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On nonmonogenic number fields defined by $x^6 + ax + b$

Anuj Jakhar and Surender Kumar

Abstract. Let q be a prime number and $K = \mathbb{Q}(\theta)$ be an algebraic number field with θ a root of an irreducible trinomial $x^6 + ax + b$ having integer coefficients. In this paper, we provide some explicit conditions on a, b for which K is not monogenic. As an application, in a special case when $a = 0$, K is not monogenic if $b \equiv 7 \bmod 8$ or $b \equiv 8 \bmod 9$. As an example, we also give a nonmonogenic class of number fields defined by irreducible sextic trinomials.

1 Introduction

In Algebraic Number Theory, it is an important problem to know whether a given algebraic number field is monogenic. This problem has been widely studied and of interest to several mathematicians (cf. [1, 2, 4–8, 10, 11]). Let K be an algebraic number field generated by a complex root θ of a monic irreducible polynomial $f(x)$ having degree n with coefficients from the ring $\mathbb{Z}$ of integers. Let $\mathbb{Z}_K$ denote the ring of algebraic integers of K. It is well-known that $\mathbb{Z}_K$ is a free abelian group of rank n. Let $\operatorname{ind}\theta$ denote the index of the subgroup $\mathbb{Z}[\theta]$ in $\mathbb{Z}_K$. If $\operatorname{ind}\alpha = 1$ for some $\alpha \in \mathbb{Z}_K$, then $\{1, \alpha, \ldots, \alpha^{n-1}\}$ is a power integral basis of $\mathbb{Z}_K$. In such a case, K is called monogenic. If there does not exist any such $\alpha \in \mathbb{Z}_K$, then K is called nonmonogenic. In 2016, Ahmad et al. [1] proved that the sextic number field generated by $m^{\frac{1}{6}}$ is not monogenic if $m \equiv 1 \bmod 4$ and $m \not\equiv \pm 1 \bmod 9$. In 2017, Gaál and Remete [8] obtained some new results on monogenity of number fields generated by $m^{\frac{1}{n}}$ with m a square free integer and $3 \le n \le 9$ by applying the explicit form of the index equation. In 2021, Yakkou and Fadil [11] studied the monogenity of number fields generated by $m^{\frac{1}{q^r}}$, where m is a square-free integer and q be a prime number. In this paper, based on prime ideal factorization, we prove some results regarding the nonmonogenity of a number field K defined by an irreducible trinomial of the type $x^6 + ax + b$ having integer coefficients. As an application of our result, we show that in the special case $a = 0$, if $b \equiv 7 \bmod 8$ or $b \equiv 8 \bmod 9$, then K is not monogenic. We illustrate our results through examples.

Throughout the paper, $\mathbb{Z}_K$ denotes the ring of algebraic integers of an algebraic number field K. For a prime number q and a nonzero m belonging to the ring $\mathbb{Z}_q$ of q-adic integers, $v_q(m)$ will be defined to be the highest power of q dividing m.

Precisely, we prove the following result.

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Theorem 1.1 *Let $K = \mathbb{Q}(\theta)$ be an algebraic number field with θ a root of an irreducible polynomial $f(x) = x^6 + ax + b$ having integer coefficients. If a and $b+1$ are both divisible by either 8 or 9, then K is not monogenic.*

The following result is an immediate consequence of the above theorem.

Corollary 1.2 *Let b be an integer. Let $f(x) = x^6 + b$ be an irreducible polynomial having a root θ and $K = \mathbb{Q}(\theta)$. If $b \equiv 7 \bmod 8$ or $b \equiv 8 \bmod 9$, then K is not monogenic.*

It may be pointed out that the above corollary also follows from the results of [4].

Theorem 1.3 *Let $K = \mathbb{Q}(\theta)$ and $f(x) = x^6 + ax + b$ be as in Theorem 1.1. Let $a \equiv 2 \bmod 4$ and $b \equiv 1 \bmod 4$. Let D denote the discriminant of $f(x)$ given by $5^5a^6 - 6^6b^5$ and $D_2 = \frac{D}{v_2(D)}$. If $v_2(D)$ is even with $D_2 \equiv 3 \bmod 4$ and $bD_2 \not\equiv 7 \bmod 8$, then K is not monogenic.*

The following corollary is an immediate consequence of Theorem 1.3.

