-nV^{(1)}))$$

and that M_n (resp. N_n) is also an \mathbf{A} -motive (resp. a dual \mathbf{A} -motive). Again, M_n and N_n are projective $\overline{K}[t, y]$ -modules of rank 1. Further, M_n is a free $\overline{K}[\tau]$ -module of rank n , and N_n is a free $\overline{K}[\sigma]$ -module of rank n .

We write down convenient bases for M_n and N_n as free $\overline{K}[\tau]$ - and $\overline{K}[\sigma]$ -modules, respectively (see [34, Prop. 3.3]). Define functions $g_i \in M_n$ for $1 \leq i \leq n$ with $\overline{\text{sgn}}(g_i) = 1$ and with divisors

$$\text{div}(g_j) = -n(V) + (n - j)(\infty) + (j - 1)(\Xi) + ([j - 1]V^{(1)} + [n - (j - 1)]V), \tag{1.9}$$

and similarly define functions $h_i \in N_n$ for $1 \leq i \leq n$, each with $\overline{\text{sgn}}(h_i) = 1$ and with divisor

$$\text{div}(h_j) = n(V^{(1)}) - (n + j)(\infty) + (j - 1)(\Xi) + (-[n - (j - 1)]V^{(1)} - [j - 1]V). \tag{1.10}$$

Then we have

$$M_n = \overline{K}[\tau]\{g_1, \dots, g_n\}, \quad N_n = \overline{K}[\sigma]\{h_1, \dots, h_n\}.$$

For $g \in N_n$, we set $m = \lfloor \text{deg}(g)/n \rfloor$ and define two maps

$$\delta_0, \delta_1 : N_n \rightarrow \overline{K}^n$$

in the following way. We write g in the $\overline{K}[\sigma]$ -basis for N_n described in (1.10),

$$g = \sum_{i=0}^m \sum_{j=1}^n b_{j,i}^{(-i)} \sigma^i h_{n-j+1}, \tag{1.11}$$

then denote $\mathbf{b}_i = (b_{1,i}, b_{2,i}, \dots, b_{n,i})^\top$, and set

$$\delta_0(g) = \mathbf{b}_0, \quad \delta_1(g) = \mathbf{b}_0 + \mathbf{b}_1 + \cdots + \mathbf{b}_m. \tag{1.12}$$

We then observe that the kernel of δ_1 equals $(\sigma - 1)N_n$ and that $N_n/(\sigma - 1)N_n \xrightarrow{\delta_1} \overline{K}^n$ is an isomorphism of \mathbb{F}_q -vector spaces. Thus, we can write the commutative diagram of \mathbb{F}_q -vector spaces

$$\begin{array}{ccc}
 N_n/(1-\sigma)N_n & \xrightarrow{\delta_1} & \overline{K}^n \\
 a \downarrow & & \downarrow \rho_a^{\otimes n} \\
 N_n/(1-\sigma)N_n & \xrightarrow{\delta_1} & \overline{K}^n,
 \end{array} \tag{1.13}$$

where the left vertical arrow is multiplication by a and the right vertical arrow is the map induced by multiplication by a , which we denote by $\rho_a^{\otimes n}$.

Definition 1.3. By [38, Prop. 2.5.8], we know that $\rho^{\otimes n}$ induces the structure of an \mathbf{A} -module on \overline{K}^n and that it satisfies the conditions of being an Anderson \mathbf{A} -module. In this way, to each curve X , for fixed n , we associate a canonical n -dimensional Anderson \mathbf{A} -module. We call $\rho := \rho^{\otimes 1}$ the canonical sign-normalized rank 1 Drinfeld module associated to X , and we call $\rho^{\otimes n}$ the n th tensor power of ρ .

Proposition 1.4. We recall the following two facts about the functions g_i and h_i from [34, §4].

- For $1 \leq i \leq n$, there exist constants $a_i, b_i \in H$ such that we can write

$$\begin{aligned}
 tg_i &= \theta g_i + a_i g_{i+1} + g_{i+2}, \\
 th_i &= \theta h_i + b_i h_{i+1} + h_{i+2}.
 \end{aligned}$$

- For the constants defined in (1), we have $a_j = b_{n-j}$ for $1 \leq j \leq n-1$ and $a_n = b_n^q$.

We can write down the matrices defining $\rho_t^{\otimes n}$ using the coefficients $a_i \in H$ from Proposition 1.4, for $n \geq 2$:

$$\rho_t^{\otimes n} := d[\theta] + E_\theta \tau := \begin{pmatrix} \theta & a_1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & \theta & a_2 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & \theta & a_3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \theta & a_{n-2} & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & \theta & a_{n-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \theta \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ a_n & 1 & 0 & \dots & 0 \end{pmatrix} \tau. \tag{1.14}$$

The t -action of the Drinfeld \mathbf{A} -module ρ is given by

$$\rho_t = \theta + x_1 \tau + \tau^2, \quad x_1 \in B,$$

(see [36, §3] for more details on this construction).

Remark 1.5. We comment by way of clarification for the reader that Formula (1.14) does reduce down to give the Drinfeld module ρ in the case of $n = 1$, but it is not intuitive how to interpret these formulas. For example, the 1's on the super-super diagonal turn into τ^2 , which is not obvious just from the formulas. For the sake of clarity, we will often state our formulas separately for the $n = 1$ and the $n \geq 2$ case in this paper. A discussion of the relationship between these cases is given in Remark 1.1 of [34]. Additionally, the case of $n = 1$ is treated exhaustively in [36], and we refer the reader to these two sources for further discussion.

The logarithm and exponential functions associated to $\rho^{\otimes n}$ will be denoted $\text{Log}_\rho^{\otimes n}$ and $\text{Exp}_\rho^{\otimes n}$, respectively, and the kernel of $\text{Exp}_\rho^{\otimes n}$ will be denoted by $\Lambda_\rho^{\otimes n}$, which we call the period lattice of $\rho^{\otimes n}$.

We recall the Tate algebra \mathbb{T} in the variable t with coefficients in \mathbb{C}_∞ as in §1.1.2,

$$\mathbb{T} = \left\{ \sum_{i=0}^{\infty} b_i t^i \in \mathbb{C}_\infty[[t]] \mid |b_i|_\infty \rightarrow 0 \right\},$$

where $|\cdot|_\infty$ is the norm on \mathbb{C}_∞ given by $|g|_\infty = q^{\deg(g)}$ and its fraction field \mathbb{L} . We now give a brief review of the functions ω_ρ , $E_{\mathbf{u}}^{\otimes n}$ and $G_{\mathbf{u}}^{\otimes n}$ defined in [34, §5-6]. Let

$$\omega_\rho = \xi^{1/(q-1)} \prod_{i=0}^{\infty} \frac{\xi q^i}{f^{(i)}} \in \mathbb{T}[y]^\times, \tag{1.15}$$

where $f \in H(t, y)$ is the shtuka function defined above, and we refer the reader to [36, Thm. 4.6] for the definition of $\xi \in H$ and for details on convergence. Observe that ω_ρ satisfies the functional equation $\omega_\rho^{(1)} = f\omega_\rho$. For $\mathbf{u} = (u_1, \dots, u_n)^\top \in \mathbb{C}_\infty^n$, define

$$E_{\mathbf{u}}^{\otimes n}(t) = \sum_{i=0}^{\infty} \text{Exp}_\rho^{\otimes n} \left(d[\theta]^{-i-1} \mathbf{u} \right) t^i, \tag{1.16}$$

$$G_{\mathbf{u}}^{\otimes n}(t, y) = E_{d[\eta]\mathbf{u}}^{\otimes n}(t) + (y + c_1 t + c_3) E_{\mathbf{u}}^{\otimes n}(t). \tag{1.17}$$

For $n = 1$ and $\mathbf{u} = u \in \mathbb{C}_\infty$, we will simplify notation by setting $E_u(t) := E_{\mathbf{u}}^{\otimes 1}(t)$ and $G_u(t, y) := G_{\mathbf{u}}^{\otimes 1}(t, y)$.

Define \mathcal{M} to be the submodule of $\mathbb{T}[y]$ consisting of all elements in $\mathbb{T}[y]$ which have a meromorphic continuation to all of $U = \text{Spec } \bar{K}[t, y]$ in the sense of [29, §4.6] (we comment that the sheaf of meromorphic functions on a rigid space is a sheaf which locally looks like \mathbb{L} , and its affine pieces can be glued together coherently). Now define the map $\text{RES}_\Xi : \mathcal{M}^n \rightarrow \mathbb{C}_\infty^n$ for a vector of functions $(z_1, \dots, z_n)^\top \in \mathcal{M}^n$ as

$$\text{RES}_\Xi((z_1, \dots, z_n)^\top) = (\text{Res}_\Xi(z_1 \lambda), \dots, \text{Res}_\Xi(z_n \lambda))^\top, \tag{1.18}$$

where λ is the invariant differential on E (defined in §1.1).

We define a map $T : \mathbb{T}[y] \rightarrow \mathbb{T}[y]^n$ by

$$T(h(t, y)) = \begin{pmatrix} h(t, y) \cdot g_1 \\ h(t, y) \cdot g_2 \\ \vdots \\ h(t, y) \cdot g_n \end{pmatrix}. \tag{1.19}$$

We collect the following facts from [34, §5-6] about the above functions.

Proposition 1.6. *We have the following properties:*

- (a) *The function $E_{\mathbf{u}}^{\otimes n}$ belongs to \mathbb{T}^n .*
- (b) *The function $G_{\mathbf{u}}^{\otimes n}$ belongs to $\mathbb{T}[y]^n$ and extends to a meromorphic function on $(\mathbb{C}_\infty \times_{\mathbb{F}_q} X) \setminus \{\infty\}$ with poles in each coordinate only at the points $\Xi^{(i)}$ for $i \geq 0$.*
- (c) *We have $\text{RES}_\Xi(G_{\mathbf{u}}^{\otimes n}) = -(u_1, \dots, u_n)^\top$.*
- (d) *If we denote $\Pi_n = -\text{RES}_\Xi(T(\omega_\rho^n))$, then $T(\omega_\rho^n) = G_{\Pi_n}^{\otimes n}$, and the period lattice of $\text{Exp}_\rho^{\otimes n}$ equals $\Lambda_\rho^{\otimes n} = \{d[a]\Pi_n \mid a \in A\}$.*
- (e) *If π_ρ is a fundamental period of the exponential function associated to ρ , and if we denote the last coordinate of $\Pi_n \in \mathbb{C}_\infty^n$ by p_n , then $p_n/\pi_\rho^n \in H \setminus \{0\}$.*

1.4. A generalization of Anderson-Thakur’s theorem on elliptic curves

Recall that $\rho : \mathbf{A} \rightarrow \mathbb{C}_\infty\{\tau\}$ is the canonical sign-normalized rank one Drinfeld module associated to the elliptic curve X constructed in the previous section and that H is the Hilbert class field of A . Let B (or \mathcal{O}_H) be the integral closure of A in H . We denote by G the Galois group $\text{Gal}(H/K)$.

We denote by $\mathcal{I}(A)$ the group of fractional ideals of A . For $I \in \mathcal{I}(A)$, denote its Artin symbol by

$$\sigma_I := (I, H/K) \in G. \tag{1.20}$$

By [32], Proposition 7.4.2 and Corollary 7.4.9, the subfield of \mathbb{C}_∞ generated by K and the coefficients of ρ_a is H . Furthermore, by [32], Lemma 7.4.5, we get

$$\forall a \in A, \quad \rho_a \in B\{\tau\}.$$

Let I be a nonzero ideal of A . We define ρ_I to be the monic element in $H\{\tau\}$ such that

$$H\{\tau\}\rho_I = \sum_{a \in I} H\{\tau\}\rho_a.$$

We have

$$\ker \rho_I = \bigcap_{a \in I} \ker \rho_a,$$

$$\rho_I \in B\{\tau\},$$

$$\deg_\tau \rho_I = \deg I.$$

We write $\rho_I = \rho_{I,0} + \dots + \rho_{I,\deg I} \tau^{\deg I}$ with $\rho_{I,\deg I} = 1$ and denote by $\psi(I) \in B \setminus \{0\}$ the constant coefficient $\rho_{I,0}$ of ρ_I . Thus, the map ψ extends uniquely into a map $\psi : \mathcal{I}(A) \rightarrow H^\times$ with the following properties (proved in [39, Prop. 3.2 and Thm. 8.5]):

- 1) for all $I, J \in \mathcal{I}(A), \psi(IJ) = \sigma_J(\psi(I))\psi(J)$,
- 2) for all $I \in \mathcal{I}(A), IB = \psi(I)B$,
- 3) for all $x \in K^\times, \psi(xA) = \frac{x}{\text{sgn}(x)}$.

Finally, for $n \in \mathbb{N}$ and $b \in B$, we define Anderson’s zeta value at n attached to ρ as follows:

$$\zeta_\rho(b, n) = \sum_{I \subseteq A} \frac{\sigma_I(b)}{\psi(I)^n} \in K_\infty. \tag{1.21}$$

By the work of Anderson (see [2], [3]), for any $b \in B$, we have

$$\exp_\rho(\zeta_\rho(b, 1)) \in B. \tag{1.22}$$

Remark 1.7. This is an example of log-algebraicity identities for Drinfeld modules. The theory began with the work of Carlitz [16] where he proved the log-algebraicity identities for the Carlitz module defined over $\mathbb{F}_q[\theta]$. Further examples for Drinfeld modules over A which are PIDs were discovered by Thakur [51, Thm. VI]. Shortly after, Anderson proved that (1.22) holds for any sign-normalized rank one Drinfeld A -module, and this is known as Anderson’s log-algebraicity theorem. For alternative proofs of this theorem, we refer the reader to [52, §8] for the $\mathbb{F}_q[\theta]$ -case, [36] for the case of elliptic curves and [6] for the general case.

The following theorem is a generalization of the celebrated Anderson-Thakur theorem for tensor powers of the Carlitz module (see [5], Theorem 3.8.3).

Theorem 1.8 (Anglès-Ngo Dac-Tavares Ribeiro [8] for any n and Green [33] for $n < q$). *Let $n \geq 1$ be an integer. Then there exists a constant $C_n \in H$ such that for $b \in B$, there exists a vector $Z_n(b) \in \mathbb{C}_\infty^n$ verifying the following properties:*

- 1) We have $\text{Exp}_\rho^{\otimes n}(Z_n(b)) \in H^n$.
- 2) The last coordinate of $Z_n(b)$ is equal to $C_n \zeta_\rho(b, n)$.

1.5. Linear relations among Anderson’s zeta values

In this section, we determine completely linear relations among Anderson’s zeta values and powers of π_ρ associated to an elliptic curve over \mathbb{F}_p . In the genus 0 case, this was done by Yu (see [55], Theorem 3.1 and [56], Theorem 4.1). His works are built on two main ingredients. The first one is Yu’s theory where he developed an analogue of Wüstholz’s analytic subgroup theorem for function fields, and the second one is the Anderson-Thakur theorem mentioned in the previous section. The main result of this section extends Yu’s work to elliptic curves.

