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DISTRIBUTION OF THE SMALLEST VISITED POINT IN A GREEDY WALK ON THE LINE

KATJA GABRYSCH,* *Uppsala University*

Abstract

We consider a greedy walk on a Poisson process on the real line. It is known that the walk does not visit all points of the process. In this paper we first obtain some useful independence properties associated with this process which enable us to compute the distribution of the sequence of indices of visited points. Given that the walk tends to $+\infty$, we find the distribution of the number of visited points in the negative half-line, as well as the distribution of the time at which the walk achieves its minimum.

Keywords: Poisson point process; greedy walk

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Secondary 60K35

1. Introduction

We consider a stationary Poisson process Π with rate 1 on the real line and define a greedy walk on Π as follows. The walk starts from point $S_0 = 0$ and always moves on the points of Π by picking the point closest to its current position that has not been visited before. In other words, the sequence of the visited points of Π of the walk $(S_n)_{n\geq 0}$ is defined recursively by

$$S_{n+1} = \arg\min\{d(X, S_n) \colon X \in \Pi \setminus \{S_0, S_1, \dots, S_n\}\}, \tag{1}$$

where d is the usual Euclidean distance. Note that the sequence $(S_n)_{n\geq 0}$ is almost surely (a.s.) well defined, i.e. there is a.s. a unique point which is arg min in the definition of the walk (1). Moreover, $0 \notin \Pi$ a.s.

The distance between the current position and the closest nonvisited point on the other side of 0 increases with each step. Using the Borel–Cantelli lemma, one can show that the walk a.s. jumps over 0 finitely many times. Therefore, $\{S_1, S_2, \dots\} \neq \Pi$ a.s. and, moreover,

$$\mathbb{P}\Big(\lim_{n\to\infty} S_n = +\infty\Big) = \mathbb{P}\Big(\lim_{n\to\infty} S_n = -\infty\Big) = \tfrac{1}{2}. \tag{2}$$

In this paper we study the distribution of the number of visited points of Π which are less than 0 and the distribution of the index of the last point in the sequence $(S_n)_{n\geq 0}$ which is less than or equal to 0. We denote these random variables by N and L, respectively.

The greedy walk is a model in queueing systems where the points of the process in our case represent positions of customers and a server (the walker) moving towards customers. Applications of such a system can be found, for example, in telecommunications and computer networks or transportation. As described in [1], the model of a greedy walk on a point process can be defined in various ways or/and on different spaces. For example, Coffman and Gilbert [2]

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* Postal address: Department of Mathematics, Uppsala University, PO Box 480, 751 06 Uppsala, Sweden. Email address: katja.gabrysch@math.uu.se

and Leskelä and Unger [5] studied a dynamic version of the greedy walk on a circle with new customers arriving to the system according to a Poisson process. Two modifications of the greedy walk model on a homogeneous Poisson process on the real line were studied by Foss *et al.* [3] and Rolla *et al.* [6]. In the first paper, the authors considered a space-time model, starting with a Poisson process at time 0 and the time and position of arrivals of new points are given by a Poisson process on the space-time half-plane. Moreover, the expected time that the walk spends at a point is 1. In this case the walk, a.s., jumps over the starting point finitely many times and the position of the walk diverges logarithmically in time. In the second paper, the points of a Poisson process were assigned one or two marks at random. The walk always moves to the point closest to the current position which still has at least one mark and then removes exactly one mark from the point. The authors showed that introducing points with two marks will force the walk to change sides infinitely many times and, thus, to visit all the points of Π. There is not much known about the behaviour of the greedy walk on a homogeneous Poisson process in higher dimensions. For example, it is an open problem whether the greedy walk on the points of a homogeneous Poisson process on $\mathbb{R}^2$ visits all points [6].

The paper is organized as follows. In Section 2 we calculate the probability that the server visits the points of Π in a particular order. The distributions of random variables N and L are studied in Section 3 and Section 4, respectively.

