COMPETITION-ORIGINATED CYCLES AND INSURANCE STRATEGIES

BY

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ABSTRACT

An insurance company entering the property and liability insurance market at the high point of the insurance cycle may decide to slash premiums to gain an advantageous market share. Such aggressive intrusion may call forth a concerted industry response, producing a severe decline in the insurance market price. This can ruin some companies, and agrees with the observation that the insurance cycles are correlated with clustered insolvencies. This paper addresses a quantitative analysis of competition-originated cycles; it explores an interplay of rational aggressive and defensive strategies in the multi-period Lundberg-type controlled risk model.

KEYWORDS

Underwriting cycles, competition, years of soft and hard market, multiperiod insurance process, solvency and subsistence.

1. INTRODUCTION

There is convincing evidence (see Pentikäinen (1988)) that the long-term variations called "business cycles" are a fundamental feature of the non-life insurance business. These business cycles are likely in all countries where there is a competitive insurance market. Insight into the driving forces of the underwriting cycles is not only a paramount theoretical problem; but it is a key to understanding the nature of the insurance market. It yields leverages for rational management and attracts permanent attention of many parties, including managers and experts in economic and actuarial studies.

There are at least two major rationales of the cyclic behavior in insurance; one is the randomness of volatile interest rates, or the random up and down swings of risk exposure. The other attributes the cycles to the interplay of insurers.

Typically, the cyclic behavior of the first kind (see e.g., Malinovskii (2009a)) depends on the factors outside the insurance industry. Tangible as irregularly oscillating underwriting process is, it is largely caused by inevitable rate-making

errors. On the other hand, the cyclic behavior of the second kind consists of alternating up and down swing phases generated by insurance industry factors such as aggressive behavior and concerted industry response.

The aggressive behavior of a particular insurer seeking a greater market share is a trigger for the competition-originated cycles. If the market recognizes the gravity of that common threat, the response is a concerted reduction of the premium rates. That starts a down swing phase of the competition-originated cycle and can ruin some companies, as premium rates gradually fall below the real costs of insurance.

That constitutes a great danger even for old established businesses and agrees with the observation that the insurance cycles are correlated with related insolvencies of certain types of insurers. For example, US industry-wide combined ratios peaked at 109% in 1975 and 117% in 1984; the insurance failure rate, or the ratio of insolvencies to total companies, peaked at 1.0% in 1975 and 1.4% in 1985 (see Feldblum (2007a) with reference to Best's Insolvency Study, Best's (1991)).

Insolvencies constitute an important driving force behind the competition-originated cycles. After elimination of exceedingly aggressive agents, or only weaker carriers, the prices increase uniformly over the industry and the upswing phase of the cycle ensues.

The impact of the competition-originated cycles on the financial strength of the property and liability insurers is paramount. The insurers may be either ruined, having their risk reserve exhausted because of too low prices, or lose their business, having a majority of clients emigrate to those competitors who offer better prices. These two principal threats deserve thorough quantitative analysis.

Much attention is paid to insurance cycles in the economic literature. Among experts and scholars exploring the driving forces behind the cycles are Venezian (1985), Cummins and Outreville (1987), Doherty and Kang (1988), Harrington and Danzon (1994), Doherty and Garven (1995), Feldblum (2007a). Feldblum's paper yields a convincing insight into the cycles of both kind; he provides as many as 109 references and it is a comprehensive account of the state of art in the field.

Much less is known about quantification of the additional risk associated with underwriting cycles. Using mostly simulation techniques, Pentikäinen (1988) and then Daykin et al. (1996) studied the relationship between the underwriting cycles and the probabilities of ruin. From the premises of the individual approach, Subramanian (1998) addressed solvency and market share balance in modeling competition in a bonus-malus framework. Further developments within the framework of dynamic financial analysis are D'Arcy et al. (1997), Kaufmann et al. (2001).

Quite a little is done for analytical investigation of the underwriting cycles. Based on the empirical data and discussion of some potential background factors of the cycles, Rantala (1988) applied the control-theoretical tools in the framework of autoregressive models. Describing the risk loading and the

claim rate by a heuristic non-random trigonometric function, Feldblum (2007b) modeled competition-originated underwriting cycles. His surplus model allows the insurer to vary the price in response to the cycles, losing or gaining market share. A more sophisticated Markovian model of the risk reserve's periodic behavior is built in Asmussen and Rolski (1994). In Malinovskii (2007) – Malinovskii (2009a) emphasis was put on harmonization of equity and solvency requirements sensitive to diverse scenarios of nature, and on the cyclic behavior of the first kind.

