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IMPLICIT RENEWAL THEORY IN THE ARITHMETIC CASE

PÉTER KEVEI,* *Technische Universität München and MTA–SZTE Analysis and Stochastics Research Group*

Abstract

We extend Goldie's implicit renewal theorem to the arithmetic case, which allows us to determine the tail behavior of the solution of various random fixed point equations. It turns out that the arithmetic and nonarithmetic cases are very different. Under appropriate conditions we obtain that the tail of the solution X of the fixed point equations $X \stackrel{\mathrm{D}}{=} AX + B$ and $X \stackrel{\mathrm{D}}{=} AX \vee B$ is $\ell(x)q(x)x^{-\kappa}$, where q is a logarithmically periodic function $q(x\mathrm{e}^h) = q(x)$, $x > 0$, with h being the span of the arithmetic distribution of $\log A$, and ℓ is a slowly varying function. In particular, the tail is not necessarily regularly varying. We use the renewal theoretic approach developed by Grincevičius (1975) and Goldie (1991).

Keywords: Perpetuity equation; maximum of perturbed random walk; implicit renewal theorem; arithmetic distribution; iterated function system

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1. Introduction

Consider the perpetuity equation

$$X \stackrel{\mathrm{D}}{=} AX + B, \tag{1}$$

where (A, B) and X on the right-hand side are independent. The tail behavior of the solution has attracted much attention since Kesten's result [20]. This result was rediscovered by Grincevičius [16], whose renewal theoretic method was developed further and applied to more general random fixed point equations by Goldie [15]. They proved the following.

Theorem 1. (Kesten–Grincevičius–Goldie theorem.) *Assume that* $A \ge 0$ *almost surely (a.s.),* $\mathbb{E}A^\kappa = 1$ *for some* $\kappa > 0$, $\mathbb{E}A^\kappa \log_+ A < \infty$, $\mathbb{E}|B|^\kappa < \infty$, *and the distribution of* $\log A$ *conditioned on* $A \ne 0$ *is nonarithmetic. Then*

$$\lim_{x\to\infty} x^\kappa \mathbb{P}\{X > x\} = c_+, \qquad \lim_{x\to\infty} x^\kappa \mathbb{P}\{X < -x\} = c_-.$$

Furthermore, if $\mathbb{P}\{Ax + B = x\} < 1$ *for all* $x \in \mathbb{R}$ *then* $c_+ + c_- > 0$.

Besides perpetuity equation (1), the best known and most investigated random fixed point equation is the maximum equation

$$X \stackrel{\mathrm{D}}{=} AX \vee B, \tag{2}$$

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* Postal address: University of Szeged, Aradi vértanúk tere 1, 6720 Szeged, Hungary.
Email address: kevei@math.u-szeged.hu

where $a \vee b = \max\{a, b\}$, $A \geq 0$, and (A, B) and X on the right-hand side are independent. This equation appears in the analysis of the maximum of a perturbed random walk. Under the same assumptions Goldie proved the same tail behavior of the solution. For theory, applications, and history of perpetuity equation (1), we refer the reader to [7] and for perturbed random walks and maximum equation (2) to [18].

Interestingly enough, the case when the distribution of $\log A$ is arithmetic was only treated by Grincevičius for the perpetuity equation, by Iksanov [18] for the maximum equation, and by Jelenković and Olvera-Cravioto [19] for more general branching type fixed point equation; see their theorems below. In each case the tail has a completely different behavior than in the nonarithmetic case. In particular, the tail is not regularly varying. Investigating the maximum of random walks, the maximum equation (2) appears with $B \equiv 1$. In this case the tail behavior was analyzed by Asmussen [3, XIII. Remark 5.4] and by Korshunov [23].

The aim of this paper is to extend Goldie's implicit renewal theorem to the arithmetic case, providing a unified approach for random fixed point equations. In Subsection 2.1 we recall the aforementioned known results. In Subsection 2.2 we treat the case when the condition $\mathbb{E}A^\kappa \log_+ A < \infty$ does not hold, while in Subsection 2.3 we deal with the case when $\mathbb{E}A^\kappa < 1$ case, but $\mathbb{E}A^t = \infty$ for $t > \kappa$. The corresponding nonarithmetic versions were treated by Kevei [22]. In each case we give the general implicit renewal theorem and then specialize it to (1) and (2). In Subsection 2.4, as an example we prove that the St. Petersburg distribution is a solution of an appropriate perpetuity equation, showing that the tail of a solution can be irregular. We also show that the set of possible functions appearing in the tail of the solution is large. Finally, in Subsection 2.5 using Alsmeyer's sandwich technique [1], we show how these results apply to iterated function systems. All the proofs are contained in Section 3.

2. Results and discussion

A random variable Y, or its distribution, is called *arithmetic* (also called centered arithmetic, or centered lattice) if $Y \in h\mathbb{Z} = \{0, \pm h, \pm 2h, \ldots\}$ a.s. for some $h > 0$. The largest such h is the span of Y. We stress the difference between arithmetic and lattice distributions, where the latter means $Y \in a + h\mathbb{Z}$ a.s. for some a, h.

Assume that $\mathbb{E}A^\kappa = 1$ for some $\kappa > 0$, which is the so-called Cramér condition (for $\log A$). Due to the multiplicative structure in (1) and (2), the key idea, which goes back to Grincevičius, is to introduce a new probability measure

$$\mathbb{P}_\kappa\{\log A \in C\} = \mathbb{E}[\mathbf{1}(\log A \in C)A^\kappa], \tag{3}$$

where C is a Borel set of $\mathbb{R}$, and $\mathbf{1}(B)$ is the indicator function of the event B, i.e. it is 1 if B holds and 0 otherwise. Under the new measure, the distribution function of $\log A$ is

$$F_\kappa(x) = \mathbb{P}_\kappa\{\log A \leq x\} = \int_{-\infty}^{x} \mathrm{e}^{\kappa y} F(\mathrm{d}y), \tag{4}$$

where $F(x) = \mathbb{P}\{\log A \leq x\}$. We use the convention $\int_a^b = \int_{(a,b]}$ for $-\infty < a < b < \infty$. Here we allow $\mathbb{P}\{A = 0\} > 0$, in which case $\mathbb{P}\{\log A = -\infty\} = \mathbb{P}\{A = 0\}$, i.e. $\log A$ is an improper random variable under the probability measure $\mathbb{P}$. However, it is a proper random variable under the new measure $\mathbb{P}_\kappa$. Note that without any further assumption on the distribution of A, we have

$$F_\kappa(-x) \leq \mathrm{e}^{-\kappa x} \quad \text{for } x > 0. \tag{5}$$

Under the new measure equations (1), (2) can be rewritten as renewal equations, where the renewal function is

$$U(x) = \sum_{n=0}^{\infty} F_\kappa^{*n}(x), \tag{6}$$

where '$*n$' denotes the usual n-fold convolution. Then the tail asymptotics can be obtained via the key renewal theorem in the arithmetic case on the whole line (note that $\log A$ can be negative). If $\mathbb{E}_\kappa \log A < \infty$, which we refer to as the 'finite mean case', the required key renewal theorem is given in [18, Proposition 6.2.6]. In the 'infinite mean case', when $\mathbb{E}_\kappa \log A = \infty$, but F_κ has regularly varying tail, we prove an infinite mean key renewal theorem in the arithmetic case in Lemma 2, which is an extension of Erickson's result [12, Theorem 3]. Finally, when Cramér's condition does not hold, i.e. $\mathbb{E}A^\kappa = \theta \in (0, 1)$, $\mathbb{E}A^t = \infty$, $t > \kappa$, one ends up with a defective renewal equation, for which a key renewal theorem is given in Lemma 3.

