



Fano quiver moduli

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Abstract. We exhibit a large class of quiver moduli spaces, which are Fano varieties, by studying line bundles on quiver moduli and their global sections in general, and work out several classes of examples, comprising moduli spaces of point configurations, Kronecker moduli, and toric quiver moduli.

1 Introduction

Moduli spaces of stable quiver representations, introduced in [16], form a particularly tractable class of moduli spaces, since they are easily constructed (as a Geometric Invariant Theory quotient for a linear representation of a reductive algebraic group), and since they arise from classification problems for linear algebra data.

In the (numerically characterized) case of fine quiver moduli spaces for acyclic quivers, which are smooth projective varieties (and rational by [27]), several favorable geometric properties of cohomological nature are well established.

Their singular cohomology is algebraic [17], they are polynomial-count [23], the Betti numbers (and the counting polynomial) can be determined explicitly [23], and their (torus-equivariant) Chow rings are tautologically generated and presented [7, 8].

In the present work, we add to these properties one of algebraic–geometric nature: under a rather mild genericity assumption, we exhibit for each indivisible dimension vector for an acyclic quiver a class of stabilities (the canonical chamber) for which the moduli space is a Fano variety, of known dimension, Picard rank, and index (see [Theorem 4.3](#)).

A similar result was announced in [6] and can also be derived from the methods of [12]. The second-named author was informed by L. Hille that he derived a similar result in [13]. In the special case of toric quiver moduli spaces, the Fano property is established in [1]. The Fano property can also be derived in the context of Mori Dream Spaces (see [Remark 3.6](#)). We prefer here a derivation, which is proprietary to the quiver setup, with the hope for adaptability to more general cases (for example, singular quiver moduli spaces).

Our result hopefully allows a fruitful interaction between the theory of Fano varieties and the theory of quiver moduli: on the one hand, it adds classes of

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arbitrarily high-dimensional, although certainly very special, Fano varieties to the existing classes of examples. In particular, it seems likely that deep properties of Fano varieties, for example, the Mukai conjecture and its various generalizations, can be verified for this class (in the spirit of such a verification for toric varieties [4], horospherical varieties [22], and symmetric varieties [10]). On the other hand, the particularly rigid properties of Fano varieties and the refined invariants available for them should facilitate the study of the geometry of these quiver moduli.

After reviewing the construction of quiver moduli spaces and their basic geometric properties in Section 2, we turn to the study of their line bundles in Section 3: first, recalling the construction of tautological bundles, we determine (under the assumption of ample stability) their Picard group and identify one chamber of their ample cone. This allows us to establish the Fano property in Section 4, by performing a Chern class computation for the class of the tangent bundle, which readily leads to the relevance of the canonical stability chamber and to the main theorem.

The second main focus of the present paper is on illustrating the generality of our assumptions by establishing several classes of examples in Section 5: we first identify all fine moduli spaces of ordered point configurations in projective spaces as Fano varieties, also discussing a subtle example related to the Segre cubic. Then, we use results of [24] to prove that all fine Kronecker moduli spaces are Fano (of nontrivial index) and verify the Kobayashi–Ochiai theorem for them purely numerically. Moreover, we exhibit a class of Fano quiver moduli spaces of arbitrary Picard rank and index, which can also be interpreted as certain torus quotients of Grassmannians, for which we verify the Mukai conjecture (see Proposition 5.4). We finally parameterize all Fano toric quiver moduli arising from our result, which allows us to exhibit particular examples of two- and three-dimensional Fano quiver moduli spaces.

2 Recollections on quiver moduli

2.1 Construction of quiver moduli

Let Q be a finite quiver with set of vertices Q_0 and arrows written $\alpha : i \rightarrow j$. We assume Q to be acyclic, that is, Q has no oriented cycles. We define the Euler form $\langle -, - \rangle_Q$ of Q on $\mathbb{Z}Q_0$ (with natural basis \mathbf{i} for $i \in Q_0$) defined by

$$\langle \mathbf{d}, \mathbf{e} \rangle = \sum_{i \in Q_0} d_i e_i - \sum_{\alpha: i \rightarrow j} d_i e_j$$

for $\mathbf{d} = \sum_{i \in Q_0} d_i \mathbf{i}$ (and similarly for \mathbf{e}).

Let $\mathbf{d} \in \mathbb{N}Q_0$ be a dimension vector. Fixing vector spaces V_i of dimension d_i for all $i \in Q_0$, let

$$R_{\mathbf{d}}(Q) = \bigoplus_{\alpha: i \rightarrow j} \text{Hom}_{\mathbb{C}}(V_i, V_j)$$

be the space of complex representations of Q of dimension vector \mathbf{d} , on which the group

$$G_{\mathbf{d}} = \prod_{i \in Q_0} \text{GL}(V_i)$$

naturally acts via base change in all V_i . Note that the diagonally embedded scalar group $\mathbb{C}^* \subset G_{\mathbf{d}}$ acts trivially.

Throughout the paper, we are going to assume that $d_i > 0$ for every $i \in Q_0$. This is no loss of generality as, otherwise, we may pass to the full subquiver consisting of the vertices for which $d_i > 0$.

We consider the dual group $(\mathbb{Z}Q_0)^*$ with its natural basis elements i taking the i th component of a dimension vector. Let $\Theta \in (\mathbb{Z}Q_0)^*$ be a linear form such that $\Theta(\mathbf{d}) = 0$ and define a point $(f_\alpha)_\alpha \in R_{\mathbf{d}}(Q)$, considered as a representation $V = ((V_i)_i, (f_\alpha)_\alpha)$ of Q , to be Θ -semistable (resp. Θ -stable) if $\Theta(\mathbf{e}) \leq 0$ (resp. $\Theta(\mathbf{e}) < 0$) for all dimension vectors \mathbf{e} of proper nonzero subrepresentations of V .

