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HIGHER GENERATING SUBGROUPS AND COHEN–MACAULAY COMPLEXES

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Abstract We show how to find higher generating families of subgroups, in the sense of Abels and Holz, for groups acting on Cohen–Macaulay complexes. We apply this to groups with a BN-pair to prove higher generation by parabolic and Levi subgroups and describe higher generating families of parabolic subgroups in $\mathrm{Aut}(F_n)$.

Keywords: higher generating subgroups; Cohen–Macaulay complexes; groups with BN-pair; Tits buildings; automorphism groups of free groups

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1. Introduction

For a group G with a family of subgroups $\mathcal{H}$, Abels and Holz in [**1**] defined the notion of 'higher generation' of G by $\mathcal{H}$. Whether or not G is highly generated by $\mathcal{H}$ depends on the connectivity properties of the nerve of the covering of G given by the set of cosets $\{gH \mid g \in G,\ H \in \mathcal{H}\}$ (for precise definitions, see § 2.1).

Abels and Holz connected higher generating families to the finiteness properties of groups; recent work in this direction can be found in [**15**]. Furthermore, in [**13**], they were used to study the BNS-invariants of right-angled Artin groups; higher generation also arises in the context of Deligne complexes [**9**, **8**, Example A.7] and braid groups [**8**]. However, the best-known example of higher generating families is probably that given in [**1**, Theorem 3.3]: the set of parabolic subgroups forms a higher generating family for any group with a BN-pair. To show this, Abels and Holz use the theory of Tits buildings.

The aim of this note is to show that with small adjustments, the result of Abels and Holz can be extended to the general setting of groups acting appropriately on Cohen–Macaulay complexes. Cohen–Macaulayness is a combinatorial property of simplicial complexes defined via local connectivity conditions (see § 2.2). Our main result is Theorem 2.10, which gives a criterion for obtaining higher generating families from group actions on Cohen–Macaulay complexes. We also give a characterization of the class of pairs $(G, \mathcal{H})$ which can be obtained that way in Theorem 2.12.

As an application of this, we construct new higher generating families. The first one is the family of Levi subgroups in groups with a BN-pair (see Theorem 3.3 and

Corollary 3.4), and the second one is the family of 'parabolic' subgroups of $\mathrm{Aut}(F_n)$, the automorphism group of the free group (see Definition 3.7 and Theorem 3.8). The corresponding Cohen–Macaulay complexes are the opposition complex and the free factor complex, respectively.

2. Definitions and general results

Throughout this text, we will often identify a simplicial complex X and its geometric realization $\|X\|$ if what is meant is clear from the context.

2.1. Higher generating subgroups

Definition 2.1. Let X be a set and $\mathcal{U}$ a collection of subsets of X such that $\mathcal{U}$ covers X. Then the *nerve* $N(\mathcal{U})$ of the cover $\mathcal{U}$ is the simplicial complex that has vertex set $\mathcal{U}$, where the vertices $U_0, \dots, U_k \in \mathcal{U}$ form a simplex if and only if $U_0 \cap \dots \cap U_k \neq \emptyset$.

Definition 2.2. Let G be a group and $\mathcal{H}$ a family of subgroups of G.

1. The collection of cosets $\mathcal{U} := \{gH \mid g \in G,\ H \in \mathcal{H}\}$ is a covering of G, and we define the *coset complex* $\mathrm{CC}(G, \mathcal{H})$ to be the nerve $N(\mathcal{U})$. This complex is endowed with a natural action of G given by left multiplication.

2. We say that $\mathcal{H}$ is *m-generating* for G if $\mathrm{CC}(G, \mathcal{H})$ is $(m-1)$-connected, i.e. $\pi_i\,\mathrm{CC}(G, \mathcal{H}) = \{1\}$ for all $i < m$.

Interesting examples of coset complexes are given by the 'coset poset' of all subgroups of a finite group, as studied in [7, 17]. The term 'higher generating subgroups' was coined by Holz in [11] and is motivated by the following. The family $\mathcal{H}$ is 1-generating for G if and only if the union of the subgroups in $\mathcal{H}$ generates G. It is 2-generating if and only if G is the free product of the subgroups in $\mathcal{H}$ amalgamated along their intersections. Roughly speaking, the latter means that the union of the subgroups generates G and that all relations that hold in G follow from relations in these subgroups. 3-generation can similarly be defined using identities among relations (see [1, 2.8]).