2. The probability to visit points in a predefined order

The points of $\Pi \cup \{0\}$ can be written in order as

$$\cdots < X_{-3} < X_{-2} < X_{-1} < X_0 = 0 < X_1 < X_2 < X_3 < \cdots .$$

For $n \in \mathbb{Z}$, let

$$Y_n = X_n - X_{n-1}.$$

Since Π is a stationary Poisson process with rate 1, $\{Y_n\}_{n\in\mathbb{Z}}$ are independent and exponentially distributed random variables with parameter 1.

Lemma 1. *Let $\{Y_i\}_{i\geq 1}$ be independent and exponentially distributed random variables with mean* 1 *and let $D_n = \sum_{i=1}^{n} Y_i$. Then*

$$\mathbb{P}(D_n < Y_{n+1}) = \frac{1}{2^n}$$

and the events $\{D_n < Y_{n+1}\}_{n\geq 1}$ are independent.

Proof. For $n = 1$ this is $\mathbb{P}(D_1 < Y_2) = \mathbb{P}(Y_1 < Y_2) = \frac{1}{2}$. Using the memoryless property of the exponential random variable Y_{n+1}, we have

$$\begin{aligned}\mathbb{P}(D_n < Y_{n+1}) &= \mathbb{P}(D_n < Y_{n+1} \mid D_{n-1} < Y_{n+1})\mathbb{P}(D_{n-1} < Y_{n+1}) \\ &= \mathbb{P}(Y_n < Y_{n+1})\mathbb{P}(D_{n-1} < Y_{n+1}) \\ &= \tfrac{1}{2}\mathbb{P}(D_{n-1} < Y_{n+1}).\end{aligned}$$

Thus, by induction, we obtain $\mathbb{P}(D_n < Y_{n+1}) = 1/2^n$.

The event $\{D_n > Y_{n+1}\}$ can be written as

$$\{D_n > Y_{n+1}\} = \left\{\frac{Y_{n+1}}{D_{n+1}} < \frac{1}{2}\right\} = \left\{\frac{D_n}{D_{n+1}} > \frac{1}{2}\right\}.$$

Therefore, in order to show the independence of the events $\{D_n > Y_{n+1}\}_{n\geq 1}$, we prove the independence of the random variables $\{D_n/D_{n+1}\}_{n\geq 1}$.

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For $i < j$, let $U_{i,j} = D_i/D_j$. For any $\ell > 0$, choose integers $i_1 < i_2 < \cdots < i_\ell$. The random variable $U_{i_1,i_2} = D_{i_1}/D_{i_2}$ is independent of D_{i_2} (see, for example, [4, Theorem 8.4.1]) and it is also independent of $D_{i_3} - D_{i_2}, \dots, D_{i_\ell} - D_{i_{\ell-1}}$. This implies that the random variable U_{i_1,i_2} is independent of the random variables $\{U_{i_j,i_{j+1}}\}_{j=2}^{\ell-1}$ since

$$U_{i_j,i_{j+1}} = \frac{D_{i_2} + \sum_{k=2}^{j-1}\left(D_{i_{k+1}} - D_{i_k}\right)}{D_{i_2} + \sum_{k=2}^{j}\left(D_{i_{k+1}} - D_{i_k}\right)} \quad \text{for } 2 \le j < \ell.$$

We can use the same argument to show that, for $2 \le k < \ell - 1$, the random variable $U_{i_k,i_{k+1}}$ is also independent of $\{U_{i_j,i_{j+1}}\}_{j=k+1}^{\ell-1}$. Therefore, the random variables $\{U_{i_j,i_{j+1}}\}_{j=1}^{\ell-1}$ are independent for any choice of ℓ and for any increasing sequence $i_1, i_2, \dots, i_\ell$ and, thus, the random variables $\{U_{n,n+1}\}_{n\ge1} = \{D_n/D_{n+1}\}_{n\ge1}$ are independent as well. □