Recall that $\mathbf{A} = \mathbb{F}_q[t, y]$, where t and y satisfy the Weierstrass equation (1.1). Following Green (see [33], Section 7), we still denote by $\rho : \mathbb{F}_q[t] \rightarrow \mathbb{C}_\infty\{\tau\}$ the Drinfeld $\mathbb{F}_q[t]$ -module induced by forgetting the y -action of the sign-normalized rank 1 Drinfeld module ρ of the previous sections. Similarly, we denote by $\rho^{\otimes n} : \mathbb{F}_q[t] \rightarrow \text{Mat}_n(\mathbb{C}_\infty)\{\tau\}$ the Anderson $\mathbb{F}_q[t]$ -module defined by forgetting the y -action. Basic properties of this Anderson module are given below.

Proposition 1.9 (Green [33], Lemmas 7.2 and 7.3).

- 1) The Anderson $\mathbb{F}_q[t]$ -module $\rho^{\otimes n} : \mathbb{F}_q[t] \rightarrow \text{Mat}_n(\mathbb{C}_\infty)\{\tau\}$ is simple in the sense of Yu (see [55, 56]).
- 2) The Anderson $\mathbb{F}_q[t]$ -module $\rho^{\otimes n} : \mathbb{F}_q[t] \rightarrow \text{Mat}_n(\mathbb{C}_\infty)\{\tau\}$ has endomorphism algebra equal to A .

We slightly generalize [33], Theorem 7.1 to obtain the following theorem which settles the problem of determining linear relations among Anderson’s zeta values and periods attached to ρ , which generalizes the work of Yu.

Theorem 1.10. *Let $\{b_1, \dots, b_h\}$ be a K -basis of H with $b_i \in B$. We consider the following sets for $m, s \geq 1$:*

$$\mathcal{R} := \{\pi_\rho^k, 0 \leq k \leq m\} \cup \{\zeta_\rho(b_i, n) : 1 \leq i \leq h, 1 \leq n \leq s \text{ such that } q - 1 \nmid n\},$$

$$\mathcal{R}' := \{\pi_\rho^k, 0 \leq k \leq m\} \cup \{\zeta_\rho(b_i, n) : 1 \leq i \leq h, 1 \leq n \leq s\}.$$

Then

- 1) The \overline{K} -vector space generated by the elements in \mathcal{R} and that generated by those in \mathcal{R}' are the same.
- 2) The elements in \mathcal{R} are linearly independent over \overline{K} .

Proof. The proof follows the same lines as Yu’s celebrated theorem [56, Th.m 4.1] (see also [33, Thm. 7.1]). We provide a proof for the convenience of the reader.

Recall that for $n \in \mathbb{N}$, the Anderson \mathbf{A} -module $\rho^{\otimes n}$ induces the structure of an $\mathbb{F}_q[t]$ -module which, by abuse of notation, we also denote by $\rho^{\otimes n}$. Also recall that h is the class number of A . We consider the product of t -modules

$$G = \mathbb{G}_a \times \left(\prod_{k=1}^m \rho^{\otimes k} \right) \times \left(\prod_{\substack{n=1 \\ q-1 \nmid n}}^s (\rho^{\otimes n})^{\oplus h} \right),$$

where we view \mathbb{G}_a in the first coordinate as a trivial t -module with the scalar A -action and with exponential function $\exp_{G_L}(z) = z$.

For $1 \leq n \leq s$, set $Z_n(b_i) = (*, \dots, *, C_n \zeta_\rho(b_i, n))^T \in \mathbb{C}_\infty^n$ to be a vector from Theorem 1.8 such that $\text{Exp}_\rho^{\otimes n}(Z_n(b_i)) \in H^n$. For $1 \leq k \leq m$, let $\Pi_k \in \mathbb{C}_\infty^k$ be a fundamental period of $\text{Exp}_\rho^{\otimes k}$ so that the bottom coordinate of Π_k is a multiple of π_ρ^k by a nonzero element of H (see [34, Thm. 6.7] for the exact multiple). Define the vector

$$\mathbf{u} = 1 \times \left(\prod_{k=1}^m \Pi_k \right) \times \left(\prod_{\substack{n=1 \\ q-1 \nmid n}}^s \prod_{i=1}^h Z_n(b_i) \right) \in G(\mathbb{C}_\infty)$$

and note $\text{Exp}_G(\mathbf{u}) \in G(H)$, where Exp_G is the exponential function on G , defined coordinatewise. Suppose, by contradiction, that there is a \bar{K} -linear relation among the $\zeta_\rho(b_i, n)$ and π_ρ^k . This implies that \mathbf{u} is contained in a $d[\mathbb{F}_q[t]]$ -invariant hyperplane of $G(\mathbb{C}_\infty)$ defined over \bar{K} . This allows us to apply [56, Thm. 3.3], which says that \mathbf{u} is in $\text{Lie}_{G'}(\bar{K})$ for a proper t -submodule $G' \subset G$. Then Proposition 1.9 together with [56, Thm. 1.3] implies that there exist $1 \leq n \leq s$ with $q - 1 \nmid n$ and a linear relation of the form

$$\sum_{i=1}^h a_i \zeta_\rho(b_i, n) + b \pi_\rho^n = 0$$

for some $a_i, b \in A$ not all zero. Since $\zeta_\rho(b_i, n) \in K_\infty$ and since $H \subset K_\infty$, this implies that $b \pi_\rho^n \in K_\infty$. Since $q - 1 \nmid n$, we know that $\pi_\rho^n \notin K_\infty$. It follows that $b = 0$ and hence $\sum_{i=1}^h a_i \zeta_\rho(b_i, n) = 0$. Since $a_i \in A$, we get

$$0 = \sum_{i=1}^h a_i \zeta_\rho(b_i, n) = \zeta_\rho \left(\sum_{i=1}^h a_i b_i, n \right).$$

We deduce that $\sum_{i=1}^h a_i b_i = 0$. Since $\{b_i\}$ is a K -basis of H , this forces $a_i = 0$ for all i . This provides a contradiction and proves the theorem. □

As explained by B. Anglès,² the following result is attributed to Carlitz and Goss (see [31, Thm. 3.2.2]) which improves [7], Theorem 5.7 and [33], Corollary 7.4.

Proposition 1.11. *Let $n \geq 1, n \equiv 0 \pmod{q-1}$ be an integer. Then for $b \in B$, we have $\zeta_\rho(b, n)/\pi_\rho^n \in K$.*

1.6. Papanikolas’s work

We review Papanikolas’ theory [47] (see also [1, 4]) and work with t -motives. Let $\bar{K}[t, \sigma]$ be the polynomial ring in variables t and σ with the rules

$$at = ta, \sigma t = t\sigma, \sigma a = a^{1/q} \sigma, \quad a \in \bar{K}.$$

Definition 1.12. An Anderson dual t -motive is a left $\bar{K}[t, \sigma]$ -module N which is free and finitely generated both as a left $\bar{K}[t]$ -module and as a left $\bar{K}[\sigma]$ -module and which satisfies

$$(t - \theta)^d N \subset \sigma N$$

for some integer d sufficiently large.

We consider $\bar{K}(t)[\sigma, \sigma^{-1}]$ the ring of Laurent polynomials in σ with coefficients in $\bar{K}(t)$.

Definition 1.13. A pre- t -motive is a left $\bar{K}(t)[\sigma, \sigma^{-1}]$ -module that is finite dimensional over $\bar{K}(t)$.

The category of pre- t -motives is abelian, and there is a natural functor from the category of Anderson dual t -motives to the category of pre- t -motives

$$N \mapsto M := \bar{K}(t) \otimes_{\bar{K}[t]} N,$$

where σ acts diagonally on M .

²Personal communication in July 2019.

We now consider pre- t -motives M which are rigid analytically trivial, which we describe here. Let $\{\mathbf{m}\} \in \text{Mat}_{r \times 1}(M)$ be a $\overline{K}(t)$ -basis of M and let $\Phi \in \text{GL}_r(\overline{K}[t])$ be the matrix representing the multiplication by σ on M :

$$\sigma(\mathbf{m}) = \Phi\mathbf{m}.$$

We recall that \mathbb{T} is the Tate algebra (Definition 1.2) in variable t with coefficients in \mathbb{C}_∞ and that \mathbb{L} is the fraction field of the Tate algebra \mathbb{T} . We say that M is rigid analytically trivial if there exists $\Psi \in \text{GL}_r(\mathbb{L})$ such that

$$\Psi^{(-1)} = \Phi\Psi.$$

We set $M^\dagger := \mathbb{L} \otimes_{\overline{K}(t)} M$ on which σ acts diagonally and define $H_{\text{Betti}}(M)$ to be the sub $\mathbb{F}_q(t)$ -vector space of the elements of M^\dagger which are fixed by σ . We call $H_{\text{Betti}}(M)$ the Betti cohomology of M . It is shown in [47, Prop. 3.3.9] that M is rigid analytically trivial if and only if the natural map

$$\mathbb{L} \otimes_{\mathbb{F}_q(t)} H_{\text{Betti}}(M) \rightarrow M^\dagger$$

is an isomorphism. We then call Ψ a rigid analytical trivialization for the matrix Φ .

The category of pre- t -motives which are rigid analytically trivial is a neutral Tannakian category over $\mathbb{F}_q(t)$ with the fiber functor ω which maps $M \mapsto H_{\text{Betti}}(M)$ (see [47], Theorem 3.3.15).

Definition 1.14. The strictly full Tannakian subcategory generated by the images of rigid analytically trivial Anderson dual t -motives is called the category of t -motives and is denoted by \mathcal{T} (see [47], Section 3.4.10). By [38], Remark 2.4.15, this category is equivalent to the category of uniformizable dual $\mathbb{F}_q[t]$ -motives given in [38], Definition 2.4.14.

By Tannakian duality, for each (rigid analytically trivial) t -motive M , the Tannakian subcategory generated by M is equivalent to the category of finite dimensional representations over $\mathbb{F}_q(t)$ of some algebraic group Γ_M called the (motivic) Galois group of the t -motive M . Further, we have a faithful representation $\Gamma_M \hookrightarrow \text{GL}(H_{\text{Betti}}(M))$, which is called the tautological representation of M .

Papanikolas proved an analogue of Grothendieck’s period conjecture which unveils a deep connection between Galois groups of t -motives and transcendence.

Theorem 1.15 (Papanikolas [47], Theorem 1.1.7). *Let M be a t -motive and let Γ_M be its Galois group. Suppose that $\Phi \in \text{GL}_n(\overline{K}(t)) \cap \text{Mat}_{n \times n}(\overline{K}[t])$ represents the multiplication by σ on M and that $\det \Phi = c(t - \theta)^s, c \in \overline{K}^\times$. If $\Psi \in \text{GL}_n(\mathbb{T})$ is a rigid analytic trivialization for Φ , then the entries of Ψ may be evaluated at θ and*

$$\text{tr. deg}_{\overline{K}} \overline{K}(\Psi(\theta)) = \dim \Gamma_M.$$

Papanikolas also shows that Γ_M equals the Galois group Γ_Ψ of the Frobenius difference equation corresponding to M (see [47], Theorem 4.5.10). This provides a method to explicitly compute the Galois groups for t -motives in many cases. This is a very powerful tool and has led to major transcendence results in the last decade. We refer the reader to [4, 17, 18, 19, 20, 21, 22, 23] for more details about transcendence applications.

Papanikolas proved that Γ_M is an affine algebraic group scheme over $\mathbb{F}_q(t)$ which is absolutely irreducible and smooth over $\overline{\mathbb{F}_q}(t)$ (see [47], Theorems 4.2.11, 4.3.1 and 4.5.10). Further, for any $\mathbb{F}_q(t)$ -algebra R , the map

$$\Gamma_M(R) \rightarrow \text{GL}(R \otimes_{\mathbb{F}_q(t)} H_{\text{Betti}}(M))$$

is given by the tautological map

$$\gamma \mapsto (1 \otimes \Psi^{-1} \mathbf{m} \mapsto (\gamma^{-1} \otimes 1) \cdot (1 \otimes \Psi^{-1} \mathbf{m})). \tag{1.23}$$

1.7. Hardouin’s work

In this section, we review the work of Hardouin [37] on unipotent radicals of Tannakian groups in positive characteristic. Let F be a field and (\mathcal{T}, ω) be a neutral Tannakian category over F with fiber functor ω . We denote by \mathbb{G}_m the multiplicative group over F . For an object $\mathcal{U} \in \mathcal{T}$, we denote by $\Gamma_{\mathcal{U}}$ the Galois group of \mathcal{U} . Let $\mathbf{1}$ be the unit object for the tensor product and \mathcal{Y} be a completely reducible object, which means that \mathcal{Y} is a direct sum of finitely many irreducible objects. We consider extensions $\mathcal{U} \in \text{Ext}^1(\mathbf{1}, \mathcal{Y})$ of $\mathbf{1}$ by \mathcal{Y} , which means that we have a short exact sequence

$$0 \rightarrow \mathcal{Y} \rightarrow \mathcal{U} \rightarrow \mathbf{1} \rightarrow 0.$$

For such an extension \mathcal{U} , the Galois group $\Gamma_{\mathcal{U}}$ of \mathcal{U} is isomorphic to the semi-direct product

$$\Gamma_{\mathcal{U}} = R_u(\mathcal{U}) \rtimes \Gamma_{\mathcal{Y}},$$

where $R_u(\mathcal{U})$ stands for the unipotent part of $\Gamma_{\mathcal{U}}$. Therefore, with the knowledge of $\Gamma_{\mathcal{Y}}$, we reduce the computation of $\Gamma_{\mathcal{U}}$ to that of its unipotent part. In [37], Hardouin proves several fundamental results which characterize $R_u(\mathcal{U})$ in terms of the extension group $\text{Ext}^1(\mathbf{1}, \mathcal{Y})$. In the next result, we keep the same notation as above, and we remind the reader that the action of a group G on a module V is isotypic if the module V is the direct sum of irreducible isomorphic G -modules.

Theorem 1.16 (Hardouin [37], Theorem 2). *Assume that*

1. every $\Gamma_{\mathcal{Y}}$ -module is completely reducible,
2. the center of $\Gamma_{\mathcal{Y}}$ contains \mathbb{G}_m ,
3. the action of \mathbb{G}_m on $\omega(\mathcal{Y})$ is isotypic,
4. $\Gamma_{\mathcal{U}}$ is reduced.

Then there exists a smallest sub-object \mathcal{V} of \mathcal{Y} such that \mathcal{U}/\mathcal{V} is a trivial extension of $\mathbf{1}$ by \mathcal{Y}/\mathcal{V} . Further, the unipotent part $R_u(\mathcal{U})$ of the Galois group $\Gamma_{\mathcal{U}}$ is equal to $\omega(\mathcal{V})$.

As a consequence, Hardouin proves the following corollary which states that algebraic relations between the extensions are exactly given by the linear relations. We continue with the above notation.

Corollary 1.17 (Hardouin [37], Corollary 1). *Let $\mathcal{E}_1, \dots, \mathcal{E}_n$ be extensions of $\mathbf{1}$ by \mathcal{Y} . Assume that*

1. every $\Gamma_{\mathcal{Y}}$ -module is completely reducible,
2. the center of $\Gamma_{\mathcal{Y}}$ contains \mathbb{G}_m ,
3. the action of \mathbb{G}_m on $\omega(\mathcal{Y})$ is isotypic,
4. $\Gamma_{\mathcal{E}_1}, \dots, \Gamma_{\mathcal{E}_n}$ are reduced.

