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Estimation of the turbulent energy production across a shock wave

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The simplest model of isotropic compressible turbulence consists of the Euler equations augmented by the equation for the turbulent energy. This model can also be viewed as the Euler equations for a continuum with two independent entropies. One of them is a conventional thermodynamic entropy, and the other is associated with the turbulent energy. The shock relations for this model are examined. It is shown that the turbulent entropy cannot exceed some critical value. We propose a closed set of Rankine–Hugoniot relations for the description of shock waves in such a medium based on this estimation.

1. Introduction

Euler equations for a compressible fluid with two independent entropies (or temperatures) appear in the description of turbulent gas flows (Mohammadi & Pironneau 1994; Wilcox 1998), in plasma physics (Zel'dovich & Raizer 2002), and in multiphase flow modelling (Kapila *et al.* 2001). Since the governing equations are hyperbolic, shock waves can appear, and a system of Rankine–Hugoniot relations is needed. The classical conservation laws of the mass, the momentum and the energy are not sufficient to close the system. An additional closure relation is needed. There is no universal approach to this problem. One of the possibilities is to look for travelling wave solutions to the Navier–Stokes-type equations connecting the gas states ahead of and behind the shock. The condition of existence of these solutions implicitly determines the missing closure relation (Berthon & Coquel 2002). Approximate jump relations for the turbulent shocks of small amplitude based on a non-conservative form of the governing equations were proposed by Forestier, Hérard & Louis (1997). Another approach is developed in Gavrilyuk, Gouin & Perepechko (1998) where the Rankine–Hugoniot relations for non-conservative two-phase flow models are obtained from the Hamilton principle of stationary action. Other approaches may exist which are specific to the physical model under consideration. For example, in plasma physics it is usually assumed that the entropy of electrons is conserved through the shock.

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Such an assumption closes the system of jump relations. This simple hypothesis, where one of the entropies does not change, is not suitable for shocks in turbulent gas flows because both entropies (one is the conventional thermodynamic entropy, and the second is the entropy associated with the turbulent energy) vary.

The closure problem for turbulent shocks is one of the key problems. Its solution is the basic ingredient for the solution of the Riemann problem for the non-conservative turbulence models as well as two-phase flow models, where the two-velocity effects are analogous to the turbulence effects. In this article we formulate a closure relation for turbulent gas flows which is analogous to the Chapman–Jouguet relation in detonation theory. We postulate that the turbulent entropy attains its maximum after the shock. As a consequence of this, the thermodynamic entropy attains its minimum. This minimax solution closes the system of jump relations.

2. A simple model of turbulent gas flows

The simplest multi-dimensional model of compressible turbulent flows can be written in the following form (Mohammadi & Pironneau 1994; Wilcox 1998):

$$\left.\begin{aligned}
&\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \boldsymbol{u}) = 0,\\
&\frac{\partial \rho \boldsymbol{u}}{\partial t} + \nabla\cdot(\rho \boldsymbol{u}\otimes\boldsymbol{u} + P\boldsymbol{I}) = 0, \quad P = p + p_T = p + (\Gamma - 1)k,\\
&\frac{\partial}{\partial t}\left(\rho\varepsilon + \frac{\rho\,|\boldsymbol{u}|^2}{2} + k\right) + \nabla\cdot\left(\boldsymbol{u}\left(\rho\varepsilon + \frac{\rho\,|\boldsymbol{u}|^2}{2} + k + P\right)\right) = 0,\\
&\frac{\partial(\rho\eta)}{\partial t} + \nabla\cdot(\rho\eta\boldsymbol{u}) = 0.
\end{aligned}\right\} \tag{2.1}$$

