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On the Non-Planarity of a Random Subgraph

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Let G be a finite graph with minimum degree r. Form a random subgraph G_p of G by taking each edge of G into G_p independently and with probability p. We prove that for any constant $\epsilon > 0$, if $p = \frac{1+\epsilon}{r}$, then G_p is non-planar with probability approaching 1 as r grows. This generalizes classical results on planarity of binomial random graphs.

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1. Introduction

Planarity is a fairly classical subject in the theory of random graphs. Erdős and Rényi in their groundbreaking paper [2] stated (re-casting their statement in the language of binomial random graphs) that a random graph $\mathbb{G}_{n,p}$ has a sharp threshold for non-planarity at $p = 1/n$ in the following sense: if $p = c/n$ and $c < 1$ then the random graph $\mathbb{G}_{n,p}$ is with high probability (w.h.p.) planar, while for $c > 1$ $\mathbb{G}_{n,p}$ is w.h.p. non-planar. The Erdős–Rényi argument for non-planarity had a certain inaccuracy, as was pointed out by Łuczak and Wierman [9], who explained how the probable non-planarity result can be obtained by other means.

The aim of this paper is to generalize this classical non-planarity result to a much wider class of probability spaces. All graphs considered in this paper are finite. For a graph $G = (V, E)$ and $0 \leqslant p \leqslant 1$ we can define the random graph $G_p = (V, E_p)$, where each $e \in E$ is independently included in E_p with probability p. When $G = K_n$, the complete graph on n vertices, G_p becomes the binomial random graph $\mathbb{G}_{n,p}$.

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Here is the main result of the present paper.

Theorem 1.1. *Let G be a finite graph with minimum degree r and let $p = \frac{1+\epsilon}{r}$, where $\epsilon > 0$ is an arbitrary constant. Then*

$$\mathbb{P}(G_p \text{ is planar}) \leqslant \theta_r,$$

where $\lim_{r\to\infty} \theta_r = 0$.

2. Proof of Theorem 1.1

Our proof rests in large part on the following simple consequence of Euler's formula.

Lemma 2.1. *Let $G = (V,E)$ be a planar graph with n vertices and m edges and girth g. Then*

$$m \leqslant \frac{g(n-2)}{g-2} < n + \frac{2}{g-2}n.$$

Proof. Let f be the number of faces of a planar embedding of G. Then we have

$$m = n + f - 2 \quad \text{and} \quad 2m \geqslant gf. \qquad \square$$

Remark. In fact, a much stronger statement (in terms of its consequences) is true: for any $\epsilon > 0$ and any integer t, there exists $g = g(\epsilon, t)$ such that any graph $G = (V,E)$ of average degree at least $2+\epsilon$ and of girth at least g contains a minor of the complete graph K_t. This was observed in particular by Kühn and Osthus in [8]. Indeed, by deleting repeatedly vertices of degree 0 or 1 and paths of vertices of degree 2 in G of length at least $2/\epsilon$, we keep average degree at least $2+\epsilon$, and eventually arrive at a subgraph G' of G of minimum degree at least 2, in which every path of degree 2 vertices has length at most $2/\epsilon$. Contracting these degree 2 paths now produces a graph G^* still with high girth (the girth went down by a factor of at most $2/\epsilon$), but the minimum degree of G^* is already at least 3. Then applying the main result of [8] to G^* gives a large complete minor in G^*, which corresponds to a large complete minor in G. Alternatively, this can be derived directly from a result of Mader [11].

Before proving Theorem 1.1 it will be instructive to prove it for the special case where $G = K_n$, *i.e.*, to show that if $p = \frac{c}{n}$ where $c > 1$ is a constant then $\mathbb{G}_{n,p}$ is non-planar with high probability.