Remark 2.3. The cosets $g_0H_0, \dots, g_kH_k$ with $g_i \in G$ and $H_i \in \mathcal{H}$ intersect non-trivially if and only if there is $g \in G$ such that

$$g_0H_0 \cap \dots \cap g_kH_k = g(H_0 \cap \dots \cap H_k).$$

Hence, the set of k-simplices of $\mathrm{CC}(G, \mathcal{H})$ is in bijection with the set

$$\{g(H_0 \cap \dots \cap H_k) \mid g \in G,\ H_i \in \mathcal{H},\ H_i \neq H_j \text{ for } i \neq j\}.$$

Assume that $\mathcal{H}$ is a finite family of subgroups of G. Then $\mathrm{CC}(G, \mathcal{H})$ has dimension $|\mathcal{H}| - 1$ and $\mathcal{H}$ itself is the vertex set of a *facet*, i.e. a maximal simplex, of the coset complex. We will write this facet as $C_{\mathcal{H}}$. This (and hence any other) facet is a *fundamental domain* for the action of G; this means that for all $0 \leq k \leq |\mathcal{H}| - 1$, the set of k-faces of $C_{\mathcal{H}}$ contains exactly one element of each G-orbit of k-simplices of $\mathrm{CC}(G, \mathcal{H})$. The following converse of this observation is due to Zaremsky.

Lemma 2.4 (see [8, Proposition A.5]). *Let G be a group acting by simplicial automorphisms on a simplicial complex X, with a single facet C as fundamental domain. Let*

$$\mathcal{P} \coloneqq \{\mathrm{Stab}_G(v) \mid v \text{ is a vertex of } C\}.$$

Then the map

$$\begin{aligned}\psi : \mathrm{CC}(G, \mathcal{P}) &\to X\\ g\,\mathrm{Stab}_G(v) &\mapsto g.v\end{aligned}$$

is an isomorphism of simplicial G-complexes.

2.2. The Cohen–Macaulay property

For the remainder of this section, let $\mathbf{k}$ be a field or the ring of integers $\mathbb{Z}$.

Definition 2.5. Let X be a simplicial complex of dimension $d < \infty$. Then X is *Cohen–Macaulay over* $\mathbf{k}$ if it is $(d-1)$-acyclic over $\mathbf{k}$, i.e. $\tilde{H}_i(X, \mathbf{k}) = \{0\}$ for all $i < d$, and the link of every s-simplex is $(d-s-2)$-acyclic over $\mathbf{k}$.

X is *homotopy Cohen–Macaulay* if it is $(d-1)$-connected and the link of every s-simplex is $(d-s-2)$-connected.

The notion of Cohen–Macaulayness over $\mathbf{k}$ was introduced in the mid-1970s and came up in the study of finite simplicial complexes via their Stanley–Reisner rings (see [19]). The homotopical version was introduced by Quillen in [14]. While it can be shown that 'being Cohen–Macaulay over $\mathbf{k}$' depends only on the geometric realization $\|X\|$ and not on its specific triangulation, the homotopical version is not a topological invariant but a property of the simplicial complex X itself. One has implications:

$$\text{homotopy CM} \Rightarrow \text{CM over } \mathbb{Z} \Rightarrow \text{CM over any field } \mathbf{k},$$

which are all strict. For more details on Cohen–Macaulayness and its connections to other combinatorial properties of simplicial complexes, see [4]. We will talk about examples of complexes having these properties in §§ 2.3 and 3.

Definition 2.6. A topological space is *d-spherical* if it is homotopy equivalent to a wedge of d-spheres; as a convention, we consider a singleton to be homotopy equivalent to a (trivial) wedge of n-spheres for all n.

Remark 2.7. By the Whitehead theorem, a d-dimensional complex is d-spherical if and only if it is $(d-1)$-connected.

