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THE MULTIPLE-PLAYER ANTE ONE GAME

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Consider a group of players playing a sequence of games. There are k players, having arbitrary initial fortunes. Each game consists of each remaining player putting 1 in a pot, which is then won (with equal probability) by one of them. Players whose fortunes drop to 0 are eliminated. Let $T^{(i)}$ be the number of games played by i, and let $T = \max_i T^{(i)}$. For the case $k = 3$, martingale stopping theory can be used to derive $E[T]$ and $E[T^{(i)}]$. When $k > 3$, we obtain upper bounds on $E[T]$ and, in the case in which all players have the same initial fortune, on $E[T^{(i)}]$. Efficient simulation methods for estimating $E[T]$ and $E[T^{(i)}]$ are discussed.

1. INTRODUCTION AND SUMMARY

Consider a group of players playing a sequence of games. Initially there are k players, with player i starting with $I_i, i = 1, \ldots, k$. Let $S = \sum_{i=1}^{k} I_i$. Each game consists of each remaining player putting 1 in a pot, which is then won (with equal probability) by one of them. Players whose fortunes drop to 0 are eliminated. Let $T^{(i)}$ be the number of games played by i, and let $T = \max_i T^{(i)}$ be the number of games needed until one of the players has all $\sum_i I_i$ units. We are interested in the quantities $E[T]$ and $E[T^{(i)}]$.

The preceding problem was noted by Engel [3], for which a formula, attributed to extensive computer simulations, was given for $E[T]$ when $k = 3$. The problem was also noted by Amano, Tromp, Vitangi and Watanabe [1], who gave some experimental results, and by Bach [2]. Bach [2] noted that a martingale approach that was used to solve a different gambling problem could also be used to obtain $E[T]$ when $k = 3$, verifying Engel's result. In this article we adopt the martingale approach of [2] and show how it can be adapted to obtain $E[T^{(i)}]$ when $k = 3$, upper bounds on $E[T]$ when $k > 3$, as well as upper bounds on $E[T^{(i)}]$ when $k > 3$ and all $I_i = S/k$. In Section 2

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we present the martingales, and in Section 3 we show how they yield $E[T]$ and $E[T^{(i)}]$ when $k = 3$. A useful bound on $E[T]$ when $k = 3$ is also presented in Section 3. Section 4 presents the upper bounds when $k > 3$. In Section 5 we consider ways of efficiently using simulation to estimate $E[T]$ and $E[T^{(i)}]$.

2. THE MARTINGALES

Let $X_i(t)$ denote players i's fortune after game i, let $W_i(t)$ be player i's winnings in game t; and let $N(t)$ denote the number of players having a positive fortune after game t. Thus,

$$X_i(t+1) = X_i(t) + W_i(t+1).$$

Let H_t denote the history of all results concerning the first t games. With $I(A)$ equal to the indicator for the event A, it follows that, given H_t,

$$W_i(t+1) = I(X_i(t) > 0) \begin{cases} N(t) - 1 & \text{with probability } \dfrac{1}{N(t)} \\ -1 & \text{with probability } 1 - \dfrac{1}{N(t)}. \end{cases}$$

Hence,

$$E[W_i(t+1)|H_t] = 0,$$
$$E[W_i^2(t+1)|H_t] = I(X_i(t) > 0)(N(t) - 1),$$
$$E[W_i^3(t+1)|H_t] = I(X_i(t) > 0)(N(t) - 1)(N(t) - 2).$$

It follows from the preceding that

$$\begin{aligned} E[X_i^2(t+1)|H_t] &= E[X_i^2(t) + 2X_i(t)W_i(t+1) + W_i^2(t+1)|H_t] \\ &= X_i^2(t) + I(X_i(t) > 0)(N(t) - 1). \end{aligned} \qquad \textbf{(1)}$$

Consequently,

$$\begin{aligned} E\left[\sum_{i=1}^{k} X_i^2(t+1)|H_t\right] &= \sum_{i=1}^{k} X_i^2(t) + (N(t) - 1)\sum_{i=1}^{k} I(X_i(t) > 0) \\ &= \sum_{i=1}^{k} X_i^2(t) + (N(t) - 1)N(t), \end{aligned}$$

which yields the following proposition.

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PROPOSITION 1:

$$Z(t) \equiv \sum_{i=1}^{k} X_i^2(t) - \sum_{u=0}^{t-1} N(u)(N(u)-1)$$

is a martingale with mean $\sum_{i=1}^{k} I_i^2$.