Here ρ is the gas density, $\boldsymbol{u}$ is the velocity field, $\boldsymbol{I}$ is the unit tensor, p is the thermodynamic pressure, p_T is the turbulent pressure, k is the volume turbulent energy, ε is the specific internal energy, Γ is the 'turbulent' polytropic exponent depending on whether the velocity fluctuations are three-dimensional ($\Gamma = 5/3$), two-dimensional ($\Gamma = 2$) or one-dimensional ($\Gamma = 3$). The specific internal energy $\varepsilon(v, p)$ is related to the thermodynamic pressure $p(v, \eta)$ by the Gibbs identity:

$$\theta \mathrm{d}\eta = \mathrm{d}\varepsilon + p\mathrm{d}v$$

where θ is the gas temperature, η is the specific entropy, and $v = 1/\rho$ is the specific volume. System (2.1) implies the following equation:

$$\frac{\partial \varkappa}{\partial t} + \boldsymbol{u}\cdot\nabla\varkappa = 0$$

where the 'turbulent entropy' $\varkappa$ is given by

$$\varkappa = \frac{k}{\rho^{\Gamma}}.$$

This quantity is analogous to the thermodynamic entropy η which is also conserved along trajectories. Formally, by introducing the turbulent entropy $\varkappa$, system (2.1) describes a medium having two entropies (or two temperatures). An example of such a medium can also be found in plasma physics where the electrons and the ions may have different entropies (different temperatures). System (2.1) is hyperbolic. In the

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one-dimensional case the characteristic slopes are

$$\lambda_{1,2} = u, \quad \lambda_{3,4} = u \pm c_T$$

where the 'turbulent' sound velocity c_T is given by

$$c_T^2 = c^2 + \Gamma(\Gamma - 1)\frac{k}{\rho} = \left.\frac{\partial p}{\partial \rho}\right|_\eta + \Gamma(\Gamma - 1)\varkappa\rho^{\Gamma-1}.$$

When shock waves appear, a system of shock relations (Rankine–Hugoniot relations) is needed. The conservation of the mass, the momentum and the energy are well-established in the analysis of shock waves. However they are not sufficient to close the system.

3. Rankine–Hugoniot relations

For a given state ahead the shock and, for example, a given shock velocity D, we have to determine the state behind the shock. We do not deal with the internal wave structure because it needs knowledge of the production and dissipation terms in the entropy equations which, *a priori*, are not known. We focus here on the determination of the turbulent entropy jump between equilibrium states. We will denote the variables ahead of the shock with the index 0; the variables behind the shock will be without indices. Conservation of the mass, the momentum and the energy are written

$$\left.\begin{aligned} &\rho(u - D) = -\rho_0 D = m, \\ &p + p_T - p_0 + m^2(v - v_0) = 0, \quad v = 1/\rho, \\ &\varepsilon(v, p) + \varepsilon_T - \varepsilon(v_0, p_0) + \tfrac{1}{2}(p + p_T + p_0)(v - v_0) = 0. \end{aligned}\right\} \tag{3.1}$$

Here $\varepsilon_T = kv$, $p_T = (\Gamma - 1)k$ and we have supposed that the initial state is at rest and turbulence free ($u_0 = 0$, $p_{T0} = 0$, $\varepsilon_{T0} = 0$). To determine the turbulent energy behind the shock, one more relation is needed. We will take an estimate of the turbulent energy (turbulent entropy) as a function of shock velocity.

We consider the internal energy as a function of (v, P), $P = p + p_T$, and rewrite jump conditions (3.1) in the form

$$\left.\begin{aligned} &E(v, P, \varkappa) - E(v_0, P_0, 0) + \tfrac{1}{2}(P + P_0)(v - v_0) = 0, \\ &P - P_0 + m^2(v - v_0) = 0. \end{aligned}\right\} \tag{3.2}$$

Here we have introduced

$$E(v, P, \varkappa) = \varepsilon(v, P - (\Gamma - 1)\varkappa v^{-\Gamma}) + \varkappa v^{-(\Gamma-1)}. \tag{3.3}$$

The second relation of (3.2) determines the Rayleigh line in the plane (P, v). We say that the turbulence is 'exothermic', if

$$\left.\frac{\partial E}{\partial \varkappa}\right|_{v,P} < 0,$$

and 'endothermic', if

$$\left.\frac{\partial E}{\partial \varkappa}\right|_{v,P} > 0.$$

Since

$$\left.\frac{\partial E}{\partial \varkappa}\right|_{v,P} = v^{-\Gamma}\left(-(\Gamma - 1)\left.\frac{\partial \varepsilon}{\partial p}\right|_v + v\right) = v^{-\Gamma}\left.\frac{\partial \varepsilon}{\partial p}\right|_v (G - (\Gamma - 1)), \quad G = v\left(\left.\frac{\partial \varepsilon}{\partial p}\right|_v\right)^{-1},$$