Corollary 1.4 *Let a, b be integers such that $a = 192r + 78$ and $b = 160r + 65$ with $r \in \mathbb{Z}$. Let $K = \mathbb{Q}(\theta)$ with θ a root of an irreducible polynomial $x^6 + ax + b$, then K is not monogenic.*

We now provide some examples of nonmonogenic number fields defined by irreducible sextic trinomials.

Example 1.5 Let q be a prime number[1] of the form $8k - 1$ with $k \in \mathbb{Z}$. Let m be an odd integer such that q divides m. Consider $f(x) = x^6 + 8m + q$. Note that $f(x)$ is irreducible over $\mathbb{Q}$ as $f(x)$ satisfies Eisenstein criterion with respect to prime q. If θ is a root of $f(x)$ and $K = \mathbb{Q}(\theta)$, then K is not monogenic in view of Theorem 1.1.

Example 1.6 Let $K = \mathbb{Q}(\theta)$ with θ a root of $f(x) = x^6 + 78x + 65$. Note that $f(x)$ satisfies Eisenstein criterion with respect to 13, hence it is irreducible over $\mathbb{Q}$. By Corollary 1.4, we see that K is not monogenic.

2 Preliminary results

Let $\mathbb{Z}_K$ denote the ring of algebraic integers of an algebraic number field $K = \mathbb{Q}(\theta)$ with θ a root of an irreducible polynomial $f(x)$ having integer coefficients. Let q be a prime number. Suppose q does not divide ind θ. Then, in 1878, Dedekind [3] proved a significant theorem which relates the decomposition of $f(x)$ modulo q with the factorization of $q\mathbb{Z}_K$ into a product of prime ideals of $\mathbb{Z}_K$. The following lemma plays an important role in the proof of Theorem 1.1, which is an immediate consequence of Dedekind's theorem. We shall denote by $\mathbb{F}_q$ the field with q elements.

[1] It is immediate from Dirichlet's theorem on primes in arithmetical progressions that there exists infinitely many primes of the form $8k - 1$, $k \in \mathbb{Z}$.

Lemma 2.1 *Let K be a number field and q be a prime number. For every positive integer f, let N_f denote the number of irreducible polynomials of $\mathbb{F}_q[x]$ of degree f and P_f denote the number of distinct prime ideals of $\mathbb{Z}_K$ lying above q having residual degree f. If $P_f > N_f$ for some f, then for every generator $\alpha \in \mathbb{Z}_K$ of K, q divides* ind α.

For a prime number q, to find the number of distinct prime ideals of $\mathbb{Z}_K$ lying above q, we shall use a weaker version of the classical Theorem of Ore. Before stating that, we first introduce the notions of Gauss valuation, ϕ-Newton polygon and q-regular where $\phi(x)$ belonging to $\mathbb{Z}_q[x]$ is a monic polynomial with $\overline{\phi}(x)$ irreducible over $\mathbb{F}_q$.

We shall denote by $v_{q,x}$ the Gauss valuation of the $\mathbb{Q}_q(x)$ of rational functions in an indeterminate x which extends the valuation v_q of $\mathbb{Q}_q$ and is defined on $\mathbb{Q}_q[x]$ by

$$v_{q,x}\left(\sum_i b_i x^i\right) = \min_i\{v_q(b_i)\}, b_i \in \mathbb{Q}_q. \tag{2.1}$$

Now, we define the notion of ϕ-Newton polygon with respect to some prime q.

Definition 2.1 Let q be a prime number and $\phi(x) \in \mathbb{Z}_q[x]$ be a monic polynomial, which is irreducible modulo q. Let $f(x) \in \mathbb{Z}_q[x]$ be a monic polynomial not divisible by $\phi(x)$ and $\sum\limits_{i=0}^{n} a_i(x)\phi(x)^i$ with $\deg a_i(x) < \deg \phi(x)$, $a_n(x) \neq 0$ be the ϕ-expansion of $f(x)$ obtained on dividing it by successive powers of $\phi(x)$. To each non-zero term $a_k(x)\phi(x)^k$, we associate the point $(n-k, v_{q,x}(a_k(x)))$ and form the set

$$P = \{(k, v_{q,x}(a_{n-k}(x))) \mid 0 \leq k \leq n, a_{n-k}(x) \neq 0\}.$$

The ϕ-Newton polygon of $f(x)$ with respect to prime q is the polygonal path formed by the lower edges along the convex hull of the points of P. The slopes of the edges are increasing when calculated from left to right.