2.1. $\mathbb{G}_{n,p}$

The non-planarity of $\mathbb{G}_{n,p}$ is already known even for $c = 1 + \omega n^{-1/3}$ provided $\omega \to \infty$ with n; see Łuczak, Pittel and Wierman [10], and see [12] for very accurate results on the probability of planarity in the critical window $p = (1 + O(n^{-1/3}))/n$. The analysis for $c = 1 + o(1)$ is quite challenging, but for constant $c > 1$ it follows simply from some well-known facts. Let G_1 be the largest connected component of $\mathbb{G}_{n,p}$ (well known to be w.h.p. the unique component of linear size for $c > 1$, the so-called giant component). It is known (see, *e.g.*, [1]) that w.h.p.

$$|V(G_1)| \sim xn \quad \text{and} \quad |E(G_1)| \sim cn(2x - x^2)/2,$$

where x is the unique solution in $(0, 1)$ to $x = 1 - e^{-cx}$. This gives

$$c = 1 + \frac{x}{2} + \frac{x^2}{3} + \cdots$$

and so if $c = 1 + \epsilon$, $\epsilon > 0$ and small, then $x = 2\epsilon - \frac{8}{3}\epsilon^2 + O(\epsilon^3)$.

Thus in this case w.h.p.

$$\frac{|E(G_1)|}{|V(G_1)|} \sim \frac{c(2 - x)}{2} = 1 + \frac{\epsilon^2}{3} + O(\epsilon^3).$$

Next let $g_0 = 10/\epsilon^2$. Then if X denotes the number of cycles in $\mathbb{G}_{n,p}$ of length at most g_0,

$$\mathbb{E}(X) \leqslant \sum_{k=3}^{g_0} n^k p^k \leqslant g_0 c^{g_0}.$$

So w.h.p. there are fewer than $\ln n$ cycles of length at most g_0. So, by removing at most $\ln n$ edges from $\mathbb{G}_{n,p}$ we obtain a subgraph G_1' with girth higher than g_0. Now

$$\frac{|E(G_1')|}{|V(G_1')|} \sim \frac{|E(G_1)|}{|V(G_1)|} \sim 1 + \frac{\epsilon^2}{3} + O(\epsilon^3) > 1 + \frac{2}{g_0 - 2}$$

for small enough ϵ. Lemma 2.1 implies that G_1' and hence G_1 are both non-planar. In fact, choosing a larger value of g_0 and then recalling the remark following the proof of Lemma 2.1 shows that $\mathbb{G}_{n,p}$ has w.h.p. an arbitrarily large complete minor.

2.2. Proof of Theorem 1.1

All asymptotic quantities are to be interpreted for $r \to \infty$, *i.e.*, if we say $\xi = \xi(r) = o(1)$ then we mean that $\limsup_{r\to\infty} |\xi| = 0$. This includes the notion of high probability. That is, if an event $\mathcal{E}$ occurs with probability $1 - \xi(r)$ where $\limsup_{r\to\infty} |\xi| = 0$, then we say that $\mathcal{E}$ occurs with high probability.

Notation. If X is a set of edges and A, B are disjoint sets of vertices, then $E_X(A, B)$ is the set of edges in X with one endpoint in A and one endpoint in B. Furthermore, $E_X(A)$ is the set of edges in X with both endpoints in A. We let $e_X(A, B) = |E_X(A, B)|$ and $e_X(A) = |E_X(A)|$.

Our strategy for proving Theorem 1.1 will be to prove the existence, w.h.p., of a subgraph which has large girth and sufficient edge density to apply Lemma 2.1. For this we will need the following lemma.

Lemma 2.2. *Let $0 < c_1, c_2 < 1$ be constants. Let $T = (V, E)$ be a tree on n vertices with maximum degree $\Delta = r^{o(1)}$. Let $F \subseteq \binom{V}{2}$ with $|F| = c_1 nr$. Form a random subset F_p of F by choosing every edge of F to belong to F_p independently and with probability $p = \frac{c_2}{r}$. Then the graph $G = T \cup F_p$ is non-planar, with probability $1 - O(r^{-1+o(1)})$.*

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Proof. Set

$$\alpha = \frac{c_1 c_2}{72} \quad \text{and} \quad A = \frac{10^{10}}{c_1^5 c_2^6}.$$

Let

$$V_0 = \{v : d_F(v) \geqslant Ar\},$$

where $d_X(v)$ is the degree of vertex v in the graph induced by $X \subseteq F$.

