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REVIEWS

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ANDREW MARKS AND SPENCER UNGER, *Borel circle squaring*, ***Annals of Mathematics***, (2017), no. 186, pp. 581–605.

The article provides a constructive Borel solution to a problem posed by Tarski in 1925 of whether a disk in $\mathbb{R}^2$ can be partitioned into finitely many pieces $A_1, \dots, A_n$ such that there are isometries $\gamma_1, \dots, \gamma_n$ of the plane so that $\gamma_1 \cdot A_1, \dots, \gamma_n \cdot A_n$ partition a square of the same area. More generally, the authors show that for every $k \geq 1$ any two bounded Borel sets $A, B \subseteq \mathbb{R}^k$ of the same positive Lebesgue measure whose boundaries have upper Minkowski dimension less than k are *Borel equidecomposable* using translations, specifically that A can be decomposed into finitely many Borel pieces $A_1, \dots, A_n$ such that there are translations $\gamma_1, \dots, \gamma_n$ so that $\gamma_1 \cdot A_1, \dots, \gamma_n \cdot A_n$ partition B. The pieces are rather simple, in the case of a disk and a square they are of complexity Σ^0_4. In 1990 Laczkovich gave a positive nonconstructive solution to Tarski's problem, and very recently Grabowski–Máthé–Pikhurko provided nonconstructive (using the axiom of choice) Lebesgue measurable and Baire measurable solutions to Tarski's problem.

The Banach–Tarski paradox says that the unit ball in $\mathbb{R}^3$ can be partitioned into finitely many pieces, which can be rearranged by isometries to form two unit balls. Moreover, Banach and Tarski showed that any two bounded sets in $\mathbb{R}^3$ with nonempty interiors are equidecomposable. The situation is very different in $\mathbb{R}^2$, since the Lebesgue measure on $\mathbb{R}^2$ can be extended to a finitely additive isometry-invariant measure on $\mathbb{R}^2$, and hence any two sets in $\mathbb{R}^2$ which are equidecomposable are of the same measure.

A *paradoxical decomposition* of an action of a group H on a set X is a finite partition $\{A_1, \dots, A_m, B_1, \dots, B_n\}$ of X together with $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n \in H$ such that each of $\{\alpha_1 \cdot A_1, \dots, \alpha_m \cdot A_m\}$ and $\{\beta_1 \cdot B_1, \dots, \beta_n \cdot B_n\}$ is a partition of X. The Banach–Tarski paradox says that the group of isometries acting on the unit ball in $\mathbb{R}^3$ admits a paradoxical decomposition. The concept of a paradoxical decomposition is strongly related to amenability and to perfect matching for graphs. An action of a group H on a set X does *not* have a paradoxical decomposition if and only if it is amenable, i.e., it admits a finitely additive invariant measure μ on all subsets of X such that $\mu(X) = 1$. Furthermore, if S is a finite symmetric set in a group H acting on a set X, we can form a bipartite graph G_S with vertex set $\{0, 1, 2\} \times X$, where there is an edge from (i, x) to (j, y) if and only if exactly one of i and j is equal to 0, and there is an $s \in S$ such that $s \cdot x = y$. Then it is not hard to see that G_S has a perfect matching if and only if the action of H on X admits a paradoxical decomposition using elements from S. Hall's marriage theorem provides a characterization of bipartite graphs which have a perfect matching.

Let us describe the main ideas of the proof in the article under review. Take bounded Borel sets $A, B \subseteq \mathbb{R}^k$ of equal measure whose boundaries have upper Minkowski dimension

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less than k. We can assume that A and B are subsets of the torus $\mathbb{T}^k$. We want to show that A and B are Borel equidecomposable by translations. Take a sufficiently large d and a generic $u \in (\mathbb{T}^k)^d$, and consider the action a_u of $\mathbb{Z}^d$ on $\mathbb{T}^k$ given by $(n_1, \ldots, n_d) \cdot_{a_u} x = n_1u_1 + \cdots + n_d u_d + x$. The important step now is to consider the graph G_{a_u} with vertex set $\mathbb{T}^k$ having an edge from x to y if and only if $y = (n_1, \ldots, n_d) \cdot_{a_u} x$, for some $n_1, \ldots, n_d$ with $\sup_i |n_i| = 1$. A crucial idea is to study Borel flows on the graph G_{a_u}. An f*-flow* on a locally finite graph G, where f is a given real-valued function on the set of vertices V of G, is a real-valued function ϕ on the set of edges of G such that $\phi(x, y) = -\phi(y, x)$ for every edge (x, y) of G, and such that for every $x \in V$, $f(x)$ is equal to the sum of all $\phi(x, y)$ such that y is adjacent to x.

Marks and Unger construct a Borel integer-valued f-flow on G_{a_u}, where $f = \chi_A - \chi_B$. We will discuss the ingredients of the construction of the flow below. The authors then turn the flow into a Borel equidecomposition of A and B as follows. First, they apply a lemma due to Gao and Jackson which says that for every n a Borel partition of $\mathbb{T}^k$ into sets which are approximately hypercubes, specifically, into sets of the form $\{(n_1, \ldots, n_d) \cdot x : 0 \leq n_i < N_i\}$, for some $x \in \mathbb{T}^k$, and $N_i = n$ or $N_i = n + 1$, $i = 1, \ldots, d$. This lemma was an important ingredient in Gao–Jackson's fundamental work establishing that a Borel action of a countable abelian group always induces a hyperfinite orbit equivalence relation. By the Laczkovich discrepancy estimates, every hypercube of side length n contains approximately $\lambda(A)n^d$ many elements of both A and B, where λ is the Lebesgue measure. For the partition taken for an appropriately large n, for any adjacent R and S in the partition, the authors consider the sum of all $\phi(x, y)$, $x \in R$, $y \in S$ and this quantity tells us exactly how many points to move from $A \cap R$ to $B \cap S$ or vice versa. After doing this each set in the partition has the same number of points of A and of B, we biject them to obtain the required equidecomposition. Marks and Unger remark that instead of the nontrivial Gao–Jackson tiling, one can use Voronoi cells together with a result of Kechris–Solecki–Todorcevic on the existence of a Borel maximal independent set in any locally finite Borel graph.

To obtain a Borel integer-valued f-flow on G_{a_u}, the authors first give an explicit construction of a Borel real-valued f-flow on G_{a_u}. Passing to an integer-valued Borel f-flow requires the Ford–Fulkerson algorithm of finite combinatorics, work of Timár on boundaries of finite sets in $\mathbb{Z}^d$, and a theorem of Gao, Jackson, Krohne, and Seward from 2015 on special types of witnesses to hyperfiniteness of free Borel actions of $\mathbb{Z}^d$. The tools used to prove the Gao–Jackson–Krohne–Seward theorem originate from the already mentioned Gao–Jackson's work.

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Aleksandra Kwiatkowska

Institut für Mathematische Logik und Grundlagenforschung, Universität Münster, and Instytut Matematyczny, Uniwersytet Wrocławski, kwiatkoa@uni-muenster.de.

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