In addition, we have that

$$\begin{aligned} E[X_i^3(t+1)|H_t] &= X_i^3(t) + 3X_i(t)E[W_i^2(t+1)|H_t] + E[W_i^3(t+1)|H_t] \\ &= X_i^3(t) + 3X_i(t)(N(t)-1) + I(X_i(t)>0)(N(t)-1)(N(t)-2). \end{aligned}$$

Summing over i gives

$$\begin{aligned} E\left[\sum_{i=1}^{k} X_i^3(t+1)|H_t\right] &= \sum_{i=1}^{k} X_i^3(t) + 3[N(t)-1] \\ &\quad \times \sum_{i=1}^{k} X_i(t) + N(t)(N(t)-1)(N(t)-2) \qquad (2) \\ &= \sum_{i=1}^{k} X_i^3(t) + 3S(N(t)-1) + N(t)(N(t)-1)(N(t)-2), \qquad (3) \end{aligned}$$

which yields the following proposition.

PROPOSITION 2:

$$V(t) \equiv \sum_{i=1}^{k} X_i^3(t) - \sum_{u=0}^{t-1}[3S(N(u)-1) + N(u)(N(u)-1)(N(u)-2)]$$

is a martingale with mean $\sum_{i=1}^{k} I_i^3$.

Now, let T_j denote the number of games that involve exactly j players, $j = 2, \ldots, k$; so $T = \sum_{j=2}^{k} T_j$ denotes the total number of games played. It follows from the martingale stopping theorem that

$$\sum_{i=1}^{k} I_i^2 = E[Z(T)] = E\left[\sum_{i=1}^{k} X_i^2(T)\right] - E\left[\sum_{u=0}^{T-1} N(u)(N(u)-1)\right]. \qquad (4)$$

Now, at time T, one of the players has a fortune of S and all the others have 0. Consequently, $\sum_{i=1}^{k} X_i^2(T) = S^2$. Moreover, as $\sum_{u=0}^{T-1} N(u)(N(u)-1)$ is the sum, over all rounds, of the number of players multiplied by that number minus 1 that are

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in involved in the round, it follows that any round that has j players contributes $j(j-1)$ to that sum. As a result, $\sum_{u=0}^{T-1} N(u)(N(u)-1) = \sum_{j=2}^{k} j(j-1)T_j$, giving

$$\sum_{i=1}^{k} I_i^2 = S^2 - \sum_{j=2}^{k} j(j-1)E[T_j]. \tag{5}$$

Similarly, applying the stopping theorem to the $V(t)$ martingale gives

$$\sum_{i=1}^{k} I_i^3 = S^3 - \sum_{j=2}^{k} [3S(j-1) + j(j-1)(j-2)]E[T_j]. \tag{6}$$

Remark: No apparently useful martingales can be obtained by raising $X_i(t+1)$ to a power higher than 3. For instance, if we raised it to the power 4, then on the right-hand side of the identity

$$E\left[\sum_i X_i^4(t+1)|H_t\right] = E\left[\sum_i (X_i(t) + D_i(t+1))^4 |H_t\right],$$

we will have the term $\sum_i X_i^2(t)E[D_i^2(t+1)|H_t] = (N(t)-1)\sum_i X_i^2(t)$, which is not convenient to work with. (The corresponding term when we raise $X_i(t+1)$ to the third power is $(N(t)-1)\sum_i X_i(t)$, which is equal to $S(N(t)-1)$.)

3. THE CASE $k = 3$

When $k = 3$, (5) and (6) give that

$$\sum_{i=1}^{3} I_i^2 = S^2 - 2E[T_2] - 6E[T_3]$$

and

$$\sum_{i=1}^{3} I_i^3 = S^3 - 3SE[T_2] - (6S+6)E[T_3].$$

Solving gives

$$E[T_2] = \frac{\sum_{i=1}^{3} I_i(I_i - 1)(S - I_i)}{S-2}$$

and

$$E[T_3] = \frac{I_1 I_2 I_3}{S-2}, \tag{7}$$

which yields the following proposition.

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PROPOSITION 3 (Engel [3] and Bach [2]): *When* $k = 3$,

$$E[T] = E[T_2] + E[T_3] = \frac{\sum_{i=1}^{3} I_i(I_i - 1)(S - I_i) + I_1 I_2 I_3}{S - 2}. \tag{8}$$

Remark: When $I_1 = I_2 = I_3 = S/3$,

$$E[T] = \frac{7S^3 - 18S^2}{27(S-2)}.$$

The following upper bound on $E[T]$ will prove useful in the next section.