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these inequalities can be expressed in terms of the Grüneisen coefficient G (Fickett & Davis 1979): for exothermic (endothermic) turbulence the Grüneisen coefficient is smaller (larger) than $\Gamma - 1$. In particular, for the ideal gas we have

$$\varepsilon(v, p) = \frac{pv}{\gamma - 1},$$

and $G = \gamma - 1$, where $\gamma > 1$ is the polytropic exponent. In terms of (P, v) variables we obtain

$$\varepsilon(v, p) = \frac{Pv}{\gamma - 1} - \frac{(\Gamma - 1)\varkappa v^{-(\Gamma-1)}}{\gamma - 1} = \varepsilon(v, P) - \frac{\Gamma - 1}{\gamma - 1}\varkappa v^{-(\Gamma-1)}$$

and

$$E(v, P, \varkappa) = \varepsilon(v, p) + \varepsilon_T(\varkappa, v) = \varepsilon(v, P) - \frac{\Gamma - \gamma}{\gamma - 1}\varkappa v^{-(\Gamma-1)}.$$

Obviously, the turbulence is exothermic if $\Gamma > \gamma$. The quantity

$$q = \frac{\Gamma - \gamma}{\gamma - 1}\varkappa v^{-(\Gamma-1)}$$

plays the role of 'heat release'. It is not a constant, its value being determined, in particular, by the shock velocity.

The Gibbs identity written in terms of the total energy $E(v, P, \varkappa)$ is

$$\theta\, \mathrm{d}\eta + v^{-(\Gamma-1)}\, \mathrm{d}\varkappa = \mathrm{d}E + P\, \mathrm{d}v \tag{3.4}$$

We call the function $H(v, P; \varkappa, v_0, P_0)$ (of centre (v_0, P_0) and intensity $\varkappa$) the *Hugoniot function of the turbulent gas flow* defined by the formula

$$H(v, P; \varkappa, v_0, P_0) = E(v, P, \varkappa) - E(v_0, P_0, 0) + \tfrac{1}{2}(P + P_0)(v - v_0).$$

For a fixed value of $\varkappa$ the Hugoniot curve $H(v, P; \varkappa, v_0, P_0) = 0$ passes through the centre (v_0, P_0) only if $\varkappa = 0$. We will suppose that

(i) the turbulence is exothermic ($\Gamma > \gamma$ for a polytropic gas);

(ii) a straight line through the centre (v_0, P_0) of negative slope intersects the Hugoniot curve $H(v, P; \varkappa, v_0, P_0) = 0$ with a given $\varkappa$ in at most two points (i.e. they can have at most one tangent point).

The ideal gas satisfies these properties.

For a fixed $\varkappa$, let us consider the variation of $H(v, P; \varkappa, v_0, P_0)$ along the Rayleigh line. Taking into account the Gibbs identity (3.4) we obtain

$$\mathrm{d}H(v, P; \varkappa, v_0, P_0) = \theta \mathrm{d}\eta + v^{-(\Gamma-1)}\, \mathrm{d}\varkappa = \theta\, \mathrm{d}\eta.$$

This equality is an analogue of that in classical gas dynamics (Courant & Friedrichs 1948). For a given slope of the Rayleigh line, the turbulent entropy cannot exceed a value $\varkappa_*$ such that the Rayleigh line is tangent to the Hugoniot curve $H(v, P; \varkappa_*, v_0, P_0) = 0$ at a point (v_*, P_*) (see figure 1 where the point (v_*, P_*) is denoted by CJ). Hence, the turbulent entropy $\varkappa$ has a *relative maximum* at point (v_*, P_*). Then, at this point $\mathrm{d}H = 0$ and, hence $\mathrm{d}\eta = 0$. Moreover, since along the Rayleigh line

$$\mathrm{d}H = \theta \mathrm{d}\eta$$

then

$$\mathrm{d}^2H = \mathrm{d}\theta\, \mathrm{d}\eta + \theta\, \mathrm{d}^2\eta.$$