Denoting by $R_{\mathbf{d}}^{\Theta\text{-st}}(Q)$ the Zariski open (semi)stable locus in $R_{\mathbf{d}}(Q)$, we consider the GIT quotient

$$M_{\mathbf{d}}^{\Theta\text{-sst}}(Q) = R_{\mathbf{d}}^{\Theta\text{-sst}}(Q) // G_{\mathbf{d}}$$

and the geometric quotient

$$M_{\mathbf{d}}^{\Theta\text{-st}}(Q) = R_{\mathbf{d}}^{\Theta\text{-st}}(Q) / G_{\mathbf{d}}.$$

We briefly review their construction following [16]. Let χ be a character of $G_{\mathbf{d}}$. A regular function $f \in \mathbb{C}[R_{\mathbf{d}}(Q)]$ is called χ -semi-invariant if $f(gM) = \chi(g)f(M)$ for all $g \in G_{\mathbf{d}}$ and all $M \in R_{\mathbf{d}}(Q)$. We consider the space $\mathbb{C}[R_{\mathbf{d}}(Q)]^{G_{\mathbf{d}}, \chi}$ of χ -semi-invariant regular functions on $R_{\mathbf{d}}(Q)$ and the graded ring

$$\mathbb{C}[R_{\mathbf{d}}(Q)]_{\chi}^{G_{\mathbf{d}}} = \bigoplus_{N \geq 0} \mathbb{C}[R_{\mathbf{d}}(Q)]^{G_{\mathbf{d}}, N \cdot \chi}.$$

To the stability Θ , we associate the character

$$\chi_{\Theta}((g_i)_i) = \prod_{i \in Q_0} \det(g_i)^{-\Theta(i)}$$

(the sign change resulting from our definition of Θ -semistability in contrast to the convention of [16]). Then,

$$M_{\mathbf{d}}^{\Theta\text{-sst}}(Q) = \text{Proj}(\mathbb{C}[R_{\mathbf{d}}(Q)]_{\chi_{\Theta}}^{G_{\mathbf{d}}}).$$

More explicitly, choosing N big enough so that $\mathbb{C}[R_{\mathbf{d}}(Q)]_{\chi_{\Theta}}^{G_{\mathbf{d}}}$ is generated by homogeneous elements f_0, \dots, f_s in $\mathbb{C}[R_{\mathbf{d}}(Q)]^{G_{\mathbf{d}}, N \cdot \chi_{\Theta}}$, the moduli space $M_{\mathbf{d}}^{\Theta\text{-sst}}(Q)$ equals the image of the map

$$f = (f_0 : \dots : f_s) : R_{\mathbf{d}}^{\Theta\text{-sst}}(Q) \rightarrow \mathbb{P}^s.$$

Finally, the moduli space $M_{\mathbf{d}}^{\Theta\text{-st}}(Q)$ equals the (open) image of $R_{\mathbf{d}}^{\Theta\text{-st}}(Q)$ under the map f .

This moduli space, if nonempty, is called the moduli space of Θ -stable representations of Q of dimension vector \mathbf{d} ; it is a connected complex algebraic manifold of complex dimension $1 - \langle \mathbf{d}, \mathbf{d} \rangle$. Its points naturally correspond to isomorphism classes $[V]$ of Θ -stable representations V of Q of dimension vector \mathbf{d} . Furthermore, the quotient map

$$R_{\mathbf{d}}^{\Theta\text{-st}}(Q) \rightarrow M_{\mathbf{d}}^{\Theta\text{-st}}(Q)$$

is a principal bundle for the group $PG_{\mathbf{d}} = G_{\mathbf{d}}/\mathbb{C}^*$.

We call the dimension vector \mathbf{d} indivisible if $\gcd(d_i : i \in Q_0) = 1$ and Θ -coprime if $\Theta(\mathbf{e}) \neq 0$ for all nonzero proper $\mathbf{e} \leq \mathbf{d}$. Then, Θ -coprimality implies indivisibility, and, conversely, an indivisible dimension vector is Θ -coprime for a sufficiently generic choice of Θ . In case \mathbf{d} is Θ -coprime, the Θ -stable and the Θ -semistable locus in $R_{\mathbf{d}}(Q)$ coincide; thus, $M_{\mathbf{d}}^{\Theta\text{-st}}(Q)$ is a connected compact complex algebraic manifold.

2.2 Stabilities

We denote by $\{\mathbf{d}, \mathbf{e}\} = \langle \mathbf{d}, \mathbf{e} \rangle - \langle \mathbf{e}, \mathbf{d} \rangle$ the antisymmetrized Euler form of Q and use it to define the stability $\{\mathbf{d}, _ \}$ of Q , which we call the canonical stability for \mathbf{d} . We can reformulate [26, Theorem 6.1] as follows: there exists a $\{\mathbf{d}, _ \}$ -stable representation of dimension vector \mathbf{d} if and only if there exists a representation V of dimension vector \mathbf{d} with trivial endomorphism ring $\text{End}_Q(V) = \mathbb{C}$.

For an indivisible dimension vector \mathbf{d} , we consider the abelian group $\text{Stab}(\mathbf{d}) = \{\Theta \in (\mathbb{Z}Q_0)^* : \Theta(\mathbf{d}) = 0\}$ of stabilities for \mathbf{d} and its associated real vector space $\text{Stab}(\mathbf{d})_{\mathbb{R}} = \text{Stab}(\mathbf{d}) \otimes_{\mathbb{Z}} \mathbb{R}$. For every nonzero proper $\mathbf{e} \leq \mathbf{d}$, we consider the hyperplane $W_{\mathbf{e}} = \{\Theta : \Theta(\mathbf{e}) = 0\} \subset \text{Stab}(\mathbf{d})_{\mathbb{R}}$, called a wall in $\text{Stab}(\mathbf{d})_{\mathbb{R}}$. For a connected component $C_{\mathbb{R}}$ of the complement

$$\text{Stab}^0(\mathbf{d})_{\mathbb{R}} = \text{Stab}(\mathbf{d})_{\mathbb{R}} \setminus \bigcup_{\mathbf{e}} W_{\mathbf{e}}$$

its closure $\overline{C_{\mathbb{R}}}$, resp. the set of integral points $\overline{C_{\mathbb{R}}} \cap \text{Stab}(\mathbf{d})$ in it, is a (convex polyhedral) cone, called a chamber in $\text{Stab}(\mathbf{d})_{\mathbb{R}}$ and in $\text{Stab}(\mathbf{d})$, respectively. Then, \mathbf{d} is Θ -coprime if and only if Θ belongs to the interior of a chamber. If \mathbf{d} is $\{\mathbf{d}, _ \}$ -coprime, we denote by C_{can} the chamber whose interior contains $\{\mathbf{d}, _ \}$ and call it the canonical chamber.

Remark 2.1 We note that this decomposition is, in general, slightly finer than the decomposition into GIT chambers, in that the set of semistable representations might stay constant even when the stability Θ crosses a wall $W_{\mathbf{e}}$. Since this phenomenon is difficult to control numerically, we prefer to work with the above chamber structure.