Let

$$\mathcal{S} = \left\{(i_n)_{n\ge0} \in \mathbb{Z}^\infty : i_0 = 0 \text{ and } i_n \in \left\{\min_{0\le j\le n-1} i_j - 1, \max_{0\le j\le n-1} i_j + 1\right\} \text{ for all } n \in \mathbb{N}\right\}.$$

If the sequence $(S_n)_{n\ge0}$ satisfies (1) then there exists a sequence $(i_n)_{n\ge0} \in \mathcal{S}$ such that $S_n = X_{i_n}$ for all $n \ge 0$. Furthermore, denote by $\pi(S_n) = i_n$ the index of the nth visited point. In the following lemma we compute the probability that the sequence $(\pi(S_n))_{n\ge0}$ is $(i_n)_{n\ge0} \in \mathcal{S}$.

Lemma 2. *Let $(i_n)_{n\ge0} \in \mathcal{S}$ and let*

$$\delta_1 = \begin{cases} 0 & \text{if } i_1 = 1, \\ 1 & \text{if } i_1 = -1, \end{cases} \quad \text{and,} \quad \text{for } n \ge 2, \quad \delta_n = \begin{cases} 0 & \text{if } |i_n - i_{n-1}| = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Then

$$\mathbb{P}((\pi(S_n))_{n\ge0} = (i_n)_{n\ge0}) = \prod_{n=1}^{\infty}\left(1 - \frac{1}{2^n}\right)^{1-\delta_n}\left(\frac{1}{2^n}\right)^{\delta_n}.$$

In other words, the random variables $\{\delta_n\}_{n\ge1}$ are independent and δ_n has a Bernoulli distribution with parameter 2^{-n}.

Proof. Let $M_n = \max_{0\le j\le n} i_j$ and let $m_n = \min_{0\le j\le n} i_j$. Moreover, define random variables $\{Z_n\}_{n\ge1}$ as

$$Z_1 = Y_0, \qquad Z_2 = Y_1, \qquad Z_n = \begin{cases} Y_{i_{n-2}+1} & \text{if } i_{n-2} > 0, \\ Y_{i_{n-2}} & \text{if } i_{n-2} < 0, \end{cases} \qquad n \ge 3.$$

Assume that the event $\{(\pi(S_j))_{j=0}^n = (i_j)_{j=0}^n\}$ occurs and, without loss of generality, assume that $i_n = M_n$. If $i_{n+1} = M_n + 1$ then $\delta_{n+1} = 0$ and

$$\begin{aligned}
&\mathbb{P}(\pi(S_{n+1}) = i_{n+1} \mid (\pi(S_j))_{j=0}^n = (i_j)_{j=0}^n) \\
&\quad = \mathbb{P}(X_{M_n+1} - X_{i_n} < X_{i_n} - X_{m_n-1} \mid (\pi(S_j))_{j=0}^n = (i_j)_{j=0}^n) \\
&\quad = \mathbb{P}\left(Y_{M_n+1} < \sum_{j=m_n}^{M_n} Y_j \;\middle|\; (\pi(S_j))_{j=0}^n = (i_j)_{j=0}^n\right) \\
&\quad = \mathbb{P}\left(Z_{n+2} < \sum_{j=1}^{n+1} Z_j \;\middle|\; (\pi(S_j))_{j=0}^n = (i_j)_{j=0}^n\right). \qquad (3)
\end{aligned}$$

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Since $\{Z_n\}_{n\geq 1}$ are independent and identically distributed exponential random variables with parameter 1 and the event $\{(\pi(S_j))_{j=0}^n = (i_j)_{j=0}^n\}$ is the intersection of the events $\{Z_{k+2} < \sum_{j=1}^{k+1} Z_j\}_{k\geq 0}^{n-1}$ or their complements, from Lemma 1, it follows that the value of (3) is $1-2^{-(n+1)}$. If $i_{n+1} = m_n - 1$ then $\delta_{n+1} = 1$ and

$$\begin{aligned}\mathbb{P}(\pi(S_{n+1}) = i_{n+1} \mid (\pi(S_j))_{j=0}^n = (i_j)_{j=0}^n) \\ = \mathbb{P}(X_{M_n+1} - X_{i_n} > X_{i_n} - X_{m_n-1} \mid (\pi(S_j))_{j=0}^n = (i_j)_{j=0}^n).\end{aligned}$$