PROPOSITION 4: *Assume* $k = 3$. *If* $S \geq 6$ *then*

$$E[T] \leq U \equiv \begin{cases} S^2/4 & \textit{if } S < 18 \\ \dfrac{7S^3 - 18S^2}{27(S-2)} & \textit{if } S \geq 18, \end{cases}$$

PROOF: With $x = I_1, y = I_2, z = I_3$, and $s = S$, the expression for $(s-2)E[T]$ can be written as

$$(s-2)E[T] \equiv f(x, y, z) = x^2(y+z) + y^2(x+z) + z^2(x+y) - 2xy - 2xz - 2yz + xyz.$$

To maximize f over $x + y + z = s$, we set the partial derivatives of the Lagrangian expression $f(x, y, z) - \lambda(x + y + z - s)$ equal to zero to obtain

$$\begin{aligned} \lambda &= 2x(y+z) + y^2 + z^2 - 2y - 2z + yz, \\ \lambda &= 2y(x+z) + x^2 + z^2 - 2x - 2z + xz, \\ \lambda &= 2z(x+y) + x^2 + y^2 - 2x - 2y + xy, \\ s &= x + y + z. \end{aligned}$$

Combining these equations in pairs show that they are equivalent to

$$\begin{aligned} (x-y)(x+y-z-2) &= 0, \\ (x-z)(x+z-y-2) &= 0, \\ (y-z)(y+z-x-2) &= 0, \\ x + y + z &= s. \end{aligned}$$

The solutions of the preceding in which all variables are nonzero are

$$x = y = z = s/3$$

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and the symmetrical versions of the solutions that have two of the variables equal to $s/2 - 1$ and the other equal to 2. Now,

$$f(s/3, s/3, s/3) = 7s^3/27 - 2s^2/3$$

and

$$f(s/2 - 1, s/2 - 1, 2) = s^3/4 - s^2/2 - s + 2,$$

showing that $f(s/3, s/3, s/3) \geq f(s/2 - 1, s/2 - 1, 2)$ when $s \geq 6$. Because

$$f(s/2, s/2, 0) = (s - 2)s^2/4 = s^3/4 - s^2/2,$$

it follows that $f(s/3, s/3, s/3) \geq f(s/2, s/2, 0)$ if and only if $s \geq 18$. ■

The approach used to prove Proposition 4 was suggested by Bach (personal communication).

When $k = 3$, we can also compute the mean number of games played by i. To begin, note that $X_i(t), t \geq 0$, is a martingale with mean I_i, yielding, by the martingale stopping theorem that

$$I_i = E[X_i(T)] = SP(X_i(T) = S).$$

Now, let $T_j^{(i)}$ denote the number of games that i plays that involve exactly j players, $j = 2, 3$, and let $T^{(i)} = \sum_{j=2}^{3} T_j^{(i)}$ denote the total number of games that i plays. Then, from (1) we see that $X_i^2(t) - \sum_{u=0}^{t-1} I(X_i(u) > 0)(N(u) - 1), t \geq 0$, is a martingale with mean I_i^2. Hence, by the martingale stopping theorem,

$$\begin{aligned} I_i^2 &= E[X_i^2(T) - \sum_{u=0}^{T-1} I(X_i(u) > 0)(N(u) - 1)] \\ &= E[X_i^2(T)] - E\left[\sum_{j=2}^{3} (j - 1)T_j^{(i)}\right] \\ &= S^2 P(X_i(T) = S) - E[T_2^{(i)}] - 2E[T_3^{(i)}] \\ &= SI_i - E[T_2^{(i)}] - 2E[T_3]. \end{aligned}$$

Equation (7) now yields the following:

PROPOSITION 5: *When* $k = 3$,

$$E[T^{(i)}] = I_i(S - I_i) - \frac{I_1 I_2 I_3}{S - 2}.$$

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4. UPPER BOUNDS WHEN $k \geq 4$

Let

$$H = S^3 - \sum_{i=1}^{k} I_i^3$$

From (6),

$$\begin{aligned} H &= \sum_{j=2}^{k} [3S(j-1) + j(j-1)(j-2)]E[T_j] \\ &\geq 3SE[T_2] + (6S+6)E[T_3] + (9S+24) \\ &\quad \times \sum_{j=4}^{k-1} E[T_j] + [3S(k-1) + k(k-1)(k-2)]E[T_k] \\ &= (9S+24)\sum_{j=2}^{k} E[T_j] - (6S+24)E[T_2] - (3S+18)E[T_3] + cE[T_k], \end{aligned} \quad \textbf{(9)}$$

where

$$c = 3S(k-1) + k(k-1)(k-2) - 9S - 24.$$

If, in the first game involving only two players, the player's fortunes are i and $S - i$, then the expected remaining number of games would be $i(S - i)$, showing that