For a nonzero stability $\Theta \in \text{Stab}(\mathbf{d})$, we finally define

$$\gcd(\Theta) = \gcd(\Theta_i \mid i \in Q_0).$$

3 Line bundles

3.1 Construction of tautological bundles on quiver moduli

We review the construction of tautological bundles of [16]. Assuming again that \mathbf{d} is Θ -coprime, and thus that \mathbf{d} is indivisible, we can choose integers a_i for $i \in Q_0$ such that

$$\sum_i a_i d_i = 1$$

and define $a \in (\mathbb{Z}Q_0)^*$ by $a(\mathbf{i}) = a_i$.

We consider the trivial bundle $R_{\mathbf{d}}^{\Theta\text{-st}}(Q) \times V_i \rightarrow R_{\mathbf{d}}^{\Theta\text{-st}}(Q)$ and $G_{\mathbf{d}}$ -linearize by defining the action on V_i by

$$(g_j)_j \cdot v_i = \prod_{j \in Q_0} \det(g_j)^{-a_j} g_j v_i.$$

By definition of a , the scalar subgroup \mathbb{C}^* then acts trivially, allowing this bundle to descend via the quotient map $R_{\mathbf{d}}^{\Theta\text{-st}}(Q) \rightarrow M_{\mathbf{d}}^{\Theta\text{-st}}(Q)$ to a bundle \mathcal{V}_i . By slight abuse of notation, we denote the above trivial bundle with its $G_{\mathbf{d}}$ -linearization just by V_i ; thus, V_i descends to \mathcal{V}_i . The bundles \mathcal{V}_i are then tautological in the following sense:

For any arrow $\alpha : i \rightarrow j$, there is a natural map $V_{\alpha} : V_i \rightarrow V_j$ given by

$$V_{\alpha}((f_{\beta})_{\beta}, v_i) = ((f_{\beta})_{\beta}, f_{\alpha}(v_i)).$$

It descends to a map of vector bundles $\mathcal{V}_{\alpha} : \mathcal{V}_i \rightarrow \mathcal{V}_j$, thus defining a representation \mathcal{V} of Q in the category of vector bundles on $M_{\mathbf{d}}^{\Theta\text{-st}}(Q)$, such that the induced quiver representation

$$(((\mathcal{V}_i)_{[V]})_i, ((\mathcal{V}_{\alpha})_{[V]})_{\alpha})$$

in the fiber over a point $[V]$ of $M_{\mathbf{d}}^{\Theta\text{-st}}$ is isomorphic to V . This, in fact, makes $M_{\mathbf{d}}^{\Theta\text{-st}}(Q)$ a fine moduli space, in the sense that any fiberwise Θ -stable representation of Q in vector bundles of rank vector \mathbf{d} arises via pullback from the above tautological family.

3.2 The Picard group of a fine quiver moduli space

Assume as before that \mathbf{d} is indivisible. The exact sequence of groups

$$1 \rightarrow \mathbb{C}^* \rightarrow G_{\mathbf{d}} \rightarrow PG_{\mathbf{d}} \rightarrow 1$$

induces an exact sequence

$$0 \rightarrow X(PG_{\mathbf{d}}) \rightarrow X(G_{\mathbf{d}}) \rightarrow X(\mathbb{C}^*)$$

of character groups. The isomorphism

$$(\mathbb{Z}Q_0)^* \rightarrow X(G_{\mathbf{d}})$$

mapping Θ to the character χ_Θ as defined above yields a diagram of abelian groups

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Stab}(\mathbf{d}) & \longrightarrow & (\mathbb{Z}Q_0)^* & \xrightarrow{\mathbf{d}} & \mathbb{Z} \longrightarrow 0 \\
 & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
 0 & \longrightarrow & X(PG_{\mathbf{d}}) & \longrightarrow & X(G_{\mathbf{d}}) & \longrightarrow & X(\mathbb{C}^*) \longrightarrow 0
 \end{array}$$

with right exact sequences, since the evaluation at \mathbf{d} admits a section induced by the element $a \in (\mathbb{Z}Q_0)^*$ chosen above.

We have an isomorphism

$$X(PG_{\mathbf{d}}) \simeq \text{Pic}^{PG_{\mathbf{d}}}(R_{\mathbf{d}}(Q))$$

given by assigning to a character χ the trivial line bundle on $R_{\mathbf{d}}(Q)$ with $PG_{\mathbf{d}}$ -linearization given by χ , denoted $L(\chi)$.

We now make the following additional assumption on \mathbf{d} :

The dimension vector \mathbf{d} is called Θ -amply stable if the codimension of the unstable locus is at least two, that is,

$$\text{codim}_{R_{\mathbf{d}}(Q)}(R_{\mathbf{d}}(Q) \setminus R_{\mathbf{d}}^{\Theta\text{-st}}(Q)) \geq 2.$$

A sufficient numerical condition for this property is stated in [24, Proposition 5.1]: \mathbf{d} is Θ -amply stable if $\langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle \leq -2$ for every nonzero proper $\mathbf{e} \leq \mathbf{d}$ such that $\Theta(\mathbf{e}) \geq 0$.

In this case, the natural restriction map

$$\text{Pic}^{PG_{\mathbf{d}}}(R_{\mathbf{d}}(Q)) \rightarrow \text{Pic}^{PG_{\mathbf{d}}}(R_{\mathbf{d}}^{\Theta\text{-st}}(Q))$$

is an isomorphism. But since $R_{\mathbf{d}}^{\Theta\text{-st}}(Q)$ is a $PG_{\mathbf{d}}$ -principal bundle over $M_{\mathbf{d}}^{\Theta\text{-st}}(Q)$, we have an isomorphism

$$\text{Pic}^{PG_{\mathbf{d}}}(R_{\mathbf{d}}^{\Theta\text{-st}}(Q)) \simeq \text{Pic}(M_{\mathbf{d}}^{\Theta\text{-st}}(Q));$$

thus, every line bundle $L(\chi)$ descends to a line bundle $\mathcal{L}(\chi)$ on $M_{\mathbf{d}}^{\Theta\text{-st}}(Q)$. Furthermore, since $M_{\mathbf{d}}^{\Theta\text{-st}}(Q)$ is smooth, taking the first Chern class gives an isomorphism

$$\text{Pic}(M_{\mathbf{d}}^{\Theta\text{-st}}(Q)) \simeq A^1(M_{\mathbf{d}}^{\Theta\text{-st}}(Q)).$$