This paper aims to give an exemplar of rigorous mathematical modeling of the competition-originated underwriting cycles within the control theoretical approach. Viewed as a "historical process" divided into distinct stages, the underwriting cycles model' prerequisite is a set of assumptions consistent with economical evidence. The goal of the paper is to set a few, among many feasible, patterns of action called strategies of competing insurers and to quantify annual "moves legally possible" which constitute admissible building blocks for any competitive strategy. Though control theoretical approach is accentuated throughout the paper, it differs from conventional optimal control set-up (i.e., minimization of an objective function subject to constrains), with a more sophisticated and more promising game theoretical approach in view.

In dealing with the insurance cycles originated by competition, practical interest of the insurance industry and academic, predominantly mathematical, interest behind the modeling is two-fold challenge. The former consists in possible practical use of a better understanding of the cycles phenomenon acquired from modeling. The later consists in solution of the academic problems such as selection of a suitable mathematical formalism.

This paper does not pretend to reveal the real economic mechanism behind the cycles and tends to analytical methods rather than e.g., numerical, based on simulation; it is inevitably biased by intensive application of a rather sophisticated Poisson–Exponential model and of corresponding finite-time ruin probability results. It focuses on the problems like "how to quantify aggressive actions and defensive reactions of interacting companies", and "how an insurance company can overcome a down swing phase of the underwriting cycle".

In Malinovskii (2007) – Malinovskii (2009a) developed is a multi-period model set for a unique insurance company whose only competitor is nature. In this paper the competitor is the falling market, and the trajectory of a "neutral", or "competing with the market" company $\mathfrak{N}$ is diagrammed as

$$\mathsf{w}_0^{\mathfrak{N}} \underbrace{\xrightarrow{\gamma_0^{\mathfrak{N}}} \mathsf{u}_0^{\mathfrak{N}} \xrightarrow{\pi_1^{\mathfrak{N}}} \mathsf{w}_0^{\mathfrak{N}}}_{\text{1-st year}} \cdots \xrightarrow{\pi_{k-1}^{\mathfrak{N}}} \mathsf{w}_{k-1}^{\mathfrak{N}} \underbrace{\xrightarrow{\gamma_{k-1}^{\mathfrak{N}}} \mathsf{u}_{k-1}^{\mathfrak{N}} \xrightarrow{\pi_k^{\mathfrak{N}}} \mathsf{w}_k^{\mathfrak{N}}}_{k\text{-th year}} \cdots, \qquad (1)$$

where $\pi_k^{\mathfrak{N}}$ refers to the k-th year probability mechanism, $\gamma_{k-1}^{\mathfrak{N}}$ to the k-th year control, $\mathsf{w}_k^{\mathfrak{N}}$ and $\mathsf{u}_{k-1}^{\mathfrak{N}}$ to the state and control variables for $\mathfrak{N}$.

Further in this paper, two insurance companies are considered, one $\mathfrak{A}$ called "aggressive", and another $\mathfrak{D}$ called "defending". The interactive trajectories

for $\mathfrak{A}$ and $\mathfrak{D}$ are diagramed (the control-oriented reader may wish to start from formalities in Section 6 below) as

$$
\begin{pmatrix} \mathsf{w}_0^{\mathfrak{A}} \\ \mathsf{w}_0^{\mathfrak{D}} \end{pmatrix} \xrightarrow{\gamma_0^{\mathfrak{A}}, \gamma_0^{\mathfrak{D}}} \underbrace{\begin{pmatrix} \mathsf{u}_0^{\mathfrak{A}} \\ \mathsf{u}_0^{\mathfrak{D}} \end{pmatrix} \xrightarrow{\pi_1^{\mathfrak{A}}, \pi_1^{\mathfrak{D}}} \begin{pmatrix} \mathsf{w}_1^{\mathfrak{A}} \\ \mathsf{w}_1^{\mathfrak{D}} \end{pmatrix}}_{\text{1-st year}} \ \ldots
\tag{2}
$$