Clearly

$$|V_0| \leqslant \frac{2c_1 n}{A}.$$

Let F' be the set of edges from F with at least one endpoint in V_0.

Case 1. $|F'| \geqslant c_1 nr/2$.

If $e_F(V_0) \geqslant c_1 nr/4$, then the Chernoff bound for the binomial distribution implies that with probability $1 - e^{-\Omega(n)}$ we have

$$e_{F_p}(V_0) \geqslant \frac{c_1 nr}{4} \cdot \frac{c_2}{r}(1 - o(1)) > \frac{c_1 c_2 n}{5} \geqslant \frac{c_2 A}{10}|V_0| > 4|V_0|.$$

In this case, the subgraph induced by $E_{F_p}(V_0)$ forms a non-planar graph. Hence we can assume from now on that F' has at least $c_1 nr/4$ edges with at most one endpoint in V_0.

Define

$$U_0 = \left\{v \notin V_0 : d_{F'}(v) \geqslant \frac{c_1 r}{8}\right\}.$$

Then

$$e_F(U_0, V_0) \geqslant |F'| - e_F(V_0) - \frac{c_1 nr}{8} \geqslant \frac{c_1 nr}{8}.$$

Now the Chernoff bound implies that with probability $1 - e^{-\Omega(n)}$ we have

$$e_{F_p}(U_0, V_0) \geqslant \frac{c_1 nr}{8} \cdot \frac{c_2}{r}(1 - o(1)) \geqslant \frac{c_1 c_2 n}{9}.$$

So if $|U_0| \leqslant \alpha n$ then

$$\frac{e_{F_p}(U_0, V_0)}{|U_0| + |V_0|} \geqslant \frac{c_1 c_2 n/9}{c_1 c_2 n/72 + 2c_1 n/A} \geqslant 4.$$

In this case, the subgraph induced by $E_{F_p}(V_0, U_0)$ forms a non-planar graph.

If $|U_0| \geqslant \alpha n$, then define a (random) subset W_0 by

$$W_0 = \{v \in U_0 : d_{F_p}(v, V_0) \geqslant 5\}.$$

The distribution of $|W_0|$ dominates $\mathrm{Bin}(|U_0|, q_1)$, where

$$\begin{aligned} q_1 &= \mathbb{P}(\mathrm{Bin}(c_1 r/8, c_2/r) \geqslant 5) \\ &\geqslant \binom{\frac{c_1 r}{8}}{5} \left(\frac{c_2}{r}\right)^5 \left(1 - \frac{c_2}{r}\right)^{c_1 r/8} \\ &\geqslant (1 - o(1)) \left(\frac{c_1 c_2}{8}\right)^5 \cdot \frac{e^{-c_1 c_2/8}}{120}. \end{aligned}$$

The Chernoff bounds then imply that with probability $1 - e^{-\Omega(n)}$ we have

$$|W_0| \geqslant \frac{|U_0| q_1}{2} \geqslant \frac{\alpha n q_1}{2},$$

and by definition $e_{F_p}(W_0, V_0) \geqslant 5|W_0|$. This is at least $4|W_0 \cup V_0|$. In this case, the subgraph induced by $E_{F_p}(V_0, W_0)$ forms a non-planar graph. This completes the analysis for Case 1.

Case 2. $|F'| \leqslant c_1 nr/2$.

Define $F'' = F \setminus F'$ and observe that $|F''| \geqslant c_1 nr/2$, and by definition the maximum degree of F'' is at most Ar.

Observe that $T \cup F''_p$ has with probability $1 - e^{-\Omega(n)}$ at least

$$n - 1 + \frac{c_1 nr}{2} \cdot \frac{c_2}{r}(1 - o(1)) > n(1 + \epsilon)$$

edges for some positive $\epsilon = \epsilon(c_1, c_2)$. It thus suffices to show that the number of 'short' cycles in $T \cup F''_p$ is $o(n)$ w.h.p., and then to use Lemma 2.1.