We conclude:

Proposition 3.1 *If \mathbf{d} is Θ -coprime and Θ -amply stable, we have a chain of isomorphisms of abelian groups*

$$\begin{array}{ccccccc}
 \text{Stab}(\mathbf{d}) & \xrightarrow{\simeq} & X(PG_{\mathbf{d}}) & \xrightarrow{\simeq} & \text{Pic}(M_{\mathbf{d}}^{\Theta\text{-st}}(Q)) & \xrightarrow{\simeq} & A^1(M_{\mathbf{d}}^{\Theta\text{-st}}(Q)) \\
 \Theta' & \mapsto & \chi_{\Theta'} & \mapsto & \mathcal{L}(\chi_{\Theta'}) & \mapsto & c_1(\mathcal{L}(\chi_{\Theta'})).
 \end{array}$$

We proceed with a stability condition Θ for which \mathbf{d} is Θ -coprime and Θ -amply stable. We want to identify the determinant bundles $\det(\mathcal{V}_i)$ of the tautological bundles \mathcal{V}_i introduced above under this identification. By the definition of the $G_{\mathbf{d}}$ -linearization of the bundle \mathcal{V}_i on $R_{\mathbf{d}}^{\Theta\text{-st}}(Q)$, its determinant bundle is linearized by the character

$$\chi((g_j)_j) = \prod_j (\det g_j)^{-a_j d_i} \cdot \det(g_i).$$

Denoting by $r : (\mathbb{Z}Q_0)^* \rightarrow \text{Stab}(\mathbf{d})$ the retraction induced by the section a , that is,

$$r(\Theta) = \Theta - \Theta(\mathbf{d}) \cdot a,$$

we thus have:

Lemma 3.2 *For all $i \in Q_0$, the following equality holds in $\text{Pic}(M_{\mathbf{d}}^{\Theta\text{-st}}(Q))$:*

$$\det(\mathcal{V}_i) = \mathcal{L}(\chi_{-r(i)}).$$

3.3 The ample cone

By slight abuse of notation, we denote the total space of a line bundle by the same symbol. For any $\chi \in X(PG_{\mathbf{d}})$, we then have a commutative square with the horizontal maps being $PG_{\mathbf{d}}$ -principal bundles and the vertical maps being bundle projections

$$\begin{array}{ccc} L(\chi) & \longrightarrow & \mathcal{L}(\chi) \\ \downarrow & & \downarrow \\ R_{\mathbf{d}}^{\Theta\text{-st}}(Q) & \longrightarrow & M_{\mathbf{d}}^{\Theta\text{-st}}(Q). \end{array}$$

A global section of $\mathcal{L}(\chi)$ then corresponds uniquely to a $PG_{\mathbf{d}}$ -equivariant section $R_{\mathbf{d}}^{\Theta\text{-st}}(Q) \rightarrow L(\chi)$ of $L(\chi)$, which determines, and is determined by, a χ -semi-invariant function on $R_{\mathbf{d}}^{\Theta\text{-st}}(Q)$. This proves:

Lemma 3.3 *If \mathbf{d} is Θ -coprime and Θ -amply stable, we have*

$$H^0(M_{\mathbf{d}}^{\Theta\text{-st}}(Q), \mathcal{L}(\chi)) \simeq \mathbb{C}[R_{\mathbf{d}}(Q)]^{G_{\mathbf{d}}, \chi}.$$

Remark 3.4 The above observation can be generalized to vector bundles that come from finite-dimensional complex representations M of $PG_{\mathbf{d}}$. We consider the trivial vector bundle $V(M)$ on $R_{\mathbf{d}}^{\Theta\text{-st}}(Q)$ with fiber M and with the $PG_{\mathbf{d}}$ -linearization given by the action of $PG_{\mathbf{d}}$ on M . This descends to a vector bundle $\mathcal{V}(M)$ on $M_{\mathbf{d}}^{\Theta\text{-st}}(Q)$. By Hartogs' theorem, global sections then identify with

$$H^0(M_{\mathbf{d}}^{\Theta\text{-st}}(Q), \mathcal{V}(M)) \simeq \text{Mor}(R_{\mathbf{d}}(Q), M)^{PG_{\mathbf{d}}},$$

the space of $PG_{\mathbf{d}}$ -invariant morphisms to M . The coordinate ring of M , regarded as a $PG_{\mathbf{d}}$ -variety, is $\mathbb{C}[M] = S(M^*)$, the symmetric algebra of the dual representation. Therefore,

$$\begin{aligned} \text{Mor}(R_{\mathbf{d}}(Q), M)^{PG_{\mathbf{d}}} &\simeq \text{Ring}(\mathbb{C}[M], \mathbb{C}[R_{\mathbf{d}}(Q)])^{PG_{\mathbf{d}}} \\ &\simeq \text{Hom}_{\mathbb{C}}(M^*, \mathbb{C}[R_{\mathbf{d}}(Q)])^{PG_{\mathbf{d}}} \\ &\simeq \text{Hom}_{PG_{\mathbf{d}}}(M^*, \mathbb{C}[R_{\mathbf{d}}(Q)]). \end{aligned}$$

If M is an irreducible representation, then the latter coincides with the M^* -isotypical component of $\mathbb{C}[R_{\mathbf{d}}(Q)]$. Note that this tallies with the previous lemma, as the $PG_{\mathbf{d}}$ -action on $\mathbb{C}[R_{\mathbf{d}}(Q)]$ is given by $(g \cdot f)(M) = f(g^{-1}M)$.

Now we turn to the ample cone of the moduli space $M_{\mathbf{d}}^{\Theta\text{-st}}(Q)$. In [11], Halic determines the ample cone and the nef cone of a GIT quotient of the form $V^{\text{st}}(G, \chi)/G$ of a vector space by a reductive group under the assumption that stability and semistability agree and that the unstable locus be of codimension at least 2. Translated to our situation, this means:

Proposition 3.5 ([11, Proposition 6.2]) *Let \mathbf{d} be Θ -coprime and Θ -amply stable, and let $\Theta' \in \text{Stab}(\mathbf{d})$.*

- (1) *The line bundle $\mathcal{L}(\chi_{\Theta'})$ on $M_{\mathbf{d}}^{\Theta\text{-st}}(Q)$ is ample if and only if $R_{\mathbf{d}}^{\Theta\text{-sst}}(Q) = R_{\mathbf{d}}^{\Theta'\text{-sst}}(Q)$.*
- (2) *The line bundle $\mathcal{L}(\chi_{\Theta'})$ on $M_{\mathbf{d}}^{\Theta\text{-st}}(Q)$ is nef if and only if $R_{\mathbf{d}}^{\Theta\text{-sst}}(Q) \subseteq R_{\mathbf{d}}^{\Theta'\text{-sst}}(Q)$.*

In particular, $\mathcal{L}(\chi_{\Theta})$ is ample.