$$
\ldots \xrightarrow{\pi_{k-1}^{\mathfrak{A}}, \pi_{k-1}^{\mathfrak{D}}} \begin{pmatrix} \mathsf{w}_{k-1}^{\mathfrak{A}} \\ \mathsf{w}_{k-1}^{\mathfrak{D}} \end{pmatrix} \xrightarrow{\gamma_{k-1}^{\mathfrak{A}}, \gamma_{k-1}^{\mathfrak{D}}} \underbrace{\begin{pmatrix} \mathsf{u}_{k-1}^{\mathfrak{A}} \\ \mathsf{u}_{k-1}^{\mathfrak{D}} \end{pmatrix} \xrightarrow{\pi_k^{\mathfrak{A}}, \pi_k^{\mathfrak{D}}} \begin{pmatrix} \mathsf{w}_k^{\mathfrak{A}} \\ \mathsf{w}_k^{\mathfrak{D}} \end{pmatrix}}_{k\text{-th year}} \ \ldots .
$$

According to this diagram (for $k = 1, 2, \ldots$), at the end of $(k-1)$-st year the state variable $\mathsf{w}_{k-1} = (\mathsf{w}_{k-1}^{\mathfrak{A}}, \mathsf{w}_{k-1}^{\mathfrak{D}})$ is observed; it is typically much more complex than a couple of surplus values. At the beginning of the k-th year the aggressive company selects, following the strategy $\gamma^{\mathfrak{A}} = \{\gamma_k^{\mathfrak{A}}, k = 0, 1, \ldots\}$, the control variable $\mathsf{u}_{k-1}^{\mathfrak{A}}$. Then, applying the strategy $\gamma^{\mathfrak{D}} = \{\gamma_k^{\mathfrak{D}}, k = 0, 1, \ldots\}$, the defending insurer uses the control rule $\gamma_{k-1}^{\mathfrak{D}}$ which yields the control variable $\mathsf{u}_{k-1}^{\mathfrak{D}}$. These control variables fix the k-th year's probability mechanism of insurance for $\mathfrak{A}$ and $\mathfrak{D}$; the transition function of this mechanism is denoted by $\pi_k = (\pi_k^{\mathfrak{A}}, \pi_k^{\mathfrak{D}})$. At the end of the k-th year, it yields the state variable $\mathsf{w}_k = (\mathsf{w}_k^{\mathfrak{A}}, \mathsf{w}_k^{\mathfrak{D}})$, and the process repeats anew.

The model (2) concatenates the annual probability mechanisms of $\mathfrak{A}$ and $\mathfrak{D}$ led by interactive control rules $\gamma_k^{\mathfrak{A}}, \gamma_k^{\mathfrak{D}}$ and may be called a dynamic, or sequential, stochastic game with imperfect information (see von Neumann and Morgenstern (1944), Owen (1982)). It means that the later player has some knowledge about the actions of earlier player and both have some knowledge about the uncontrolled environment; albeit very little knowledge.

Paramount in that framework is modeling the annual mechanisms of insurance π_k and selecting sensible rational offensive and defensive control strategies. In Malinovskii (2007) and Malinovskii (2009a) the annual mechanism of insurance is modeled by a homogeneous diffusion model. In Malinovskii (2008a) and Malinovskii (2008b) a homogeneous Lundberg-type (Poisson-Exponential) control model is applied. In this paper a non-homogeneous Lundberg-type (Poisson-Exponential) control model, with a non-homogeneous claims arrival process and with a variable portfolio size, is considered.

Having specified the annual mechanism of insurance, it is necessary to keep track of how the information is revealed in time. It is known (see §1 of Chapter 1 in Gihman and Skorokhod (1979)) that under certain mild regularity conditions the couple $\boldsymbol{\pi} = \{(\pi_k^{\mathfrak{A}}, \pi_k^{\mathfrak{D}}), k = 1, 2, \ldots\}$ and $\boldsymbol{\gamma} = \{(\gamma_k^{\mathfrak{A}}, \gamma_k^{\mathfrak{D}}), k = 0, 1, \ldots\}$ is sufficient background for a rigorous definition of the controlled random sequence on a probability space $(\Omega, \mathcal{F}, \mathbf{P}^{\pi, \gamma})$.

The rest of the paper is arranged as follows.