We deduce, again from Lemma 1, that the probability above is $2^{-(n+1)}$. Thus, we can write

$$\mathbb{P}(\pi(S_{n+1}) = i_{n+1} \mid (\pi(S_j))_{j=0}^n = (i_j)_{j=0}^n) = \left(1 - \frac{1}{2^{n+1}}\right)^{1-\delta_{n+1}} \left(\frac{1}{2^{n+1}}\right)^{\delta_{n+1}},$$

and the claim of the lemma follows. □

Corollary 1. *The expected number of times the sequence $(S_n)_{n\geq 1}$ changes sign is $\frac{1}{2}$.*

Proof. The sequence changes sign after visiting point S_n, $n \geq 1$, if $\delta_{n+1} = 1$. Since $\{\delta_n\}_{n\geq 1}$ are independent random variables with $\mathbb{P}(\delta_n = 1) = 2^{-n}$, the expected number of times the sequence changes sign is

$$\mathbb{E}\left(\sum_{n=2}^{\infty} \delta_n\right) = \sum_{n=2}^{\infty} \mathbb{P}(\delta_n = 1) = \sum_{n=2}^{\infty} \frac{1}{2^n} = \frac{1}{2}.$$ □

Remark 1. Corollary 1 implies that the number of times the sequence $(S_n)_{n\geq 0}$ changes sign is a.s. finite and, thus, the walk a.s. does not visit all the points of Π. This is another way to prove the fact that $\{S_1, S_2, \dots\} \neq \Pi$ a.s.

3. Distribution of the random variable N

In this section we study the distribution of the random variable N, the number of visited points of Π which are less than $S_0 = 0$. We can write $N = -\min_{n\geq 0} \pi(S_n)$. From (2), we know that N is a defective random variable with $\mathbb{P}(N = \infty) = \frac{1}{2}$ and the law of N, when N is finite, is given in the following theorem.

Theorem 1. *Let*

$$C(k, \ell) = \sum_{\substack{1\leq i_1 < j_1 < i_2 < j_2 < \cdots < i_\ell < j_\ell \\ \sum_{m=1}^{\ell}(j_m - i_m) = k}} \frac{1}{(2^{i_1}-1)(2^{j_1}-1)\cdots(2^{i_\ell}-1)(2^{j_\ell}-1)} \quad \text{for } k, \ell \geq 1.$$

Then,

$$\mathbb{P}(N = 0) = \prod_{n=1}^{\infty}\left(1 - \frac{1}{2^n}\right)$$

and, for $k \geq 1$,

$$\mathbb{P}(N = k) = \prod_{n=1}^{\infty}\left(1 - \frac{1}{2^n}\right) \sum_{\ell=1}^{k} C(k, \ell).$$

Moreover, for $k \geq 0$*,*

$$\mathbb{P}(N = k) = \frac{c}{2^k} + O\bigg(\frac{1}{2^{2k}}\bigg),$$

where c *is a positive constant that does not depend on* k*.*

Proof. If $N = 0$, the walk visits points $X_0, X_1, X_2, \dots$ successively. This implies that $(\pi(S_n))_{n\geq 0} = (n)_{n\geq 0}$ and the sequence $(\delta_n)_{n\geq 0}$, defined in Lemma 2, is identically 0. Now, the result follows directly from Lemma 2.