For the convenience of the reader, let us prove the part of the proposition that is most relevant for us: we show that $\mathcal{L}(\chi_{\Theta'})$ is ample provided that $R_{\mathbf{d}}^{\Theta\text{-sst}}(Q) = R_{\mathbf{d}}^{\Theta'\text{-sst}}(Q)$. The line bundle $\mathcal{L}(\chi_{\Theta'})$ being ample means that for large enough N , generators f_0, \dots, f_s of

$$H^0(M_{\mathbf{d}}^{\Theta\text{-st}}(Q), \mathcal{L}(\chi_{\Theta'})^{\otimes N}) \simeq \mathbb{C}[R_{\mathbf{d}}(Q)]^{G_{\mathbf{d}}, N, \chi_{\Theta'}}$$

descend from $R_{\mathbf{d}}^{\Theta\text{-st}}(Q)$ via the quotient map to a closed embedding

$$(f_0 : \dots : f_s) : M_{\mathbf{d}}^{\Theta\text{-st}}(Q) \rightarrow \mathbb{P}^s.$$

But, by the construction of quiver moduli, we know that, assuming N to be large enough to generate $\mathbb{C}[R_{\mathbf{d}}(Q)]_{\chi_{\Theta'}}^{G_{\mathbf{d}}}$, the image of this map is precisely $M_{\mathbf{d}}^{\Theta'\text{-st}}(Q)$. Ampleness of $\mathcal{L}(\chi_{\Theta'})$ is thus equivalent to such functions inducing a closed immersion

$$M_{\mathbf{d}}^{\Theta\text{-st}}(Q) \rightarrow M_{\mathbf{d}}^{\Theta'\text{-st}}(Q)$$

and thus an isomorphism, since both spaces are irreducible and projective. This is certainly the case if the open sets $R_{\mathbf{d}}^{\Theta\text{-st}}(Q)$ and $R_{\mathbf{d}}^{\Theta'\text{-st}}(Q)$ coincide, which, in turn, is fulfilled if the semistable loci with respect to Θ and Θ' agree.

Remark 3.6 Under the assumption of the previous proposition, a result of Hu and Keel [15, Theorem 2.3] shows that $M_{\mathbf{d}}^{\Theta\text{-st}}(Q)$ is a Mori Dream Space. Note that Maslovic shows the Mori Dream Space Property for quiver moduli in [19] even without the hypothesis of \mathbf{d} being amply stable for Θ . Theorem 2.3 of [15] further identifies Mori chambers with GIT chambers. In order to determine the ample cone of $M_{\mathbf{d}}^{\Theta\text{-st}}(Q)$, it hence suffices to find a stability condition Θ' for which $\mathcal{L}(\chi_{\Theta'})$ is ample. The ample cone is then necessarily the GIT chamber that contains Θ' . Therefore, the argument given in the proof above actually fully proves item (1) of Proposition 3.5.

4 Fano quiver moduli

4.1 Computation of the (anti)canonical class

We would like to express the class of the determinant bundle $\det \mathcal{T}$ in $\text{Pic}(M_d^{\Theta\text{-st}}(Q))$, resp. in $A^1(M_d^{\Theta\text{-st}}(Q))$, where \mathcal{T} is the tangent bundle. To do this, note that the fiber of \mathcal{T} in a point $[V] \in M_d^{\Theta\text{-st}}(Q)$ is

$$\mathcal{T}_{[V]} \simeq \text{Ext}_Q^1(V, V).$$

Let $[V]$ be represented by $V = (f_\alpha)_\alpha \in R_d(Q)$. There is an exact sequence

$$0 \rightarrow \text{End}_Q(V, V) \rightarrow \bigoplus_{i \in Q_0} \text{End}(V_i) \xrightarrow{\phi} \bigoplus_{\alpha: i \rightarrow j} \text{Hom}(V_i, V_j) \rightarrow \text{Ext}_Q^1(V, V) \rightarrow 0,$$

where $\phi((x_i)_i) = (x_j f_\alpha - f_\alpha x_i)_\alpha$. This sequence is a consequence of the standard projective resolution of V . Note that the map ϕ is precisely the derivative of the map $G_d \rightarrow R_d^{\Theta\text{-st}}(Q)$, which sends $g \mapsto gV$. The endomorphism ring of a stable representation reduces to the scalars. By globalizing the above exact sequence, we obtain an exact sequence of vector bundles

$$0 \rightarrow \mathcal{O} \rightarrow \bigoplus_{i \in Q_0} \mathcal{V}_i^* \otimes \mathcal{V}_i \rightarrow \bigoplus_{\alpha: i \rightarrow j} \mathcal{V}_i^* \otimes \mathcal{V}_j \rightarrow \mathcal{T} \rightarrow 0.$$

Remark 4.1 The above sequence is a special case of the following sequence (see, e.g., [5, Section 4.1]). Let a reductive group G act linearly on a finite-dimensional vector space V and let χ be a character of G . Suppose that there are no properly semistable points for χ and that G acts freely on the stable locus $V^{\text{st}}(G, \chi)$. Then, we obtain a short exact sequence

$$0 \rightarrow \mathfrak{g} \times^G V^{\text{st}}(G, \chi) \rightarrow V \times^G V^{\text{st}}(G, \chi) \rightarrow \mathcal{T}_{V^{\text{st}}(G, \chi)/G} \rightarrow 0$$

of vector bundles on the quotient $V^{\text{st}}(G, \chi)/G$.

To calculate the anticanonical class, we perform a Chern class calculation in the Chow group $A^1(M_d^{\Theta\text{-st}}(Q))$, made possible by the above isomorphism to the Picard group, using the following properties of first Chern classes:

- c_1 is additive on short exact sequences;
- $c_1(\det \mathcal{E}) = c_1(\mathcal{E})$;
- $c_1(\mathcal{O}) = 0$;
- $c_1(\mathcal{E}^*) = -c_1(\mathcal{E})$;
- $c_1(\mathcal{E} \otimes \mathcal{F}) = \text{rk}(\mathcal{F}) \cdot c_1(\mathcal{E}) + \text{rk}(\mathcal{E}) \cdot c_1(\mathcal{F})$.