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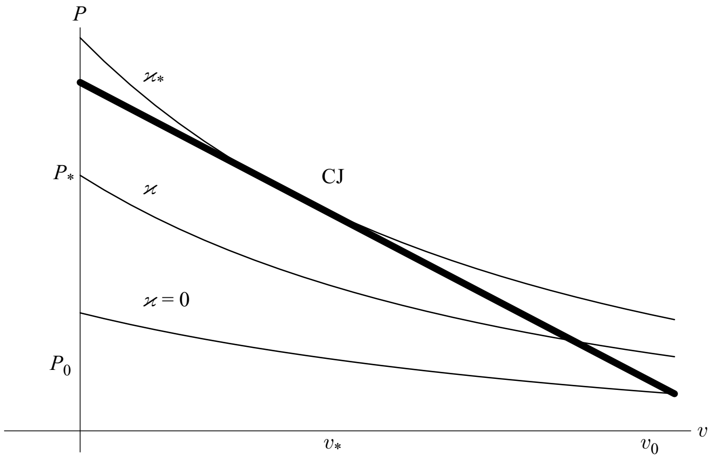

FIGURE 1. The turbulent entropy attains its relative maximum in the shock while the thermodynamic entropy attains its relative minimum. In this figure, it corresponds to the point CJ which is the point where the Hugoniot curve $H(v, P; \varkappa_*, v_0, P_0) = 0$ is tangent to the Rayleigh line (shown in bold) passing through (v_0, P_0). $\varkappa$ and $\varkappa_*$ indicate the Hugoniot curves corresponding to an intermediate value of $\varkappa$ and the critical value $\varkappa_*$.

Since

$$\mathrm{d}^2 H < 0$$

at point (v_*, P_*), then

$$\mathrm{d}^2 \eta < 0.$$

Hence, the thermodynamic entropy has a *relative minimum* at point (v_*, P_*). The shock front is supersonic with respect to the state of the gas ahead of the front, and sonic with respect to the state of the gas behind the front.

For a given value of the shock velocity D (or a given value of $m^2 = \rho_0^2 D^2$) the maximal value of the entropy $\varkappa = \varkappa_*$ can be attained. The hypothesis that $\varkappa = \varkappa_*$ behind the shock would be a reasonable assumption to close the Rankine–Hugoniot system.

Remark This minimax principle was partially used earlier by Saurel, Gavrilyuk & Renaud (2003) in the problem of shock–bubble interaction. This complex multi-dimensional phenomenon was described by a one-dimensional two-velocity model with several entropies (two thermodynamic entropies and two turbulent entropies). They assumed that velocity and pressure relaxation effects are responsible for the increase of the turbulent entropies (which appeared due to the Richtmeyer–Meshkov instability). The thermodynamic entropies did not change during this process. This hypothesis was in a perfect agreement with multi-dimensional simulations.

4. Numerical results

4.1. *Qualitative properties of the CJ point*

We have considered a one-dimensional case with three-dimensional velocity fluctuations ($\Gamma = 5/3$) and have chosen the polytropic gas with $\gamma = 4/3$. The problem to be solved is to determine for a given value of $m^2 > \frac{4}{3}\rho_0 P_0$ the value of $\varkappa = \varkappa_*$ such