$$E[T_2] \leq \frac{S^2}{4}.$$

Moreover, it follows from Proposition 4 that

$$E[T_2] + E[T_3] \leq U.$$

In addition, clearly $E[T_k] \geq M \equiv \min_i I_i$. Hence, from (9),

$$\begin{aligned} (9S+24)E[T] &\leq H + (6S+24)E[T_2] + (3S+18)E[T_3] - cE[T_k] \\ &= H + (3S+6)E[T_2] + (3S+18)(E[T_2] + E[T_3]) - cE[T_k] \\ &\leq H + (3S+6)\frac{S^2}{4} + (3S+18)U - cM; \end{aligned}$$

that is,

$$E[T] \leq \frac{H + (3S+6)\frac{S^2}{4} + (3S+18)U - cM}{9S+24}, \quad \textbf{(10)}$$

where

$$U \equiv \begin{cases} S^2/4 & \text{if } S < 18 \\ \dfrac{7S^3 - 18S^2}{27(S-2)} & \text{if } S \geq 18. \end{cases}$$

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Remark: If all $I_i = I = S/k$, the right-hand side of the preceding is roughly

$$S^2 \frac{1 - 1/k^2 + 3/4 + 7/9}{9} \approx \left(0.2802 - \frac{1}{9k^2}\right) S^2 \approx 0.2808\, S^2$$

when $S > 18$ and is roughly $0.2778S^2$ when $S \leq 18$.

Another upper bound can be obtained by first subtracting (5) from (6). With

$$C = S^3 - \sum_i I_i^3 - S^2 + \sum_i I_i^2$$

and

$$d = 3S(k-1) + k(k-1)(k-3) - 9S - 12,$$

this yields

$$\begin{aligned} C &= (3S-2)E[T_2] + 6SE[T_3] \\ &\quad + \sum_{j=4}^{k-1} [3S(j-1) + j(j-1)(j-3)]E[T_j] + [d + 9S + 12]E[T_k] \\ &\geq (9S+12)E[T] - (3S+2)E[T_2] - (3S+12)(E[T_2] + E[T_3]) + dE[T_k] \end{aligned}$$

Hence,

$$E[T] \leq \frac{C + (3S+2)\frac{S^2}{4} + (3S+12)U - dM}{9S+12}. \qquad \textbf{(11)}$$

Example 1: If $k = 4$ and $I_i \equiv 5$, then (10) yields the bound $E[T] \leq 107.51$, whereas (11) gives $E[T] \leq 107.45$.

Example 2: Suppose there are four players with initial fortunes $3, 2, 2$, and 2. Conditioning on the results of the first two games gives

$$\begin{aligned} E[T] &= E[T|1,1]\frac{1}{16} + E[T|1,2]\frac{6}{16} + E[T|2,2]\frac{3}{16} + E[T|2,3]\frac{6}{16} \\ &= 2 + 20\frac{6}{16} + 8\frac{3}{16} + E[T(4,4,1)]\frac{6}{16} \\ &= 2 + \frac{144}{16} + \frac{136}{7}\frac{6}{16} = 18.2857. \end{aligned}$$

Inequality (10) gives $E[T] \leq 21.5$, whereas (11) gives $E[T] \leq 21.4516$. (The incorrect answer $350612/69969 \approx 5.01$ was given in [3].)

We can also derive an upper bound on $E[T^{(i)}]$, the mean number of games played by i, when all k initial fortunes are S/k. First note that

$$E[T_j^{(i)}|T_j] = \frac{j}{k}T_j,$$

yielding that

$$E[T_j^{(i)}] = \frac{j}{k}E[T_j].$$

Therefore,

$$E[T^{(i)}] = \frac{1}{k}\sum_{j=2}^{k} jE[T_j]. \tag{12}$$

Hence, from (6),

$$\begin{aligned} H &= \sum_{j=2}^{k}[3S(j-1)+j(j-1)(j-2)]E[T_j] \\ &\geq 3SE[T_2]+(6S+6)E[T_3] \\ &\quad + \frac{9S+24}{4}\sum_{j=4}^{k-1} jE[T_j] + [3S(k-1)+k(k-1)(k-2)]E[T_k] \\ &= \frac{9S+24}{4}\sum_{j=2}^{k} jE[T_j] - \frac{3S+24}{2}E[T_2] - \frac{3S+48}{4}E[T_3] + CE[T_k], \end{aligned} \tag{13}$$

where

$$C = 3S(k-1)+k(k-1)(k-2) - \frac{9S+24}{4}k.$$

Hence, when $S \geq 18$,

$$\begin{aligned} \frac{9S+24}{4}\sum_{j=2}^{k} jE[T_j] &\leq H + \frac{3S}{4}E[T_2] + \frac{3S+48}{4}(E[T_2]+E[T_3]) - CE[T_k] \\ &\leq H + \frac{3S^3}{16} + \frac{(3S+48)U}{4} - \frac{CS}{k}. \end{aligned}$$

From (12), we obtain

$$E[T^{(i)}] \leq \frac{4}{k(9S+24)}\left[H + \frac{3S^3}{16} + \frac{(3S+48)U}{4} - \frac{CS}{k}\right].$$

When S is large, the preceding upper bound is roughly $0.6142S^2/k$.