An advantage of a complex that is Cohen–Macaulay over one that is merely spherical is that it allows for inductive methods using its local structure. We will make use of this in the proof of the following lemma.

Lemma 2.8. *Let X be a d-dimensional complex and let $X_s \coloneqq \|X\| \setminus \|X^{(s)}\|$ denote the complement of the s-skeleton of $\|X\|$. The following hold true.*

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1. *If X is Cohen–Macaulay over* $\mathbf{k}$, *the homology with* $\mathbf{k}$*-coefficients of* X_s *is concentrated in dimension* $d-s-1$, *i.e.* $\tilde{H}_i(X_s, \mathbf{k})$ *is trivial if* $i \neq d-s-1$.

2. *If X is homotopy Cohen–Macaulay,* X_s *is* $(d-s-1)$*-spherical.*

Proof. The proofs of the two statements are completely parallel and will be done by induction on s. Setting $X_{-1} := \|X\|$, the statements hold for $s=-1$ as $\|X\|$ itself is assumed to be $(d-1)$-acyclic or $(d-1)$-connected, respectively. For all s, the space X_{s-1} is the union of X_s and the open s-simplices of $\|X\|$, so we will successively adjoin these simplices to X_s while keeping track of the homotopy type. Assume that we have already constructed X' as the union of X_s and a set of open s-simplices of $\|X\|$. Then for every s-simplex σ in $\|X\|$ that is not contained in X', there is an open contractible neighbourhood U of the interior of σ in $X'' := X' \cup \mathring{\sigma}$ such that $U \cap X' = U \setminus \mathring{\sigma}$ is homotopy equivalent to the link of σ in X. As X is Cohen–Macaulay, this link is $(d-s-2)$-acyclic in the homological and $(d-s-2)$-connected in the homotopical setting. This means that X'' can be constructed by gluing together X' and U, which is contractible, along the open subset $U \setminus \mathring{\sigma}$, which is $(d-s-2)$-acyclic or $(d-s-2)$-connected. Hence, the inclusion $X' \hookrightarrow X''$ induces for all $i \leq d-s-2$ an isomorphism on homology groups $\tilde{H}_i(\cdot, \mathbf{k})$ or homotopy groups π_i, respectively.

By induction, we can conclude that if X is Cohen–Macaulay over $\mathbf{k}$, we have $\{0\} = \tilde{H}_i(X_{s-1}, \mathbf{k}) \cong \tilde{H}_i(X_s, \mathbf{k})$, and if it is homotopy Cohen–Macaulay, we have $\{1\} = \pi_i(X_{s-1}) \cong \pi_i(X_s)$ for $i \leq d-s-2$. Noting that the complement of the s-skeleton of any simplicial complex of dimension d is homotopy equivalent to a complex of dimension $(d-s-1)$ (contract all the simplices of dimension $(s+1)$ to their barycentres), the result follows. □

A simplicial complex is called *pure* if all of its facets have the same dimension. A pure simplicial complex X is called a *chamber complex* (or *strongly connected*) if every pair of facets $\sigma, \tau \in X$ can be connected by a sequence of facets $\sigma = \tau_1, \dots, \tau_k = \tau$ such that for all $1 \leq i \leq k$, the intersection of τ_i and τ_{i+1} is a face of codimension 1. The facets of a chamber complex are also called *chambers*.

Remark 2.9. Every Cohen–Macaulay complex is pure and a chamber complex (see e.g. [**4**, Proposition 11.7]). The preceding lemma is a generalization of this well-known fact in the following sense. Let X be a pure, d-dimensional simplicial complex, $d \geq 1$. Define a graph Γ whose vertices are given by the facets of X and where two vertices are joined by an edge if and only if the corresponding facets intersect in a face of codimension 1. The graph Γ, which is also called the *chamber graph of* X, is homotopy equivalent to the complement of the $(d-2)$-skeleton of X. Furthermore, X is a chamber complex if and only if Γ is connected, which is equivalent to $\tilde{H}_0(\Gamma) = \{0\}$. So if we assume that X is Cohen–Macaulay, Lemma 2.8 implies that it is a chamber complex.