For constants $\ell, t = O(1)$, let us estimate the expected number of cycles of length ℓ in $T \cup F''_p$ having t edges from T. We choose an initial vertex v in n ways, then decide about the placement of the edges of T in the cycle in $O(1)$ ways. We thus get a sequence $P_1 * P_2 \cdots * P_{t+1}$, where the stars correspond to edges from T and P_i is a path of length ℓ_i in F'' for $i = 1, 2, \ldots, t+1$. Now, a path of length ℓ_i in F'', starting from a given point, can be chosen in at most $(Ar)^{\ell_i}$ ways, an edge from T from a given vertex can be chosen in at most $\Delta(T) = r^{o(1)}$ ways, and finally the last path of length ℓ_{t+1}, connecting two already chosen vertices, can be chosen in at most $(Ar)^{\ell_{t+1}-1}$ ways. Altogether the number of such cycles in $T \cup F''$ is $n \cdot r^{o(1)} \cdot O(r^{\ell_1 + \cdots + \ell_{t+1} - 1})$. The probability of such a cycle surviving in $T \cup F''_p$ is $O(r^{-(\ell_1 + \cdots + \ell_{t+1})})$. We thus expect $O(n/r^{1-o(1)})$ such cycles. Summing over all choices of ℓ and t we get that the expected number of cycles of length $O(1)$ in $T \cup F''_p$ is $O(n/r^{1-o(1)})$, and thus the Markov inequality implies that we get fewer than $n/\ln r$ cycles, with probability $1 - O(r^{-1+o(1)})$. By choosing ℓ sufficiently large and deleting one edge from each cycle of length at most ℓ, we get a graph of large constant girth, with n vertices and at least $(1 + \epsilon/2)n$ edges – which is non-planar, by Lemma 2.1. □

We now set about using the above lemma. We let $G_p = G_1 \cup G_2$, where $G_i = G_{p_i}, i = 1, 2$, and $p_1 = \frac{1+\epsilon/2}{r}$ and $(1 - p_1)(1 - p_2) = 1 - p$ so that $p_2 = \frac{\epsilon + O(\epsilon^2)}{2r}$.

2.3. Proof outline

Before going into concrete details we provide a short outline of the proof. We start by probing relatively few vertices and their incident random edges until we find a vertex v whose degree in G_1 is at least $d = \ln^{1/2} r$. The immediate neighbourhood of v in G_1 is large enough to support the growth of (some version of) the breadth-first search tree T_k from v until it accumulates about $i_0 = \ln^3 r$ vertices, while its frontier S_k is of size $\Theta(\epsilon)|T_k|$.

From this point on, we proceed iteratively, at each iteration looking at the current tree T_k, its frontier S_k and the edges of G touching S_k. If many of these edges go back to T_k, we can sprinkle them in G_2 and apply Lemma 2.2 to argue that the resulting random graph is w.h.p. non-planar. Otherwise, many of the edges touching S_k leave T_k, which allows us to expose them in G_1 and to add yet another layer of substantial size to the current tree, while controlling its maximum degree, and to proceed to the next iteration. This growth process cannot go on forever, as G is finite, and thus it eventually collapses, with the first alternative above being applicable, thus resulting in a non-planar graph with high probability.

2.4. Initial tree growth

We begin by repeatedly choosing a vertex $v \in V$ and analysing a restricted breadth-first search (RBFS) from v until we succeed in obtaining a certain condition: see (2.1) below. Basically, we need to find v which has sufficiently many neighbours in G_1. So, let $S_0 = \{v\}$. In general, let

$$S_{i+1} = \bigcup_{w \in S_i} (RN(w) \setminus (T_i \cup B)),$$

where

- $B, |B| = o(r)$, is a set of vertices that has already been rejected by our search,
- $T_i = \bigcup_{j=0}^{i} S_j$,
- $RN(w)$ denotes the first $\mathrm{Bin}(r_1, p_1)$, $r_1 = r - O(i_0 \ln r)$, neighbours of w in G_1, where

$$i_0 = \ln^3 r,$$

  By 'first' we mean that $V(G) = [n]$ for some integer n. Thus we try the first r_1 G-neighbours of a vertex w, in numerical value, to see if they are neighbours of w in G_1. The edges found will be part of a subgraph H_1, and we only keep the first edge found for each vertex added. In this way, H_1 will be a tree.