If $N = k \geq 1$, the set of indices of visited points is $\{\pi(S_0), \pi(S_1), \pi(S_2), \dots\} = \{-k, -k+1, -k+2, \dots\}$. Then there is ℓ, $1 \leq \ell \leq k$, and sequences $i_1, i_2, \dots, i_\ell$ and $j_1, j_2, \dots, j_\ell$ such that $0 < i_1 < j_1 < i_2 < j_2 < \cdots < i_\ell < j_\ell$ and the sequence $(S_n)_{n\geq 0}$ is negative when $i_m \leq n \leq j_m - 1$, $m \in \{1, 2, \dots, \ell\}$ and nonnegative otherwise. Since the walk visits exactly k points on the left, we have $\sum_{m=1}^{\ell}(j_m - i_m) = k$, $\delta_n = 1$ for $n \in \{i_1, i_2, \dots, i_\ell, j_1, j_2, \dots, j_\ell\}$ and $\delta_n = 0$ otherwise.

Therefore, from Lemma 2, we obtain

$$\begin{aligned}
&\mathbb{P}(N = k)\\
&\quad= \sum_{\substack{(i_n)_{n\geq 0}\in\mathcal{S}\\ \{i_0,i_1,i_2,\dots\}=\{-k,-k+1,-k+2,\dots\}}} \mathbb{P}((\pi(S_n))_{n\geq 0} = (i_n)_{n\geq 0})\\
&\quad= \sum_{\ell=1}^{k} \sum_{\substack{1\leq i_1<j_1<i_2<j_2<\cdots<i_\ell<j_\ell\\ \sum_{m=1}^{\ell}(j_m-i_m)=k}} \prod_{i\notin\{i_1,\dots,i_\ell,j_1,\dots,j_\ell\}} \bigg(1 - \frac{1}{2^i}\bigg) \prod_{i\in\{i_1,\dots,i_\ell,j_1,\dots,j_\ell\}} \frac{1}{2^i}\\
&\quad= \prod_{n=1}^{\infty}\bigg(1 - \frac{1}{2^n}\bigg) \sum_{\ell=1}^{k} \sum_{\substack{1\leq i_1<j_1<i_2<j_2<\cdots<i_\ell<j_\ell\\ \sum_{m=1}^{\ell}(j_m-i_m)=k}} \frac{1}{(2^{i_1}-1)(2^{j_1}-1)\cdots(2^{i_\ell}-1)(2^{j_\ell}-1)}\\
&\quad= \prod_{n=1}^{\infty}\bigg(1 - \frac{1}{2^n}\bigg) \sum_{\ell=1}^{k} C(k, \ell).
\end{aligned}$$

In order to find the asymptotic value of the expression above, we first obtain an upper bound for $2^k C(k, \ell)$ by using the inequalities $1/(2^k - 1) \leq 2/2^k$ and $1/(4^k - 1) \leq 2/4^k$. Thus,

$$\begin{aligned}
&2^k C(k, \ell)\\
&\quad= \sum_{\substack{1\leq i_1,i_2,\dots,i_\ell\\ 1\leq k_1,k_2,\dots,k_{\ell-1}\\ \sum_{i=1}^{\ell-1}k_i<k}} \frac{2^k}{(2^{i_1}-1)(2^{i_1+k_1}-1)\cdots(2^{i_1+\cdots+i_\ell+k_1+\cdots+k_{\ell-1}}-1)(2^{i_1+\cdots+i_\ell+k}-1)}\\
&\quad\leq \sum_{\substack{1\leq i_1,i_2,\dots,i_\ell\\ 1\leq k_1,k_2,\dots,k_{\ell-1}\\ \sum_{i=1}^{\ell-1}k_i<k}} \frac{2^{2\ell}}{2^{2\ell i_1}2^{2(\ell-1)i_2}\dots 2^{2i_\ell}2^{2(\ell-1)k_1}2^{2(\ell-2)k_2}\dots 2^{2k_{\ell-1}}}\\
&\quad= \frac{2^{2\ell}}{(4^\ell-1)(4^{\ell-1}-1)^2(4^{\ell-2}-1)^2\cdots(4-1)^2}
\end{aligned}$$

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$$\begin{aligned}&\le \frac{2^{2\ell}2^{2\ell-1}}{4^{\ell^2}}\\&= \frac{1}{2^{2\ell^2-4\ell+1}}.\end{aligned} \tag{4}$$