2.1. Finite mean case

Our assumptions on A are the following:

$$\begin{gathered} A \geq 0, \qquad \mathbb{E}A^\kappa = 1 \text{ for some } \kappa > 0, \qquad \mathbb{E}A^\kappa \log_+ A < \infty, \\ \text{and } \log A \text{ conditioned on } A \neq 0 \text{ is arithmetic with span } h. \end{gathered} \tag{7}$$

Lemma 2.2 in [15] implies that $\mathbb{E}_\kappa \log A = \mathbb{E}A^\kappa \log A =: \mu > 0$. Moreover, (5) implies that $\mathbb{E}_\kappa[(\log A)_-]^2 < \infty$. Therefore, the renewal function U in (6) is well defined; see Theorem 2.1 in [21].

For a real function f, the set of its continuity points is denoted by C_f. For $\kappa > 0$ and $h > 0$ introduce the notation

$$\mathcal{Q}_{\kappa,h} = \{q \colon (0,\infty) \to [0,\infty) \colon x^{-\kappa} q(x) \text{ is nonincreasing}, \ q(x\mathrm{e}^h) = q(x), \text{ for all } x > 0\}.$$

In all the statements below, a function $q \in \mathcal{Q}_{\kappa,h}$ appears in the tail asymptotics. Note that $q \in \mathcal{Q}_{\kappa,h}$ is either strictly positive or identically 0. The following result is a special case of Theorem 3.7 by Jelenković and Olvera-Cravioto [19] (with $N \equiv 1$ and nonnegative A) for general branching type random fixed point equations. This result is the arithmetic counterpart of Goldie's implicit renewal theorem [15, Theorem 2.3]. We note that there is an extra moment condition on X in [19, Theorem 3.7]. For completeness, and to show that in this special case the extra moment condition is not necessary, we provide the sketch of the proof.

Theorem 2. (Jelenković and Olvera-Cravioto [19, Theorem 3.7].) *Assume that (7) holds and, for a random variable X,*

$$\int_0^\infty y^{\kappa-1} |\mathbb{P}\{X > y\} - \mathbb{P}\{AX > y\}| \, \mathrm{d}y < \infty, \tag{8}$$

where A and X are independent. Then there exists a function $q \in \mathcal{Q}_{\kappa,h}$ such that, for $x \in C_q$,

$$\lim_{n\to\infty} x^\kappa \mathrm{e}^{\kappa n h} \mathbb{P}\{X > x\mathrm{e}^{nh}\} = q(x). \tag{9}$$

Moreover, if

$$\sum_{j\in\mathbb{Z}} \mathrm{e}^{\kappa(x+jh)} |\mathbb{P}\{X > \mathrm{e}^{x+jh}\} - \mathbb{P}\{AX > \mathrm{e}^{x+jh}\}| < \infty \quad \textit{for each } x \in \mathbb{R}, \tag{10}$$

then (9) holds for all $x > 0$.

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Whenever q is continuous, the exact tail asymptotic can be determined. For later use, we state this statement allowing an extra slowly varying function.

Lemma 1. *Assume that, for a random variable X,*

$$\lim_{n\to\infty} \ell(x\mathrm{e}^{nh})(x\mathrm{e}^{nh})^{\kappa}\mathbb{P}\{X > x\mathrm{e}^{nh}\} = q(x) \quad \textit{for every } x > 0,$$

where $q \in \mathcal{Q}_{\kappa,h}$ is nonzero and continuous, and ℓ is slowly varying. Then

$$\mathbb{P}\{X > x\} \sim \frac{q(x)}{\ell(x)x^{\kappa}} \quad \textit{as } x \to \infty.$$

In Proposition 1 below, we show that the set of possible q functions is large. Indeed, q can be constant, which corresponds to a regularly varying tail, and also can be nonconstant continuous.

The oscillating behavior in Theorem 2 appears in the theory of semistable and max-semistable laws, and in the theory of smoothing transformation. If $\kappa \in (0, 2)$ then $q(x)x^{-\kappa}$ is exactly the tail of the Lévy measure of a semistable law with $q \in \mathcal{Q}_{\kappa,h}$. For $\kappa > 0$, the function $\exp\{-q(x)x^{-\kappa}\}$, $x > 0$, is a max-semistable distribution function. For more in this direction we refer the reader to [25]–[27].

The smoothing transformation is closely related to our setup. Consider the fixed point equation

$$X \overset{\mathrm{D}}{=} A_1X_1 + \cdots + A_NX_N, \tag{11}$$

where $N \ge 1$, $X_1, \dots, X_N$ are independent and identically distributed (i.i.d.) copies of X, $A_1, \dots, A_N$ are general (not necessarily i.i.d.) nonnegative random variables, and the A and X are independent. Durrett and Liggett [11] gave necessary and sufficient conditions for the existence of the solution of (11). Assuming existence, let φ be the Laplace transform of the solution. In [11, Theorem 2] it is shown that, under appropriate conditions, $1 - \varphi(t) \sim t^{\alpha}h(t)$ as $t \to 0$ for some $\alpha \in (0, 1]$, where h is a logarithmically periodic function. For results on the nonhomogeneous equation $X \overset{\mathrm{D}}{=} A_1X_1 + \cdots + A_NX_N + B$, we refer the reader to Jelenković and Olvera-Cravioto [19]. For more results and references, see [2], in particular, Corollary 2.3 and Theorem 3.3. It is not clear how the tail behavior can be inferred from these results.

Finally, we mention that functions of the form $f(x) = p(x)\mathrm{e}^{\lambda x}$, $\lambda \in \mathbb{R}$, where p is a periodic function, are the solutions of certain integrated Cauchy functional equations; see [24].

Consider the perpetuity equation (1). We present Grincevičius's result in the arithmetic case below. The slight improvement is the positivity of q, which follows from Goldie's argument [15, p. 157] combined with Theorem 1.3.8 of [18].

Theorem 3. (Grincevičius [16, Theorem 2].) *Assume that (7) holds and $\mathbb{E}|B|^{\kappa} < \infty$. Let X be the unique solution of (1). Then there exist functions $q_1, q_2 \in \mathcal{Q}_{\kappa,h}$ such that*

$$\begin{aligned} \lim_{n\to\infty} x^{\kappa}\mathrm{e}^{\kappa nh}\mathbb{P}\{X > x\mathrm{e}^{nh}\} &= q_1(x), \qquad x \in C_{q_1}, \\ \lim_{n\to\infty} x^{\kappa}\mathrm{e}^{\kappa nh}\mathbb{P}\{X < -x\mathrm{e}^{nh}\} &= q_2(x), \qquad x \in C_{q_2}. \end{aligned} \tag{12}$$

If $\mathbb{P}\{Ax + B = x\} < 1$ for any $x \in \mathbb{R}$ then $q_1(x) + q_2(x) > 0$.

Grincevičius also showed that (12) holds for all $x \in \mathbb{R}$ if $B \ge 0$ a.s.

The corresponding maximum equation was treated by Iksanov [18].

Theorem 4. (Iksanov [18, Theorem 1.3.8].) *Assume that (7) holds and $\mathbb{E}|B|^\kappa < \infty$. Let X be the unique solution of (2). Then there exists a function $q \in \mathcal{Q}_{\kappa,h}$ such that, for any $x > 0$,*

$$\lim_{n\to\infty} x^\kappa \mathrm{e}^{\kappa nh} \mathbb{P}\{X > x\mathrm{e}^{nh}\} = q(x). \tag{13}$$

If $B \geq 0$ a.s. and $\mathbb{P}\{B > 0\} > 0$ then $q(x) > 0$.

In [18], this theorem was stated under the additional condition that $B > 0$ a.s. In the context of [18] this condition automatically holds since $B = \mathrm{e}^\eta$ for some random variable η.

Note the difference between the two theorems. In the case of (2) it is possible to show that the stronger condition (10) holds (see the proof of [18, Theorem 1.3.8]), while in the perpetuity case (1) one only has the weaker condition (8).