2.3. Higher generation by actions on Cohen–Macaulay complexes

We now want to combine Lemma 2.4 with the observations of the preceding subsection in order to obtain higher generating families of subgroups for groups acting on Cohen–Macaulay complexes. Our main result here is as follows.

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Theorem 2.10. *Let G be a group acting by simplicial automorphisms on a simplicial complex X, with a single facet C as fundamental domain. If X is homotopy Cohen–Macaulay and has dimension d, the set*

$$\mathcal{P}_k := \{\mathrm{Stab}_G(\sigma) \mid \sigma \text{ is a } k\text{-dimensional face of } C\}$$

is $(d-k)$-generating for all $0 \leq k \leq d$. Furthermore, the corresponding coset complex $\mathrm{CC}(G, \mathcal{P}_k)$ is $(d-k)$-spherical.

Proof. By Lemma 2.4, we can identify X with the coset complex $\mathrm{CC}(G, \mathcal{P}_0)$.

As C is a fundamental domain for the action of G, the stabilizer of a k-face F of C is equal to the intersection of the stabilizers of all the vertices of F. Hence, the elements of $\mathcal{P}_k$ are given by all the intersections of $(k+1)$ pairwise distinct elements from $\mathcal{P}_0$.

By Remark 2.3, the vertices of $\mathrm{CC}(G, \mathcal{P}_k)$ are in one-to-one correspondence with the k-simplices of $\mathrm{CC}(G, \mathcal{P}_0) \cong X$. Moreover, a set of vertices in $\mathrm{CC}(G, \mathcal{P}_k)$ forms a simplex if and only if the corresponding k-simplices in X are all faces of one common facet. It follows that the geometric realization $\|\mathrm{CC}(G, \mathcal{P}_k)\|$ is homotopy equivalent to $\|Y\|$, where Y is the induced subcomplex of the barycentric subdivision $\mathcal{B}(X)$ whose vertices are the barycentres of all simplices of X that have dimension greater or equal to k.

The complex $\|Y\|$ is homotopy equivalent to the complement of the $(k-1)$-skeleton of $\|X\|$. As X is Cohen–Macaulay, we can use Lemma 2.8 to conclude that $\mathrm{CC}(G, \mathcal{P}_k)$ is $(d-k)$-spherical. This finishes the proof. □

Before we apply this theorem to obtain higher generating families of subgroups for specific examples in the next section, we now characterize the class of pairs $(G, \mathcal{H})$ which can be obtained using Theorem 2.10. By Lemma 2.4, the conditions of Theorem 2.10 are fulfilled if and only if $\mathrm{CC}(G, \mathcal{P}_0)$ is homotopy Cohen–Macaulay. We will give an alternative characterization of this condition for coset complexes.

A pure simplicial complex X of dimension d is called *coloured* (or *completely balanced*) if there is a map $c : X^{(0)} \to \{0, \ldots, d\}$ restricting to a bijection on each facet. In this setting, for each $J \subseteq \{0, \ldots, d\}$, let X_J be the induced subcomplex of X with vertex set $c^{-1}(J)$. As stated below, the following result is due to Walker.

Theorem 2.11 (see [5, Theorem 5.2, 4, Theorem 11.14]). *Let X be a pure d-dimensional coloured complex. Then X is Cohen–Macaulay over* $\mathbf{k}$ *if and only if X_J is $(|J|-2)$-acyclic over* $\mathbf{k}$ *for every $J \subseteq \{0, \ldots, d\}$. It is homotopy Cohen–Macaulay if and only if X_J is $(|J|-2)$-connected for every $J \subseteq \{0, \ldots, d\}$.*

Every finite-dimensional coset complex is a pure simplicial complex which can be given a colouring

$$c : \mathrm{CC}(G, \{H_0, \ldots, H_d\}) \to \{0, \ldots, d\}$$

by setting $c(gH_i) := i$. Hence, the following is an immediate consequence of Theorem 2.11.