Section 2 contains basic assumptions and outlines the price competition-originated cycles.

Section 3 deals with a Lundberg-type collective model of the annual probability mechanism of insurance for an individual company with varying portfolio size. That section lays a background for comprehensive modeling and for quantitative analysis of the interactive competitive strategies.

Section 4 is devoted to quantification of a few types of annual controls which are the building blocks in diverse multi-period competitive insurance business models. Their quantification is a major challenge of this paper. Seeking for transparency, we impose rather restrictive regularity conditions appropriate for Poisson-Exponential risk model. These conditions may be largely weakened at the price of using more complicated auxiliary results which abound in risk theory literature (see for example Asmussen (2000), Malinovskii (1994), Malinovskii (1996b), Malinovskii (1998), Malinovskii (2000)).

Section 5 deals with a neutral company's multi-period strategic modeling.

Section 6 glances at a competitive companies' interactive multi-period strategic modeling.

Section 7 contains some further results on quantification of annual controls.

2. Assumptions and outline of competition-originated cycles

The rationale behind the assumptions of this paper lies in the interplay of three main parties: (i) insurers (clustered in most of this paper as "aggressive company" $\mathfrak{A}$, "defending company" $\mathfrak{D}$ and "the rest of the market"), (ii) insureds, more or less liable to migration, and (iii) regulators, more or less lenient. In particular, by market price the insurers' price averaged over "the rest of the market" is largely meant.

Basic assumption (Assumption 3 in Section 2.3) postulates a persistent self-reinforcing trend triggered by a price aggression of $\mathfrak{A}$. Its rationale lies in the analysis of the intrinsic factors which induce "the rest of the market" to follow $\mathfrak{A}$ with a certain time lag, of one year in our case. Recall (see Soros (1994), p. 44) that "to put matters into perspective, we may classify events into two categories: humdrum, everyday events that are correctly anticipated by the participants and do not provoke a change in their perceptions, and unique, historical events that affect the participants' bias and lead to further changes. The first kind of event is susceptible to equilibrium analysis, the second is not: it can be understood only as part of a historical process". According to that classification, unfolded price aggression is considered to be a "historical event", and our concern is to envisage its consequences.

Other basic assumptions (Assumptions 1 and 2 of Section 2.1) of "market-homogeneous risk" and "time-homogeneous risk" are more technical. They do not hold for many practically interesting cases and will be relaxed in a future research.

It has to be mentioned straightforwardly that the set of assumptions in this paper is not a unique one conformable with practice.

2.1. Years of soft and hard market and annual risk

The concept of a market-price rate being constant within each insurance year is paramount. It is consistent with the insurance practice, as price is a major stipulation of the insurance policy not liable to voluntarily variations within the period of time specified in the contract.

DEFINITION 2.1 (Market price). Insurance price rate P^M prevailing on the market in a certain insurance year is called *market price*.

It is known that the insurance market comprises better and worse risks; and the race for the better risks is a paramount component of competitive marketing (see Subramanian (1998)). In this paper we admit the following simplified assumption.

ASSUMPTION 1 (Market-homogeneous risk). *For each insurance year, the annual risks are market-homogeneous, i.e., the claim sizes are identically distributed over the whole insurance market.*

DEFINITION 2.2 (Years of soft and hard market). The ratio $\kappa = P^M / EY > 0$, where EY is the averaged losses, is called *year's index*. The insurance year is called year of *soft market*, if $\kappa < 1$. The insurance year is called year of *hard market*, if $\kappa > 1$.

Though a particular insurer may keep its individual price P above or below the averaged losses EY, and above or below the market price P^M, regardless of the year's index, short-time and long-time consequences of the "row against the flood" are important for its safety.

Assumption 1 refers to the annual risks. Market homogeneity does not mean that the claim sizes may not differ in the different insurance years. Moreover, it is sensible to assume that claim sizes decrease (in a certain probabilistic sense) in the down swing phases of the insurance cycle because of more careful, and increase in the upswing phases because of less careful, underwriting. Though the time-inhomogeneous set-up is therefore paramount, this paper admits the following simplified assumption.