2.2. Infinite mean case

Now we assume that F_κ in (4) belongs to the domain of attraction of an α-stable law with $\alpha \in (0, 1]$, that is,

$$1 - F_\kappa(x) =: \overline{F}_\kappa(x) = \frac{\ell(x)}{x^\alpha}, \tag{14}$$

where ℓ is a slowly varying function. Furthermore, we assume that the mean is infinite if $\alpha = 1$. Introduce the truncated expectation

$$m(x) = \int_0^x \overline{F}_\kappa(y)\,\mathrm{d}y. \tag{15}$$

Simple properties of regularly varying functions imply that $m(x) \sim \ell(x)x^{1-\alpha}/(1-\alpha)$ for $\alpha \neq 1$, and m is slowly varying for $\alpha = 1$. Recall U from (6) and set $u_n = U(nh) - U(nh-)$. Note that $U(x) < \infty$ for all $x \in \mathbb{R}$, since the random walk $(S_n = \log A_1 + \cdots + \log A_n)_{n\geq 1}$ drifts to ∞ under $\mathbb{P}_\kappa$ and $\mathbb{E}_\kappa[(\log A)_-]^2 < \infty$ by (5); see Theorem 2.1 in [21]. In this case, the Blackwell theorem states only that $u_n \to 0$. The so-called strong renewal theorem (SRT) yields the exact rate, namely,

$$\lim_{n\to\infty} u_n m(nh) = hC_\alpha, \qquad C_\alpha = \frac{\sin(\alpha\pi)}{(1-\alpha)\pi}, \tag{16}$$

with the convention $C_1 = 1$. The first infinite mean SRT in the arithmetic case was shown by Garsia and Lamperti [14], who proved that (16) holds for $\alpha \in (\frac{1}{2}, 1)$, and under some extra assumptions, for $\alpha \leq \frac{1}{2}$. Their results were extended to the nonarithmetic case by Erickson [12], who also showed (16) for $\alpha = 1$; see [12, Equation (2.4)]. Necessary and sufficient conditions for the SRT for general random variables were obtained by Caravenna and Doney [8]. It turned out that, if (14) holds with $\alpha \in (0, \frac{1}{2}]$ then (16) holds if and only if

$$\lim_{\delta\to 0} \limsup_{x\to\infty} x\overline{F}_\kappa(x) \int_1^{\delta x} \frac{1}{y\overline{F}_\kappa(y)^2} F_\kappa(x - \mathrm{d}y) = 0. \tag{17}$$

It was also shown in [8] that, for $\alpha > \frac{1}{2}$, (17) automatically holds. It was pointed out in [22, Appendix] that their result extends to our case, where the random variable is not necessarily positive but the left tail is exponential.

Summarizing, our assumptions on A are the following:

$$\begin{aligned} &A \geq 0, \qquad \mathbb{E}A^\kappa = 1, \qquad \text{(14) and (17) hold for } F_\kappa \text{ for some } \kappa > 0 \text{ and } \alpha \in (0, 1], \\ &\qquad\qquad \text{and } \log A \text{ conditioned on } A \neq 0 \text{ is arithmetic with span } h. \end{aligned} \tag{18}$$

Recall the definition of m from (15), and that m is regularly varying.

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Theorem 5. *Assume that (18) holds and, for a random variable X,*

$$\int_0^\infty y^{\kappa+\delta-1}|\mathbb{P}\{X > y\} - \mathbb{P}\{AX > y\}|\,\mathrm{d}y < \infty \quad \textit{for some } \delta > 0, \tag{19}$$

where A and X are independent. Then there exists a function $q \in \mathcal{Q}_{\kappa,h}$ such that

$$\lim_{n\to\infty} m(nh)\, x^\kappa \mathrm{e}^{\kappa nh}\mathbb{P}\{X > x\mathrm{e}^{nh}\} = q(x), \qquad x \in C_q. \tag{20}$$

Since m is regularly varying, $m(\log x)$ is slowly varying, and $m(\log x + nh) \sim m(nh)$ as $n \to \infty$. For a continuous nonzero function q, (20) and Lemma 1 imply that

$$\mathbb{P}\{X > x\} \sim \frac{q(x)}{x^\kappa m(\log x)} \quad \text{as } x \to \infty.$$

As in Theorem 2, it is possible to give a stronger condition, similar to (10), which implies that (20) holds for all $x > 0$. However, in the corresponding key renewal theorem below (Lemma 2), besides summability a growth condition is also needed. Therefore, the resulting stronger condition would be unnatural and it would not be clear how to check its validity either for the perpetuity equation (1) or for the maximum equation (2).

The maximum and perpetuity results are the following.

Theorem 6. *Assume that (18) holds and $\mathbb{E}|B|^\nu < \infty$ for some $\nu > \kappa$. Let X be the unique solution of (2). Then there exists a function $q \in \mathcal{Q}_{\kappa,h}$ such that*

$$\lim_{n\to\infty} m(nh)x^\kappa \mathrm{e}^{\kappa nh}\mathbb{P}\{X > x\mathrm{e}^{nh}\} = q(x), \qquad x \in C_q.$$

If $B \ge 0$ a.s. and $\mathbb{P}\{B > 0\} > 0$ then $q(x) > 0$.

In the special case $B \equiv 1$ this theorem was obtained by Korshunov [23, Theorem 2].

Theorem 7. *Assume that (18) holds and $\mathbb{E}|B|^\nu < \infty$ for some $\nu > \kappa$. Let X be the unique solution of (1). Then there exist functions $q_1, q_2 \in \mathcal{Q}_{\kappa,h}$ such that*

$$\lim_{n\to\infty} m(nh)x^\kappa \mathrm{e}^{\kappa nh}\mathbb{P}\{X > x\mathrm{e}^{nh}\} = q_1(x), \qquad x \in C_{q_1},$$

$$\lim_{n\to\infty} m(nh)x^\kappa \mathrm{e}^{\kappa nh}\mathbb{P}\{X < -x\mathrm{e}^{nh}\} = q_2(x), \qquad x \in C_{q_2}.$$

If $\mathbb{P}\{Ax + B = x\} < 1$ for any $x \in \mathbb{R}$ then $q_1(x) + q_2(x) > 0$.

Note that we only state convergence in continuity points in both cases.

2.3. Beyond Cramér's condition

Assume now that $\mathbb{E}A^\kappa = \theta < 1$ for some $\kappa > 0$, and $\mathbb{E}A^t = \infty$ for any $t > \kappa$. In this case, the definition of the new measure is

$$\mathbb{P}_\kappa\{\log A \in C\} = \theta^{-1}\mathbb{E}[\mathbf{1}(\log A \in C)A^\kappa],$$

where C is a Borel set of $\mathbb{R}$, and F_κ is defined accordingly. The assumption that $\mathbb{E}A^t = \infty$ for all $t > \kappa$ means that F_κ is heavy tailed. The same renewal method leads now to a *defective* renewal equation. To analyze the asymptotic behavior of the resulting equation we extend the results in [4, Section 6] to the arithmetic case.

Assume that H is the distribution function of an arithmetic random variable with span h. Let $p_k = H(kh) - H(kh-)$, $k \in \mathbb{Z}$, and $p_k^{*2} = (H * H)(kh) - (H * H)(kh-)$. Then H is h-subexponential, $H \in \mathcal{S}_h$, if $p_{n+1} \sim p_n$ and $p_n^{*2} \sim 2p_n$ as $n \to \infty$. (According to the terminology introduced in [4] for distributions on $[0, \infty)$ and in [13, Section 4.7] for distributions on $\mathbb{R}$, these distributions are $(0, h]$-subexponential.) In order to use a slight extension of [4, Theorem 5] we need the additional natural assumption that $\sup_{k \geq n} p_k = O(p_n)$ as $n \to \infty$. Although in [4] the distributions were concentrated on $(0, \infty)$, the results remain true in our setup due to the extra growth assumption. We refer the reader to [22, Appendix].