Example 2.2 Let $f(x) = (x+5)^4 - 5$. Here, take $\phi(x) = x$. Then the x-Newton polygon of $f(x)$ with respect to prime 2 consists of only one edge joining the points $(0,0)$ and $(4,2)$ with the lattice point $(2,1)$ lying on it (see Figure 1).

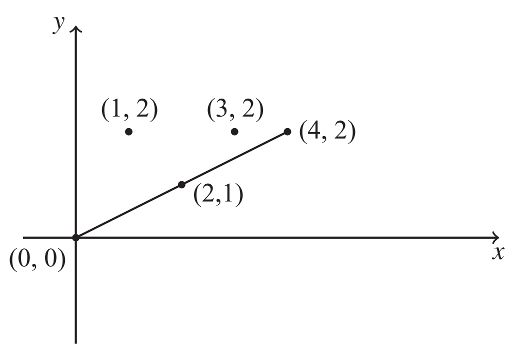

Figure 1: x-Newton polygon of $f(x)$ with respect to prime 2.

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Definition 2.2 Let $\phi(x) \in \mathbb{Z}_q[x]$ be a monic polynomial which is irreducible modulo a prime q having a root α in the algebraic closure $\widetilde{\mathbb{Q}}_q$ of $\mathbb{Q}_q$. Let $f(x) \in \mathbb{Z}_q[x]$ be a monic polynomial not divisible by $\phi(x)$ having degree a multiple of $\deg \phi(x)$ with $\phi(x)$-expansion $\phi(x)^n + a_{n-1}(x)\phi(x)^{n-1} + \cdots + a_0(x)$. Suppose that the ϕ-Newton polygon of $f(x)$ with respect to q consists of a single edge, say S having positive slope denoted by $\frac{d}{e}$ with d, e coprime, i.e.,

$$\min\left\{\frac{v_{q,x}(a_{n-i}(x))}{i} \mid 1 \le i \le n\right\} = \frac{v_{q,x}(a_0(x))}{n} = \frac{d}{e},$$

so that n is divisible by e, say $n = et$ and $v_{q,x}(a_{n-ej}(x)) \ge dj$ for $1 \le j \le t$. Thus, the polynomial $\frac{a_{n-ej}(x)}{q^{dj}} = b_j(x)$ (say) has coefficients in $\mathbb{Z}_q$ and hence $b_j(\alpha) \in \mathbb{Z}_q[\alpha]$ for $1 \le j \le t$. The polynomial $T(y)$ in an indeterminate y defined by $T(y) = y^t + \sum_{j=1}^{t} \overline{b_j}(\overline{\alpha}) y^{t-j}$ having coefficients in $\mathbb{F}_q[\overline{\alpha}]$ is said to be the polynomial associated to $f(x)$ with respect to (ϕ, S); here, the field $\mathbb{F}_q[\overline{\alpha}]$ is isomorphic to the field $\frac{\mathbb{F}_q[x]}{\langle\overline{\phi}(x)\rangle}$.

Example 2.3 Consider $f(x) = (x+5)^4 - 5$. Then, as in Example 2.2, the x-Newton polygon of $f(x)$ with respect to prime 2 consists of only one edge joining the points $(0,0)$ and $(4,2)$ with the lattice point $(2,1)$ lying on it. With notations as in the above definition, we see that $e = 2$, $d = 1$ and the polynomial associated to $f(x)$ with respect to (x, S) is $T(y) = y^2 + y + \overline{1}$ belonging to $\mathbb{F}_2[y]$.

The following definition extends the notion of associated polynomial when $f(x)$ is more general.