ASSUMPTION 2 (Time-homogeneous risk). *Market-homogeneous risks are time-homogeneous, i.e., within consecutive insurance years the annual claim sizes are independent and identically distributed.*

In particular, the averaged losses EY are assumed time-invariant and identical for all the companies on the market. That assumption is simplistic, allows easier mathematics, and will be removed in a future research.

2.2. Strategic goals

Besides the ultimate strategic goals which largely coincide with rather vague directives of solvency, profitability and equity, the *strategic goals* of insurer may be structured as follows.

- COMPETITIVE GOALS. In particular, that may be "to win a market share", or "to win a market share and not be ruined", or "to win a market share and to ruin the competitors", or "not lose a market share", or "not lose a market share and not be ruined", or "not lose a market share and to ruin the competition", within a particular time span.
- PEACEFUL, OR PROFIT-SEEKING GOALS. Largely, that may be "to increase profit" within a particular time span, which implies "not to be ruined".

The main resources required in order to be able to compete are capital and market share. The main competitive advantage lies in a competent and timely configuration of resources. For example, large market share may not only be an advantage but also a disadvantage, depending on the market conditions.

Besides underwriting, the main insurer's maneuvering available to achieve the strategic goals is redistribution (win or lose) of the market shares. Technically, that comprises numerous marketing techniques such as:

- price decrease (increase) aimed to increase (decrease) market share,
- price increase (decrease) aimed to increase (decrease) capital,
- straightforward increase (decrease) of market share by selling or purchasing a part of the insurance portfolio,
- straightforward increase (decrease) of capital by means of capital lending or borrowing.

A clue to the competition-originated cycles lies in the antagonistic strategies of the companies endeavoring to achieve diverse – competitive or profit-seeking – strategic goals:

Goal A: redistribution (defense or conquest) of the market shares,
Goal B: profitable operations.

Each of these goals may be set at the discretion of a particular company, but it is essential that they dominate in turns over the whole insurance market and impel the individual insurers to make allowance for that.

2.3. Genesis of a cycle: concerted industry's response

Formalize the intuitive idea that an aggressive price slash may call forth a decline of the market price due to the industry concerted response.

DEFINITION 2.3. Call *aggressive* a company $\mathfrak{A}$ whose strategic goals are market share gains by means of severe reduction of the insurance price.

We assume that market share gains are equivalent to the portfolio growth due to immigration. Plainly, the intention of $\mathfrak{A}$ to increase its market share would require a premium $P_k^{\mathfrak{A}}$ below the market price P_k^M within a series of insurance years, $k = 1, 2, \ldots$. Otherwise a strong incentive for customers to change to $\mathfrak{A}$ will not be created.

Introduce an important notion of *concerted industry response*. Though the averaged losses $\mathrm{E}Y$ are assumed fixed in this paper, and the annual market price P^M is established by the whole market, the influence of a particular insurer on the whole market may either be negligible, or not. Concerted industry response refers to the latter case.

ASSUMPTION 3 (Aggression calls forth a concerted industry response). *Assume that an aggressive company $\mathfrak{A}$ persistently seeks a larger market share, and reduces its prices below the current market price over a series of insurance years. Assume that the industry matches these prices after one year. Thus, in the years of hard market*

$$P_1^M > P_1^{\mathfrak{A}} = P_2^M > P_2^{\mathfrak{A}} = P_3^M \ldots > \mathrm{E}Y, \tag{3}$$

and in the years of soft market

$$\mathrm{E}Y > P_1^M > P_1^{\mathfrak{A}} = P_2^M > P_2^{\mathfrak{A}} = P_3^M > \ldots . \tag{4}$$

It appears from (3) that the company $\mathfrak{A}$, while operating in the years of hard market, consistently and aggressively reduces its annual prices, trading its premium income for market share, while (4) means that $\mathfrak{A}$ is waging an open aggression[1]. Aggressively reducing prices below the costs of insurance, it aggravates the entire situation on the market, rendering profitable operations impossible.

Behavior of that kind is conventional for intruders entering the market at the high point of the insurance cycle. Having a large exogenous capital, they aim to seize a large market share and to win a leading position by drastically reducing prices. The apparent ease of entry into the insurance market is a prerequisite for aggressive companies which seek greater market share by applying these means.

A necessary condition of the industry's concerted response outlined in Eq. (3), (4) is recognition that $\mathfrak{A}$ constitutes a grave common threat. It amounts to a gradual but synchronous drift from profitable operations (i.e., from Goal B) to the market share protection (i.e., to Goal A).