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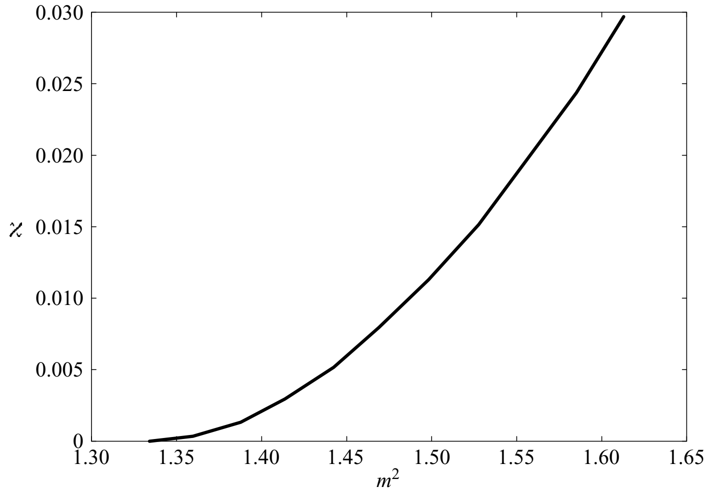

FIGURE 2. The critical turbulent entropy as a function of m^2 is shown in dimensionless variables $\varkappa/(P_0 v_0^{5/3})$ and $m^2 v_0/P_0$ in the vicinity of $m^2 v_0/P_0 = 4/3$. A nonlinear behaviour is observed.

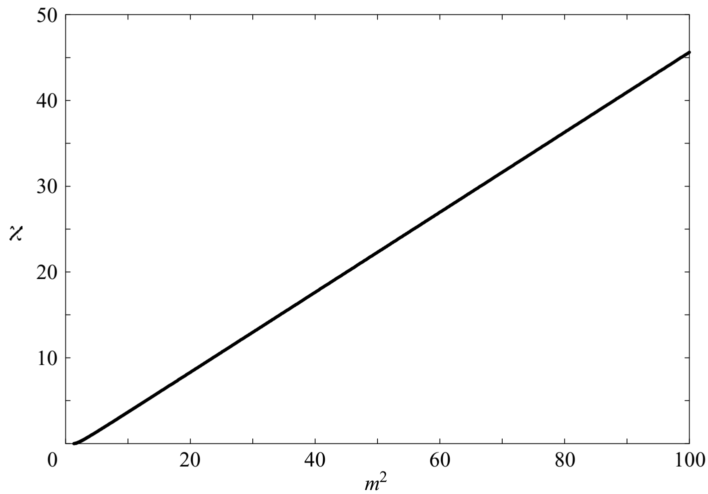

FIGURE 3. As figure 2 but for large m^2. We see that this dependence is linear for large m^2.

that the following curves are tangent:

$$P = \frac{P_0(7v_0 - v) + 2\varkappa v^{-2/3}}{7v - v_0}, \quad P - P_0 + m^2(v - v_0) = 0.$$

The first curve is the Hugoniot curve for $\Gamma = 5/3$ and $\gamma = 4/3$, and the second is the Rayleigh line. The unknown quantity

$$q = \varkappa v^{-2/3}$$

plays the role of a non-constant 'heat release'. We show the dependence of $\varkappa_*$ on m^2 in figures 2 and 3. The dependence of the turbulent entropy for shocks of small amplitude (for m^2 in the vicinity of $\frac{4}{3}\rho_0 P_0$) is nonlinear (figure 2); it increases very

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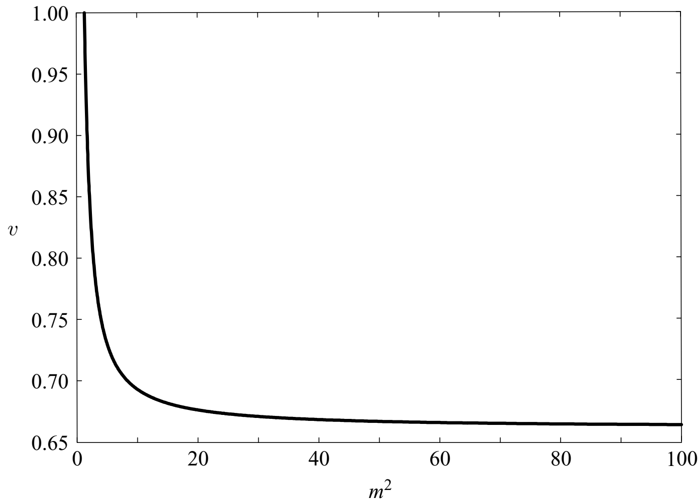