Let

$$c_\ell = \sum_{1\le i_1,i_2,\dots,i_\ell} \frac{1}{2^{i_1+\cdots+i_\ell}} \sum_{1\le k_1,k_2,\dots,k_{\ell-1}} \frac{1}{(2^{i_1}-1)(2^{i_1+k_1}-1)\cdots(2^{i_1+\cdots+i_\ell+k_1+\cdots+k_{\ell-1}}-1)}.$$

Similarly as in (4) one can show that $c_\ell \le 1/(2^{2\ell^2-4\ell+2})$ and, moreover,

$$\begin{aligned}
&|c_\ell - 2^k C(k,l)|\\
&\quad\le \sum_{1\le i_1,i_2,\dots,i_\ell} \frac{1}{2^{i_1+\cdots+i_\ell}(2^{i_1+\cdots+i_\ell+k}-1)}\\
&\qquad\times \sum_{\substack{1\le k_1,k_2,\dots,k_{\ell-1}\\ \sum_{i=1}^{\ell-1}k_i<k}} \frac{1}{(2^{i_1}-1)(2^{i_1+k_1}-1)\cdots(2^{i_1+\cdots+i_\ell+k_1+\cdots+k_{\ell-1}}-1)}\\
&\qquad+ \sum_{1\le i_1,i_2,\dots,i_\ell} \frac{1}{2^{i_1+\cdots+i_\ell}}\\
&\qquad\quad\times \sum_{\substack{1\le k_1,k_2,\dots,k_{\ell-1}\\ \sum_{i=1}^{\ell-1}k_i\ge k}} \frac{1}{(2^{i_1}-1)(2^{i_1+k_1}-1)\cdots(2^{i_1+\cdots+i_\ell+k_1+\cdots+k_{\ell-1}}-1)}\\
&\quad\le \frac{1}{2^k}\sum_{1\le i_1,i_2,\dots,i_\ell}\sum_{1\le k_1,k_2,\dots,k_{\ell-1}} \frac{2^{2\ell}}{2^{(2\ell+1)i_1}2^{(2\ell-1)i_2}\cdots 2^{3i_\ell}2^{2(\ell-1)k_1}2^{2(\ell-2)k_2}\cdots 2^{2k_{\ell-1}}}\\
&\qquad+ \sum_{1\le i_1,i_2,\dots,i_\ell}\sum_{\substack{1\le k_1,k_2,\dots,k_{\ell-1}\\ \sum_{i=1}^{\ell-1}k_i\ge k}} \frac{2^{2\ell-1}}{2^{2\ell i_1}2^{2(\ell-1)i_2}\cdots 2^{2i_\ell}2^{2(\ell-1)k_1}2^{2(\ell-2)k_2}\cdots 2^{2k_{\ell-1}}}\\
&\quad\le \frac{1}{2^k}\frac{1}{2^{2\ell^2-3\ell+1}}\\
&\qquad+ \frac{1}{2^{2k}}\frac{2^{2\ell-1}}{(4^\ell-1)(4^{\ell-1}-1)\cdots(4-1)} \sum_{1\le k_1,k_2,\dots,k_{\ell-2}} \frac{2}{2^{2(\ell-2)k_1}2^{2(\ell-3)k_2}\cdots 2^{2k_{\ell-2}}}\\
&\quad\le \frac{1}{2^k}\frac{1}{2^{2\ell^2-3\ell+1}} + \frac{1}{2^{2k}}\frac{1}{2^{2\ell^2-6\ell+4}}\\
&\quad= O\bigg(\frac{1}{2^k}\frac{1}{2^{\ell^2}}\bigg).
\end{aligned} \tag{5}$$