Definition 2.3 Let $\phi(x), \alpha$ be as in Definition 2.2. Let $g(x) \in \mathbb{Z}_q[x]$ be a monic polynomial not divisible by $\phi(x)$ such that $\overline{g}(x)$ is a power of $\overline{\phi}(x)$. Let $\lambda_1 < \cdots < \lambda_k$ be the slopes of the edges of the ϕ-Newton polygon of $g(x)$ and S_i denote the edge with slope λ_i. In view of a classical result proved by Ore (cf. [9, Theorem 1.1]), we can write $g(x) = g_1(x)\cdots g_k(x)$, where the ϕ-Newton polygon of $g_i(x) \in \mathbb{Z}_q[x]$ has a single edge, say S_i' which is a translate of S_i. Let $T_i(y)$ belonging to $\mathbb{F}_q[\overline{\alpha}][y]$ denote the polynomial associated to $g_i(x)$ with respect to (ϕ, S_i') described as in Definition 2.2. For convenience, the polynomial $T_i(y)$ will be referred to as the polynomial associated to $g(x)$ with respect to (ϕ, S_i). The polynomial $g(x)$ is said to be q-regular with respect to ϕ if $T_i(y)$ is irreducible in the algebraic closure of $\mathbb{F}_q$, $1 \le i \le k$. In general, if $f(x)$ belonging to $\mathbb{Z}_q[x]$ is a monic polynomial and $\overline{f}(x) = \overline{\phi}_1(x)^{e_1}\cdots\overline{\phi}_r(x)^{e_r}$ is its factorization modulo q into irreducible polynomials with each $\phi_i(x)$ belonging to $\mathbb{Z}_q[x]$ monic and $e_i > 0$, then by Hensel's Lemma there exist monic polynomials $f_1(x), \ldots, f_r(x)$ belonging to $\mathbb{Z}_q[x]$ such that $f(x) = f_1(x)\cdots f_r(x)$ and $\overline{f}_i(x) = \overline{\phi}_i(x)^{e_i}$ for each i. The polynomial $f(x)$ is said to be q-regular (with respect to $\phi_1, \ldots, \phi_r$) if each $f_i(x)$ is q-regular with respect to ϕ_i.

We now state a weaker version of Theorem 1.2 of [9].

Theorem 2.4 *Let $L = \mathbb{Q}(\xi)$ be an algebraic number field with ξ satisfying a monic irreducible polynomial $g(x) \in \mathbb{Z}[x]$ and q be a prime number. Let $\overline{\phi}_1(x)^{e_1} \cdots \overline{\phi}_r(x)^{e_r}$ be the factorization of $g(x)$ modulo q into a product of powers of distinct irreducible polynomials over $\mathbb{F}_q$ with each $\phi_i(x) \neq g(x)$ belonging to $\mathbb{Z}[x]$ monic. Assume that the ϕ_i-Newton polygon of $g(x)$ has k_i edges, say S_{ij} having slopes $\lambda_{ij} = \frac{d_{ij}}{e_{ij}}$ with* $\gcd(d_{ij}, e_{ij}) = 1$ *for $1 \leq j \leq k_i$. If $g(x)$ is q-regular with respect to $\phi_1, \ldots, \phi_r$, then*

$$q\mathbb{Z}_L = \prod_{i=1}^{r} \prod_{j=1}^{k_i} \wp_{ij}^{e_{ij}},$$

where $\wp_{ij}$s are distinct prime ideals of $\mathbb{Z}_L$ having residual degree $\deg \phi_i(x) \times \deg T_{ij}(y)$ *and $T_{ij}(y)$ is the polynomial associated to $g(x)$ with respect to (ϕ_i, S_{ij}), $1 \leq j \leq k_i$.*