If $\mathcal{E}_1, \dots, \mathcal{E}_n$ are $\text{End}(\mathcal{Y})$ -linear independent in $\text{Ext}^1(\mathbf{1}, \mathcal{Y})$, then the unipotent radical of the Galois group $\Gamma_{\mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_n}$ of the direct sum $\mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_n$ is isomorphic to $\omega(\mathcal{Y})^n$.

We remark that we will apply this theorem and its corollary in §3.6 to the t -motives $\mathcal{X}_{\rho}^{\otimes n}$ and $\mathcal{X}_n(b)$, which are defined in §2.2 and §3.2.

2. Constructing t -motives connected to periods

From now on, we investigate the problem of determining algebraic relations between special zeta values and periods attached to ρ , the canonical, sign-normalized rank 1 Drinfeld module attached to X of the previous section. For the Carlitz module, this was done by Chang and Yu [23] using the machinery of

Papanikolas [47] (see Section 1.6). For our setting, we will also need the results of Hardouin [37] (see Section 1.7).

2.1. The t -motive associated to ρ

We follow the construction given by Chang-Papanikolas [18], Sections 3.3 and 3.4. Recall that B is the integral closure of A in the Hilbert class field H . We consider $\rho : \mathbb{F}_q[t] \rightarrow B\{\tau\}$ from Definition 1.3 as a Drinfeld $\mathbb{F}_q[t]$ -module of rank 2 by forgetting the y action. We recall

$$\rho_t = \theta + x_1\tau + \tau^2, \quad x_1 \in B$$

(see [36, §3] for more details on this construction). For $\mathbf{u} = u \in \mathbb{C}_\infty$, we denote the associated Anderson generating function $E_u(t) := E_{\mathbf{u}}^{\otimes 1}(t)$ given by Equation (1.16). This function extends meromorphically to all of \mathbb{C}_∞ with simple poles at $t = \theta^{q^i}$, $i \geq 0$. Further, it satisfies the functional equation

$$\rho_t(E_u(t)) = \exp_\rho(u) + tE_u(t).$$

In other words, we have

$$\theta E_u(t) + x_1 E_u(t)^{(1)} + E_u(t)^{(2)} = \exp_\rho(u) + tE_u(t).$$

Now we fix an $\mathbb{F}_q[t]$ -basis $u_1 = \pi_\rho$ and $u_2 = \eta\pi_\rho$ of the period lattice Λ_ρ of ρ . We set $E_i := E_{u_i}$ for $i = 1, 2$. We define the following matrices:

$$\begin{aligned} \Phi_\rho &= \begin{pmatrix} 0 & 1 \\ t - \theta & -x_1^{(-1)} \end{pmatrix} \in \text{Mat}_{2 \times 2}(\overline{K}[t]), & \Upsilon &= \begin{pmatrix} E_1 & E_1^{(1)} \\ E_2 & E_2^{(1)} \end{pmatrix} \in \text{Mat}_{2 \times 2}(\mathbb{T}), \\ \Theta &= \begin{pmatrix} 0 & t - \theta \\ 1 & -x_1 \end{pmatrix} \in \text{Mat}_{2 \times 2}(\overline{K}[t]), & V &= \begin{pmatrix} x_1 & 1 \\ 1 & 0 \end{pmatrix} \in \text{Mat}_{2 \times 2}(\overline{K}). \end{aligned}$$

Then we set

$$\Psi_\rho := V^{-1}(\Upsilon^{(1)})^{-1}.$$

Since $V^{(-1)}\Phi_\rho = \Theta V$ and $\Upsilon^{(1)} = \Upsilon\Theta$, we get

$$\Psi_\rho^{(-1)} = \Phi_\rho \Psi_\rho.$$

2.2. The t -motive associated to $\rho^{\otimes n}$

Let $n \geq 2$ be an integer. Recall the definition of $\rho^{\otimes n}$ from Definition 1.3. By forgetting the y -action, the Anderson \mathbf{A} -module $\rho^{\otimes n}$ can be considered as an Anderson $\mathbb{F}_q[t]$ -module given by

$$\rho_t^{\otimes n} = d[\theta] + E_\theta\tau$$

(see (1.14) for explicit formulas of $d[\theta]$ and E_θ). For $\mathbf{u} = (u_1, \dots, u_n)^T \in \text{Mat}_{n \times 1}(\mathbb{C}_\infty)$, recall the associated Anderson generating function $E_{\mathbf{u}}^{\otimes n}(t)$ given by Equation (1.16). It satisfies the functional equation

$$\rho_t(E_{\mathbf{u}}^{\otimes n}(t)) = \text{Exp}_\rho^{\otimes n}(\mathbf{u}) + tE_{\mathbf{u}}^{\otimes n}(t).$$

As n is fixed throughout this section, to simplify notation, we will suppress the dependence on n and denote the coordinates of

$$E_{\mathbf{u}}^{\otimes n}(t) := (E_{\mathbf{u},1}, \dots, E_{\mathbf{u},n})^\top, \tag{2.1}$$

and similarly for other vector valued functions. Recall the definitions of the functions h_i and the coefficients a_i and b_i from Proposition 1.4. For $1 \leq i \leq n$, we set

$$\Theta_i = \begin{pmatrix} 0 & 1 \\ t - \theta & -a_i \end{pmatrix}, \quad \phi_i = \begin{pmatrix} 0 & 1 \\ t - \theta & -b_i \end{pmatrix}.$$

From Proposition 1.4, we have

$$\begin{pmatrix} h_{i+1} \\ h_{i+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ t - \theta & -b_i \end{pmatrix} \begin{pmatrix} h_i \\ h_{i+1} \end{pmatrix} = \phi_i \begin{pmatrix} h_i \\ h_{i+1} \end{pmatrix}.$$

We define

$$\Phi_\rho^{\otimes n} = \phi_n \dots \phi_1 = \begin{pmatrix} 0 & 1 \\ t - \theta & -b_n \end{pmatrix} \dots \begin{pmatrix} 0 & 1 \\ t - \theta & -b_1 \end{pmatrix} \in \text{Mat}_{2 \times 2}(\overline{K}[t]).$$

It follows that

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}^{(-1)} = \begin{pmatrix} h_{n+1} \\ h_{n+2} \end{pmatrix} = \Phi_\rho^{\otimes n} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.$$

Now we fix $\mathbf{u}_1 = \Pi_n$ and $\mathbf{u}_2 = d[\eta]\Pi_n$ such that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a basis of the period lattice $\Lambda_\rho^{\otimes n}$ of $\rho^{\otimes n}$, where Π_n is defined in Proposition 1.6(d). We denote by $E_i^{\otimes n} = E_{\mathbf{u}_i}^{\otimes n}$ for $i = 1, 2$. When the dimension n is fixed, we will often drop the $\otimes n$ from our notation to avoid clutter. Then

$$\rho_t(E_i^{\otimes n}) = \text{Exp}_\rho^{\otimes n}(\mathbf{u}_i) + tE_i^{\otimes n} = tE_i^{\otimes n}.$$

If we set

$$\Theta = \Theta_n \dots \Theta_1 = \begin{pmatrix} 0 & 1 \\ t - \theta & -a_n \end{pmatrix} \dots \begin{pmatrix} 0 & 1 \\ t - \theta & -a_1 \end{pmatrix} \in \text{Mat}_{2 \times 2}(\overline{K}[t]),$$

and

$$\Upsilon = \begin{pmatrix} E_{1,1} & E_{2,1} \\ E_{1,2} & E_{2,2} \end{pmatrix} \in \text{Mat}_{2 \times 2}(\mathbb{T}), \tag{2.2}$$

then we obtain

$$\Upsilon^{(1)} = \Theta \Upsilon.$$

We define

$$V = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \in \text{Mat}_{2 \times 2}(\overline{K}).$$

Note that V is symmetric and

$$V^{(-1)} = \begin{pmatrix} b_n & 1 \\ 1 & 0 \end{pmatrix}.$$

We claim that

$$V^{(-1)}\Phi_\rho^{\otimes n} = (\Theta^\top)V. \tag{2.3}$$

In fact, recall from Proposition 1.4 that $b_i = a_{n-i}$ for $1 \leq i \leq n - 1$ and $a_n = b_n^q$. It is clear that for any $x \in \mathbb{C}_\infty$, we have

$$\begin{pmatrix} t - \theta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ t - \theta & x \end{pmatrix} = \begin{pmatrix} 0 & t - \theta \\ 1 & x \end{pmatrix} \begin{pmatrix} t - \theta & 0 \\ 0 & 1 \end{pmatrix}.$$

Then the claim follows immediately:

$$\begin{aligned} V^{(-1)}\Phi_\rho^{\otimes n} &= \begin{pmatrix} b_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ t - \theta & -b_n \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ t - \theta & -b_1 \end{pmatrix} \\ &= \begin{pmatrix} t - \theta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ t - \theta & -b_{n-1} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ t - \theta & -b_1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & t - \theta \\ 1 & -b_{n-1} \end{pmatrix} \begin{pmatrix} t - \theta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ t - \theta & -b_{n-2} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ t - \theta & -b_1 \end{pmatrix} \\ &= \cdots \\ &= \begin{pmatrix} 0 & t - \theta \\ 1 & -b_{n-1} \end{pmatrix} \cdots \begin{pmatrix} 0 & t - \theta \\ 1 & -b_1 \end{pmatrix} \begin{pmatrix} t - \theta & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & t - \theta \\ 1 & -a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & t - \theta \\ 1 & -a_{n-1} \end{pmatrix} \begin{pmatrix} t - \theta & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & t - \theta \\ 1 & -a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & t - \theta \\ 1 & -a_{n-1} \end{pmatrix} \begin{pmatrix} 0 & t - \theta \\ 1 & -a_n \end{pmatrix} \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \\ &= (\Theta^\top)V. \end{aligned}$$

We set

$$\Psi_\rho^{\otimes n} := V^{-1}((\Upsilon^\top)^{(1)})^{-1} \in \text{Mat}_{2 \times 2}(\mathbb{L}).$$

Thus, we get

$$(\Psi_\rho^{\otimes n})^{(-1)} = \Phi_\rho^{\otimes n} \Psi_\rho^{\otimes n}.$$

Remark 2.1. From the previous discussion, we have

$$(\Psi_\rho^{\otimes n})^{-1} = (\Upsilon^\top)^{(1)}V = \begin{pmatrix} a_n E_{1,1}^{(1)} + E_{1,2}^{(1)} & E_{1,1}^{(1)} \\ a_n E_{2,1}^{(1)} + E_{2,2}^{(1)} & E_{2,2}^{(1)} \end{pmatrix}.$$

By direct calculations (see Lemma 3.3), we show that

$$[\Psi_\rho^{\otimes n}]_{i,1}^{-1}(\theta) = \mathbf{u}_{i,n} \in K\pi_\rho^n,$$

where $\mathbf{u}_{i,n}$ is the n th coordinate of the period \mathbf{u}_i for $i = 1, 2$.

Remark 2.2. As $E_{i,j} \in \mathbb{T}$ for $1 \leq i, j \leq 2$ by Proposition 1.6, it follows that $(\Psi_\rho^{\otimes n})^{-1} \in \text{Mat}_{2 \times 2}(\mathbb{T})$. By [35, Remark 3.1, Part 2], it implies that $(\Psi_\rho^{\otimes n})^{-1} \in \text{Mat}_{2 \times 2}(\mathbb{T}_\theta)$.

2.3. Galois groups

We denote by $\mathcal{X}_\rho^{\otimes n}$ the pre t -motive associated to $\rho^{\otimes n}$. The following proposition gives some basic properties of this pre t -motive (compare to [18], Theorem 3.5.4).

Proposition 2.3.

- 1) The pre t -motive $\mathcal{X}_\rho^{\otimes n}$ is a t -motive.
- 2) The t -motive $\mathcal{X}_\rho^{\otimes n}$ is pure.
- 3) Its Galois group $\Gamma_{\mathcal{X}_\rho^{\otimes n}}$ is $\text{Res}_{K/\mathbb{F}_q[t]}\mathbb{G}_{m,K}$. In particular, it is a torus.

Proof. For Part 1, since $\mathcal{X}_\rho^{\otimes n}$ is an A -motive of rank 1, it follows from [38], Proposition 2.3.24, Part b that $\mathcal{X}_\rho^{\otimes n}$ is uniformizable as an A -motive. This implies that $\mathcal{X}_\rho^{\otimes n}$ is uniformizable as an $\mathbb{F}_q[t]$ -motive. Thus, Part 1 follows immediately.

For Part 2, if $n = 1$, then ρ is a Drinfeld $\mathbb{F}_q[t]$ -module. By [38], Example 2.2.5, we know that ρ is pure. Thus, the motive N (viewed as a t -motive) is pure, which implies N_n is also pure as a t -motive, by [38], Proposition 2.3.11(e). Hence, we get the purity of $\mathcal{X}_\rho^{\otimes n}$.

We now prove Part 3. To calculate the Galois group $\Gamma_{\mathcal{X}_\rho^{\otimes n}}$ associated to the t -motive $\mathcal{X}_\rho^{\otimes n}$, we will use [41], Theorem 5.1.2. We claim that the t -motive $\mathcal{X}_\rho^{\otimes n}$ verifies all the conditions of this theorem. In fact, it is a pure uniformizable dual $\mathbb{F}_q[t]$ -motive thanks to Part 2. Further, it has complex multiplication since $\text{End}_{\mathbb{C}_\infty}(\mathcal{X}_\rho^{\otimes n}) = \text{End}_{\mathbb{C}_\infty}(\rho^{\otimes n}) = A$ by [33], Lemma 7.3. Thus, we apply [41], Theorem 5.1.2 to the t -motive $\rho^{\otimes n}$ to obtain

$$\Gamma_{\mathcal{X}_\rho^{\otimes n}} = \text{Res}_{K/\mathbb{F}_q[t]}\mathbb{G}_{m,K}. \quad \square$$

Remark 2.4. 1) We note that the Galois group associated to the A -motive $\rho^{\otimes n}$ is also equal to $\text{Res}_{K/\mathbb{F}_q[t]}\mathbb{G}_{m,K}$ (see [38], Example 2.3.29).

2) We should mention a similar result of Pink and his collaborators that completely determines the Galois group of a Drinfeld A -module (see [38], Theorem 2.6.3 and also [19], Theorem 3.5.4 for more details). It states that if M is the t -motive associated to a Drinfeld A -module defined over \overline{K} , then

$$\Gamma_M = \text{Cent}_{\text{GL}(H_{\text{Betti}}(M))} \text{End}_{\mathbb{C}_\infty}(M).$$

Proposition 2.5. *The entries of $\Psi_\rho^{\otimes n}$ are regular at $t = \theta$, and we have*

$$\text{tr. deg}_{\overline{K}} \overline{K}(\Psi_\rho^{\otimes n}(\theta)) = \dim \Gamma_{\mathcal{X}_\rho^{\otimes n}}.$$

Proof. By [47], Proposition 3.3.9 (c) and Section 4.1.6, there exists a matrix $U \in \text{GL}_2(\mathbb{F}_q(t))$ such that $\widetilde{\Psi} = \Psi_\rho^{\otimes n}U$ is a rigid analytic trivialization of $\Phi_\rho^{\otimes n}$ and $\widetilde{\Psi} \in \text{GL}_2(\mathbb{T})$. By [4], Proposition 3.1.3, the entries of $\widetilde{\Psi}$ are entire functions in the variable t . Thus, in particular, the entries of $\widetilde{\Psi}$ and U^{-1} are regular at $t = \theta$. This implies that the entries of $\Psi_\rho^{\otimes n}$ are regular at $t = \theta$.