Introduce the notation

$$p_n = F_\kappa(nh) - F_\kappa(nh-).$$

Our assumptions on A are the following:

$$\begin{gathered} A \geq 0, \qquad \mathbb{E}A^\kappa = \theta < 1, \qquad \kappa > 0, \qquad \sup_{k \geq n} p_k = O(p_n) \quad \text{as } n \to \infty, \\ F_\kappa \in \mathcal{S}_h, \qquad \text{and } \log A \text{ conditioned on } A \neq 0 \text{ is arithmetic with span } h. \end{gathered} \tag{21}$$

Theorem 8. *Assume that (19) and (21) hold for some $\delta > 0$. Then there exists a function $q \in \mathcal{Q}_{\kappa,h}$ such that*

$$\lim_{n \to \infty} p_n^{-1} x^\kappa \mathrm{e}^{\kappa nh} \mathbb{P}\{X > x\mathrm{e}^{nh}\} = q(x), \qquad x \in C_q. \tag{22}$$

For a possible stronger version of (22), which holds for all $x \in \mathbb{R}$, see the comment after Theorem 5.

Whenever q is continuous, Lemma 1 gives tail asymptotics as before.

The corresponding maximum and perpetuity results are the following.

Theorem 9. *Assume that (21) holds and $\mathbb{E}|B|^\nu < \infty$ for some $\nu > \kappa$. Let X be the unique solution of (2). Then there exists a function $q \in \mathcal{Q}_{\kappa,h}$ such that*

$$\lim_{n \to \infty} p_n^{-1} x^\kappa \mathrm{e}^{\kappa nh} \mathbb{P}\{X > x\mathrm{e}^{nh}\} = q(x), \qquad x \in C_q.$$

If $B \geq 0$ a.s. and $\mathbb{P}\{B > 0\} > 0$ then $q(x) > 0$.

Theorem 10. *Assume that (21) holds and $\mathbb{E}|B|^\nu < \infty$ for some $\nu > \kappa$. Let X be the unique solution of (1). Then there exist functions $q_1, q_2 \in \mathcal{Q}_{\kappa,h}$ such that*

$$\lim_{n \to \infty} p_n^{-1} x^\kappa \mathrm{e}^{\kappa nh} \mathbb{P}\{X > x\mathrm{e}^{nh}\} = q_1(x), \qquad x \in C_{q_1},$$
$$\lim_{n \to \infty} p_n^{-1} x^\kappa \mathrm{e}^{\kappa nh} \mathbb{P}\{X < -x\mathrm{e}^{nh}\} = q_2(x), \qquad x \in C_{q_2}.$$

Moreover, if $\mathbb{P}\{Ax + B = x\} < 1$ for any $x \in \mathbb{R}$ then $q_1(x) + q_2(x) > 0$.

2.4. The set of possible *q* functions

The formula for the function $q(x)$ in our results (see the proof of Theorem 2 below) is complicated and implicit, since it contains the tail of the solution X. Therefore, one might think that $q(x) \equiv c$ and the tail is simply $\ell(x)x^{-\kappa}$, with a slowly varying function ℓ, as in the nonarithmetic case. We first provide an explicit example which shows that this is not the case, i.e. the function q can be nonconstant.

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Example 1. The St. Petersburg game is defined as follows. Peter tosses a fair coin until it lands heads and pays 2^k ducats to Paul if this happens at the kth toss. If X denotes Paul's winning then $\mathbb{P}\{X = 2^k\} = 2^{-k}$, $k = 1, 2, \ldots$. The distribution function of X is

$$\mathbb{P}\{X \le x\} = \begin{cases} 1 - \dfrac{2^{\{\log_2 x\}}}{x}, & x \ge 2, \\ 0, & x < 2, \end{cases}$$

where $\lfloor x \rfloor = \max\{k \in \mathbb{Z} \colon k \le x\}$ is the usual (lower) integer part of x, $\lceil x \rceil = -\lfloor -x \rfloor$ stands for the upper integer part, and $\{x\} = x - \lfloor x \rfloor$ is the fractional part. We note that this distribution does not belong to the domain of attraction of any stable law, since the function $2^{\{\log_2 x\}}$ is not slowly varying at ∞. For further properties and history of the St. Petersburg games we refer the reader to [5] and [9], and the references therein.

We show that X is the solution of a perpetuity equation, where the joint distribution of (A, B) in (1) satisfies the following:

$$\begin{aligned} &\mathbb{P}\{A = 0, B = 2^k\} = 2^{-2k}, \qquad k = 1, 2, \ldots, \\ &\mathbb{P}\{A = 2^\ell, B = 0\} = 2^{-(2\ell+1)}, \qquad \ell = 0, 1, \ldots. \end{aligned} \tag{23}$$

Indeed, assume that X is independent of (A, B). Then, for $k \ge 1$,

$$\begin{aligned} \mathbb{P}\{AX + B = 2^k\} &= \sum_{\ell=0}^{k-1} \mathbb{P}\{A = 2^\ell, B = 0\}\mathbb{P}\{X = 2^{k-\ell}\} + \mathbb{P}\{A = 0, B = 2^k\} \\ &= \sum_{\ell=0}^{k-1} 2^{-(2\ell+1)} 2^{-(k-\ell)} + 2^{-2k} \\ &= 2^{-k-1} 2(1 - 2^{-k}) + 2^{-2k} \\ &= 2^{-k}. \end{aligned}$$

Moreover, $\log A$ conditioned on A being nonzero is arithmetic with span $h = \log 2$, and

$$\mathbb{E}A = \sum_{k=0}^{\infty} \mathbb{P}\{A = 2^k\} 2^k = 1, \qquad \mathbb{E}A \log_+ A < \infty, \qquad \mathbb{E}B < \infty.$$

That is, the conditions of Theorem 3 are satisfied with $\kappa = 1$. In this special case we see that $q(x) = 2^{\{\log_2 x\}}$.

What simplifies the analysis of the perpetuity equation with (A, B) in (23) is that $AB = 0$ a.s. It is worth mentioning that whenever $AB = 0$ and $B \ge 0$ a.s., the solutions of perpetuity equation (1) and maximum equation (2) take the same form $X = A_1 \ldots A_{N-1} B_N$ for appropriate geometrically distributed N (see the proof of Proposition 1 for more details). In particular, the St. Petersburg distribution is the solution of (2) with (A, B) in (23).

Now we generalize this example and show that the set of all possible q functions in the tail asymptotics of the solutions of (1) and (2) contains the set of right-continuous nonzero functions in $\mathcal{Q}_{\kappa,h}$.

Proposition 1. *Let $q, q_1, q_2 \in \mathcal{Q}_{\kappa,h}$, for some $h > 0, \kappa > 0$, be right-continuous functions such that $q \ne 0$ and $q_1 + q_2 \ne 0$. Then there exists (A, B) satisfying the conditions of*

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Theorem 3 such that, for the tail of the unique solution of (1), the asymptotic (12) holds with the prescribed q_1, q_2. Furthermore, there exists (A, B) satisfying the conditions of Theorem 4 such that, for the tail of the unique solution of (2), the asymptotic (13) holds with the prescribed q.

The corresponding statements hold in the cases treated in Subsections 2.2 and 2.3.

In the proof of this statement we give an explicit construction of (A, B). In fact, for $\kappa = 1$ and $h = \log 2$, the distribution of A is (almost) the same as in the example above, and only the distribution of B depends on q. When $q(x) \equiv q$ is constant, Lemma 1 implies that the tail of the solution X is regularly varying. An explicit example is given in the proof of Proposition 1.

However, for general (A, B) it seems very difficult to determine q. It would be interesting to know what conditions on (A, B) imply that q is constant, or q is continuous, but these questions do not seem to be tractable with our methods.

2.5. Iterated function systems

In this subsection we show that using Alsmeyer's sandwich method [1] our results extend naturally to a more general framework.