Our initial aim in RBFS is to find a smallest k such that

$$i_0 \leqslant |T_k| \leqslant 2i_0 \quad \text{and} \quad \frac{|S_k|}{|T_k|} \in \left[\frac{\epsilon}{4}, \frac{3\epsilon}{4}\right]. \tag{2.1}$$

Let $d = \ln^{1/2} r$. We first look for v such that $d \leqslant |S_1| \leqslant \ln^2 r$. This is quite simple. Let $l_0 = (2d)^d = o(r)$ and suppose that we have already examined $v_1, v_2, \ldots, v_l$, $l \leqslant l_0$, without success. We choose $v \notin B_l = \{v_1, v_2, \ldots, v_l\}$ and examine the first $r - o(r)$ neighbours of v that are not in B_l. The probability that v has at least d neighbours in G_1 is greater than

$$\binom{r - o(r)}{d} p_1^d (1 - p_1)^{r - o(r)} \geqslant d^{-d}.$$

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So, the probability we have not found v with large enough degree after l_0 trials is less than $(1-d^{-d})^{l_0} = o(1)$. Furthermore, the probability that v has more than $\log^2 r$ neighbours is less than $\binom{r}{\ln^2 r} p_1^{\ln^2 r} \leqslant e^{-\ln^2 r}$. We can therefore assume that we can find a suitable v with $d \leqslant |S_1| \leqslant \ln^2 r$, where $B = B_l$ is of size $o(r)$.

Suppose now that $S_i, T_i, i \geqslant 1$ do not satisfy (2.1) and that $|T_i| \leqslant 2i_0$. We observe first that the distribution of the size of S_{i+1} is dominated by $\mathrm{Bin}((r-o(r))|S_i|, p_1)$. In fact we bound $|S_{i+1}|$ from above by the number of edges from S_i to S_{i+1}. We examine the first $r_1 = r - o(r)$ G-neighbours of each v in S_i and include an edge vw in our count if the edge vw is in G_1. Therefore

$$\mathbb{P}(|S_{i+1}| \geqslant (1+2\epsilon/3)s \mid |S_i| = s) \leqslant e^{-\Omega(\epsilon^2 s)}. \tag{2.2}$$

We can also argue that $|S_{i+1}|$ dominates a binomial $\mathrm{Bin}(|S_i|(r-o(r)), p_1)$. The $o(r)$ term here differs from the one used in the upper bound. We will have to exclude edges to those G-neighbours that have already been placed in S_{i+1} and to those G-neighbours in B_l. Because we are looking for a lower bound which is less than i_0, we can claim to get at least the result of $|S_i|(r-o(r))$ trials with success probability p_1. Therefore

$$\mathbb{P}(|S_{i+1}| \leqslant (1+\epsilon/3)s \mid |S_i| = s) \leqslant e^{-\Omega(\epsilon^2 s)}. \tag{2.3}$$

So we can assume that $\alpha_1 \geqslant d$ and $|S_i|/|S_{i-1}| = \alpha_i \in [(1+\epsilon/3), (1+2\epsilon/3)]$ for $i \geqslant 2$. And then

$$\frac{|S_i|}{|T_i|} = \frac{\alpha_1\alpha_2\cdots\alpha_i}{1+\alpha_1+\alpha_1\alpha_2+\cdots+\alpha_1\alpha_2\cdots\alpha_i}. \tag{2.4}$$

The expression (2.4) is minimized (resp. maximized) by putting $\alpha_i = (1+\epsilon/3)$ (resp. $= (1+2\epsilon/3)$) for $i \geqslant 2$. It follows that w.h.p.

$$\frac{|S_i|}{|T_i|} = \frac{\alpha_1\theta^{i-1}(\theta-1)}{\theta-1+\alpha_1(\theta^i-1)} \tag{2.5}$$

for some $\theta \in [(1+\epsilon/3), (1+2\epsilon/3)]$.