Further, since $\overline{K}(\Psi_\rho^{\otimes n}(\theta)) = \overline{K}(\widetilde{\Psi}(\theta))$, by Theorem 1.15, we have

$$\text{tr. deg}_{\overline{K}} \overline{K}(\Psi_\rho^{\otimes n}(\theta)) = \text{tr. deg}_{\overline{K}} \overline{K}(\widetilde{\Psi}(\theta)) = \dim \Gamma_{\mathcal{X}_\rho^{\otimes n}}.$$

The proof is finished. □

2.4. Endomorphisms of t -motives

We write down explicitly the endomorphism of $\mathcal{X}_\rho^{\otimes n}$ given by $y \in A$. In [34], Proposition 4.2 gives a formula for yg_i for the basis elements g_i of Proposition 1.4. Using the same strategy, and replacing the g_i by h_i , a short argument shows that there exist $y_i, z_i \in H$ such that for $1 \leq i \leq n$, we have

$$yh_i = \eta h_i + y_i h_{i+1} + z_i h_{i+2} + h_{i+3}.$$

We deduce that there exists $M_y \in \text{Mat}_{2 \times 2}(\overline{K}[t])$ such that the endomorphism y expressed in terms of the basis $\{h_1, h_2\}$ is represented by M_y :

$$y \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = M_y \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.$$

Since $A \cong \mathbb{F}_q[t] + y\mathbb{F}_q[t]$, using the formulas above for any element $a \in A$, we can find a matrix $M_a \in \text{Mat}_{2 \times 2}(\overline{K}[t])$ such that the endomorphism induced by multiplication by a on $\mathcal{X}_\rho^{\otimes n}$ in the basis $\{h_1, h_2\}$ is represented by M_a :

$$a \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = M_a \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.$$

Lemma 2.6. *With the above notation, the $(1, 1)$ th coordinate $M_{a,1,1}|_{t=\theta} \in K^\times$, and similarly, $M_{a,2,1}|_{t=\theta} = 0$.*

Proof. (Compare to [18], Proposition 4.1.1) Specializing the above equality at Ξ and recalling that $h_1(\Xi) \neq 0$ and $h_2(\Xi) = 0$ by (1.10), we obtain $M_{a,1,1}|_{t=\theta} = a(\theta) \in K^\times$ and $M_{a,2,1}|_{t=\theta} = 0$. \square

3. Constructing t -motives connected to logarithms

3.1. Logarithms attached to ρ

We keep the notation of Section 2.1. Following Chang-Papanikolas (see [18], Section 4.2), for $u \in \mathbb{C}_\infty$ with $\exp_\rho(u) = v \in \overline{K}$, we consider E_u the associated Anderson generating function associated to u given by (1.16). We set

$$h_v = \begin{pmatrix} v \\ 0 \end{pmatrix} \in \text{Mat}_{2 \times 1}(\overline{K}), \quad \Phi_v = \begin{pmatrix} \Phi_\rho & 0 \\ h_v^\top & 1 \end{pmatrix} \in \text{Mat}_{3 \times 3}(\overline{K}[t]).$$

We define

$$g_v = V \begin{pmatrix} -E_u^{(1)} \\ -E_u^{(2)} \end{pmatrix} = \begin{pmatrix} -(t - \theta)E_u - v \\ -E_u^{(1)} \end{pmatrix} \in \text{Mat}_{2 \times 1}(\mathbb{T}),$$

$$\Psi_v = \begin{pmatrix} \Psi_\rho & 0 \\ g_v^\top \Psi_\rho & 1 \end{pmatrix} \in \text{Mat}_{3 \times 3}(\mathbb{T}).$$

Then we get

$$\Phi_\rho^\top g_v^{(-1)} = g_v + h_v,$$

and

$$\Psi_v^{(-1)} = \Phi_v \Psi_v.$$

The associated pre-motive \mathcal{X}_v is, in fact, a t -motive in the sense of Papanikolas as is proved in [19, §4.2].

For $v := \exp_\rho(\zeta_\rho(b, 1)) \in H$, we call the corresponding t -motive \mathcal{X}_v the zeta motive associated to $\zeta_\rho(b, 1)$.

3.2. Logarithms attached to $\rho^{\otimes n}$

We now switch to the case for $n \geq 2$ and use freely the notation of Section 2.2. We fix $\mathbf{u} \in \text{Mat}_{n \times 1}(\mathbb{C}_\infty)$ with $\text{Exp}_\rho^{\otimes n}(\mathbf{u}) = \mathbf{v} \in \text{Mat}_{n \times 1}(\overline{K})$, and we consider $E_{\mathbf{u}}^{\otimes n}$ the Anderson generating function associated to \mathbf{u} given by Equation (1.16). Recall that

$$\rho_t(E_{\mathbf{u}}^{\otimes n}(t)) = \text{Exp}_\rho^{\otimes n}(\mathbf{u}) + tE_{\mathbf{u}}^{\otimes n}(t) = \mathbf{v} + tE_{\mathbf{u}}^{\otimes n}(t). \tag{3.1}$$

Thus, we get

$$\begin{pmatrix} E_{\mathbf{u},i+1} \\ E_{\mathbf{u},i+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ t - \theta & -a_i \end{pmatrix} \begin{pmatrix} E_{\mathbf{u},i} \\ E_{\mathbf{u},i+1} \end{pmatrix} + \begin{pmatrix} 0 \\ -v_i \end{pmatrix} = \Theta_i \begin{pmatrix} E_{\mathbf{u},i} \\ E_{\mathbf{u},i+1} \end{pmatrix} + \begin{pmatrix} 0 \\ -v_i \end{pmatrix}.$$

We define the vector $f_{\mathbf{v}} := (f_{\mathbf{v},1}, f_{\mathbf{v},2})^\top$ given by

$$f_{\mathbf{v}} = \Theta_n \cdots \Theta_2 \begin{pmatrix} 0 \\ v_1 \end{pmatrix} + \Theta_n \cdots \Theta_3 \begin{pmatrix} 0 \\ v_2 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ v_n \end{pmatrix}. \tag{3.2}$$

It follows that

$$\begin{pmatrix} E_{\mathbf{u},1} \\ E_{\mathbf{u},2} \end{pmatrix}^{(1)} = \Theta \begin{pmatrix} E_{\mathbf{u},1} \\ E_{\mathbf{u},2} \end{pmatrix} - \begin{pmatrix} f_{\mathbf{v},1} \\ f_{\mathbf{v},2} \end{pmatrix}.$$

Here, we recall

$$\Theta = \begin{pmatrix} 0 & 1 \\ t - \theta & -a_n \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ t - \theta & -a_1 \end{pmatrix}$$

and

$$\Upsilon = \begin{pmatrix} E_{1,1} & E_{2,1} \\ E_{1,2} & E_{2,2} \end{pmatrix} \in \text{Mat}_{2 \times 2}(\mathbb{T}).$$

They verify

$$\Upsilon^{(1)} = \Theta \Upsilon.$$

We set

$$\Theta_{\mathbf{v}} = \begin{pmatrix} \Theta & -(f_{\mathbf{v},1}, f_{\mathbf{v},2})^\top \\ 0 & 1 \end{pmatrix} \in \text{Mat}_{3 \times 3}(\overline{K}[t]), \quad \Upsilon_{\mathbf{v}} = \begin{pmatrix} \Upsilon & (E_{\mathbf{u},1}, E_{\mathbf{u},2})^\top \\ 0 & 1 \end{pmatrix} \in \text{Mat}_{3 \times 3}(\mathbb{T}). \tag{3.3}$$

Then we get

$$\Upsilon_{\mathbf{v}}^{(1)} = \Theta_{\mathbf{v}} \Upsilon_{\mathbf{v}}.$$

Now we are ready to construct the associated rigid analytic trivialization. Recall that

$$V = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that V is symmetric. We set

$$\begin{aligned} W &= \text{diag}(V, 1) = \begin{pmatrix} V & 0 \\ 0 & 1 \end{pmatrix} \in \text{GL}_3(\overline{K}), \\ \Phi_{\mathbf{v}} &= (W^{(-1)})^{-1} (\Theta_{\mathbf{v}}^\top) W = \begin{pmatrix} \Phi_{\rho}^{\otimes n} & 0 \\ (h_{\mathbf{v},1}, h_{\mathbf{v},2}) & 1 \end{pmatrix} \in \text{Mat}_{3 \times 3}(\overline{K}[t]), \\ \Psi_{\mathbf{v}} &= W^{-1} ((\Upsilon_{\mathbf{v}}^\top)^{(1)})^{-1} = \begin{pmatrix} \Psi_{\rho}^{\otimes n} & 0 \\ (g_{\mathbf{v},1}, g_{\mathbf{v},2}) \Psi_{\rho}^{\otimes n} & 1 \end{pmatrix} \in \text{Mat}_{3 \times 3}(\mathbb{T}). \end{aligned}$$

Remark 3.1. Note that by direct calculations, we obtain

$$(g_{\mathbf{v},1}, g_{\mathbf{v},2}) \Psi_{\rho}^{\otimes n} = -(a_n E_{\mathbf{u},1}^{(1)} + E_{\mathbf{u},2}^{(1)}, E_{\mathbf{u},1}^{(1)}).$$

Further, we will show below that $g_{\mathbf{v},1}(\theta) = u_n - v_n$, where u_n and v_n are the last coordinate of \mathbf{u} and \mathbf{v} , respectively (see Lemma 3.3).

Thus, we get

$$\Psi_{\mathbf{v}}^{(-1)} = \Phi_{\mathbf{v}} \Psi_{\mathbf{v}}.$$

The associated pre-motive $\mathcal{X}_{\mathbf{v}}$ is in fact a t -motive because it is an extension of two t -motives (see for example [38], Lemma 2.3.25).

3.3. Period calculations

In this section, we show explicitly how to obtain the periods and zeta values discussed in the previous section from evaluations of the entries of the rigid analytic trivialization $\Psi_{\mathbf{v}}$, for fixed \mathbf{u} and \mathbf{v} as in the previous section.

Lemma 3.2. *Let $E_{\mathbf{u}}^{\otimes n}$ be defined as above. Then*

$$\text{Res}_{\theta}(E_{\mathbf{u}}^{\otimes n} dt) = \begin{pmatrix} \text{Res}_{\theta}(E_{\mathbf{u},1} dt) \\ \vdots \\ \text{Res}_{\theta}(E_{\mathbf{u},n} dt) \end{pmatrix} = -\mathbf{u}.$$

Proof. Write $\mathbf{u} = (u_1, \dots, u_n)^{\top} \in \mathbb{C}_{\infty}^n$. As in the proof of [34, Prop. 6.5], we have the identity

$$E_{\mathbf{u}}^{\otimes n} = \sum_{j=0}^{\infty} Q_j \left(d[\theta]^{(j)} - t \text{Id}_n \right)^{-1} \mathbf{u}^{(j)},$$

and we find that only the $j = 0$ term contributes to the residue. Then, using the cofactor expansion of $(d[\theta]^{(j)} - t \text{Id}_n)^{-1}$ from [34, Pg. 26], we find that

$$\text{Res}_{\theta}(E_{\mathbf{u}}^{\otimes n} dt) = \begin{pmatrix} \text{Res}_{\theta} \left(\left(\frac{u_1}{\theta-t} + r_1(t) \right) dt \right) \\ \vdots \\ \text{Res}_{\theta} \left(\left(\frac{u_n}{\theta-t} + r_n(t) \right) dt \right) \end{pmatrix} = - \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix},$$

where $r_i(t)$ is some function in powers of $(\theta - t)^k$ for $k \leq -2$, and hence does not contribute to the residue (see [34, (57)] and preceding discussion for more details). □

Lemma 3.3. *Let $E_{\mathbf{u}}^{\otimes n} = (E_{\mathbf{u},1}, \dots, E_{\mathbf{u},n})^{\top}$ as above and let a_i be the defining coefficients for the Drinfeld t -module action as in Proposition 1.4. Then if we write $\text{Exp}_{\rho}^{\otimes n}(\mathbf{u}) := \mathbf{v} = (v_1, \dots, v_n)^{\top}$,*

$$a_n E_{\mathbf{u},1}^{(1)}(\theta) + E_{\mathbf{u},2}^{(1)}(\theta) = -u_n + v_n.$$

Proof. As in (3.1), we have that

$$\rho_t^{\otimes n}(E_{\mathbf{u}}^{\otimes n}) = (d[\theta] + E_{\theta}\tau)(E_{\mathbf{u}}^{\otimes n}) = \text{Exp}_{\rho}^{\otimes n}(\mathbf{u}) + tE_{\mathbf{u}}^{\otimes n}.$$

Rearranging terms gives

$$E_{\theta}(E_{\mathbf{u}}^{\otimes n})^{(1)} = (t - d[\theta])E_{\mathbf{u}}^{\otimes n} + \text{Exp}_{\rho}^{\otimes n}(\mathbf{u}).$$

Both sides of the above equation are regular at $t = \theta$ in each coordinate, so evaluating at $t = \theta$, using the formula for E_θ and taking the last coordinate gives

$$a_n E_{\mathbf{u},1}^{(1)}(\theta) + E_{\mathbf{u},2}^{(1)}(\theta) = E_{\mathbf{u},n}(t - \theta)|_{t=\theta} + v_n.$$

Finally, from our analysis in Lemma 3.2, we see that $E_{\mathbf{u},n}$ has a simple pole at $t = \theta$, and thus the right-hand side of the above equation is simply the residue at $t = \theta$. Thus,

$$a_n E_{\mathbf{u},1}^{(1)}(\theta) + E_{\mathbf{u},2}^{(1)}(\theta) = \text{Res}_\theta(E_{\mathbf{u},n} dt) + v_n = -u_n + v_n. \quad \square$$

Proposition 3.4. *With all notation as above, the elements π_ρ^n and u_n are contained in $\overline{K}(\Psi_{\mathbf{v}}(\theta))$.*

Proof. As above, we have that $\overline{K}(\Psi_{\mathbf{v}}(\theta)) = \overline{K}(\Upsilon_{\mathbf{v}}(\theta))$. Then note that

$$E_{i,j}^{(1)}(\theta), E_{\mathbf{u},i}^{(1)}(\theta) \in \overline{K}(\Upsilon_{\mathbf{v}}(\theta))$$

for $1 \leq i, j \leq 2$ by construction. Then by Lemma 3.3, we see that the last coordinates of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}$ are contained in $\overline{K}(\Upsilon_{\mathbf{v}}(\theta))$, where $\mathbf{u}_1, \mathbf{u}_2$ are an $\mathbb{F}_q[t]$ -basis for the period lattice of $\rho^{\otimes n}$ and \mathbf{u} is a vector such that $\text{Exp}_\rho^{\otimes n}(\mathbf{u}) = \mathbf{v} \in \overline{K}$. From [34, Thm. 6.7], we know that the last coordinate of Π_n is an algebraic multiple of π_ρ^n , and thus it follows that π_ρ^n and u_n are contained in $\overline{K}(\Psi_{\mathbf{v}}(\theta))$. \square

We next prove a lemma about the linear relations between the entries of $\overline{K}(\Psi_\rho^{\otimes n}(\theta))$. Let π_ρ and Π_n be generators of the period lattices of \exp_ρ and $\text{Exp}_\rho^{\otimes n}$, respectively, as \mathbf{A} -modules. Then recall that $\{\pi_\rho, \eta\pi_\rho\}$ and $\{\Pi_n, d[y]\Pi_n\}$ are bases for Λ_ρ and $\Lambda_\rho^{\otimes n}$, respectively, as $\mathbb{F}_q[t]$ -modules. Also for $\mathbf{u} \in \mathbb{C}_\infty^n$, we recall the definition of $G_{\mathbf{u}}$ from 1.17 and define

$$\overline{G}_{\mathbf{u}}^{\otimes n} = -yE_{\mathbf{u}}^{\otimes n} + E_{\eta\mathbf{u}}^{\otimes n},$$

and note that this equals $[-1]G_{\mathbf{u}}^{\otimes n}$, where $[-1]$ represents the inverse of the group law on the elliptic curve X . We will denote the coordinates of $G_{\mathbf{u}}^{\otimes n} := (G_{\mathbf{u},1}, \dots, G_{\mathbf{u},n})$ and similarly for $\overline{G}_{\mathbf{u}}$.