The Markov chain $(X_n)_{n\in\mathbb{N}}$ is an *iterated function system of i.i.d. Lipschitz maps* (IFS) if $X_{n+1} = \Psi(\theta_{n+1}, X_n)$, $n \in \mathbb{N}$, where $\theta, \theta_1, \theta_2, \ldots$ are i.i.d. random vectors in $\mathbb{R}^d$, $d \geq 1$, the initial value X_0 is independent of the sequence $\theta_1, \theta_2 \ldots$, and $\Psi\colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is a measurable function, which is Lipschitz continuous in the second argument, i.e. for all ϑ there exists $L_\vartheta > 0$ such that, for all $x, y \in \mathbb{R}$,

$$|\Psi(\vartheta, x) - \Psi(\vartheta, y)| \leq L_\vartheta |x - y|.$$

For theory and examples (and for a more general definition) we refer the reader to [1], [7, Section 5], and [10].

Under general conditions, the stationary solution of the IFS exists and satisfies the random fixed point equation

$$X \stackrel{\mathrm{D}}{=} \Psi(\theta, X), \tag{24}$$

where θ and X on the right-hand side are independent. Therefore, the corresponding implicit renewal theorem works and we obtain a tail asymptotic for the solution X. The crucial difficulty here is the same as in the nonarithmetic case (see the remark after Theorem 2.3 in [15]), namely, to determine whether q is nonzero or not. For (1) and (2) there are reasonably good sufficient conditions for the strict positivity of the function q (of the constant, in the arithmetic case). The main idea in [1] is to find lower and upper bounds for Ψ such that

$$Ax \vee B = F(\theta, x) \leq \Psi(\theta, x) \leq G(\theta, x) = Ax + B'$$

holds a.s. with some (random) A, B, B'. Now, if (A, B) and (A, B') satisfy the conditions of Theorem 4 and 3, respectively, then the tail of the solution X of (24) is of order $x^{-\kappa}$. In particular, Theorems 5.3 and 5.4 in [1] hold in the arithmetic case.

Finally, we mention that there is no need to restrict ourselves to the finite mean case. Assuming that (18) or (21) hold, the corresponding version of Theorems 5.3 and 5.4 in [1] hold. The same results hold in the nonarithmetic case treated in [22].

3. Proofs

First we prove Lemma 1 and Proposition 1, since they are independent of the rest of the proofs.

Proof of Lemma 1. We show that every sequence $x_n \uparrow \infty$ contains a subsequence x_{n_k} such that

$$\lim_{k\to\infty} \ell(x_{n_k}) x_{n_k}^{\kappa} q^{-1}(x_{n_k}) \mathbb{P}\{X > x_{n_k}\} = 1.$$

This is equivalent to the statement.

Let us write $x_n = z_n \mathrm{e}^{l_n h}$ with

$$z_n = \exp\biggl(h\biggl\{\frac{\log x_n}{h}\biggr\}\biggr), \qquad l_n = \biggl\lfloor \frac{\log x_n}{h} \biggr\rfloor.$$

Since $z_n \in [1, \mathrm{e}^h)$ by the Bolzano–Weierstrass theorem, there is a subsequence n_k such that $\lim_{k\to\infty} z_{n_k} = \lambda \in [1, \mathrm{e}^h]$. To ease the notation, we write n for n_k. For any $\varepsilon > 0$, there is an n_ε such that $|z_n - \lambda| \le \varepsilon$ for $n \ge n_\varepsilon$. Therefore, also using (9) and the uniform convergence theorem for slowly varying functions (see [6, Theorem 1.2.1]),

$$\begin{aligned}
&\limsup_{n\to\infty} \ell(x_n) x_n^{\kappa} \mathbb{P}\{X > x_n\} \\
&\quad = \limsup_{n\to\infty} \ell(z_n \mathrm{e}^{l_n h}) z_n^{\kappa} \mathrm{e}^{\kappa l_n h} \mathbb{P}\{X > z_n \mathrm{e}^{l_n h}\} \\
&\quad \le \limsup_{n\to\infty} \biggl(\frac{\lambda+\varepsilon}{\lambda-\varepsilon}\biggr)^{\kappa} \ell((\lambda-\varepsilon)\mathrm{e}^{l_n h})(\lambda-\varepsilon)^{\kappa} \mathrm{e}^{\kappa l_n h} \mathbb{P}\{X > (\lambda-\varepsilon)\mathrm{e}^{l_n h}\} \qquad (25) \\
&\quad = \biggl(\frac{\lambda+\varepsilon}{\lambda-\varepsilon}\biggr)^{\kappa} q(\lambda-\varepsilon).
\end{aligned}$$

The same argument yields the corresponding lower bound. Since $\varepsilon > 0$ is arbitrary, we obtain

$$q(\lambda+) \le \liminf_{n\to\infty} \ell(x_n) x_n^{\kappa} \mathbb{P}\{X > x_n\} \le \limsup_{n\to\infty} \ell(x_n) x_n^{\kappa} \mathbb{P}\{X > x_n\} \le q(\lambda-). \qquad (26)$$

Now the continuity of q implies the statement. □

Note that (26) holds for general q. Indeed, in (25) for any λ, one can choose $\varepsilon > 0$ arbitrarily small such that $\lambda \pm \varepsilon$ is a continuity point of q.

Proof of Proposition 1. First we prove the statement in the finite mean case. Motivated by the St. Petersburg example, we assume that $h = \log 2$ and $\kappa = 1$. Moreover, we only prove the statement for the right tail. The general case follows easily from this.

Let H be a distribution function, such that $H(1-) = 0$, $H(2-) = 1$. Let the joint distribution of (A, B) be the following:

$$\begin{aligned}
&\mathbb{P}\{A = 2^{\ell}, B = 0\} = (1-2p)p^{\ell}, && \ell = 0, 1, \ldots, \\
&\mathbb{P}\{A = 0, B \le x\} = \frac{p}{1-p} H(x), && p \in \bigl(0, \tfrac{1}{2}\bigr).
\end{aligned} \qquad (27)$$

It is easy to check that (A, B) satisfies the conditions of Theorem 3 with $\kappa = 1$, $h = \log 2$. Let $(A, B), (A_1, B_1), \ldots$ be i.i.d. random vectors with distribution given in (27). Since $AB = 0$ a.s., the solution of the perpetuity equation (1) can be written as

$$X = B_1 + A_1 B_2 + A_1 A_2 B_3 + \cdots = A_1 A_2 \cdots A_{N-1} B_N, \qquad (28)$$

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where $N = \min\{i : A_i = 0\}$ has geometric distribution with parameter $\mathbb{P}\{A = 0\} = p/(1-p)$, i.e.

$$\mathbb{P}\{N = k\} = \frac{p}{1-p}\left(\frac{1-2p}{1-p}\right)^{k-1}, \quad k = 1, 2, \ldots.$$

From (28), we also see that the solutions of (1) and of (2) are the same. Conditioning on the event $N = k$, the variables $A_1, \ldots, A_{k-1}, B_k$ are independent, $A_1, \ldots, A_{k-1}$ have common distribution $\mathbb{P}\{A = 2^\ell \mid A \neq 0\} = (1-p)p^\ell$, $\ell = 0, 1, 2, \ldots$, and B_k has distribution function H. To ease the notation, we introduce the i.i.d. sequence $Y, Y_1, Y_2, \ldots$ independent of the sequence $(A_i, B_i)_{i\in\mathbb{N}}$, such that $\mathbb{P}\{Y = \ell\} = (1-p)p^\ell$, $\ell = 0, 1, 2, \ldots$, and set $S_k = Y_1 + \cdots + Y_k$. Let $x > 1$ and write $x = 2^n z$ with $n = \lfloor \log_2 x \rfloor$, $z = 2^{\{\log_2 x\}}$. Since $B \in [1, 2)$, we have

$$\begin{aligned}
\mathbb{P}\{X > x\} &= \mathbb{P}\{A_1 A_2 \cdots A_{N-1} B_N > x\} \\
&= \sum_{k=1}^{\infty} \mathbb{P}\{N = k\} \mathbb{P}\{A_1 A_2 \cdots A_{k-1} B_k > x | N = k\} \\
&= \sum_{k=1}^{\infty} \mathbb{P}\{N = k\} (\mathbb{P}\{S_{k-1} \geq n + 1\} + \mathbb{P}\{S_{k-1} = n\}[1 - H(z)]) \\
&= \mathbb{P}\{S_{N-1} \geq n + 1\} + \mathbb{P}\{S_{N-1} = n\}[1 - H(z)]. \qquad (29)
\end{aligned}$$