Thus we will achieve (2.1) w.h.p. Here we use two facts. (i) The sum of the failure probabilities in (2.2) and (2.3) is bounded by $\sum_{s \geqslant d} e^{-\Omega(\epsilon^2 s)} = o(1)$. (ii) We have assumed that $|S_1| = o(i_0)$, which means that the value of k in (2.1) is $\omega(1)$, which in turn means that we need only consider large i in (2.5). In this case the ratio in (2.5) is asymptotically $\theta^{-1}(\theta-1)$.

2.5. Remaining tree growth

Let us consider the current tree T_k, which is of size $\Omega(i_0)$, and its frontier S_k of size $|S_k| = s_k = \Theta(\epsilon|T_k|)$. Choose $r_1 = r - o(r)$ arbitrary edges incident to each vertex of S_k, denote the obtained set by E_k, $|E_k| \leqslant rs_k$. If E_k has $\Theta(rs_k)$ edges inside $V(T_k)$, then sprinkling the edges of E_k with probability p_2 produces w.h.p. a non-planar graph on $V(T_k)$ by Lemma 2.2.

We can therefore assume that E_k has at least $(1-\frac{\epsilon}{10})rs_k$ edges between S_k and $V \setminus T_k$.

Let $V_0 = \{v \notin T_k : d_{E_k}(v, S_k) \geqslant r \ln r\}$. Clearly, $|V_0| \leqslant s_k/\ln r$. If E_k has at least $\epsilon r s_k/10$ edges between S_k and V_0, then in the random subset of E_k, formed by taking each edge independently and with probability p_1, there is w.h.p. a set W_0 of $|W_0| = \Theta(s_k)$ vertices

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$v \in S_k$, whose degrees η_v into V_0 are at least three. Indeed, there will be at least $\epsilon s_k/20$ vertices S_k' in S_k that have at least $\epsilon r/20$ neighbours in V_0. Each vertex in S_k' has a probability at least

$$\xi = \binom{\epsilon r/20}{3} p_1^3 (1-p_1)^{\epsilon r/20-3} \geqslant \epsilon^3 10^{-10}$$

of having $\eta_v \geqslant 3$, and these events are independent. Thus w.h.p. $|W_0| \geqslant |S_k'|\xi/2$, and the bipartite subgraph of G_p induced by W_0, V_0 has more than $2(|W_0| + |V_0|)$ edges and so is non-planar. We can assume therefore that E_k has at least $(1-\frac{\epsilon}{5})rs_k$ edges between S_k and $V_1 = V \setminus (T_k \cup V_0)$. Denote this set of edges by F_k.

Form a random subgraph R_k of F_k by taking each edge independently and with probability p_1.

(P1) Then the Chernoff bound implies that with probability $1 - e^{-\Omega(s_k)}$, $|R_k| \geqslant (1+\frac{\epsilon}{5})s_k$.

(P2) Furthermore, we will show next that with probability $1 - \epsilon_1(r)$ at most $2s_k/\ln r$ of these edges are incident with vertices in $V_2 \subseteq S_k$ whose degree in R_k is more than $\ln\ln r$.

The value of $\epsilon_1(r) = \epsilon_1'(r) + \epsilon_1''(r)$ is obtained from (2.6) and (2.7) below. Indeed, if $v \in S_k$ then

$$\mathbb{P}(d_{R_k}(v) \geqslant \ln\ln r) \leqslant \mathbb{P}(\mathrm{Bin}(r,p_1) \geqslant \ln\ln r) \leqslant q_2 = \left(\frac{2e}{\ln\ln r}\right)^{\ln\ln r}$$

and

$$\mathbb{P}(d_{R_k}(v) \geqslant \ln r) \leqslant \mathbb{P}(\mathrm{Bin}(r,p_1) \geqslant \ln r) \leqslant q_3 = \left(\frac{2e}{\ln r}\right)^{\ln r}.$$