Recall that Ψ_ρ and $\Psi_\rho^{\otimes n}$ are the 2×2 rigid analytic trivialization matrices from §2.1–2.2. The following lemma gives \overline{K} -linear relations between the entries of these matrices evaluated at $t = \theta$ and thus allows us to find a smaller set of generators for the following fields, which notably include powers of the period.

Lemma 3.5. *We have the following facts:*

- for $n = 1$, we have $\overline{K}(\Psi_\rho(\theta)) = \overline{K}(E_{\pi_\rho}^{(1)}(\theta), E_{\pi_\rho}^{(2)}(\theta))$.
- for $n \geq 2$, we have $\overline{K}(\Psi_\rho^{\otimes n}(\theta)) = \overline{K}(E_{\Pi_n,1}^{(1)}(\theta), E_{\Pi_n,2}^{(1)}(\theta))$.

Consequently, for each $n \geq 1$, we have $\overline{K}(\Psi_\rho^{\otimes n}(\theta)) = \overline{K}(\pi_\rho^n, W_n)$ for quantities $W_n \in \mathbb{C}_\infty$.

Proof. To ease the exposition, we will assume that $p = \text{char}(\mathbb{F}_q) \geq 3$, so that we may assume the elliptic curve X has Weierstrass equation given by $y^2 = t^3 + at + b$ with $a, b \in \mathbb{F}_q$ and such that inversion on X is given by $[-1] : (t, y) \mapsto (t, -y)$. The lemma is also true for $p = \text{char}(\mathbb{F}_q) = 2$, and the proof is similar, but calculations are more cumbersome. Particularly, the negation map is more complicated, and this requires more sophisticated analysis. Additionally, we give the full details for the case $n \geq 2$. The details for the case for $n = 1$ are similar, and we leave them to the interested reader. By our definition of $G_{\Pi_n}^{\otimes n}$, we can write

$$\begin{pmatrix} y & 1 & 0 & 0 \\ -y & 1 & 0 & 0 \\ 0 & 0 & y & 1 \\ 0 & 0 & -y & 1 \end{pmatrix} \begin{pmatrix} E_{\Pi_n,1}^{(1)} \\ E_{d[y]\Pi_n,1}^{(1)} \\ E_{\Pi_n,2}^{(1)} \\ E_{d[y]\Pi_n,2}^{(1)} \end{pmatrix} = \begin{pmatrix} G_{\Pi_n,1}^{(1)} \\ \overline{G}_{\Pi_n,1}^{(1)} \\ G_{\Pi_n,2}^{(1)} \\ \overline{G}_{\Pi_n,2}^{(1)} \end{pmatrix},$$

and, in particular, we note that the above matrix is invertible in $\text{Mat}_{4 \times 4}(\mathbf{K})$. By inverting the above matrix, this allows us to write each of the Anderson generating functions $E_{d[y]\Pi_n, k}^{(1)}$ for $k = 1, 2$ as a \mathbf{K} -linear combination of the functions $G_{\Pi_n, k}^{(1)}$ and $\overline{G}_{\Pi_n, k}^{(1)}$. Further, from Proposition 1.6(d), we get the formula $g_2 \cdot G_{\Pi_n, 1} = g_1 \cdot G_{\Pi_n, 2}$. Note that the functions $g_i \in \overline{K}(t, y)$. Taken together, these two facts allow us to write the functions $E_{d[\eta]\Pi_n, 1}^{(1)}(\theta)$ and $E_{d[\eta]\Pi_n, 2}^{(1)}(\theta)$ as \overline{K} -linear combinations of $E_{\Pi_n, 1}^{(1)}(\theta)$ and $E_{\Pi_n, 2}^{(1)}(\theta)$. As $\overline{K}(\Psi_\rho^{\otimes n}(\theta)) = \overline{K}(\Upsilon(\theta))$ (Υ is defined in (2.2)), and this allows us to conclude that $\overline{K}(\Psi_\rho^{\otimes n}(\theta)) = \overline{K}(E_{\Pi_n, 1}^{(1)}(\theta), E_{\Pi_n, 2}^{(1)}(\theta))$. Finally, setting $\mathbf{u} = \Pi_n$ and $\mathbf{v} = 0$ in Lemma 3.3 shows that a \overline{K} -linear combination of $E_{\Pi_n, 1}^{(1)}(\theta)$ and $E_{\Pi_n, 2}^{(1)}(\theta)$ equals the last coordinate of Π_n , which is an algebraic multiple of Π_n^n by Proposition 1.6(e). It remains to take an additional linearly independent combination of $E_{\Pi_n, 1}^{(1)}(\theta)$ and $E_{\Pi_n, 2}^{(1)}(\theta)$, which we set equal to W_n , to finish the proof. \square

3.4. An application of Hartl-Juschka’s work to period calculations

In this section, we maintain the notation from the previous section of $\mathbf{u}_1, \mathbf{u}_2$ being a generating set for the period lattice $\Lambda_\rho^{\otimes n}$ and $\mathbf{u} \in \mathbb{C}_\infty^n$. We wish to apply Corollaries 2.5.23 and 2.5.24 from Hartl and Juschka [38] (originally due to Anderson in unpublished work) to give a more conceptual method for period calculations with an aim towards generalizing these arguments to curves of arbitrary genus. We restate [38, Cors. 2.5.23, 2.5.24] here for the convenience of the reader, but we first translate their notation into our setting. By definition, σ acts by the matrix $\Phi_\rho^{\otimes n}$ on a $\mathbb{C}_\infty[t]$ -basis $\{h_1, h_2\}$ of N_n . Explicitly, for $z \in N_n$, we express $z = ah_1 + bh_2$ with $a, b \in \mathbb{C}_\infty[t]$ and we get

$$\sigma(z) = \sigma(ah_1 + bh_2) = \sigma\left((a, b) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\right) = (a, b)^{(-1)} \Phi_\rho^{\otimes n} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.$$

Thus, we see that if we view N_n as a free $\mathbb{C}_\infty[t]$ -module, then σ acts by inverse twisting and right multiplication by $\Phi_\rho^{\otimes n}$. Transposing, we get a left multiplication:

$$\sigma \begin{pmatrix} a \\ b \end{pmatrix} = \left(\Phi_\rho^{\otimes n}\right)^\top \begin{pmatrix} a \\ b \end{pmatrix}^{(-1)}, \quad a, b \in \mathbb{F}_q[t].$$

By [45, Lemma 3.4.1], δ_0 extends to $N_n \otimes_{\mathbb{C}_\infty[t]} \mathbb{T}_\theta$, where \mathbb{T}_θ is a Tate algebra of functions with radius of convergence $|\theta|_\infty$ (see §1.1.2).

Corollary 3.6. (This is [38, Cor. 2.5.23 and 2.5.24]) *Let N_n and $\Phi_\rho^{\otimes n}$ be as above. Further, let $\mathbf{w} \in \mathbb{T}_\theta^2$ satisfy*

$$(\sigma - 1)(\mathbf{w}) = (\Phi_\rho^{\otimes n})^\top \mathbf{w}^{(-1)} - \mathbf{w} = \mathbf{z} \in N_n. \tag{3.4}$$

Then

$$\text{Exp}_\rho^{\otimes n}(\delta_0(\mathbf{w} + \mathbf{z})) = \delta_1(\mathbf{z}).$$

Further, if $\mathbf{z} = 0$, then $\delta_0(\mathbf{w}) \in \Lambda_\rho^{\otimes n}$ and the set of all such \mathbf{w} forms a spanning set for the periods.

So, we wish to look for vectors $\mathbf{w} \in \mathbb{T}_\theta^2$ which satisfy (3.4) for some $z \in N_n$.

Lemma 3.7. *For Υ as in (2.2), we have*

$$(\Phi_\rho^{\otimes n})^\top \left(V^\top \Upsilon^{(1)}\right)^{(-1)} = V^\top \Upsilon^{(1)}.$$

Proof. From (2.3), we have that $(\Phi_\rho^{\otimes n}) = (V^{(-1)})^{-1}\Theta^\top V$, and then substituting in gives

$$\begin{aligned} (\Phi_\rho^{\otimes n})^\top (V^\top \Upsilon^{(1)})^{(-1)} &= ((V^{(-1)})^{-1}\Theta^\top V)^\top (V^\top \Upsilon^{(1)})^{(-1)} \\ &= V^\top \Theta ((V^{(-1)})^{-1})^\top (V^\top \Upsilon^{(1)})^{(-1)} \\ &= V^\top (\Theta \Upsilon) \\ &= V^\top (\Upsilon^{(1)}). \end{aligned} \quad \square$$

We comment that $V^\top \Upsilon^{(1)} = ((\Psi_\rho^{\otimes n})^{-1})^\top \in \text{Mat}_{2 \times 2}(\mathbb{T}_\theta)$ by Remark 2.2, but to save on notation, we shall denote $P := V^\top \Upsilon^{(1)} \in \text{Mat}_{2 \times 2}(\mathbb{T}_\theta)$ and denote the columns of P by $P_i \in \mathbb{T}_\theta^2$. Thus, for $i = 1, 2$, we have $(\Phi_\rho^{\otimes n})^\top P_i^{(-1)} - P_i = 0$, and therefore the vectors P_i satisfy the conditions of Lemma 3.6. So we get

$$\text{Exp}_\rho^{\otimes n}(\delta_0(P_i + 0)) = \delta_1(0) = 0. \tag{3.5}$$

Lemma 3.8. For $\mathbf{u} \in \text{Mat}_{n \times 1}(\mathbb{C}_\infty)$ with $\text{Exp}_\rho^{\otimes n}(\mathbf{u}) = \mathbf{v} \in \text{Mat}_{n \times 1}(\overline{K})$, we let $E_{\mathbf{u}}^{\otimes n}$ be the Anderson generating function (with coordinates denoted as in (2.1)) associated to \mathbf{u} and let $E_{\mathbf{u},*} = (E_{\mathbf{u},1}, E_{\mathbf{u},2})^\top$ and $P_{\mathbf{u}} := V^\top E_{\mathbf{u},*}^{(1)} \in \mathbb{T}_\theta^2$. Then $P_{\mathbf{u}}$ satisfies the conditions of Lemma 3.6; that is,

$$(\Phi_\rho^{\otimes n})^\top (P_{\mathbf{u}}) - P_{\mathbf{u}} = V^\top f_{\mathbf{v}},$$

where $f_{\mathbf{v}}$ is the vector defined in Section 3.2 and $V^\top f_{\mathbf{v}} \in \overline{K}[t]^2$.

Proof. As stated as in Section 3.2, we find that $\Theta E_{\mathbf{u},*} = E_{\mathbf{u},*}^{(1)} + f_{\mathbf{v}}$. We then calculate that

$$\begin{aligned} (\Phi_\rho^{\otimes n})^\top (V^\top E_{\mathbf{u},*}^{(1)})^{(-1)} &= V^\top \Theta ((V^{(-1)})^{-1})^\top (V^{(-1)})^\top E_{\mathbf{u},*} \\ &= V^\top \Theta E_{\mathbf{u},*} \\ &= V^\top E_{\mathbf{u},*}^{(1)} + V^\top f_{\mathbf{v}}. \end{aligned} \quad \square$$

Thus, $P_{\mathbf{u}}$ satisfies the conditions for Lemma 3.6, and we can write

$$\text{Exp}_\rho^{\otimes n}(\delta_0(P_{\mathbf{u}} + V^\top f_{\mathbf{v}})) = \delta_1(V^\top f_{\mathbf{v}}). \tag{3.6}$$

Proposition 3.9. For a fixed n , with all notation as above, the quantities π_ρ^n and u_n are contained in $\overline{K}(\Psi_{\mathbf{v}}(\theta))$.

Proof. By definition, we have that $\overline{K}(\Psi_{\mathbf{v}}(\theta)) = \overline{K}(\Upsilon_{\mathbf{v}}(\theta))$, and we further see that $\overline{K}(\Upsilon_{\mathbf{v}}(\theta)) = \overline{K}(P(\theta), E_{\mathbf{u},1}^{(1)}(\theta), E_{\mathbf{u},2}^{(1)}(\theta))$, for $P \in \text{Mat}_{2 \times 2}(\mathbb{T}_\theta)$ defined in the proof of Lemma 3.7. Lemma 3.6 implies that $\delta_0(P_i)$ for $i = 1, 2$ is in the period lattice, and we deduce that the vectors $\delta_0(P_i)$ must form a generating set for the period lattice over $\mathbb{F}_q[t]$. This implies that some \mathbf{A} -linear combination of $\delta_0(P_1)$ and $\delta_0(P_2)$ (via the $\text{Lie}_\rho^{\otimes n}(\mathbf{A})$ -action of \mathbf{A}) equals Π_n , the fundamental period. Since the bottom coordinate of Π_n is a \overline{K} -multiple of π_ρ^n , it implies that $\pi_\rho^n \in \overline{K}(\Psi_{\mathbf{v}}(\theta))$.

We now perform a similar analysis on Equation (3.6). To proceed, we need to better understand the vector $f_{\mathbf{v}} := (f_{\mathbf{v},1}, f_{\mathbf{v},2})^\top$ for \mathbf{v} chosen as in Lemma 3.8.