We thus find

$$c_1\left(\bigoplus_{i \in Q_0} \mathcal{V}_i^* \otimes \mathcal{V}_i\right) = \sum_i (-d_i c_1(\mathcal{V}_i) + d_i c_1(\mathcal{V}_i)) = 0,$$

$$c_1\left(\bigoplus_{\alpha: i \rightarrow j} \mathcal{V}_i^* \otimes \mathcal{V}_j\right) = \sum_{\alpha: i \rightarrow j} (-d_j c_1(\mathcal{V}_i) + d_i c_1(\mathcal{V}_j)),$$

and thus, using the above exact sequence,

$$\begin{aligned} c_1(\mathcal{T}) &= c_1\left(\bigoplus_{\alpha:i \rightarrow j} \mathcal{V}_i^* \otimes \mathcal{V}_j\right) \\ &= \sum_{\alpha:i \rightarrow j} (-d_j c_1(\mathcal{V}_i) + d_i c_1(\mathcal{V}_j)) \\ &= \sum_i (\langle \mathbf{i}, \mathbf{d} \rangle - \langle \mathbf{d}, \mathbf{i} \rangle) c_1(\mathcal{V}_i). \end{aligned}$$

Since

$$c_1(\mathcal{V}_i) = c_1(\det(\mathcal{V}_i)) = c_1(\mathcal{L}(\chi_{-r(i)})),$$

we thus find $\det(\mathcal{T}) = \mathcal{L}(\chi_{\Theta'})$ for

$$\Theta' = \sum_i (\langle \mathbf{i}, \mathbf{d} \rangle - \langle \mathbf{d}, \mathbf{i} \rangle) (-r(i)) = -r(\langle _, \mathbf{d} \rangle - \langle \mathbf{d}, _ \rangle) = r(\langle \mathbf{d}, _ \rangle) = \langle \mathbf{d}, _ \rangle,$$

since r acts as the identity on linear forms in $\text{Stab}(\mathbf{d})$. We have thus proved:

Proposition 4.2 *In $\text{Pic}(M_{\mathbf{d}}^{\Theta-st})(Q)$, we have*

$$\det(\mathcal{T}) = \mathcal{L}(\chi_{\langle \mathbf{d}, _ \rangle}).$$

Applying the above criterion for ampleness, we can thus derive our main result, summarizing our findings:

Theorem 4.3 *If \mathbf{d} is $\langle \mathbf{d}, _ \rangle$ -coprime and $\langle \mathbf{d}, _ \rangle$ -amply stable, then the moduli space $M_{\mathbf{d}}^{\langle \mathbf{d}, _ \rangle-st}(Q)$ is a smooth irreducible projective Fano variety of dimension $1 - \langle \mathbf{d}, \mathbf{d} \rangle$, Picard rank $|Q_0| - 1$, and index $\gcd(\langle \mathbf{d}, _ \rangle)$, which is rational and has only algebraic cohomology.*

5 Classes of examples

5.1 Subspace quivers—moduli of point configurations in projective space

We consider the quiver S_m with vertices i_1, \dots, i_m, j and arrows $i_k \rightarrow j$ for $k = 1, \dots, m$, called the m -subspace quiver. For a fixed $d \geq 2$, we consider the dimension vector

$$\mathbf{d} = \sum_k \mathbf{i}_k + d\mathbf{j}.$$

We have

$$\Theta = \langle \mathbf{d}, _ \rangle = d \sum_k i_k - m\mathbf{j};$$

thus, \mathbf{d} is Θ -coprime if m and d are coprime. In this case, the moduli space $M_{\mathbf{d}}^{\Theta-st}(S_m)$ equals the moduli space of stable ordered configurations of m points in \mathbb{P}^{d-1} modulo, the natural $\text{PGL}_d(\mathbb{C})$ -action. Here, (p_1, \dots, p_m) is called stable if the linear subspace spanned by any k of these points has dimension at least $dk/m - 1$. This moduli space

is nonempty and not a single point if and only if $m - 1 > d$, in which case it is of dimension

$$(d - 1)(m - d - 1).$$

We can easily verify that \mathbf{d} is $\{\mathbf{d}, _ \}$ -amply stable with the numerical sufficient criterion: suppose that $\mathbf{e} \leq \mathbf{d}$, and thus $\mathbf{e} = \sum_k \mathbf{i}_k + e\mathbf{j}$ for a subset $K \subset \{1, \dots, m\}$ and $e \leq d$. Then, $\Theta(\mathbf{e}) > 0$ if and only if $|K| > me/d$. We calculate

$$\langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle_{S_m} = (e - |K|)(d - e) < -\frac{m - d}{d}e(d - e) \leq 0.$$

Thus, assuming that $\langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle \geq -1$, we find

$$e = d - 1 \text{ and } |K| = e + 1 = d.$$

By assumption on K , this yields $d > m(d - 1)/d$ or, equivalently,

$$d^2 > m(d - 1) \geq d^2 - 1.$$

We thus find $m = d + 1$, contradicting the assumption.

Corollary 5.1 *For given coprime m and d such that $m - 1 > d \geq 2$, the moduli space $(\mathbb{P}^{d-1})_{st}^m / \text{PGL}_d(\mathbb{C})$ is Fano of dimension $(d - 1)(m - d - 1)$, with Picard rank m and index one.*

To examine these Fano varieties in low dimension, we can additionally assume $m > 2d$ using the duality [25, Theorem 3.1]. For $d = 2$ and $m = 5$, we thus find a del Pezzo surface of Picard rank 5, which is, therefore, isomorphic to the blowup of the projective plane in four general points. We will see below how all del Pezzo surfaces of lower Picard rank (which are already toric) can be realized as quiver moduli. For $d = 2$ and $m = 7$, we find a four-dimensional Fano variety with Picard rank 7 and fourth Betti number $b_4 = 22$. For $d = 3$ and $m = 7$, we find a six-dimensional Fano variety with Picard rank 7, $b_4 = 29$, and $b_6 = 64$, using the Betti number formulas of [25].