Thus the number of edges in R_k that are incident with $v \in V_2$ is bounded by $\mathrm{Bin}(s_k, q_2)\ln r + \mathrm{Bin}(s_k, q_3)r$. We observe that because $s_k \geqslant \epsilon i_0/8 \gg \ln^2 r$ we can write

$$\mathbb{P}(\mathrm{Bin}(s_k, q_2) \geqslant s_k/\ln^2 r) \leqslant \epsilon_1'(r) = (e\ln^2 r\, q_2)^{s_k/\ln^2 r} \tag{2.6}$$

and

$$\mathbb{P}(\mathrm{Bin}(s_k, q_3) \geqslant s_k/r^2) \leqslant \epsilon_1''(r) = \begin{cases} r^2 q_3 & 0 \leqslant s_k < r^3, \\ (er^2\, q_3)^{s_k/r^2} & s_k \geqslant r^3. \end{cases} \tag{2.7}$$

Let N_k be the set of neighbours of S_k defined by edges in R_k. We observe that $|N_k|$ is the sum of independent Bernoulli random variables. We consider two cases depending on the value of $\mathbb{E}(|N_k|)$ with respect to the random set R_k. Splitting the argument in this way will not condition R_k or N_k.

Case 1. $\mathbb{E}(|N_k|) \geqslant \left(1 + \frac{\epsilon}{10}\right)s_k$.

We first observe that

$$\mathbb{P}\left(|N_k| \leqslant \left(1 + \frac{\epsilon}{20}\right)s_k\right) \leqslant e^{-\epsilon^2 s_k/1000}. \tag{2.8}$$

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We therefore assume that

$$|N_k| \geqslant \left(1+\frac{\epsilon}{20}\right)s_k.$$

R_k contains a subset R_k' of size $v_k = \left(1+\frac{\epsilon}{25}\right)s_k$ such that the degrees of all the vertices in S_k with respect to R_k' are at most $\ln\ln r$, and every vertex outside T_k is incident to at most one edge from R_k' and there are v_k vertices outside T_k incident to an edge in R_k'. We obtain this by removing edges incident with V_2 and by then deleting edges incident with N_k to get degree at most one. Use R_k' to form the next frontier of size $(1+\frac{\epsilon}{25})s_k$, composed of the endpoints of the edges of R_k' outside S_k. Proceed to the next round.

Case 2. $\mathbb{E}(|N_k|) \leqslant \left(1+\frac{\epsilon}{10}\right)s_k$.

(Q1) $|N_k| \leqslant (1+\frac{\epsilon}{8})s_k$ with high probability. Indeed,

$$\mathbb{P}\left(|N_k| \geqslant \left(1+\frac{\epsilon}{8}\right)s_k\right) \leqslant e^{-\epsilon^2 s_k/5000}.$$

(Q2) ($\equiv$P1) $|R_k| \geqslant (1+\frac{\epsilon}{5})s_k$ with high probability. Indeed, $|R_k| = \text{Bin}(|F_k|, p_1)$ and so

$$\mathbb{P}\left(|R_k| \leqslant \left(1+\frac{\epsilon}{5}\right)s_k\right) \leqslant e^{-\epsilon^2 s_k/1200}$$

for small $\epsilon > 0$.

(Q3) There are $o(s_k)$ short cycles in $T_k \cup R_k$ with high probability. For this calculation we consider the graph Γ_k induced by the edges in $E(T_k) \cup F_k$. This has vertex set $V(T_k) \cup N_k$. Here the expectation calculation is quite similar to that of the lemma. We use the fact that V_0 has been excluded, and therefore all relevant vertices outside of T_k have their degrees into S_k bounded by $r\ln r$. Also, all degrees in T_k are $r^{o(1)}$ by our construction.