Since $c_\ell \le 1/(2^{2\ell^2-4\ell+2})$ then $\sum_{\ell=k+1}^{\infty} c_\ell \le 1/(2^{2k^2-1})$. This, together with (5), gives

$$2^k\mathbb{P}(N=k) = \prod_{n=1}^{\infty}\bigg(1-\frac{1}{2^n}\bigg)\sum_{\ell=1}^{k}2^kC(k,\ell) = \prod_{n=1}^{\infty}\bigg(1-\frac{1}{2^n}\bigg)\sum_{\ell=1}^{k}c_\ell + O\bigg(\frac{1}{2^k}\bigg) = c + O\bigg(\frac{1}{2^k}\bigg),$$

where $c = \prod_{n=1}^{\infty}(1-1/2^n)\sum_{\ell=1}^{\infty}c_\ell$. □

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4. Distribution of the random variable L

In this section we study the distribution of the index of the last point in the sequence $(S_n)_{n\geq 0}$, which is less than or equal to $S_0 = 0$; that is, the distribution of the random variable $L = \max\{n\colon S_n \leq 0\}$. Again, from (2), we have $\mathbb{P}(L = \infty) = \frac{1}{2}$. Just as in the previous section, we find the exact distribution of this random variable as well as an asymptotic result.

Theorem 2. *We have* $\mathbb{P}(L = 0) = \prod_{n=1}^{\infty}(1 - 1/2^n)$ *and, for* $k \geq 1$,

$$\mathbb{P}(L = k) = \frac{1}{2^{k+2}} \prod_{n=k+2}^{\infty}\bigg(1 - \frac{1}{2^n}\bigg).$$

Moreover, for $k \geq 0$,

$$\mathbb{P}(L = k) = \frac{1}{2^{k+2}} + O\bigg(\frac{1}{2^{2k}}\bigg).$$

Proof. The value of $\mathbb{P}(L = 0)$ follows directly from Theorem 1 since $\{N = 0\} = \{L = 0\}$.

If $L = k$, $k \geq 1$, for the sequence $(\pi(S_n))_{n\geq 0} \in \mathcal{S}$ it holds that $\pi(S_k) < 0$ and $\pi(S_{k+\ell}) > 0$ for all $\ell \geq 1$. Thus, for the corresponding sequence $(\delta_n)_{n\geq 1}$, which was defined in Lemma 2, we have $\delta_{k+1} = 1$ and $\delta_{k+\ell+1} = 0$ for all $\ell \geq 1$. The only constraint for the first k members in both sequences, other than the constraints given by the definition of the sequences, is that $\pi(S_k) < 0$. This has probability $\frac{1}{2}$ due to symmetry. Thus, by Lemma 2, we have

$$\begin{aligned}\mathbb{P}(L = k) &= \sum_{\substack{(i_n)_{n\geq 0}\in\mathcal{S} \\ i_k<0,\, i_{k+\ell}>0 \text{ for } \ell\geq 1}} \mathbb{P}((\pi(S_n))_{n\geq 0} = (i_n)_{n\geq 0}) \\ &= \frac{1}{2} \sum_{\delta_1,\delta_2,\dots,\delta_k\in\{0,1\}} \prod_{n=1}^{k}\bigg(1 - \frac{1}{2^n}\bigg)^{1-\delta_n}\bigg(\frac{1}{2^n}\bigg)^{\delta_n} \frac{1}{2^{k+1}} \prod_{n=k+2}^{\infty}\bigg(1 - \frac{1}{2^n}\bigg) \\ &= \frac{1}{2^{k+2}} \prod_{n=k+2}^{\infty}\bigg(1 - \frac{1}{2^n}\bigg).\end{aligned}$$

Furthermore, we can write

$$\mathbb{P}(L = k) = \frac{1}{2^{k+2}}\bigg(1 - \sum_{n=k+2}^{\infty}\frac{1}{2^n} + \cdots\bigg) = \frac{1}{2^{k+2}}\bigg(1 - \frac{1}{2^{k+1}} + \cdots\bigg) = \frac{1}{2^{k+2}} + O\bigg(\frac{1}{2^{2k}}\bigg). \qquad \square$$

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