3 Proof of Theorems 1.1, 1.3

Proof of Theorem 1.1 We first consider the case when 8 divides both a and $b+1$. In this case, $f(x) \equiv (x^2+x+1)^2(x+1)^2 \bmod 2$. Set $\phi_1(x) = x^2+x+1$ and $\phi_2(x) = x+1$. The ϕ_1-expansion of $f(x)$ is given by

$$f(x) = (x^2+x+1)^3 - 3x(x^2+x+1)^2 + (2x-2)(x^2+x+1) + ax + b + 1.$$

Using the fact that $\min\{v_2(a), v_2(b+1)\} \geq 3$, it is easy to see that the ϕ_1-Newton polygon of f has two edges of positive slope, say S_1 and S_2 joining the point $(1,0)$ with $(2,1)$ and the point $(2,1)$ with $(3, \min\{v_2(a), v_2(b+1)\})$. The polynomial associated to $f(x)$ with respect to (ϕ_1, S_i) is linear for $i = 1,\ 2$. Note that the ϕ_2-expansion of $f(x)$ is given by

$$\begin{aligned} f(x) = {} & (x+1)^6 - 6(x+1)^5 + 15(x+1)^4 - 20(x+1)^3 + 15(x+1)^2 \\ & + (a-6)(x+1) + (-a+b+1). \end{aligned}$$

It can be easily verified that the ϕ_2-Newton polygon of f has two edges of positive slope, say S_1' and S_2' joining the point $(4,0)$ with $(5,1)$ and the point $(5,1)$ with $(6, v_2(-a+b+1))$ respectively. The polynomial associated to $f(x)$ with respect to (ϕ_2, S_i) is linear for $i = 1,\ 2$. So $f(x)$ is 2-regular with respect to $\phi_1,\ \phi_2$. Hence, applying Theorem 2.4, we see that there are two distinct prime ideals of $\mathbb{Z}_K$ lying above 2 having residual degree two each. Since there is only one irreducible polynomial over $\mathbb{F}_2$ of degree two, in view of Lemma 2.1, 2 divides ind α for every generator $\alpha \in \mathbb{Z}_K$. So K is not monogenic.

Now consider the case when 9 divides both a and $b+1$. In this case, $f(x) \equiv (x-1)^3(x+1)^3 \bmod 3$. Set $\phi_1(x) = x-1$ and $\phi_2(x) = x+1$. It is easy to check that the ϕ_1-expansion of $f(x)$ is given by

$$\begin{aligned} f(x) = {} & (x-1)^6 + 6(x-1)^5 + 15(x-1)^4 + 20(x-1)^3 + 15(x-1)^2 \\ & + (a+6)(x-1) + (a+b+1). \end{aligned}$$

Since 9 divides both a and $b+1$, we have $a+6 \equiv 6 \bmod 9$ and $a+b+1 \equiv 0 \bmod 9$. Therefore, it can be easily seen that the ϕ_1-Newton polygon of $f(x)$ has two edges of positive slope. The first edge, say S_1 is the line segment joining the point $(3,0)$

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with $(5,1)$ and the second edge, say S_2 is the line segment joining the point $(5,1)$ with $(6, v_3(a+b+1))$. The polynomial associated to $f(x)$ with respect to $(x-1, S_i)$ is linear for $i = 1,\ 2$.

Recall that the ϕ_2-expansion of $f(x)$ is given by

$$f(x) = (x+1)^6 - 6(x+1)^5 + 15(x+1)^4 - 20(x+1)^3 + 15(x+1)^2 + (a-6)(x+1) + (-a+b+1).$$

One can verify that the ϕ_2-Newton polygon of $f(x)$ has two edges of positive slope. The first edge, say S_1' is the line segment joining the point $(3,0)$ with $(5,1)$ and the second edge, say S_2' is the line segment joining the point $(5,1)$ with $(6, v_3(-a+b+1))$. The polynomial associated to $f(x)$ with respect to $(x+1, S_i')$ is linear for $i = 1,\ 2$.