Details. For constants $\ell, t = O(1)$ let us estimate the expected number of cycles of length ℓ in Γ_k having t edges from T_k. We choose an initial vertex v in $O(s_k)$ ways, then decide about the placement of the edges of T_k in the cycle in $O(1)$ ways. We thus get a sequence $P_1 * P_2 \cdots * P_{t+1}$, where the stars correspond to edges from T_k and P_i is a path of length ℓ_i for $i = 1, 2, \ldots, t+1$ using edges in F_k. Now, a path of length ℓ_i using edges in F_k, starting from a given point, can be chosen in at most $(r\ln r)^{\ell_i}$ ways, an edge from T_k from a given vertex can be chosen in at most $\Delta(T_k) = r^{o(1)}$ ways, and finally the last path of length ℓ_{t+1}, connecting two already chosen vertices, can be chosen in at most $(r\log r)^{\ell_{t+1}-1}$ ways. Altogether the number of such cycles in Γ is $s_k \cdot r^{o(1)} \cdot \tilde{O}(r^{\ell_1+\cdots+\ell_{t+1}-1})$ ways. The probability of such a cycle surviving in $T_k \cup R_k$ is $O(r^{-(\ell_1+\cdots+\ell_{t+1})})$. We thus expect $O(s_k/r^{1-o(1)})$ such cycles. Summing over all choices of ℓ and t we get that the expected number of cycles of length $O(1)$ in $T \cup F_p''$ is $O(s_k/r^{1-o(1)})$, and thus the Markov inequality implies that we get fewer than $s_k/\ln r$ cycles, with probability $1 - O(r^{-1+o(1)})$.

By choosing ℓ sufficiently large and removing edges from the short cycles (length $\leqslant \ell$) leaves a graph of average degree $2 + \Theta(\epsilon)$ and without short cycles. This is non-planar by Lemma 2.1.

As a final note in proof, we argue about the probability that this construction fails. We have seen that the initial tree growth in Section 2.4 succeeds with high probability. The success of the remaining tree growth rests on the probabilities in (P1, P2) being high enough. These events need to happen multiple times, whereas other events are only required to occur once.

For (P1) and (2.8) we verify that $\sum_{t\geqslant i_0} e^{-\Omega(t)} = o(1)$, and for (P2) we verify that

$$\sum_{t\geqslant i_0}(e\log^2 r\, q_2)^{-t/\log^2 r} + \sum_{t=0}^{r^3} r^2 q_3 + \sum_{t\geqslant r^3}(er^2\, q_3)^{t/r^2} = o(1).$$

3. Concluding remarks

We have proved that for every finite graph G of minimum degree $r \gg 1$, a random subgraph G_p of G, with $p = p(r) = \frac{1+\epsilon}{r}$ and $\epsilon > 0$ being an arbitrary small constant, is w.h.p. non-planar. This generalizes the classical non-planarity results for binomial random graphs $\mathbb{G}_{n,p}$. It should be noted that for a statement of such generality we cannot hope to have a matching lower bound on $p(r)$. Indeed, if G is a collection of, say, 2^{r^3} vertex-disjoint cliques K_{r+1}, then for any constant $c > 0$, the random subgraph G_p, $p = c/r$, retains w.h.p. one of the cliques K_{r+1} in full and is thus w.h.p. non-planar.

Notice that our proof, with fairly straightforward and simple adjustments, shows in fact that under the conditions of Theorem 1.1 the random subgraph G_p is typically not only non-planar, but has a complete minor of arbitrarily large constant size. This can be obtained by employing the remark following Lemma 2.1. It would be interesting to determine the largest $t = t(r)$ such that under the same conditions the random graph G_p has w.h.p. a minor of a complete graph K_t. For the case of binomial random graphs $\mathbb{G}_{n,p}$, Fountoulakis, Kühn and Osthus showed [3] that for any $c > 1$, the random graph $\mathbb{G}_{n,p}$ with $p = c/n$ has w.h.p. a complete minor of order $\sqrt{n}$. (See also [5] for results for other values of $p = p(n)$, and [4] for results on random regular graphs and for $\mathbb{G}_{n,p}$ in the slightly supercritical regime.)

The main theorem of this paper can be viewed as yet another contribution to a growing sequence of results about properties of random subgraphs of graphs of given minimum degree. We can mention here Krivelevich and Sudakov [6], who showed that if G is a finite graph of minimum degree r and $p = \frac{1+\epsilon}{r}$, then the random graph G_p contains w.h.p. a path of length linear in r, and also Krivelevich, Lee and Sudakov [7], who proved that under the same assumptions on the base graph G and when taking $p = \frac{(1+o(1))\ln r}{r}$, the random graph G_p contains w.h.p. a path of length at least r, in both cases substantially generalizing classical results about binomial random graphs. One can certainly anticipate more results of this type appearing in the near future.

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