Resisting the aggression largely consists in matching the aggressive company $\mathfrak{A}$, even in reducing premiums below the real costs of insurance, until the aggressor becomes exhausted or extinct. Unless the assaulted company $\mathfrak{D}$

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[1] As the market remains hard, the year-by-year decrease of P^M (see Eq. (3)) is annoying but endurable. It is a sensible maneuver for those who wish to protect their market share by reducing emigration to $\mathfrak{A}$. As soon as $P^{\mathfrak{A}}$ is set (see Eq. (4)) such that $P^M > \mathrm{E}Y > P^{\mathfrak{A}}$, it may be called "casus belli" by the whole market. From that time on, a company defending its market share by mere drastic reduction of premiums, can do it no more. Unless it possesses a huge capital, its portfolio reduction becomes *constrained*. Indeed, that — below-cost insurance-deficient premiums — negatively affects the insurer's solvency. It is such the larger the insurer's portfolio is; because claims are no longer matched by premium income, a defending company imprudently charges an exceedingly low premium, while its portfolio remains large, and is likely to be ruined within a short time interval.

is "too big to fail"', a forestalling *defensive evolution* is the only option that makes it possible to engage in further defensive actions.

DEFINITION 2.4. Call *mobilized* a company structured to seek market share redistribution (Goal A), rather than to perform profitable operations (Goal B). Call *demobilized* a company structured to perform profitable operations (Goal B) rather than to seek market share redistribution (Goal A).

By defensive evolution we will largely mean switching of a company from demobilized to mobilized condition. In the sequel we quantify the *constrained* defensive evolution in the framework of a Lundberg-type risk model. In particular, we formalize the observation that, as premiums are below marginal costs, *the larger is a company's market share the greater are the cumulative losses*, and the excessive market share must be dropped down.

To return to profitable operations (i.e., to Goal B) requires eliminating the aggressive companies in either way[2] and carrying out demobilization. It agrees with the fundamental observation of Feldblum (2001), who emphasized that "insolvencies are not just a by-product of dismal earnings; they are a driving force behind the cycles".

2.4. Quarters linked in a cycle

Each cycle typically consists of one downswing and one upswing phase. Bearing in mind years of hard and soft markets, specify the following successive quarters, which may be degenerate.

QUARTER DH (Downswing at hard market). A sequence of insurance years with market prices $P_1^M > P_2^M > P_3^M > \dots > EY$ is called Quarter DH of the insurance cycle.

Though there may be "anticipation of competition", profitable operations are predominant in this quarter. Most companies, except aggressive and over-cautious ones, which forestall the approaching competition, are demobilized.

QUARTER DS (Downswing at soft market). A sequence of insurance years with market prices $EY > P_1^M > P_2^M > P_3^M > \dots$ is called Quarter DS of the insurance cycle.

The beginning of that quarter is marked by "outbreak of open competition" triggered by an aggressive company. The necessity to switch from the profitable operations (from Goal B) to defense (to Goal A) compels most companies to become mobilized as quickly as possible. Those who are late are more liable to quick ruin. The companies ready for defense wage "close contest" price competition. Capital is wasted by all companies, uniformly over the market. Started as "war of maneuver", this quarter ends as "trench warfare".

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[2] By means of contenting or destruction.

QUARTER US (Upswing at soft market). A sequence of insurance years with market prices $P_1^M < P_2^M < P_3^M < \ldots < \mathrm{E}Y$ is called Quarter US of the insurance cycle.

Since losses of all parties are heavy, the prices in this quarter gradually rise to the real costs of insurance. In the case of all-or-none competition, a "struggle of attrition" may occur. It means spiral competition, when the former aggressor has wasted its capital and becomes lenient, while the formerly defending company preserved enough capital to reduce premiums and knock the competitor down. The capital of both contenders expends over each spiral convolution, as the rôle of aggressor and defender may be reversed many times.

QUARTER UH (Upswing at hard market). A sequence of insurance years with market prices $\mathrm{E}Y < P_1^M < P_2^M < P_3^M < \ldots$ is called Quarter UH of the insurance cycle.