Theorem 2.12. *Let G be a group and $\mathcal{H}$ a finite family of subgroups of G.*

1. $\mathrm{CC}(G, \mathcal{H})$ *is Cohen–Macaulay over* $\mathbf{k}$ *if and only if for all $\mathcal{H}' \subseteq \mathcal{H}$, the coset complex $\mathrm{CC}(G, \mathcal{H}')$ is $(|\mathcal{H}'|-2)$-acyclic over* $\mathbf{k}$.

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2. $\mathrm{CC}(G, \mathcal{H})$ *is homotopy Cohen–Macaulay if and only if every* $\mathcal{H}' \subseteq \mathcal{H}$ *is* $(|\mathcal{H}'|-1)$*-generating for* G.

Being a coset complex imposes rather strong restrictions. In addition to being coloured, every such complex is endowed with a facet transitive group action. One might ask whether in this setting, Cohen–Macaulayness implies already stronger combinatorial conditions such as shellability. A finite complex is *shellable* if and only if the set of its facets admits a sufficiently nice ordering, called a *shelling*; for the precise definition, see [4, § 11.2]. In general, being shellable is strictly stronger than being homotopy Cohen–Macaulay. Buildings form a class of coset complexes which are shellable (see § 3 and [3]). The following example, however, shows that there are also coset complexes which are Cohen–Macaulay over $\mathbb{Z}$ but are not homotopy Cohen–Macaulay, and so in particular are not shellable.

Let Alt_5 be the alternating group on the set $\{1, 2, 3, 4, 5\}$ and consider the following subgroups:

$$H_1 \coloneqq \mathrm{Stab}_{\mathrm{Alt}_5}(\{2\}),$$
$$H_2 \coloneqq N_{\mathrm{Alt}_5}(\langle(1, 2, 3, 4, 5)\rangle),$$
$$H_3 \coloneqq N_{\mathrm{Alt}_5}(\langle(1, 3, 5)\rangle),$$

where $\mathrm{Stab}_{\mathrm{Alt}_5}$ and N_{Alt_5} denote the stabilizer and normalizer in Alt_5. The group H_1 is isomorphic to Alt_4, and H_2 and H_3 are isomorphic to the dihedral groups D_5 and D_3, respectively. Let $\mathcal{H} \coloneqq \{H_1, H_2, H_3\}$. The coset complex $\mathrm{CC}(\mathrm{Alt}_5, \mathcal{H})$ has dimension 2 and consists of 21 vertices, 80 edges and 60 two-simplices. This complex was first found by Oliver; an explicit description of it as a coset complex can be found in [16]. For further details and a picture, see [12, § 7.3]; note that $\mathrm{CC}(\mathrm{Alt}_5, \mathcal{H})$ is isomorphic to the complex N_0 in [12].

Lemma 2.13. *The coset complex* $\mathrm{CC}(\mathrm{Alt}_5, \mathcal{H})$ *is Cohen–Macaulay over* $\mathbb{Z}$ *but is not homotopy Cohen–Macaulay.*

Proof. In [12], Lutz shows that $\|\mathrm{CC}(\mathrm{Alt}_5, \mathcal{H})\|$ is homeomorphic to a cell complex Q obtained by taking the boundary of a dodecahedron and identifying opposite pentagons by a coherent twist of $\pi/5$. The complex Q arises in triangulations of the Poincaré homology 3-sphere Σ^3. It is $\mathbb{Z}$-acyclic and one has $\pi_1(Q) \cong \pi_1(\Sigma^3)$ (see [6, p. 57]). As this fundamental group is non-trivial, Q and therefore $\mathrm{CC}(\mathrm{Alt}_5, \mathcal{H})$ cannot be homotopy Cohen–Macaulay.