We compute the probabilities $\mathbb{P}\{S_{N-1} = n\}$. By the independence of N and the Y, after some straightforward calculation, we have, for $s \in [0, 1]$,

$$\mathbb{E} s^{S_{N-1}} = \frac{1}{2(1-p)} + \frac{1-2p}{2(1-p)} \sum_{k=1}^{\infty} \frac{s^k}{2^k}.$$

That is,

$$\mathbb{P}\{S_{N-1} = k\} = \begin{cases} \dfrac{1}{2(1-p)}, & k = 0, \\ \dfrac{1-2p}{2(1-p)} 2^{-k}, & k = 1, 2, \ldots. \end{cases} \qquad (30)$$

Thus, $\mathbb{P}\{S_{N-1} \geq n + 1\} = ((1-2p)/2(1-p))2^{-n}$, and so continuing (29), we have

$$\mathbb{P}\{X > x\} = x^{-1} \frac{1-2p}{2(1-p)} 2^{\{\log_2 x\}} [2 - H(2^{\{\log_2 x\}})]. \qquad (31)$$

Let us choose now a right-continuous $q \in \mathcal{Q}_{1,\log 2}$ such that $q(2-) \in (0, 1)$, otherwise q is arbitrary. Let us choose p and H in (27) as

$$p = 1 - [2 - q(2-)]^{-1}, \qquad H(y) = \begin{cases} 0, & y < 1, \\ 2 - \dfrac{2(1-p)}{1-2p} \dfrac{q(y)}{y}, & y \in [1, 2), \\ 1, & y \geq 2. \end{cases}$$

Since $q(y)/y$ is nonincreasing and right-continuous, this is a distribution function. Substituting this back into (31), we see that the tail is as stated.

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To eliminate the condition $q(2-) \in (0, 1)$, one only has to note that if $q(x)$ corresponds to (A, B) then $cq(x/c)$ corresponds to (A, cB), $c > 0$. Thus, the proof is complete in the finite mean case.

The proof in the infinite mean case is similar, so we only sketch it. Again, we work with $\kappa = 1$ and $h = \log 2$.

The arising difficulty is that we cannot determine the explicit probabilities in (30). In order to determine the asymptotics of these probabilities, we apply Theorem 6 with a specific choice of (A, B). Fix $\alpha \in (0, 1)$, and let the distribution of (A, B) be

$$\begin{aligned} \mathbb{P}\{A = 2^\ell, B = 0\} &= c_1 2^{-\ell}(\ell+1)^{-(\alpha+1)}, \qquad \ell = 0, 1, \dots, \\ \mathbb{P}\{A = 0, B = 1\} &= c_2, \end{aligned} \tag{32}$$

where $c_1 = (\sum_{\ell=0}^\infty (\ell+1)^{-(\alpha+1)})^{-1}$ and $c_2 = 1 - c_1 \sum_{\ell=0}^\infty 2^{-\ell}(\ell+1)^{-(\alpha+1)}$. It is easy to check that the conditions of Theorem 6 are satisfied. Therefore, for some $\overline{q} \in \mathcal{Q}_{1,\log 2}$,

$$\frac{c_1 \log 2}{\alpha(1-\alpha)} n^{1-\alpha} 2^n \mathbb{P}\{\overline{X} > x2^n\} \to \frac{\overline{q}(x)}{x}, \qquad x \in C_{\overline{q}},$$

where $\overline{X} = A_1 \cdots A_{N-1}$ is the unique solution of (2) and N is a geometric random variable with parameter c_2; see (28). Note that $\overline{X} \in \{1, 2, 4, 8, \dots\}$, which means that the left-hand side is constant for $x \in (1, 2)$. Thus, the right-hand side must be constant too, i.e. $\overline{q}(x) = c_3 2^{\{\log_2 x\}}$ for some $c_3 > 0$. This readily implies that

$$\mathbb{P}\{\overline{X} > 2^n\} \sim \mathbb{P}\{\overline{X} = 2^n\} \sim \frac{c_3 \alpha(1-\alpha)}{c_1 \log 2} 2^{-n} n^{\alpha-1}. \tag{33}$$

After these preliminaries, we modify the definition of the distribution of (A, B) in (32) as

$$\mathbb{P}\{A = 2^\ell, B = 0\} = c_1 2^{-\ell}(\ell+1)^{-(\alpha+1)}, \quad \ell = 0, 1, \dots, \qquad \mathbb{P}\{A = 0, B \le x\} = c_2 H(x),$$

where H is a distribution function such that $H(1-) = 0$ and $H(2-) = 1$. Following the lines and using the notation of the proof in the finite mean case, we obtain (29), where, by (33),

$$\mathbb{P}\{S_{N-1} = n\} \sim \frac{c_3 \alpha(1-\alpha)}{c_1 \log 2} 2^{-n} n^{\alpha-1}.$$

The rest of the proof is the same, so we omit it.

The proof of the proposition under the conditions of Theorem 9 follows similarly, and it is left to the interested reader. □

In particular, with the choice of

$$H(y) = \begin{cases} 0, & y \le 1, \\ 2 - \dfrac{2}{y}, & y \in [1, 2], \\ 1, & y \ge 2, \end{cases}$$

in (27) we obtain $\mathbb{P}\{X > x\} = (2 - 1/(1-p))x^{-1}$, $x > 2$, which is regularly varying.

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Proof of Theorem 2. We follow [16, Theorem 2] and [15, Theorem 2.3].

Introduce the notation

$$\psi(x) = \mathrm{e}^{\kappa x}[\mathbb{P}\{X > \mathrm{e}^x\} - \mathbb{P}\{AX > \mathrm{e}^x\}], \qquad f(x) = \mathrm{e}^{\kappa x}\mathbb{P}\{X > \mathrm{e}^x\}.$$

From the definition of ψ, using the independence of X and A, we easily obtain the renewal equation

$$f(x) = \psi(x) + \mathbb{E}_\kappa f(x - \log A), \tag{34}$$

where $\mathbb{E}_\kappa$ stands for the expectation under the measure $\mathbb{P}_\kappa$ defined in (3); see the proof of Theorem 3.2 in [15], or the proof of Theorem 2.1 in [22]. Introduce the smoothing of g as

$$\widehat{g}(s) = \int_{-\infty}^{s} \mathrm{e}^{-(s-x)} g(x)\,\mathrm{d}x.$$

Applying this transform to both sides of (34), we obtain the renewal equation

$$\widehat{f}(s) = \widehat{\psi}(s) + \mathbb{E}_\kappa \widehat{f}(s - \log A). \tag{35}$$

For the solution, we have (see again the proof of [22, Theorem 2.1])

$$\widehat{f}(s) = \int_{\mathbb{R}} \widehat{\psi}(s-y) U(\mathrm{d}y), \tag{36}$$

where $U(x) = \sum_{n=0}^{\infty} F_\kappa^{*n}(x)$ is the renewal function from (6).