An interesting (non)example arises here, too, which shows that the assumption of our main result cannot be relaxed much: for the moduli space of six points in the projective line, the dimension vector \mathbf{d} is not Θ -coprime for the canonical stability Θ . The semistable moduli space is isomorphic to the Segre cubic by [14], which is indeed a singular projective Fano threefold with 10 isolated singularities. Slightly deforming the stability Θ to a new stability Θ^+ as in [9] yields a small desingularization of the Segre cubic, which cannot be Fano: a Fano threefold of Picard rank 6 is automatically isomorphic to the product of the projective line and the degree five del Pezzo surface, thus contains a two-nilpotent element in second cohomology. But in [9], it is shown that this does not hold for this desingularization, disproving the Fano property.

5.2 Generalized Kronecker quivers—Kronecker moduli

Let K_m be the quiver with vertices i and j and $m \geq 1$ arrows from i to j . Let $\mathbf{d} = d\mathbf{i} + e\mathbf{j}$ be a dimension vector, and let

$$\Theta = \{\mathbf{d}, _ \} = mei - mdj$$

be the canonical stability. As in the previous example, Θ -coprimality of \mathbf{d} is equivalent to d and e being coprime. If $m = 1$ or $m = 2$, the only nonempty moduli spaces $K_{(d,e)}^{(m)} = M_{\mathbf{d}}^{\Theta\text{-st}}(Q)$ are single points or a projective line (for $m = 2$ and $d = 1 = e$); thus, we assume $m \geq 3$. In this case, if nonempty, the moduli space is of dimension

$$mde - d^2 - e^2 + 1.$$

The Θ -ample stability of \mathbf{d} is proved in [24, Proposition 6.2]. We thus find:

Corollary 5.2 *For given $m \geq 3$ and coprime d and e , the Kronecker moduli space $K_{(d,e)}^{(m)}$, if nonempty, is a Fano variety of dimension $mde - d^2 - e^2 + 1$, Picard rank 1, and index m .*

A theorem of Kobayashi and Ochiai [18] asserts that a smooth projective Fano variety X of dimension n and index q satisfies $q \leq n + 1$ and equality holds if and only if $X \simeq \mathbb{P}^n$. This led Mukai [21] to a conjecture on the relation between the rank, the index, and the dimension of a Fano variety. This conjecture was later refined by Bonaveroet al. [3]. It states the following. Let X be a smooth projective Fano variety of dimension n , let r denote the rank of the Picard group of X , and let p be the minimal positive integer such that $-K_X \cdot C = p$ for a rational curve $C \subset X$, called the pseudo-index of X . It is then conjectured that

$$r \cdot (p - 1) \leq n,$$

and equality is satisfied if and only if X is isomorphic to $(\mathbb{P}^{p-1})^r$. In the original Mukai conjecture, the pseudo-index is replaced by the index. Note that $q \leq p \leq n + 1$; for the right-hand estimate, see Mori [20]. It would be interesting to know if the Mukai conjecture holds for Fano quiver moduli (see the following section for a first example). For Kronecker moduli, the rank of the Picard group is 1, so the conjecture is true by the Kobayashi–Ochiai theorem. We illustrate how to confirm this statement entirely numerically in this case.

Proposition 5.3 *If nonempty, the moduli space $K_{(d,e)}^{(m)}$ has at least dimension $m - 1$ and is isomorphic to \mathbb{P}^{m-1} if equality holds.*

So, assume that $K_{(d,e)}^{(m)}$ is nonempty. We will prove that

$$mde - d^2 - e^2 + 1 \geq m - 1,$$

and that equality holds if and only if $d = 1 = e$. By standard reductions (see the proof of [24, Proposition 6.2]), we can assume that $d \leq e \leq md/2$. Writing $e = \alpha d$ for $\alpha \in \mathbb{Q}$ with $1 \leq \alpha \leq m/2$, we thus have to prove that

$$(m\alpha - 1 - \alpha^2)d^2 \geq m - 2,$$

and that equality holds if and only if $d = 1 = e$. The quadratic function

$$\alpha \mapsto m\alpha - 1 - \alpha^2$$

being strictly monotonously increasing in the interval $[1, m/2]$, we have

$$m\alpha - 1 - \alpha^2 \geq m - 2,$$

with equality only for $\alpha = 1$, and thus

$$(m\alpha - 1 - \alpha^2)d^2 \geq (m - 2)d^2 \geq m - 2,$$

with equality only if $d = 1$ and $\alpha = 1$, which implies $d = 1 = e$ by coprimality of d and e .

5.3 A class of Fano quiver moduli of arbitrary rank and index

For fixed $m, k \geq 1$, we consider the quiver $S_m^{(k)}$ with vertices i_1, \dots, i_m, j and k arrows from each vertex i_l to j , for $l = 1, \dots, m$ (a thickened version of the m -subspace quiver). For a fixed $d \geq 1$, we consider the dimension vector

$$\mathbf{d} = \sum_k \mathbf{i}_k + d\mathbf{j}.$$

We have

$$\Theta = \{\mathbf{d}, _ \} = k \left(d \sum_l i_l - m j \right);$$

thus, \mathbf{d} is again Θ -coprime if m and d are coprime, which we assume from now on.

A representation of $S_m^{(k)}$ is given by a tuple $(v_{i,j} \mid i = 1, \dots, m, j = 1, \dots, k)$ of vectors in \mathbb{C}^d , which are considered up to the diagonal action of $\text{GL}_d(\mathbb{C})$, and up to the action of an m -torus T acting by

$$(t_i)_i \cdot (v_{i,j})_{i,j} = (t_i v_{i,j})_{i,j}.$$

Such a tuple is stable if the $v_{i,j}$ span \mathbb{C}^d and, for every nonempty proper subset I of $\{1, \dots, m\}$, the span U_I of the $v_{i,j}$ for $i \in I$ has dimension strictly bigger than $d|I|/m$. Assuming $d \leq km$, we can choose the vectors $v_{i,j}$ sufficiently independent such that, for all nonempty proper I , the space U_I has dimension $k|I|$ if $k|I| \leq d$ and equals \mathbb{C}^d otherwise. In both cases, the estimate for stability is trivially fulfilled, proving that there exists a stable point if $d \leq km$.

Arranging the vectors $v_{i,j}$ into a $d \times km$ -matrix and noting that the quotient of the open set of highest rank matrices by the $\text{GL}_d(\mathbb{C})$ -action is isomorphic to the Grassmannian Gr_d^{km} , we see that the moduli space $M_{\mathbf{d}}^{\Theta\text{-st}}(S_m^{(k)})$ admits an interpretation, as the quotient of an open set of stable points in $\text{Gr}_d(\mathbb{C}^k \otimes \mathbb{C}^m)$ by the action of the torus T induced from its natural action on \mathbb{C}^m .