It remains to show that $\mathrm{CC}(\mathrm{Alt}_5, \mathcal{H})$ is Cohen–Macaulay over $\mathbb{Z}$. By Theorem 2.12, it suffices to show that for all $\mathcal{H}' \subseteq \mathcal{H}$, the complex $\mathrm{CC}(\mathrm{Alt}_5, \mathcal{H}')$ is $(|\mathcal{H}'|-2)$-acyclic. For $\mathcal{H}' = \mathcal{H}$, this is true as Q is $\mathbb{Z}$-acyclic, and for $|\mathcal{H}'| = 1$, there is nothing to show. Hence, one only needs to check that for all two-element subsets $\mathcal{H}'$ of $\mathcal{H}$, the corresponding subcomplex of $\mathrm{CC}(\mathrm{Alt}_5, \mathcal{H})$ is connected. This can easily be verified, e.g. by using Figure 7.5 of [12]. □

A further question in the same direction which might be interesting to consider is whether every coset complex that is *homotopy* Cohen–Macaulay is already shellable.

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A counterexample to that (if existent) would have to be a pure, completely balanced simplicial complex with a facet-transitive group action that is homotopy Cohen–Macaulay but not shellable. To us, it seems likely that such a complex exists, but we are not currently aware of any examples.

3. Applications

In what follows, we give three applications of Theorem 2.10. All of them are either directly or in spirit connected to the theory of buildings. The first two examples make direct use of this theory; definitions and background material needed for these subsections can be found in [**2**]. The third application concerns higher generating families of subgroups in $\mathrm{Aut}(F_n)$, the automorphism group of the free group.

3.1. Parabolic subgroups and buildings

Our first application recovers [**1**, Theorem 3.3] of Abels and Holz. We will be brief here and refer to their text for further details.

Let G be a group with a BN-pair. Denote by Δ the corresponding building and by $Ch(\Delta)$ the set of its chambers. If the corresponding Weyl group W has rank r, this building is homotopy equivalent to a wedge of $(r-1)$-spheres by the Solomon–Tits theorem (see [**18**]); it is in fact contractible if W is infinite. The link of a simplex of dimension k in Δ is again a building of rank $r-k-1$, which implies that Δ is homotopy Cohen–Macaulay.

The action of G is transitive on the chambers of Δ, so we can apply Theorem 2.10 to deduce that for any choice of chamber $C \in Ch(\Delta)$, the family $\mathcal{P}_k$ of stabilizers of the k-dimensional faces of C is $(r-1-k)$-generating for G. If we take C to be the 'fundamental' chamber associated with the Borel subgroup B, these stabilizers are exactly the *standard parabolic subgroups of rank* $r-k-1$. Hence we get the following.

Theorem 3.1 (see [1, Theorem 3.3]). *The family of rank-m standard parabolic subgroups is m-generating for G.*

3.2. Levi subgroups and the opposition complex

To show that the families of standard parabolic subgroups in a group G with a BN-pair are higher generating, we only needed to use the chamber transitivity of the action of G on the associated building. However, this action is known to satisfy stronger transitivity conditions; we will exploit these to find other families of higher generating subgroups in this subsection.

Let Δ be a spherical building. The chamber distance $d(-,-)$ induces an *opposition relation* op between chambers of Δ which is defined by

$$C \,\mathrm{op}\, C' :\Leftrightarrow d(C,C') = \max\{d(C_1,C_2) \mid C_1, C_2 \in Ch(\Delta)\}.$$

This opposition relation can be extended to arbitrary simplices $\sigma, \sigma' \in \Delta$ of equal dimension by saying that σ is opposite to σ' if and only if the following holds true.

> *For every chamber $C \geq \sigma$ in Δ, there is a chamber $C' \geq \sigma'$ such that $C \,\mathrm{op}\, C'$, and for every chamber $C' \geq \sigma'$, there is a chamber $C \geq \sigma$ such that $C \,\mathrm{op}\, C'$.*

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Using this opposition relation, one can define a new complex from Δ as follows.

Definition 3.2. The *opposition complex* $\mathrm{Opp}(\Delta)$ is the simplicial complex whose simplices are of the form (σ,σ') with $\sigma,\sigma'\in\Delta$, $\sigma\,\mathrm{op}\,\sigma'$, where the face relation is given by

$$(\tau,\tau')\leq(\sigma,\sigma'):\Leftrightarrow\tau\leq\sigma\quad\text{and}\quad\tau'\leq\sigma'.$$

$\mathrm{Opp}(\Delta)$ has the same dimension as Δ, and it was shown to be homotopy Cohen–Macaulay by von Heydebreck in [20]. The complex is pure and its facets are given by pairs (C,C') of opposite chambers $C,C'\in Ch(\Delta)$.