In order to apply the key renewal theorem in the lattice case (Proposition 6.2.6 in [18]), we have to check that $\sum_{j\in\mathbb{Z}} |\widehat{\psi}(x+jh)| < \infty$ for any $x \in \mathbb{R}$. This follows from the direct Riemann integrability of $\widehat{\psi}$, which is proved in the course of the proof of [22, Theorem 2.1]. For completeness and since we need the same calculation (without $|\cdot|$) we give a proof here. Using Fubini's theorem, after some calculation, we have, for any $x \in \mathbb{R}$,

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} |\widehat{\psi}(x+jh)| &\le \sum_{j\in\mathbb{Z}} \int_{-\infty}^{\infty} \mathbf{1}(x+jh \ge y)\mathrm{e}^{-(x+jh-y)} |\psi(y)|\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty} \frac{1}{1-\mathrm{e}^{-h}} \mathrm{e}^{-(x-y)-h\lceil (y-x)/h \rceil} |\psi(y)|\,\mathrm{d}y \\
&\le \frac{1}{1-\mathrm{e}^{-h}} \int_{-\infty}^{\infty} |\psi(y)|\,\mathrm{d}y \\
&< \infty.
\end{aligned}$$

Therefore, we may apply the key renewal theorem to obtain

$$\lim_{n\to\infty} \widehat{f}(s+nh) = C(s), \tag{37}$$

where, using the same calculation as above,

$$\begin{aligned}
C(s) &= \frac{h}{\mu} \sum_{j\in\mathbb{Z}} \widehat{\psi}(s+jh) \\
&= \frac{h}{\mu}\frac{1}{1-\mathrm{e}^{-h}} \int_{-\infty}^{\infty} \mathrm{e}^{-(s-y)-h\lceil (y-s)/h \rceil} \psi(y)\,\mathrm{d}y \\
&= \frac{h}{\mu}\frac{1}{1-\mathrm{e}^{-h}} \int_{-\infty}^{\infty} \mathrm{e}^{-h\{(s-y)/h\}} \psi(y)\,\mathrm{d}y,
\end{aligned} \tag{38}$$

with $\mu = \mathbb{E}_\kappa \log A = \mathbb{E} A^\kappa \log A < \infty$.

We 'unsmooth' (37) the same way as in [16]. Using the definition of $\widehat{f}$, multiplying by e^s, we obtain, from (37), that, for any $0 < s_1 \leq s_2$,

$$\lim_{n\to\infty} \mathrm{e}^{-nh} \int_{\mathrm{e}^{s_1}\mathrm{e}^{nh}}^{\mathrm{e}^{s_2}\mathrm{e}^{nh}} u^\kappa \mathbb{P}\{X > u\}\,\mathrm{d}u = \mathrm{e}^{s_2}C(s_2) - \mathrm{e}^{s_1}C(s_1).$$

Changing variables yields

$$\lim_{n\to\infty} \int_{\mathrm{e}^{s_1}}^{\mathrm{e}^{s_2}} (y\mathrm{e}^{nh})^\kappa \mathbb{P}\{X > y\mathrm{e}^{nh}\}\,\mathrm{d}y = \mathrm{e}^{s_2}C(s_2) - \mathrm{e}^{s_1}C(s_1). \tag{39}$$

Since this holds for any $s_1 \leq s_2$, it follows that the integrand is bounded, thus, there is a subsequence $n_k \uparrow \infty$ and a function q such that $(y\mathrm{e}^{n_k h})^\kappa \mathbb{P}\{X > y\mathrm{e}^{n_k h}\} \to q(y)$ for any $y \in C_q$. As a limit of nonincreasing functions, $q(y)y^{-\kappa}$ is nonincreasing. Moreover, from (39), we see that

$$\int_{e^{s_1}}^{e^{s_2}} q(y)\,\mathrm{d}y = \mathrm{e}^{s_2}C(s_2) - \mathrm{e}^{s_1}C(s_1), \tag{40}$$

which determines q uniquely at its continuity points. The uniqueness of q readily implies that $(y\mathrm{e}^{nh})^\kappa \mathbb{P}\{X > y\mathrm{e}^{nh}\} \to q(y)$ holds for the whole sequence of natural numbers whenever $y \in C_q$. From the latter, we obtain the multiplicative periodicity $q(\mathrm{e}^h y) = q(y)$. Since $y^{-\kappa}q(y)$ is nonincreasing, the set of discontinuity points of q is at most countable. Thus, the first statement is completely proved.

Assume now that $\sum_{j\in\mathbb{Z}} |\psi(x + jh)| < \infty$ for any $x \in \mathbb{R}$. Then there is no need for the smoothing. Indeed, we may apply the key renewal theorem directly for (34) and we obtain

$$\lim_{n\to\infty} x^\kappa \mathrm{e}^{\kappa nh} \mathbb{P}\{X > x\mathrm{e}^{nh}\} = q(x) := \frac{h}{\mu} \sum_{j\in\mathbb{Z}} \psi(\log x + jh),$$

which is exactly the statement. The fact that $q \in \mathcal{Q}_{\kappa,h}$ follows easily. □

Remark 1. Note that (40) implies that $q(v) = (vC(\log v))'$ Lebesgue almost everywhere, from which $q(x) \equiv 0$ if and only if $vC(\log v)$ is constant. Since

$$vC(\log v) = \frac{h}{\mu(1 - \mathrm{e}^{-h})} \int_{-\infty}^{\infty} \mathrm{e}^{h\lfloor(\log v - y)/h\rfloor} \mathrm{e}^y \psi(y)\,\mathrm{d}y,$$

we see that if ψ is nonnegative then $q(x) \equiv 0$ if and only if $\psi(y) \equiv 0$. This readily implies the positivity of the function q when $B \geq 0$ a.s. and $\mathbb{P}\{B > 0\} > 0$ in the case of both the perpetuity equation (1) and the maximum equation (2).

Proof of Theorem 3. We only have to show that $q_1(x) + q_2(x) > 0$. Goldie's argument [15, p. 157] shows that it is enough to prove the positivity of the function for the maximum of the corresponding random walk. This was shown in [18, Theorem 1.3.8]. □

Proof of Theorem 5. Recall the notation from the proof of Theorem 2. Exactly the same way as in the previous proof, we obtain the renewal equation (35), which has a unique bounded solution (36). We want to apply the key renewal theorem in the infinite mean case. In order to do so, we first have to prove such a result.

The following simple lemma is the arithmetic analogue of [12, Theorem 3], [17, Proposition 6.4.2] and [22, Lemma 2.2]. We note that the statement holds under a less restrictive condition on the left tail; see [17, Proposition 6.4.2]. However, for our purposes this weaker version is sufficient.

Lemma 2. *Assume that (16) and (18) hold. Let z be a function such that $\sum_{j\in\mathbb{Z}} |z(x+jh)| < \infty$ for any $x \in \mathbb{R}$ and $z(x) = O(|x|^{-1})$ as $|x| \to \infty$. Then*

$$\lim_{n\to\infty} m(nh) \int_{\mathbb{R}} z(x+nh-y)U(\mathrm{d}y) = hC_\alpha \sum_{j\in\mathbb{Z}} z(x+jh).$$

Proof. We have

$$\begin{aligned}\int_{\mathbb{R}} z(x+nh-y)U(\mathrm{d}y) &= \sum_{j\in\mathbb{Z}} z(x+nh-jh)u_j \\ &= \sum_{k\in\mathbb{Z}} z(x+kh)u_{n-k} \\ &= \bigg(\sum_{k\le 0} + \sum_{1\le k\le n} + \sum_{k>n}\bigg) z(x+kh)u_{n-k} \\ &= I_1 + I_2 + I_3.\end{aligned}$$

Recall that m in (15) is regularly varying with parameter $1-\alpha$ and nondecreasing. For I_1,

$$m(nh)I_1 = \sum_{k\le 0} z(x+kh)m((n-k)h)u_{n-k} \frac{m(nh)}{m((n-k)h)} \to hC_\alpha \sum_{k\le 0} z(x+kh),$$

since the summands converge and $m(nh)/m((n-k)h) \le 1$, thus, Lebesgue's dominated convergence theorem applies. To handle I_2 let $1 > \delta > 0$ be arbitrarily small. Then, from the Potter bounds [6, Theorem 1.5.6], we obtain $m(nh)/m((n-k)h) \le 2\delta^{-1}$ for large enough n and $k \le (1-\delta)n$, thus, by Lebesgue's dominated convergence theorem,

$$\sum_{k=1}^{(1-\delta)n} z(x+kh)m((n-k)h)u_{n-k} \frac{m(nh)}{m((n-k)h)} \to hC_\alpha \sum_{k\ge 1} z(x+kh) \quad \text{as } n \to \infty.$$