We recall from Section 2.2 that

$$\Theta_i = \begin{pmatrix} 0 & 1 \\ t - \theta & a_i \end{pmatrix}, \quad \phi_i = \begin{pmatrix} 0 & 1 \\ t - \theta & b_i \end{pmatrix}, \quad V = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}.$$

If we denote

$$X = \begin{pmatrix} t - \theta & 0 \\ 0 & 1 \end{pmatrix},$$

then we have the following equalities from (2.3):

$$\Theta_n^\top V = V^{(-1)}\phi_n = X, \quad \Theta_{n-i}^\top X = X\phi_i, \quad 1 \leq i \leq n-1.$$

By the definition of f_v (3.2), we find that

$$V^\top f_v = V^\top \Theta_n \cdots \Theta_2 \begin{pmatrix} 0 \\ v_1 \end{pmatrix} + V^\top \Theta_n \cdots \Theta_3 \begin{pmatrix} 0 \\ v_2 \end{pmatrix} + \cdots + V^\top \begin{pmatrix} 0 \\ v_n \end{pmatrix}.$$

Then, in anticipation of calculating $\delta_0(V^\top f_v)$, using the above equalities, we find that

$$\begin{aligned} (h_1, h_2)V^\top f_v &= (h_1, h_2) \left(V^\top \Theta_n \cdots \Theta_2 \begin{pmatrix} 0 \\ v_1 \end{pmatrix} + V^\top \Theta_n \cdots \Theta_3 \begin{pmatrix} 0 \\ v_2 \end{pmatrix} + \cdots + V^\top \begin{pmatrix} 0 \\ v_n \end{pmatrix} \right) \\ &= ((0, v_1)\Theta_2^\top \cdots \Theta_n^\top V + \cdots + (0, v_{n-1})\Theta_n^\top V + (0, v_n)V) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \\ &= ((0, v_1)X\phi_{n-2} \cdots \phi_1 + \cdots + (0, v_{n-2})X\phi_1 + (0, v_{n-1})X + (0, v_n)V) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}. \end{aligned}$$

Now recall that (see Section 2.2)

$$\begin{pmatrix} h_{i+1} \\ h_{i+2} \end{pmatrix} = \phi_i \begin{pmatrix} h_i \\ h_{i+1} \end{pmatrix}.$$

Since $(0, v_i)X = (0, v_i)$, it follows that

$$\begin{aligned} (h_1, h_2)V^\top f_v &= ((0, v_1)X\phi_{n-2} \cdots \phi_1 + \cdots + (0, v_{n-2})X\phi_1 + (0, v_{n-1})X + (0, v_n)V) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \\ &= (0, v_1) \begin{pmatrix} h_{n-1} \\ h_n \end{pmatrix} + \cdots + (0, v_{n-2}) \begin{pmatrix} h_2 \\ h_3 \end{pmatrix} + (0, v_{n-1}) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + (v_n, 0) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \\ &= v_1 h_n + \cdots + v_n h_1. \end{aligned}$$

Thus, we find that

$$\delta_0(V^\top f_v) = \delta_1(V^\top f_v) = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \mathbf{v}.$$

Then, returning to (3.6), we find that

$$\text{Exp}_\rho^{\otimes n}(\delta_0(P_u) + \mathbf{v}) = \mathbf{v} = \text{Exp}_\rho^{\otimes n}(\mathbf{u}).$$

Thus, the two quantities in the exponential functions in the above equality differ by a period, so there exists some $a \in A$ such that

$$\delta_0(P_u) + \mathbf{v} = \mathbf{u} + d[a]\Pi_n,$$

where $d[a]$ denotes the action of a under $\text{Lie}(\rho^{\otimes n})$. By our above analysis of δ_0 , we conclude that the bottom coordinate α of $\delta_0(P_u)$ is equal to

$$\alpha = a_n E_{\mathbf{u},1}^{(1)}(\theta) + E_{\mathbf{u},2}^{(1)}(\theta) = u_n - v_n + a\pi_\rho^n.$$

Since we have proved above that $\pi_\rho^n \in \overline{K}(\Psi_v(\theta))$ and since $v_n \in \overline{K}$, it follows that $u_n \in \overline{K}(\Psi_v(\theta))$ as well. □

3.5. Independence in $\text{Ext}_{\mathcal{T}}^1(\mathbf{1}, \mathcal{X}_{\rho}^{\otimes n})$

For $b \in B$ and $\mathbf{v} := \text{Exp}_{\rho}^{\otimes n}(Z_n(b)) \in \text{Mat}_{n \times 1}(H)$, where $Z_n(b)$ is the log-algebraic vector of Theorem 1.8, we call the corresponding t -motive $\mathcal{X}_n(b) := \mathcal{X}_{\mathbf{v}}$ the zeta t -motive associated to $\zeta_{\rho}(b, n)$. We also denote by $\Phi_n(b) := \Phi_{\mathbf{v}}$ and $\Psi_n(b) := \Psi_{\mathbf{v}}$ the corresponding matrices. As in §3.2, we put

$$\Phi_{\mathbf{v}} = \begin{pmatrix} \Phi_{\rho}^{\otimes n} & 0 \\ (h_{\mathbf{v},1}, h_{\mathbf{v},2}) & 1 \end{pmatrix} \in \text{Mat}_{3 \times 3}(\overline{K}[t]).$$

We have a short exact sequence

$$0 \longrightarrow \mathcal{X}_{\rho}^{\otimes n} \longrightarrow \mathcal{X}_n(b) \longrightarrow \mathbf{1} \longrightarrow 0,$$

where $\mathbf{1}$ is the trivial pre- t -motive, equal to $\overline{K}[t]$ with σ -action given by the inverse Frobenius twist.

We follow closely [19], Section 4.2. The group $\text{Ext}_{\mathcal{T}}^1(\mathbf{1}, \mathcal{X}_{\rho}^{\otimes n})$ has the structure of a K -vector space by pushing along $\mathcal{X}_{\rho}^{\otimes n}$. With the above notation, for $a \in A$ whose corresponding matrix is $M_a \in \text{Mat}_{2 \times 2}(\overline{K}[t])$ as in §2.4, the extension $a_*\mathcal{X}_{\mathbf{v}}$ is represented by the matrix

$$\begin{pmatrix} \Phi_{\rho}^{\otimes n} & 0 \\ (h_{\mathbf{v},1}, h_{\mathbf{v},2})M_a & 1 \end{pmatrix} \in \text{Mat}_{3 \times 3}(\overline{K}[t]).$$

We will show the following proposition (compare to [18], Theorem 4.4.2):

Proposition 3.10. *Suppose that $\mathbf{u}_1, \dots, \mathbf{u}_m \in \text{Mat}_{n \times 1}(\mathbb{C}_{\infty})$ with $\text{Exp}_{\rho}^{\otimes n}(\mathbf{u}_i) = \mathbf{v}_i \in \text{Mat}_{n \times 1}(\overline{K})$. If $\pi_{\rho}^n, \mathbf{u}_{1,n}, \dots, \mathbf{u}_{m,n}$ are linearly independent over K , then the classes of $\mathcal{X}_{\mathbf{v}_i}$ ($1 \leq i \leq m$) in $\text{Ext}_{\mathcal{T}}^1(\mathbf{1}, \mathcal{X}_{\rho}^{\otimes n})$ are linearly independent over K .*

Proof. The proof follows closely that of [18], Theorem 4.4.2. Suppose that there exist $e_1, \dots, e_m \in K$ not all zero so that $N = e_{1*}X_1 + \dots + e_{m*}X_m$ is trivial in $\text{Ext}_{\mathcal{T}}^1(\mathbf{1}, \mathcal{X}_{\rho}^{\otimes n})$. We can suppose that for $1 \leq i \leq m$, e_i belongs to A and is represented by $M_i \in \text{Mat}_{2 \times 2}(\overline{K}[t])$. Then the extension $N = e_{1*}X_1 + \dots + e_{m*}X_m$ is represented by $h_{\mathbf{v}_1}M_1 + \dots + h_{\mathbf{v}_m}M_m$ and we have

$$\begin{aligned} \Phi_N &= \begin{pmatrix} \Phi_{\rho}^{\otimes n} & 0 \\ \sum_{i=1}^m h_{\mathbf{v}_i} M_i & 1 \end{pmatrix} \in \text{Mat}_{3 \times 3}(\overline{K}[t]), \\ \Psi_N &= \begin{pmatrix} \Psi_{\rho}^{\otimes n} & 0 \\ (\sum_{i=1}^m g_{\mathbf{v}_i} M_i) \Psi_{\rho}^{\otimes n} & 1 \end{pmatrix} \in \text{Mat}_{3 \times 3}(\mathbb{T}). \end{aligned}$$

Since this extension is trivial in $\text{Ext}_{\mathcal{T}}^1(\mathbf{1}, \mathcal{X}_{\rho}^{\otimes n})$, there exists a matrix

$$\gamma = \begin{pmatrix} \text{Id}_2 & 0 \\ (\gamma_1, \gamma_2) & 1 \end{pmatrix} \in \text{Mat}_{3 \times 3}(\overline{K}[t])$$

such that $\gamma^{(-1)}\Phi_N = \text{diag}(\Phi_{\rho}^{\otimes n}, 1)\gamma$. By [47], Section 4.1.6, there exists

$$\delta = \begin{pmatrix} \text{Id}_2 & 0 \\ (\delta_1, \delta_2) & 1 \end{pmatrix} \in \text{Mat}_{3 \times 3}(\mathbb{F}_q(t))$$

such that $\gamma\Psi_N = \text{diag}(\Psi_{\rho}^{\otimes n}, 1)\delta$. It follows that

$$(\gamma_1, \gamma_2) + \left(\sum_{i=1}^m g_{\mathbf{v}_i} M_i \right) = (\delta_1, \delta_2)(\Psi_{\rho}^{\otimes n})^{-1}.$$

Specializing the first coordinates at $t = \theta$ and recalling that by Lemma 2.6, we have $(M_i)_2(\theta) \in K^\times$ and $(M_i)_{2,1}(\theta) = 0$, we obtain

$$\gamma_1(\theta) + \sum_{i=1}^m g_{v_i,1}(\theta)(M_i)_2(\theta) = \delta_1(\theta)[(\Psi_\rho^{\otimes n})^{-1}]_2(\theta) + \delta_2(\theta)[(\Psi_\rho^{\otimes n})^{-1}]_{2,1}(\theta).$$

By Lemma 3.3, we have $g_{v_i,1}(\theta) = \mathbf{u}_{i,n} - v_{i,n}$ and $[(\Psi_\rho^{\otimes n})^{-1}]_2(\theta), [(\Psi_\rho^{\otimes n})^{-1}]_{2,1}(\theta) \in K\pi_\rho^n$. Since $(M_i)_2(\theta) \in K^\times$ (see Section 2.4), we get a nontrivial \bar{K} -linear relation between $1, \mathbf{u}_{1,n}, \dots, \mathbf{u}_{m,n}, \pi_\rho^n$. By [33], page 29 (proof of Theorem 7.1), it implies a nontrivial K -linear relation between $\mathbf{u}_{1,n}, \dots, \mathbf{u}_{m,n}, \pi_\rho^n$. Thus, we get a contradiction. \square

3.6. An application of Hardouin’s work

We now apply Hardouin’s work to our context to determine the Galois groups of the t -motives defined in the previous sections. We work with the neutral Tannakian category of t -motives \mathcal{T} over $F = \mathbb{F}_q(t)$ defined in § 1.6 endowed with the fiber functor $\omega : M \mapsto H_{\text{Betti}}(M)$. By the proof of [33], Lemma 7.2, we know that $\mathcal{X}_\rho^{\otimes n}$ is irreducible. Then we consider this irreducible object $\mathcal{Y} = \mathcal{X}_\rho^{\otimes n}$ and extensions of $\mathbf{1}$ by $\mathcal{X}_\rho^{\otimes n}$. Hardouin’s work turns out to be a powerful tool and allows us to prove the proposition below which generalizes the results of Papanikolas [47] (for the Carlitz module C) and Chang-Yu [23] (for the tensor powers $C^{\otimes n}$ of the Carlitz module).

Proposition 3.11. *Let $n \geq 1$ be an integer with $(q - 1) \nmid n$ and b be an element in B . Then the unipotent radical of $\Gamma_{\mathcal{X}_n(b)}$ is equal to the $\mathbb{F}_q(t)$ -vector space $\mathbb{F}_q(t)^2$ of dimension 2. In particular,*

$$\dim \Gamma_{\mathcal{X}_n(b)} = \dim \Gamma_{\mathcal{X}_\rho^{\otimes n}} + 2 = 4.$$

Proof. We claim that the assumptions of Theorem 1.16 are satisfied for the t -motive $\mathcal{X}_\rho^{\otimes n}$ since

1. By Proposition 2.3, the Galois group $\Gamma_{\mathcal{X}_\rho^{\otimes n}}$ of $\mathcal{X}_\rho^{\otimes n}$ is a torus. Thus $\Gamma_{\mathcal{X}_\rho^{\otimes n}}$ is completely reducible (compare to [19], Corollary 3.5.7).
2. It is clear that the center of $\Gamma_{\mathcal{X}_\rho^{\otimes n}}$ contains $\mathbb{G}_{m, \mathbb{F}_q(t)}$.
3. The action of $\mathbb{G}_{m, \mathbb{F}_q(t)}$ on $H_{\text{Betti}}(\mathcal{X}_\rho^{\otimes n})$ is isotypic. In fact, the weights are all equal to n .
4. The Galois groups of t -motives are reduced (see [47] and also [38], Proposition 2.6.2 for A -motives).

We apply Theorem 1.16 to the t -motive $\mathcal{X}_n(b)$ which is an extension of $\mathbf{1}$ by $\mathcal{X}_\rho^{\otimes n}$. Thus there exists a sub-object \mathcal{V} of $\mathcal{X}_\rho^{\otimes n}$ such that $\mathcal{X}_n(b)/\mathcal{V}$ is a trivial extension of $\mathbf{1}$ by $\mathcal{X}_\rho^{\otimes n}/\mathcal{V}$. As $\mathcal{X}_\rho^{\otimes n}$ is irreducible, either $\mathcal{V} = 0$ or $\mathcal{V} = \mathcal{X}_\rho^{\otimes n}$.

We claim that $\mathcal{V} = \mathcal{X}_\rho^{\otimes n}$. In fact, suppose that $\mathcal{V} = 0$. We deduce that $\mathcal{X}_n(b)$ is a trivial extension of $\mathbf{1}$ by $\mathcal{X}_\rho^{\otimes n}$. It follows that π_ρ^n and $\zeta_\rho(b, n)$ are linearly dependent over K . We get a contradiction by Proposition 3.10 and Theorem 1.10.

Since $\mathcal{V} = \mathcal{X}_\rho^{\otimes n}$, by Theorem 1.16, the unipotent radical of the Galois group $\Gamma_{\mathcal{X}_n(b)}$ is equal to $H_{\text{Betti}}(M)(\mathcal{X}_\rho^{\otimes n})$ that is an $\mathbb{F}_q[t]$ -vector space of dimension 2. The Theorem follows immediately. \square

As a consequence, we obtain a generalization of [23], Theorem 4.4.