Every building Δ comes with a map

$$\delta : Ch(\Delta)\times Ch(\Delta)\to W,$$

where W is the Weyl group of Δ. This function is called the *Weyl distance function* (of Δ), and it is related to the gallery distance as follows:

$$d(C,C')=l_S(\delta(C,C')),$$

where l_S denotes the Coxeter length function on W. If a group acts by type-preserving automorphisms on Δ, we say that the action is *Weyl transitive* if for each $w\in W$, the action is transitive on the set of order pairs of chambers (C,C') with $\delta(C,C')=w$.

Theorem 3.3. *Let G be a group acting Weyl transitively by type-preserving automorphisms on a spherical building Δ of dimension d. Choose any pair (C,C') of opposite chambers $C,C'\in Ch(\Delta)$. Then the set*

$$\mathcal{P}_k:=\{\mathrm{Stab}_G(\sigma)\cap\mathrm{Stab}_G(\sigma')\mid\sigma,\sigma'\,k\text{-dimensional faces of }C,C';\ \sigma\,\mathrm{op}\,\sigma'\}$$

is $(d-k)$-generating for G.

Proof. As the action of G on Δ preserves distances and adjacency relations, it induces a simplicial action on $\mathrm{Opp}(\Delta)$ given by

$$g.(\sigma,\sigma'):=(g.\sigma,g.\sigma').$$

We claim that the simplex $(C,C')\in\mathrm{Opp}(\Delta)$ is a fundamental domain for this action of G. Because Δ is spherical, its Weyl group W is finite and has a unique element w_S of maximal length. Hence, two chambers $D,D'\in Ch(\Delta)$ are opposite to each other if and only if $\delta(D,D')=w_S$, and by Weyl transitivity, G acts transitively on such pairs of opposite chambers. This implies that the set of vertices of (C,C') contains a representative of each G-orbit of vertices in $\mathrm{Opp}(\Delta)$. Furthermore, the type of any vertex of the chamber C is preserved by all the elements of G. Hence, no two distinct vertices of (C,C') lie in the same G-orbit, which proves that this facet is indeed a fundamental domain.

As a consequence, Theorem 2.10 shows that the set $\mathcal{P}_k$ of stabilizers of k-simplices in $\mathrm{Opp}(\Delta)$ is $(d-k)$-generating. Since a k-simplex in $\mathrm{Opp}(\Delta)$ is a pair (σ,σ') of k-simplices $\sigma,\sigma'\in\Delta$, this finishes the proof. □

In particular, the conditions of the preceding theorem are fulfilled in the following situation. If G is a group having a BN-pair of rank r with finite Weyl group $W=\langle S\rangle$,

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it acts Weyl transitively on the associated spherical building. The chambers associated with B and $B^- = w_S B w_S$ are opposite to each other and, after setting $C := B$ and $C' := B^-$, the family $\mathcal{P}_k$ defined in Theorem 3.3 is the set of *standard rank-*$(r-k-1)$ *Levi subgroups*. We state this as follows.

Corollary 3.4. *Let* (G, B, N, S) *be a Tits system with finite Weyl group. Then the family of standard rank-*m *Levi subgroups is* m*-generating for* G.

Example 3.5. As an illustration, we spell out the following special case of this result. If Δ is the flag complex of proper subspaces of the vector space $\mathbf{k}^n$, i.e. a building of type A_{n-1}, the opposition complex $\mathrm{Opp}(\Delta)$ is the complex with vertex set

$$\{(U, U') \mid U,\ U' \text{ are proper subspaces of } \mathbf{k}^n \text{ and } U \oplus U' = \mathbf{k}^n\},$$

in which $(U_0, U_0'), \ldots, (U_k, U_k')$ form a simplex if and only if (possibly after reordering) one has $U_0 < U_1 < \cdots < U_k$ and $U_0' > U_1' > \cdots > U_k'$.

Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbf{k}^n$. The flags

$$C := \langle e_1 \rangle < \langle e_1, e_2 \rangle < \cdots < \langle e_1, \ldots, e_{n-1} \rangle \quad \text{and}$$
$$C' := \langle e_2, \ldots, e_n \rangle > \langle e_3, \ldots, e_n \rangle > \cdots > \langle e_n \rangle$$

form opposite chambers of Δ. The building Δ has dimension $n-2$ and $\mathrm{GL_n}(\mathbf{k})$ acts Weyl transitively on it. The corresponding family of stabilizers $\mathcal{P}_k$ with $0 \le k \le n-3$ consists of all subgroups of the form

$$\begin{pmatrix} \mathrm{GL_{n_1}}(\mathbf{k}) & 0 & \cdots & 0 \\ 0 & \mathrm{GL_{n_2}}(\mathbf{k}) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \mathrm{GL_{n_{k+2}}}(\mathbf{k}) \end{pmatrix} \le \mathrm{GL_n}(\mathbf{k}).$$

So the number of blocks in the corresponding matrices is $k+2$, and the n_i are natural numbers such that $\sum_{i=1}^{k+2} n_i = n$. These are exactly the standard rank-$(n-2-k)$ Levi subgroups of $\mathrm{GL_n}(\mathbf{k})$, and by Theorem 3.3 this family is $(n-2-k)$-generating.

3.3. Parabolics in $\mathrm{Aut}(F_n)$ and the free factor complex

Hatcher and Vogtmann in [10] defined a simplicial complex associated with $\mathrm{Aut}(F_n)$, the automorphism group of the free group on n letters. It is defined similarly to and shares many properties with the building associated with $\mathrm{GL_n}(\mathbb{Z}) = \mathrm{Aut}(\mathbb{Z}^n)$.

Definition 3.6. A subgroup A of F_n is called a *free factor* if there is a subgroup $B \le F_n$ such that F_n can be written as a free product $F_n = A * B$.

The *free factor complex* FC_n is the simplicial complex whose vertices are proper free factors of F_n, where the vertices $H_0, \ldots, H_k$ form a simplex if and only if they form a flag $H_0 \le H_1 \le \cdots \le H_k$.

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FC_n is a chamber complex of dimension $n-2$ and comes with a simplicial action of $\mathrm{Aut}(F_n)$ given by $g.(H_0 < H_1 < \cdots < H_k) := g(H_0) < g(H_1) < \cdots < g(H_k)$. A fundamental domain for this action is given by any maximal flag $H_0 < \cdots < H_{n-2}$ of free factors in F_n.

Definition 3.7. Fix a basis $b_1, \dots, b_n$ of F_n. This gives rise to a 'standard flag'

$$C := \langle b_1 \rangle < \langle b_1, b_2 \rangle < \cdots < \langle b_1, \dots, b_{n-1} \rangle$$

of free factors in F_n. Now, analogous to the situation in buildings, we define a *standard rank-m parabolic subgroup* to be the stabilizer of a sub-flag of C that has length $n-m-1$. To match the numbering of § 2.3, we use the corank to number the parabolic subgroups and define $\mathcal{P}_{n-m-2}$ to be the set of standard rank-m parabolics.

Again, we use Theorem 2.10 to show the following.

Theorem 3.8. *For all $1 \le m \le n-2$, the family $\mathcal{P}_{n-m-2}$ of standard rank-m parabolic subgroups is m-generating for* $\mathrm{Aut}(F_n)$. *The corresponding coset complex* $\mathrm{CC}(\mathrm{Aut}(F_n), \mathcal{P}_{n-m-2})$ *is m-spherical.*

Proof. As noted above, $\mathrm{Aut}(F_n)$ acts on the free factor complex with the facet $C = \langle b_1 \rangle < \langle b_1, b_2 \rangle < \cdots < \langle b_1, \dots, b_{n-1} \rangle$ as a fundamental domain. The standard rank-m parabolics are exactly the stabilizers of the $(n-m-2)$-faces of C. Since Hatcher and Vogtmann showed that FC_n is homotopy Cohen–Macaulay (see [**10**, § 4]), the statement is an immediate consequence of Theorem 2.10. □

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