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Remark: Consider the case where all $I_i = I = S/k$ and suppose that whenever one of the players has all S units, the money is redistributed, with each player again receiving I, and play begins anew. Letting A be the average number of players in a game, it follows from renewal reward process theory that

$$A = \frac{E[\sum_{i=1}^{k} T^{(i)}]}{E[T]} = \frac{kE[T^{(1)}]}{E[T]}.$$

When $k = 3$, results of Section 3 yield

$$A = \frac{15I^3 - 12I^2}{7I^3 - 6I^2} \approx 15/7 \quad \text{when } I \geq 2.$$

Now, suppose we use our upper bounds on $E[T]$ and $E[T^{(i)}]$ as approximations. This yields, for $k > 3$, that

$$A \approx \frac{0.6142S^2}{0.2808S^2} = 2.18732.$$

If we let N be a random variable such that $P(N = j)$ is equal to the proportion of games that are played with j players, then $E[N] = A$ and the Markov inequality yields

$$P(N \geq 3) = P(N - 2 \geq 1) \leq E[N] - 2 \approx 0.18732$$

suggesting that most games are played with only two players. (The exact proportion of games that would involve more than two players when $k = 3$ is $I^3/(7I^3 - 6I^2)$.) As experimental evidence reported in [1] indicated that $E[T]$ is about $k^2I^2/4$, it appears that at the moment when only two players remain, each of their respective fortunes tends, with high probability, to be close to $kI/2$.

5. EFFICIENT SIMULATION PROCEDURES

Suppose all $I_i = I = S/k$. To estimate $E[T]$ and $E[T^{(1)}]$ by simulation, one can start by stratifying on N, equal to the number of simultaneous games won by the initial winner; that is, $N = j$ means that the same player wins the first j games, and if $j < I$, that a different player wins game $j + 1$. With $T(j)$ equal to the number of games when the k players have initial fortunes $I + j(k-1) - 1, I - j + k - 1, I - j - 1, \ldots,$ and $I - j - 1$, this gives

$$\begin{aligned} E[T] &= \sum_{j=1}^{I-2} E[T|N = j](1/k)^{j-1}(1 - 1/k) \\ &\quad + E[T|N = I-1](1/k)^{I-2}(1 - 1/k) + I(1/k)^{I-1} \\ &= \sum_{j=1}^{I-2} (j + 1 + E[T(j)]) \frac{k-1}{k^j} + [I + k(kI - k)] \frac{k-1}{k^{I-1}} + I \frac{1}{k^{I-1}}. \end{aligned}$$

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Simulation can now be used to estimate the quantities $E[T(j)]$, $j = 1, \ldots, I-2$. In doing the simulation, we suggest that the determination of which player wins a game be accomplished as follows. If there are currently r active players, with current fortunes $f_1 \geq f_2 \geq \cdots \geq f_r$, then a random number U should be generated, and if $(i-1)/r < U \leq i/r$, then the player whose current fortune is f_i is the winner of that game. Because a small value of U corresponds to a winning game of a player with a larger fortune, there should be a negative dependence between T and the generated values of the random numbers. For this reason, using the antithetic versions of the random numbers used in a run (i.e., if $U_1, U_2, \ldots$ are used in a run, then the next run should use $1-U_1, 1-U_2, \ldots$) should result in a variance reduction over using an independent stream of random numbers.

We suggest the same variance reductions be employed when estimating $E[T^{(1)}]$. However, one should not solely estimate $E[T^{(1)}]$ but should in each run, estimate all of the quantities $E[T^{(i)}]$, with the estimate of $E[T^{(1)}]$ being the average of the estimates of the $E[T^{(i)}], i = 1, \ldots, k$. Such an estimator would have a smaller variance than just using the estimator of $E[T^{(1)}]$ because the former estimator can be regarded as a conditional expectation estimator that conditions on the order statistics of the estimators of $E[T^{(i)}], i = 1, \ldots, k$.

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