FIGURE 4. The specific volume at the CJ point as a function of m^2 is shown in dimensionless variables v_*/v_0 and $m^2 v_0/P_0$. Asymptotically, this critical value tends to a constant. This is a reason why for large m^2 the product $\varkappa_* v_*^{-2/3}$ also behaves linearly with respect to m^2.

slowly with m^2. For large values of m^2 the turbulent entropy $\varkappa_*$ behaves linearly with respect to m^2 (figure 3). Since the specific volume at the CJ point does not depend on m^2 for large m^2 (figure 4), the same conclusion will hold for $q = \varkappa_* v_*^{-2/3}$. This implies some interesting consequences. In detonation theory with $q = \text{const}$ and $\gamma = \text{const}$ it is known that the CJ velocity is proportional to $\sqrt{q}$ (Fickett & Davis 1979):

$$D \approx \sqrt{q}.$$

We have shown that the same behaviour takes place for large values of m^2. The only difference is in using this formula: for shocks it determines q for a given D, and for detonation waves it determines D for a given q.

4.2. *Comparison with DNS data*

In order to check the validity of the present model we compare its predictions with a direct numerical simulation of turbulent flows. The flow configuration retained is the simplest one: a one-dimensional periodic system of gas layers of different densities (figure 5) where a shock is created by a piston propagating with a constant velocity. Quite rapidly, a stationary dispersive shock wave forms. Reflections at the material interfaces are responsible for the turbulent behaviour of the gas. Since the velocity fluctuations are one-dimensional, $\Gamma = 3$. The turbulent energy k is calculated by

$$k = \frac{\overline{\rho u^2}}{2} - \frac{(\overline{\rho u})^2}{2\overline{\rho}}$$

where for any function f its average is defined by

$$\overline{f}(x) = \frac{1}{L}\int_{x-L/2}^{x+L/2} f(z)\,\mathrm{d}z.$$

For simplicity, we have taken the layers to have the same width H. It was checked that the results do not depend on L if it varies between $2H$ and $10H$. In this computation we have taken $L = 4H$. The initial data were chosen as follows: $\rho_1 = \rho_0(1-\delta)$,

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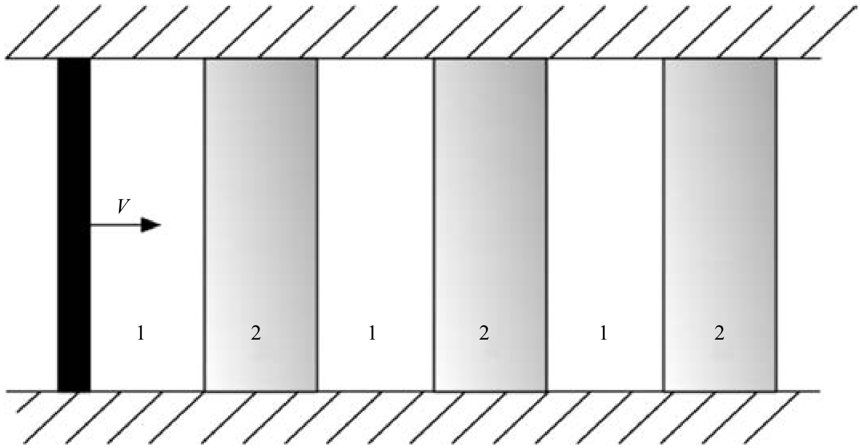

FIGURE 5. A piston propagates to the right with a constant velocity V in a periodic system of gas layers of different densities ρ_1 and ρ_2.