Corollary 3.12. *Let $n \geq 1$ be an integer. Then for any $b \in B$, the quantities π_ρ and $\zeta_\rho(b, n)$ are algebraically independent over \bar{K} .*

Proof. We find that $\bar{K}(\Psi_\mathbf{v}) = \bar{K}(Y_\mathbf{v})$, where $Y_\mathbf{v}$ is defined in (3.3). A priori, the field $\bar{K}(Y_\mathbf{v})$ has 6 non-trivial generators given by evaluations of various Anderson generating functions. However, from Lemma 3.5 we see that $\bar{K}(Y_\mathbf{v}) = \bar{K}(\pi_\rho^n, W_n, E_{\mathbf{u},1}(\theta), E_{\mathbf{u},2}(\theta))$, where $\mathbf{u} = Z_n(b)$ is the vector from Theorem 1.8. Finally, by Lemma 3.3 and Proposition 3.4, we conclude that for some $Y_n \in \mathbb{C}_\infty$, we have $\bar{K}(\pi_\rho^n, W_n, E_{\mathbf{u},1}(\theta), E_{\mathbf{u},2}(\theta)) = \bar{K}(\pi_\rho^n, W_n, \zeta_\rho(b, n), Y_n)$. The corollary is then a direct consequence of Proposition 3.11 and Theorem 1.15. \square

We obtain the following theorem which could be considered as a partial generalization of [19], Theorem 5.1.5 in our context.

Theorem 3.13. *Suppose that $\mathbf{u}_1, \dots, \mathbf{u}_m \in \text{Mat}_{n \times 1}(\mathbb{C}_\infty)$ with $\text{Exp}_\rho^{\otimes n}(\mathbf{u}_i) = \mathbf{v}_i \in \text{Mat}_{n \times 1}(\overline{K})$. If $\pi_\rho^n, \mathbf{u}_{1,n}, \dots, \mathbf{u}_{m,n}$ are linearly independent over K , then they are algebraically independent over \overline{K} .*

Proof. By Proposition 3.10, we deduce that the classes of $\mathcal{X}_{\mathbf{v}_i}$ ($1 \leq i \leq n$) in $\text{Ext}_{\mathcal{T}}^1(\mathbf{1}, \mathcal{X}_\rho^{\otimes n})$ are linearly independent over K . By Corollary 1.17, the unipotent part of the Galois group of the direct sum $\mathcal{X}_{\mathbf{v}_1} \oplus \dots \oplus \mathcal{X}_{\mathbf{v}_n}$ is of dimension $2n$. Thus the Theorem follows immediately from Theorem 1.15. \square

4. Algebraic relations among Anderson’s zeta values

4.1. Direct sums of t -motives

Let $m \in \mathbb{N}, m \geq 1$. To study Anderson’s zeta values $\zeta_\rho(b, n)$ for $1 \leq n \leq m$ and π_ρ simultaneously, we set

$$S := \{n \in \mathbb{N} : 1 \leq n \leq m \text{ such that } p \nmid n \text{ and } (q - 1) \nmid n\},$$

and consider the direct sum t -motive

$$\mathcal{X}(b) := \bigoplus_{n \in S} \mathcal{X}_n(b)$$

and define block diagonal matrices

$$\Phi(b) := \bigoplus_{n \in S} \Phi_n(b), \quad \Psi(b) := \bigoplus_{n \in S} \Psi_n(b).$$

Then $\Phi(b)$ represents multiplication by σ on $\mathcal{X}(b)$ and $\Psi(b)$ is a rigid analytic trivialization of $\Phi(b)$. We would like to understand the Galois group $\Gamma_{\mathcal{X}(b)}$ of the t -motive $\mathcal{X}(b)$ and to calculate the dimension of this Galois group.

We first have

$$\Gamma_{\mathcal{X}(b)} \subseteq \bigoplus_{n \in S} \Gamma_{\mathcal{X}_n(b)} = \bigoplus_{n \in S} \begin{pmatrix} \text{Res}_{K/\mathbb{F}_q[t]} \mathbb{G}_{m,K} & 0 \\ * & 1 \end{pmatrix}$$

For $n = 1$, the t -motive $\mathcal{X}_1(b)$ contains \mathcal{X}_ρ . It follows that \mathcal{X}_ρ is also contained in $\mathcal{X}(b)$. We consider $\mathcal{T}_{\mathcal{X}(b)}$ and $\mathcal{T}_{\mathcal{X}_\rho}$, the strictly full Tannakian subcategories of the category \mathcal{T} of t -motives which are generated by $\mathcal{X}(b)$ and \mathcal{X}_ρ respectively. Thus we get a functor from $\mathcal{T}_{\mathcal{X}_\rho}$ to $\mathcal{T}_{\mathcal{X}(b)}$. By Tannakian duality, we have a surjective map of algebraic groups over $\mathbb{F}_q(t)$

$$\pi : \Gamma_{\mathcal{X}(b)} \twoheadrightarrow \Gamma_{\mathcal{X}_\rho} = \text{Res}_{K/\mathbb{F}_q[t]} \mathbb{G}_{m,K}$$

where we have the last equality by Proposition 2.3. By Equation (1.23), this map π is in fact the projection on the upper left-most corner of elements of $\Gamma_{\mathcal{X}(b)}$. We denote by $U(b)$ the kernel of π . It follows that $U(b)$ is contained in the unipotent group

$$U := \bigoplus_{n \in S} \begin{pmatrix} \text{Id}_2 & 0 \\ * & 1 \end{pmatrix}.$$

We prove the following result similar to [23], Section 4.3.

Proposition 4.1. *We keep the previous notation. Then we have*

$$U(b) = \bigoplus_{n \in \mathcal{S}} \begin{pmatrix} \text{Id}_2 & 0 \\ * & 1 \end{pmatrix}.$$

Proof. In fact, the strategy of Chang-Yu (see [23], Section 4.3) based on a weight argument indeed carries over without much modification. For completeness, we sketch a proof of this Proposition.

We introduce a $\mathbb{G}_{m, \mathbb{F}_q(t)}$ -action on $U(b)$ and on the direct sum of unipotent groups

$$U = \bigoplus_{n \in \mathcal{S}} \begin{pmatrix} \text{Id}_2 & 0 \\ * & 1 \end{pmatrix}.$$

On the matrix indexed by $n \in \mathcal{S}$, it is defined by

$$a \cdot \begin{pmatrix} \text{Id}_2 & 0 \\ \mathbf{u} & 1 \end{pmatrix} \mapsto \begin{pmatrix} \text{Id}_2 & 0 \\ a^n \mathbf{u} & 1 \end{pmatrix}, \quad a \in \mathbb{G}_{m, \mathbb{F}_q(t)}.$$

Note that this action on $U(b)$ agrees with the conjugation of $\mathbb{G}_{m, \mathbb{F}_q(t)}$ on $U(b)$.

For each $n \in \mathcal{S}$, we recall that

$$\Gamma_{\mathcal{X}_n(b)} = \begin{pmatrix} \text{Res}_{K/\mathbb{F}_q[t]} \mathbb{G}_{m, K} & 0 \\ * & 1 \end{pmatrix}.$$

We denote by $U_n(b)$ the unipotent part of this Galois group. Thus

$$U_n(b) = \begin{pmatrix} \text{Id}_2 & 0 \\ * & 1 \end{pmatrix}$$

and we have a short exact sequence

$$1 \rightarrow U_n(b) \rightarrow \Gamma_{\mathcal{X}_n(b)} \rightarrow \text{Res}_{K/\mathbb{F}_q[t]} \mathbb{G}_{m, K} \rightarrow 1.$$

Since $\mathcal{X}_n(b)$ is contained in $\mathcal{X}(b)$, by Tannakian duality, we obtain a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & U(b) & \longrightarrow & \Gamma_{\mathcal{X}(b)} & \longrightarrow & \text{Res}_{K/\mathbb{F}_q[t]} \mathbb{G}_{m, K} \longrightarrow 1 \\ & & \varphi_n \downarrow & & \varphi_n \downarrow & & \chi_n \downarrow \\ 1 & \longrightarrow & U_n(b) & \longrightarrow & \Gamma_{\mathcal{X}_n(b)} & \longrightarrow & \text{Res}_{K/\mathbb{F}_q[t]} \mathbb{G}_{m, K} \longrightarrow 1. \end{array}$$

Here the middle vertical arrow is surjective by Tannakian duality and the map χ_n is the character $a \mapsto a^n$. We deduce that the induced map $U(b) \rightarrow U_n(b)$ is also surjective.

We suppose now that $U(b)$ is of codimension $r > 0$ in U . We identify U with the product

$$U \simeq \prod_{n \in \mathcal{S}} \mathbb{G}_{a, \mathbb{F}_q(t)}^2.$$

Chang and Yu proved that there exist an integer $n \in \mathcal{S}$ and a set J of r double indices ij with $i \in \mathcal{S}$ and $j \in \{1, 2\}$ such that if we denote by $W_{(J)}$ the linear subspace of U of codimension r consisting of points (x_{ij}) satisfying $x_{ij} = 0$ whenever $ij \in J$, then $W_{(J)} \cap U_n(b) \subseteq U_n(b)$ and the composed map

$$f_n : W_{(J)} \hookrightarrow U(b) \xrightarrow{\varphi_n} U_n(b)$$

is surjective.

Recall that for $k \in \mathcal{S}$, the action of $\mathbb{G}_{m, \mathbb{F}_q(t)}$ on $U_k(b)$ is of weight k . Since $p \nmid n$, by [23], Lemma 4.7, f_n maps $W_{(J)} \cap U_k(b)$ to zero for all $k \neq n$ in \mathcal{S} . Thus, it maps $W_{(J)} \cap U_n(b)$ onto $U_n(b)$ which has strictly greater dimension. We obtain a contradiction.

As a consequence, we get $U(b) = U$ as required. The proof is complete. □

4.2. Algebraic relations among Anderson’s zeta values

As an immediate consequence of Proposition 4.1, we see that the radical unipotent of $\Gamma_{\mathcal{X}(b)}$ has dimension $2|\mathcal{S}|$, and $\Gamma_{\mathcal{X}(b)}$ itself has dimension $2|\mathcal{S}| + 2$. By Theorem 1.15, we deduce the following theorem.

Theorem 4.2. *Let $b \in B$. Then the elements of the set*

$$\{\pi_\rho\} \cup \{\zeta_\rho(b, n), 1 \leq n \leq m \text{ such that } p \nmid n \text{ and } (q - 1) \nmid n\}$$

are algebraically independent over \overline{K} .

We present a slight generalization of the above theorem by taking account of the p -power relations. Let $\{b_1, \dots, b_h\}$ be a K -basis of H with $b_i \in B$. Since the extension H/K is separable, it follows that for any $b \in B$, we can write $b = a_1 b_1^{p^m} + \dots + a_h b_h^{p^m}$ with $a_1, \dots, a_h \in K$. Thus, we get

$$\zeta_\rho(b, p^m n) = \sum_{I \subseteq A} \frac{\sigma_I(b)}{u_I^{p^m n}} = \sum_{i=1}^h a_i \left(\sum_{I \subseteq A} \frac{\sigma_I(b_i)}{u_I^n} \right)^{p^m} = \sum_{i=1}^h a_i \zeta_\rho(b_i, n)^{p^m}.$$

Theorem 4.3. *Let $\{b_1, \dots, b_h\}$ be a K -basis of H with $b_i \in B$. We consider the set*

$$\mathcal{A} = \{\pi_\rho\} \cup \{\zeta_\rho(b_i, n) : 1 \leq i \leq h, 1 \leq n \leq m \text{ such that } q - 1 \nmid n \text{ and } p \nmid n\}.$$

Then the elements of \mathcal{A} are algebraically independent over \overline{K} .

Proof. The proof of Theorem 4.3 follows identically to that of Theorem 4.2. □

5. Algebraic relations among Goss’s zeta values

In this section, we investigate algebraic relations among Goss’s zeta values. This section owes its very existence to B. Anglès. In particular, the proofs of Proposition 5.2 and Corollary 5.4 are due to him. For more details about the theory of L series and Goss’s zeta values, we refer the interested reader to [32], Section 8.

5.1. Goss’s map

We set $\pi := t/y$, which is a uniformizer of K_∞ . Set $\pi_1 = \pi$, and for $n \geq 2$, choose $\pi_n \in \overline{K}_\infty^\times$ such that $\pi_n^n = \pi_{n-1}$. If $z \in \mathbb{Q}$, $z = \frac{m}{n}$ for some $m \in \mathbb{Z}, n \geq 1$, we set

$$\pi^z := \pi_n^m.$$

Let $\overline{\mathbb{F}_q}$ be the algebraic closure of \mathbb{F}_q in \overline{K}_∞ and let

$$U_\infty := \{x \in \overline{K}_\infty, v_\infty(x - 1) > 0\}.$$

Then $\overline{K}_\infty^\times = \pi^{\mathbb{Q}} \times \overline{\mathbb{F}}_q^\times \times U_\infty$. Therefore, if $x \in \overline{K}_\infty^\times$, one can write in a unique way

$$x = \pi^{v_\infty(x)} \operatorname{sgn}(x) \langle x \rangle, \quad \operatorname{sgn}(x) \in \overline{\mathbb{F}}_q^\times, \langle x \rangle \in U_\infty.$$

Let $I \in \mathcal{I}(A)$. Then there exists an integer $h \geq 1$ such that $I^h = xA$, $x \in K^\times$. We set $\langle I \rangle := \langle x \rangle^{\frac{1}{h}} \in U_\infty$. Then one shows (see [32], Section 8.2) that the map called Goss’s map

$$\begin{aligned} [\cdot]_A : \mathcal{I}(A) &\rightarrow \overline{K}_\infty^\times \\ I &\mapsto \langle I \rangle \pi^{-\frac{\deg I}{d_\infty}} \end{aligned}$$

is a group homomorphism such that

$$\forall x \in K^\times, \quad [xA]_A = \frac{x}{\operatorname{sgn}(x)}.$$

Observe that for all $I \in \mathcal{I}(A)$, we have $\operatorname{sgn}([I]_A) = 1$.

Let E/K be a finite extension and let O_E be the integral closure of A in E . Let $\mathcal{I}(O_E)$ be the group of non-zero fractional ideals of O_E . We denote by $N_{E/K} : \mathcal{I}(O_E) \rightarrow \mathcal{I}(A)$ the group homomorphism such that if \mathfrak{P} is a maximal ideal of O_E and $P = \mathfrak{P} \cap A$, we have

$$N_{E/K}(\mathfrak{P}) = P \left[\frac{O_E}{\mathfrak{P}} : \frac{A}{P} \right].$$

Note that if $\mathfrak{P} = xO_E, x \in E^\times$, then $N_{E/K}(\mathfrak{P}) = N_{E/K}(x)A$, where $N_{E/K} : E \rightarrow K$ also denotes the usual norm map.

5.2. Goss’s zeta functions and Goss’s zeta values

We recall the definition of Goss’s zeta functions introduced in [32], Chapter 8. Let $\mathbb{S}_\infty = \mathbb{C}_\infty^\times \times \mathbb{Z}_p$ be the Goss ‘complex plane’. The group action of \mathbb{S}_∞ is written additively. Let $I \in \mathcal{I}(A)$ and $s = (x; y) \in \mathbb{S}_\infty$; we set

$$I^s := \langle I \rangle^y x^{\deg I} \in \mathbb{C}_\infty^\times.$$

We have a natural injective group homomorphism: $\mathbb{Z} \rightarrow \mathbb{S}_\infty, j \mapsto s_j = \left(\pi^{-\frac{j}{d_\infty}}, j \right)$. Observe that $I^{s_j} = [I]_A^j$.