As special cases, we find the moduli spaces of point configurations from above for $k = 1$, Grassmannians Gr_d^k for $m = 1$, and powers of projective spaces $(\mathbb{P}^{k-1})^m$ for $d = 1$ or $d = mk - 1$.

In general, the moduli space is of dimension

$$1 - m - d^2 + kmd = (km - 1 - d)(d - 1) + (k - 1)m;$$

thus, it is not empty and not a single point if $d < km$, and additionally, $2 \leq d \leq m - 1$ in the case $k = 1$ treated before; we assume these estimates for d from now on.

We can then verify Θ -ample stability: suppose given a dimension vector $\mathbf{e} \leq \mathbf{d}$ such that $\Theta(\mathbf{e}) > 0$ and $\langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle \geq -1$. Then, $\mathbf{e} = \sum_{k \in I} i_k + e\mathbf{j}$ for a subset I of $\{1, \dots, m\}$ and an $e \leq d$. Denoting $n = |I|$, the assumptions on \mathbf{e} translate to $dn > em$ and $(kn - e)(d - e) \leq 1$. The first inequality $dn > em$ easily implies that both factors in the second inequality are positive, and thus $kn - 1 = e = d - 1$. Then, $dn > em$ reads $d \cdot d/k > (d - 1)m$, and thus $d^2 > (d - 1)mk$, which is equivalent to

$$(d - 1)(mk - 1 - d) \leq 0.$$

Both factors being non-negative, we arrive at $d = 1$ or $d = mk - 1$. But, we also have $k|d$, and thus $k = 1$, which is the trivial case we excluded above. We thus find:

Proposition 5.4 For $m, k, d \geq 1$ such that d and m are coprime, $d \leq mk - 1$, and $d \neq 1, m - 1$ in case $k = 1$, the moduli space $Gr_d(\mathbb{C}^k \otimes \mathbb{C}^m)^{st}/T$ is a Fano variety of dimension

$$(km - 1 - d)(d - 1) + (k - 1)m,$$

Picard rank m , and index k , and it verifies the Mukai conjecture.

It remains to verify the Mukai conjecture: our dimension formula indeed yields the estimate

$$m(k - 1) \leq (km - 1 - d)(d - 1) + (k - 1)m,$$

and equality holds if $d = 1$ or $d = km - 1$, in which case, the moduli space is a power of projective space.

5.4 Toric quiver moduli

On the variety of representations $R_d(Q)$, there is an action of a torus T_1 of rank equal to the number of arrows in Q by scaling the linear maps,

$$(t_\alpha)_\alpha (f_\alpha)_\alpha = (t_\alpha f_\alpha)_\alpha,$$

which commutes with the base change action of G_d , thus inducing an action on $M_d^{\Theta-st}(Q)$. If $\mathbf{d} = \mathbf{1} := \sum_{i \in Q_0} \mathbf{i}$, this endows $M_1^{\Theta-st}(Q)$ with the structure of a toric variety. We make our conditions for the moduli space being Fano explicit in this case.

Since Q is acyclic, we can assume that $Q_0 = \{i_1, \dots, i_n\}$ is indexed such that existence of an arrow $\alpha : i_k \rightarrow i_l$ implies $k < l$. We denote by $a_{k,l}$ the number of arrows from k to l . For subsets $K, L \subset [n] = \{1, \dots, n\}$, we abbreviate

$$a_{K,L} = \sum_{k \in K} \sum_{l \in L} a_{k,l}.$$

The canonical stability $\Theta = \{\mathbf{1}, _ \}$ is given by

$$\Theta_i = a_{i[n]} - a_{[n]i}.$$

Θ -coprimality then easily translates into the condition $a_{K[n]} \neq a_{[n]K}$ for all proper nonempty subsets K , which simplifies to

$$a_{K\bar{K}} \neq a_{\bar{K}K}$$

for all proper nonempty K , where $\bar{K} = [n] \setminus K$. Finally, the numerical condition for ample stability reads

$$\max(a_{K\bar{K}}, a_{\bar{K}K}) \geq 2.$$

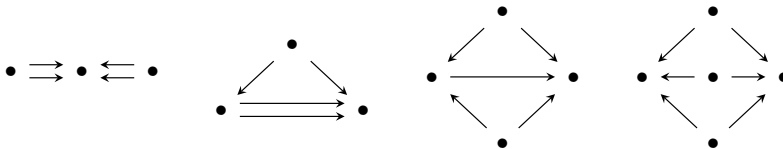
We thus find:

Proposition 5.5 *Given non-negative integers a_{kl} for $1 \leq k < l \leq n$ such that*

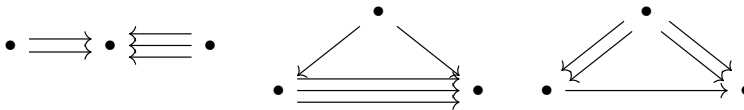
$$a_{K\bar{K}} \neq a_{\bar{K}K} \text{ and } \max(a_{K\bar{K}}, a_{\bar{K}K}) \geq 2$$

for all proper nonempty subsets $K \subset [n]$, the moduli space $M_1^{\{\mathbf{1}, _ \}}\text{-st}(Q)$ for the quiver with vertices i_1, \dots, i_n and a_{kl} arrows from vertex i_k to i_l is a smooth projective toric Fano variety of dimension $a_{[n][n]} - n + 1$, Picard rank $n - 1$, and index $\gcd(a_{k[n]} - a_{[n]k} \mid k = 1, \dots, n)$.

With this result, we can find the del Pezzo surfaces $\mathbb{P}^1 \times \mathbb{P}^1$, $\text{Bl}_1\mathbb{P}^2$, $\text{Bl}_2\mathbb{P}^2$, and $\text{Bl}_3\mathbb{P}^2$ by the respective toric quiver moduli



We can also realize the toric Fano threefolds of Picard rank 2 $\mathbb{P}^1 \times \mathbb{P}^2$, $\text{Bl}_p\mathbb{P}^3$ (blowup in a point), and $\text{Bl}_l\mathbb{P}^3$ (blowup in a line), respectively, by



The two blowups can be distinguished by considering the index; we note that a fourth class of such Fano varieties, namely $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2(2)} \oplus \mathcal{O}_{\mathbb{P}^2})$, cannot be realized using quiver moduli.

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