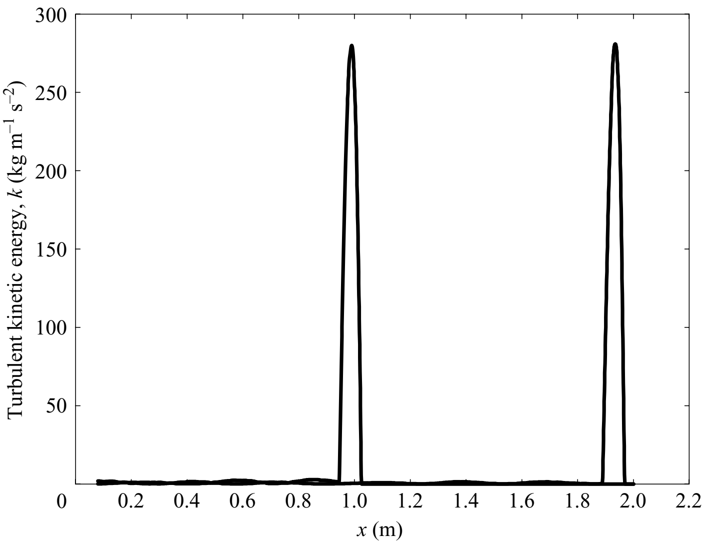

FIGURE 6. The turbulent energy k as a function of x is shown for two time instants. We see that the wave is stationary (its profile does not change with time) and is very localized. The parameters of the computation are: piston velocity $V = 50\,\mathrm{m\,s^{-1}}$, $H = 0.02\,\mathrm{m}$. The computational domain contains 100 gas layers.

$\rho_2 = \rho_0(1+\delta)$, $\delta = 0.1$, $\rho_0 = 1\,\mathrm{kg\,m^{-3}}$, $p_0 = 10^5\,\mathrm{kg\,m^{-1}\,s^{-2}}$, $\gamma_1 = \gamma_2 = 4/3$. The computation was performed with 30 grid cells in each gas layer. A second-order variant of the Godunov scheme was used. Typical behaviour of the turbulent energy k is shown in figure 6 for two different time instants. The turbulent energy k is localized at the shock front. Its maximal value does not vary in this time interval which confirms that a stationary shock is formed. Computations were done for different values of the piston velocities: 25, 50, 100 and 200 $\mathrm{m\,s^{-1}}$. They correspond to Mach numbers 1.06, 1.1, 1.2 and 1.4 calculated for stationary shock profiles (here the Mach number M is defined by $M^2 = 3m^2 v_0/(4P_0)$). The comparison is shown in figure 7. The theoretical value of $\varkappa$ (curve) is always greater than numerical data obtained by DNS (squares).

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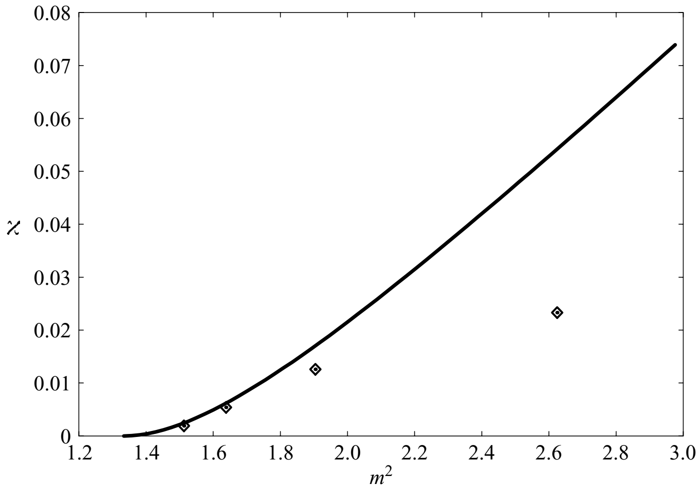

FIGURE 7. The critical turbulent entropy as a function of m^2 is shown in dimensionless variables $\varkappa_*/(P_0 v_0^3)$ and $m^2 v_0/P_0$ for $m^2 v_0/P_0 > 4/3$ ($M^2 > 1$). The theoretical values (curve) are always an upper bound for the DNS data (squares). For the Mach numbers near one this bound is almost exact.

This confirms that our estimation is an upper bound. We also see that for the Mach numbers near one, small density perturbations are sufficient ($\delta = 0.1$) to attain the maximal value of $\varkappa$. For larger Mach numbers some deviation from the theoretical curve is observed. An interesting question arises: what is the 'optimal' configuration of initial perturbations giving the maximal value of the turbulent entropy? This question needs further study.

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