Let E/K be a finite extension and let O_E be the integral closure of A in E . Let \mathfrak{I} be a non-zero ideal of E . We have

$$\forall j \in \mathbb{Z}, \quad N_{E/K}(\mathfrak{I})^{s_j} = \left[\frac{O_E}{\mathfrak{I}} \right]_A^j.$$

Letting $s \in \mathbb{S}_\infty$, the following sum converges in \mathbb{C}_∞ (see [32], Theorem 8.9.2):

$$\zeta_{O_E}(s) := \sum_{d \geq 0} \sum_{\substack{\mathfrak{I} \in \mathcal{I}(O_E), \mathfrak{I} \subset O_E, \\ \deg(N_{E/K}(\mathfrak{I})) = d}} N_{E/K}(\mathfrak{I})^{-s}.$$

The function $\zeta_{O_E} : \mathbb{S}_\infty \rightarrow \mathbb{C}_\infty$ is called the *zeta function attached to O_E and $[\cdot]_A$* . Observe that

$$\forall j \in \mathbb{Z}, \quad \zeta_{O_E}(j) := \zeta_{O_E}(s_j) = \sum_{d \geq 0} \sum_{\substack{\mathfrak{S} \in \mathcal{I}(O_E), \mathfrak{S} \subset O_E, \\ \deg(N_{E/K}(\mathfrak{S}))=d}} \left[\frac{O_E}{\mathfrak{S}} \right]_A^{-j}.$$

In particular,

$$\zeta_{O_E}(1) = \prod_{\mathfrak{P}} \left(1 - \frac{1}{\left[\frac{O_E}{\mathfrak{P}} \right]_A} \right)^{-1} \in \overline{K_\infty}^\times,$$

where \mathfrak{P} runs through the maximal ideals of O_E .

Recall that $\rho : A \rightarrow B\{\tau\}$ is the sign-normalized rank one Drinfeld module given in Section 1.3, where B is the integral closure of A in H , the Hilbert class field. By [6], Proposition 2.1, the following product converges to an element in $U_\infty \cap K_\infty^\times$:

$$L_A(\rho/O_E) := \prod_{\mathfrak{P}} \frac{[\text{Fitt}_A(O_E/\mathfrak{P})]_A}{[\text{Fitt}_A(\rho(O_E/\mathfrak{P}))]_A},$$

where \mathfrak{P} runs through the maximal ideals of O_E .

We have the following crucial fact (see [6], Proposition 3.4) which provides a deep connection between the special L -values and the Goss’s zeta value at 1.

Proposition 5.1. *Let E/K be a finite extension such that $H \subset E$. Then*

$$L_A(\rho/O_E) = \zeta_{O_E}(1).$$

5.3. Relations with Anderson’s zeta values

Recall that B is the integral closure of A in H , the Hilbert class field of K . Let z be an indeterminate over K_∞ and recall that $\mathbb{T}_z(K_\infty)$ denotes the Tate algebra in the variable z with coefficients in K_∞ . Recall that

$$H_\infty = H \otimes_K K_\infty,$$

$$\mathbb{T}_z(H_\infty) = H \otimes_K \mathbb{T}_z(K_\infty).$$

For $n \in \mathbb{Z}$, we set

$$Z_B(n; z) = \sum_{d \geq 0} \sum_{\substack{\mathfrak{S} \in \mathcal{I}(B), \mathfrak{S} \subset B, \\ \deg(N_{H/K}(\mathfrak{S}))=d}} \left[\frac{B}{\mathfrak{S}} \right]_A^{-n} z^d.$$

Then, by [32], Theorem 8.9.2, for all $n \in \mathbb{Z}$, $Z_B(n; \cdot)$ defines an entire function on \mathbb{C}_∞ , and

$$\forall n \in \mathbb{N}, \quad Z_B(-n; z) \in A[z].$$

Observe that

$$\forall n \in \mathbb{Z}, \quad Z_B(n; z) \in \mathbb{T}_z(K_\infty)$$

and

$$\forall n \geq 1, \quad Z_B(n; z) = \prod_{\mathfrak{P}} \left(1 - \frac{z^{\deg(N_{H/K}(\mathfrak{P}))}}{\left[\frac{B}{\mathfrak{P}} \right]_A^n} \right)^{-1} \in \mathbb{T}_z(K_\infty)^\times.$$

Finally, we note that

$$Z_B(n; 1) = \zeta_B(n).$$

Recall that $G = \text{Gal}(H/K)$. Then $G \simeq \text{Gal}(H(z)/K(z))$ acts on $\mathbb{T}_z(H_\infty)$. We denote by $\mathbb{T}_z(H_\infty)[G]$ the non-commutative group ring where the commutation rule is given by

$$\forall h, h' \in \mathbb{T}_z(H_\infty), \forall g, g' \in G, \quad hg \cdot h'g' = hg(h')gg'.$$

Let $n \in \mathbb{Z}$. One can show (see [6], Lemma 3.5) that the following infinite sum converges in $\mathbb{T}_z(H_\infty)[G]$:

$$\mathcal{L}(\rho/B; n; z) := \sum_{d \geq 0} \sum_{\substack{I \in \mathcal{I}(A), I \subset A, \\ \deg I = d}} \frac{z^{\deg I}}{\psi(I)^n} \sigma_I.$$

Furthermore, for all $n \geq 1$, we have

$$\mathcal{L}(\rho/B; n; z) = \prod_P \left(1 - \frac{z^{\deg P}}{\psi(P)^n} \sigma_P \right)^{-1} \in (\mathbb{T}_z(H_\infty)[G])^\times$$

and for all $n \leq 0$,

$$\mathcal{L}(\rho/B; n; z) \in B[z][G].$$

Note that

$$\zeta_\rho(\cdot, n) = \mathcal{L}(\rho/B; n; 1) \in (H_\infty[G])^\times.$$

We observe that $\mathcal{L}(\rho/B; n; z)$ induces a $\mathbb{T}_z(K_\infty)$ -linear map $\mathcal{L}(\rho/B; n; z) : \mathbb{T}_z(H_\infty) \rightarrow \mathbb{T}_z(H_\infty)$. Since $\mathbb{T}_z(H_\infty)$ is a free $\mathbb{T}_z(K_\infty)$ -module of rank $[H : K]$ (recall that $\mathbb{T}_z(K_\infty)$ is a principal ideal domain), $\det_{\mathbb{T}_z(K_\infty)} \mathcal{L}(\rho/B; n; z)$ is well-defined. We also observe that $\zeta_\rho(\cdot, n)$ induces a K_∞ -linear map $\zeta_\rho(\cdot, n) : H_\infty \rightarrow H_\infty$, and we denote by $\det_{K_\infty} \zeta_\rho(\cdot, n)$ its determinant. Recall that $\text{ev} : \mathbb{T}_z(H_\infty) \rightarrow H_\infty$ is the H_∞ -linear map given by

$$\forall f \in \mathbb{T}_z(H_\infty), \quad \text{ev}(f) = f|_{z=1}.$$

Observe that if $\{e_1, \dots, e_h\}$ is a K -basis of H/K (recall that $h = [H : K]$), then

$$H_\infty = \bigoplus_{i=1}^h K_\infty e_i,$$

$$\mathbb{T}_z(H_\infty) = \bigoplus_{i=1}^h \mathbb{T}_z(K_\infty) e_i.$$

We deduce that

$$\det_{K_\infty} \zeta_\rho(\cdot, n) = \text{ev}(\det_{\mathbb{T}_z(K_\infty)} \mathcal{L}(\rho/B; n; z)).$$

By [6], Theorem 3.6, we have

$$\det_{\mathbb{T}_z(K_\infty)} \mathcal{L}(\rho/B; n; z) = Z_B(n; z).$$

In particular,

$$\det_{K_\infty} \zeta_\rho(\cdot, n) = \zeta_B(n).$$

5.4. Algebraic relations among Goss’s zeta values

The class number $\text{Cl}(A)$ of A equals to the number of rational points $X(\mathbb{F}_q)$ on the elliptic curve X and also to the degree of extension $[H : K]$. For a prime ideal \mathfrak{p} of A of degree 1 corresponding to an \mathbb{F}_q -rational point on X , we denote by \mathfrak{p}_+ the subset of elements in \mathfrak{p} of sign 1 and consider the sum (compare to [36], Section 6 and [33], Section 6):

$$\zeta_A(\mathfrak{p}, n) = \sum_{\substack{a \in \mathfrak{p}^{-1}, \\ \text{sgn}(a)=1}} \frac{1}{a^n}, \quad n \in \mathbb{N}.$$

We will see that the sums $\zeta_A(\mathfrak{p}, n)$ where \mathfrak{p} runs through the set \mathcal{P} of prime ideals of A of degree 1 are the elementary blocks in the study of Goss’s zeta values on elliptic curves. For the rest of this section, it will be convenient to slightly modify these sums as follows.

Proposition 5.2. *Let $n \in \mathbb{N}$. For $\sigma \in G = \text{Gal}(H/K)$, we set*

$$\zeta_A(\sigma, n) := \sum_{d \geq 0} \sum_{\substack{I \in \mathcal{I}(A), I \subset A, \\ \text{deg}(I)=d, \\ \sigma_I = \sigma}} \frac{1}{[I]_A^n}.$$

Then the elements $\zeta_A(\sigma, n)$ indexed by $\sigma \in G$ are algebraically independent over \overline{K} .

Proof. Let $\sigma \in \text{Gal}(H/K)$ and \mathfrak{p} be the corresponding ideal in \mathcal{P} such that $\sigma_{\mathfrak{p}} = \sigma$. We get

$$\zeta_A(\sigma, n) = \sum_{\substack{I \in \mathcal{I}(A), I \subset A, \\ \sigma_I = \sigma}} \frac{1}{[I]_A^n} = \frac{1}{[\mathfrak{p}]_A^n} \sum_{\substack{a \in \mathfrak{p}^{-1}, \\ \text{sgn}(a)=1}} \frac{1}{a^n} = \frac{1}{[\mathfrak{p}]_A^n} \zeta_A(\mathfrak{p}, n),$$

and

$$\sum_{\substack{I \in \mathcal{I}(A), I \subset A, \\ \sigma_I = \sigma}} \frac{1}{\psi(I)^n} = \frac{1}{\psi(\mathfrak{p})^n} \sum_{\substack{a \in \mathfrak{p}^{-1}, \\ \text{sgn}(a)=1}} \frac{1}{a^n} = \frac{1}{\psi(\mathfrak{p})^n} \zeta_A(\mathfrak{p}, n).$$

Thus, we obtain

$$\zeta_A(\sigma, n) = \frac{\psi(\mathfrak{p})^n}{[\mathfrak{p}]_A^n} \sum_{\substack{I \in \mathcal{I}(A), I \subset A, \\ \sigma_I = \sigma}} \frac{1}{\psi(I)^n}.$$

Note that $\frac{\psi(\mathfrak{p})^n}{[\mathfrak{p}]_A^n}$ belongs to \overline{K}^\times . It follows that for $b \in B$, we can express

$$\zeta_\rho(b, n) = \sum_{\sigma \in G} a_\sigma(b) \zeta_A(\sigma, n)$$

with some coefficients $a_\sigma(b) \in \overline{K}$.

By Theorem 4.3, if $\{b_1, \dots, b_h\} \subset B$ is a K -basis of H , then the elements $\zeta_\rho(b_i, n)$ ($1 \leq i \leq h$) are algebraically independent over \bar{K} . By the above discussion and the fact that

$$|\zeta_\rho(b_i, n), 1 \leq i \leq h| = |\zeta_A(\sigma, n), \sigma \in G| = [H : K],$$

the Proposition follows immediately. □

Let U be the p -Sylow subgroup of G where p is the characteristic of \mathbb{F}_q . We set $\Delta := G/U = \text{Gal}(F/K)$ where $F = H^U$. We write $p^s = |U|$ and set

$$\widehat{G} = \text{Hom}(G, \overline{\mathbb{F}}_q^\times) = \text{Hom}(\Delta, \overline{\mathbb{F}}_q^\times) \simeq \Delta,$$

with $|\Delta| \in \mathbb{Z}_p^\times$.

For $\delta \in \Delta$, we set

$$Z(n, \delta) = \sum_{\substack{I \in \mathcal{I}(A), I \subset A, \\ (I, F/K) = \delta}} \frac{1}{[I]_A^n} \in \bar{K}_\infty.$$

We see easily that

$$Z(n, \delta) = \sum_{\sigma \equiv \delta \pmod{U}} \zeta_A(\sigma, n).$$

By Proposition 5.2, $Z(n, \delta), \delta \in \Delta$ are algebraically independent over \bar{K} .

Let $\chi \in \widehat{G}$, and we consider the value at 1 of Goss L -series attached to χ given by

$$L(n, \chi) = \sum_{\delta \in \Delta} \chi(\delta) Z(n, \delta) = \sum_{I \in \mathcal{I}(A), I \subset A} \frac{\chi((I, F/K))}{[I]_A^n},$$

where $(\cdot, F/K)$ is the Artin map. It is clear that for all $\delta \in \Delta$,

$$Z(n, \delta) = \frac{1}{|\Delta|} \sum_{\chi \in \widehat{G}} \chi(\delta)^{-1} L(n, \chi).$$

The above discussion combined with Theorem 4.3 implies immediately a transcendental result for Goss's zeta values.

Theorem 5.3. *Let $m \in \mathbb{N}, m \geq 1$. Then the special values of Goss L -series*

$$G_n = \{\pi_\rho\} \cup \{L(n, \chi) : \chi \in \widehat{G}, 1 \leq n \leq m \text{ such that } q - 1 \nmid n \text{ and } p \nmid n\}.$$

are algebraically independent over \bar{K} .

As a direct consequence, we obtain the following corollary:

Corollary 5.4. *Let $m \in \mathbb{N}, m \geq 1$. Let L be an extension of K such that $L \subset H$. We consider the following set:*

$$\mathcal{G}_L = \{\pi_\rho\} \cup \{\zeta_{O_L}(n) : 1 \leq n \leq m \text{ such that } q - 1 \nmid n \text{ and } p \nmid n\}.$$

Then the elements of \mathcal{G}_L are algebraically independent over \bar{K} .

Remark 5.5. 1) When $L = K$, we have shown that $\zeta_A(1)$ is transcendental over K , which gives an affirmative answer to an old question of D. Goss ³

³Personal communication in Spring 2016

2) When $L = H$, the above Theorem states that $\zeta_B(1)$ is transcendental over K . It answers positively to [10], Problem 4.1 in this case. Note that our proof is highly nontrivial.

Proof of Corollary 5.4. Let p^k be the exact power of p that divides $[L : K]$ and let $N = \text{Gal}(F/F \cap L) \subseteq \Delta$. We have (see for example [32], Section 8.10):

$$\zeta_{O_L}(n) = \left(\prod_{\chi \in \widehat{G}, \chi(N)=\{1\}} L(n, \chi) \right)^{p^k}.$$

Thus, Corollary 5.4 follows from Theorem 5.3. \square

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