Thus $f(x)$ is 3-regular with respect to $\phi_1,\ \phi_2$. Hence, applying Theorem 2.4, we see that there exist four distinct prime ideals of $\mathbb{Z}_K$ lying above 3 having residual degree one. Since there are only three irreducible polynomials of $\mathbb{F}_3[x]$ of degree 1, by Lemma 2.1, K is not monogenic. ∎

Proof of Theorem 1.3 By hypothesis, we have $a \equiv 2 \bmod 4,\ b \equiv 1 \bmod 4$. In this case, $f(x) \equiv (x^2+x+1)^2(x+1)^2 \mod 2$. Set $\phi_1(x) = x^2+x+1$. The ϕ_1-expansion of $f(x)$ is given by

$$f(x) = (x^2+x+1)^3 - 3x(x^2+x+1)^2 + (2x-2)(x^2+x+1) + ax + b + 1. \tag{3.1}$$

Since $\min\{v_2(a), v_2(b+1)\} = 1$, the ϕ_1-Newton polygon of $f(x)$ with respect to prime 2 has a single edge of positive slope, say S joining the point $(1,0)$ with $(3,1)$. The polynomial associated to $f(x)$ with respect to (ϕ_1, S) is linear.

Since $a \equiv 2 \bmod 4,\ b \equiv 1 \bmod 4$, we have $v_2(D) \geq 8$. Since $v_2(D)$ is even, denote $\frac{v_2(D)-6}{2}$ by u. Consider a rational number $\delta = \frac{2^u - 3b}{5a_2}$ with $a_2 = \frac{a}{2}$. Note that $v_2(\delta) = 0$. Now set $\phi_2(x) = x - \delta$. The ϕ_2-expansion of $f(x)$ is given by

$$f(x) = (x-\delta)^6 + 6\delta(x-\delta)^5 + 15\delta^2(x-\delta)^4 + 20\delta^3(x-\delta)^3 + 15\delta^4(x-\delta)^2 + f'(\delta)(x-\delta) + f(\delta). \tag{3.2}$$

We claim that

$$v_2(f(\delta)) = 2u+2,\ v_2(f'(\delta)) = u+1. \tag{3.3}$$

Substituting $\delta = \frac{2^u-3b}{5a_2}$ in $f(\delta) = \delta^6 + a\delta + b$, we have

$$5^6 a_2^6 f(\delta) = (2^u - 3b)^6 + a(2^u-3b)(5a_2)^5 + b(5a_2)^6;$$

the above equation on applying binomial theorem and rearranging terms can be rewritten as

$$5^6 a_2^6 f(\delta) = 2^{6u} - 9\cdot 2^{5u+1}b + 135b^2 2^{4u} - (3b)^3\cdot 5\cdot 2^{3u+2} + 15(3b)^4 2^{2u} + (b - 2^{u+1})(3^6 b^5 - 5^5 a_2^6).$$

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Dividing the above equation by 2^{2u} and taking congruence modulo 8, we see that

$$\frac{5^6 a_2^6 f(\delta)}{2^{2u}} \equiv -bD_2 - 1 \bmod 8. \tag{3.4}$$

By hypothesis we have $a \equiv 2 \bmod 4$, $b \equiv 1 \bmod 4$, $D_2 \equiv 3 \bmod 4$ and $bD_2 \not\equiv 7 \bmod 8$. So by (3.4), we have $v_2(f(\delta)) = 2u + 2$. Now substituting $x = \delta$ in the relation $6f(x) - xf'(x) = 5ax + 6b$ and keeping in mind that $5a\delta + 6b = 2^{u+1}$ together with the fact $v_2(f(\delta)) = 2u + 2$, we see that $v_2(f'(\delta)) = u + 1$. This proves our claim.

Using (3.2) and (3.3), one can see that the ϕ_2-Newton polygon of f has a single edge of positive slope, say S' joining the points $(4, 0)$ and $(6, 2u + 2)$ with the lattice point $(5, u + 1)$ lying on it. The polynomial associated to $f(x)$ with respect to (ϕ_2, S') is $Y^2 + Y + \bar{1}$ which has no repeated roots. Therefore, $f(x)$ is 2-regular with respect to ϕ_1, ϕ_2. Applying Theorem 2.4, we see that there are two distinct prime ideals of $\mathbb{Z}_K$ lying above 2 having residual degree two each. Since there is only one irreducible polynomial over $\mathbb{F}_2$ of degree two, by Lemma 2.1, K is not monogenic. ∎

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Department of Mathematics, Indian Institute of Technology (IIT) Bhilai, Raipur, Chhattisgarh 492015, India
e-mail: anujjakhar@iitbhilai.ac.in surenderk@iitbhilai.ac.in