Furthermore, noting that $U(y) \sim \sin(\pi\alpha)/(\pi\alpha)\ y^\alpha/\ell(y)$ as $y \to \infty$, for some $c > 0$ we have

$$\sum_{k=(1-\delta)n}^{n} |z(x+kh)|m(nh)u_{n-k} \le \sup_{y>0} y|z(y)| \frac{m(nh)}{nh} U(\delta nh) \le c\delta^\alpha.$$

Since $\delta > 0$ is arbitrarily small, we obtain

$$\lim_{n\to\infty} m(nh)I_2 = hC_\alpha \sum_{k\ge 1} z(x+kh).$$

Finally, for I_3,

$$m(nh) \sum_{k>n} |z(x+kh)|u_{n-k} \le \sup_{y>0} y|z(y)|U(0) \frac{m(nh)}{nh} \to 0.$$

This completes the proof of Lemma 2. □

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Continuing the proof of Theorem 5, in the proof of Theorem 2.1 [22] it was shown that under our conditions

$$\widehat{\psi}(s) = O(\mathrm{e}^{-\delta s}) \quad \text{as } s \to \infty \tag{41}$$

for some $\delta > 0$. Therefore, the condition of Lemma 2 is satisfied, from which

$$\lim_{n\to\infty} m(nh)\widehat{f}(s+nh) = C(s) := hC_\alpha \sum_{j\in\mathbb{Z}} \widehat{\psi}(s+jh) \tag{42}$$

with the same C as in (38).

Using the definition of $\widehat{f}$, multiplying by e^s, we obtain, from (42) that, for any $0 < s_1 \le s_2$,

$$\lim_{n\to\infty} m(nh)\mathrm{e}^{-nh} \int_{\mathrm{e}^{s_1}\mathrm{e}^{nh}}^{\mathrm{e}^{s_2}\mathrm{e}^{nh}} u^\kappa \mathbb{P}\{X > u\}\,\mathrm{d}u = \mathrm{e}^{s_2}C(s_2) - \mathrm{e}^{s_1}C(s_1).$$

Changing variables yields

$$\lim_{n\to\infty} \int_{\mathrm{e}^{s_1}}^{\mathrm{e}^{s_2}} m(nh)(y\mathrm{e}^{nh})^\kappa \mathbb{P}\{X > y\mathrm{e}^{nh}\}\,\mathrm{d}y = \mathrm{e}^{s_2}C(s_2) - \mathrm{e}^{s_1}C(s_1).$$

As in the previous proof, this implies that $m(nh)(y\mathrm{e}^{nh})^\kappa \mathbb{P}\{X > y\mathrm{e}^{nh}\} \to q(y)$ holds for the whole sequence of natural numbers whenever $y \in C_q$ with some q, which satisfies the stated properties. □

Proof of Theorems 6 and 7. We only have to prove that the assumptions imply the integrability condition in Theorem 5. This is carried out in the proof of Theorems 1.1 and 1.2 in [22].

Remark 1 implies that $q(x) > 0$ in Theorem 6. Now, the strict positivity of $q_1(x) + q_2(x)$ follows again from Goldie's argument [15, p.157] and from the just proved positivity of q in Theorem 6. □

Before the proof of Theorem 8, we need a key renewal theorem in the arithmetic case for defective distribution functions. The following statement is an extension to the arithmetic case of Theorem 5(i) in [4]. Recall p_n from Theorem 8.

Lemma 3. *Assume that (21) holds,* $\sum_{j\in\mathbb{Z}} |z(x+jh)| < \infty$ *for any* $x \in \mathbb{R}$, *and that, as* $n \to \infty$, $\sup_{x\in[0,h]} z(x+nh) = o(p_n)$. *Let* $U(x) = \sum_{n=0}^\infty (\theta F_\kappa)^{*n}(x)$. *Then*

$$\lim_{n\to\infty} p_n^{-1} \int_{\mathbb{R}} z(x+nh-y)U(\mathrm{d}y) = \frac{\theta}{(1-\theta)^2} \sum_{j\in\mathbb{Z}} z(x+jh).$$

Proof. Note that Proposition 12 in [4] holds in our case. Therefore,

$$u_n = U(nh) - U(nh-) \sim \frac{\theta}{(1-\theta)^2}[F_\kappa(nh) - F_\kappa(nh-)] = \frac{\theta}{(1-\theta)^2} p_n. \tag{43}$$

Since $\lim_{n\to\infty} p_n/p_{n+1} = 1$, there is a sequence $\ell_n < n/2$ tending to ∞ such that

$$\lim_{n\to\infty} \frac{\max_{|\ell|\le\ell_n} |u_n - u_{n+\ell}|}{u_n} = 0.$$

Therefore,

$$\sum_{|\ell|\le\ell_n} z(x+\ell h)u_{n-\ell} \sim u_n \sum_{\ell\in\mathbb{Z}} z(x+\ell h) \sim \frac{\theta}{(1-\theta)^2} p_n \sum_{\ell\in\mathbb{Z}} z(x+\ell h).$$

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Thus, we only have to show that the remaining terms are $o(p_n)$. For $\ell \le -\ell_n$, using the fact that $\max_{k\ge n} p_k = O(p_n)$, we obtain

$$\sum_{\ell \le -\ell_n} z(x+\ell h) u_{n-\ell} = O(p_n) o(1).$$

Using $z(x+nh) = o(p_n)$, (43), and Proposition 2 in [4], we have

$$\sum_{\ell=\ell_n}^{n-\ell_n} z(x+\ell h) u_{n-\ell} = o(1) \sum_{\ell=\ell_n}^{n-\ell_n} p_\ell p_{n-\ell} = o(p_n).$$

Finally, $z(x+nh) = o(p_n)$ and $\max_{k\ge n} p_k = O(p_n)$ imply that

$$\sum_{\ell > n-\ell_n} z(x+\ell h) u_{n-\ell} = o(p_n),$$

and the proof is complete. □

Proof of Theorem 8. Following the same steps as in the proof of Theorem 2, we obtain

$$\widehat{f}(s) = \int_{\mathbb{R}} \widehat{\psi}(s-y) U(\mathrm{d}y),$$

with the defective renewal function $U(x) = \sum_{n=0}^{\infty} (\theta F_\kappa)^{*n}(x)$. Since $\theta < 1$, we have $U(\mathbb{R}) = (1-\theta)^{-1} < \infty$.

As in (41), we have $\widehat{\psi}(x) = O(\mathrm{e}^{-\delta x})$ for some $\delta > 0$. The h-subexponentiality of F_κ implies that $p_n \sim p_{n+1}$, thus, by [13, Lemma 2.17], $p_n \mathrm{e}^{\lambda n} \to \infty$ for any $\lambda > 0$. Therefore, $\sup_{x\in[0,h]} \widehat{\psi}(x+nh) = o(p_n)$. That is, the condition of Lemma 3 holds, and we obtain the asymptotic

$$\lim_{n\to\infty} p_n^{-1} \widehat{f}(s+nh) = \frac{\theta}{(1-\theta)^2} \sum_{j\in\mathbb{Z}} \widehat{\psi}(s+jh).$$

The rest of the proof follows in exactly the same way as in the proof of Theorem 5. □

Proof of Theorems 9 and 10. Again, the integrability condition in Theorem 8 follows from the proofs of Theorems 1.3 and 1.4 in [22]. The positivity of the functions follows as before, completing the proof. □

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