This quarter is marked by the elimination of weaker carriers. Those who are ruined go away, and the victors go the spoils. Profitable operations in a quiet market yield prosperity, which is liable to attract anew envious intruders.

2.5. Types of annual price control and price ratios

Consider different control price options available for particular insurers in the years of hard and soft markets.

DEFINITION 2.5 (Insurer's price on hard market). In the year of hard market, set

$$\mathcal{P}^h = \mathcal{P}^h_{\text{CL-SW}} \cup \mathcal{P}^h_{\text{CW-SW}} \cup \mathcal{P}^h_{\text{CW-SL}},$$

where $\mathcal{P}^h_{\text{CL-SW}} = \{P : 0 < P \leqslant \mathrm{E}Y < P^M\}$, $\mathcal{P}^h_{\text{CW-SW}} = \{P : \mathrm{E}Y < P < P^M\}$ and $\mathcal{P}^h_{\text{CW-SL}} = \{P : P \geqslant P^M > \mathrm{E}Y\}$. The prices[3] $P \in \mathcal{P}^h_{\text{CL-SW}}$ are called *capital losing - share winning* (CL-SW) control, the prices $P \in \mathcal{P}^h_{\text{CW-SW}}$ are called *capital winning - share winning* (CW-SW) control, the prices $P \in \mathcal{P}^h_{\text{CW-SL}}$ are called *capital winning - share losing* (CW-SL) control.

DEFINITION 2.6. (Insurer's price on soft marke). In the year of soft market, set

$$\mathcal{P}^s = \mathcal{P}^s_{\text{CL-SW}} \cup \mathcal{P}^s_{\text{CL-SL}} \cup \mathcal{P}^s_{\text{CW-SL}},$$

where $\mathcal{P}^s_{\text{CL-SW}} = \{P : 0 < P \leqslant P^M < \mathrm{E}Y\}$, $\mathcal{P}^s_{\text{CL-SL}} = \{P : P^M < P < \mathrm{E}Y\}$ and $\mathcal{P}^s_{\text{CW-SL}} = \{P : P \geqslant \mathrm{E}Y > P^M\}$. The prices[4] $P \in \mathcal{P}^s_{\text{CL-SW}}$ are called *capital losing -*

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[3] The price $P = \mathrm{E}Y$ yields *capital conserving - share winning* (CC-SW) control, the price $P = P^M$ yields *capital winning - share conserving* (CW-SC) control.

[4] The price $P = \mathrm{E}Y$ yields *capital conserving - share losing* (CC-SL) control, the price $P = P^M$ yields *capital losing - share conserving* (CL-SC) control.

TABLE 2.1

PRICE CONTROLS IN DIFFERENT QUARTERS

Quarters	Types of price control		
DH	CL-SW	CW-SW	CW-SL
DS	CL-SW	CL-SL	CW-SL
US	CL-SW	CL-SL	CW-SL
UH	CL-SW	CW-SW	CW-SL

share winning (CL-SW) control, the prices $P \in \mathcal{P}^s_{\text{CL-SL}}$ are called *capital losing - share losing* (CL-SL) control, the prices $P \in \mathcal{P}^s_{\text{CW-SL}}$ are called *capital winning - share losing* (CW-SL) control.

REMARK 2.1. Feldblum (2007b) applies the following terminology: CL-SC or CW-SC controls are called *maintaining market share* (MMS) controls, CC-SL or CC-SW controls are called *conserving capital* (CC) controls, and all other controls are called *mixed* controls.

DEFINITION 2.7 (Price ratios). The ratios

$$g(P) = \frac{P}{\text{E}Y} > 0, \; d(P) = \frac{P}{P^M} > 0$$

are called *price to real costs of insurance* ratio and *price to market price* ratio. For brevity, one may call $g(P)$ *price-to-cost* and $d(P)$ *price-to-market* ratio.

REMARK 2.2. In the year of hard market, as $\kappa > 1$ or $\text{E}Y < P^M$, which is equivalent to the inequality $d(P) < g(P)$, the implication $P \in \mathcal{P}^h_{\text{CL-SW}}$ is equivalent to $d(P) < g(P) \leqslant 1$, the implication $P \in \mathcal{P}^h_{\text{CW-SW}}$ is equivalent to $d(P) < 1 < g(P)$, and the implication $P \in \mathcal{P}^h_{\text{CW-SL}}$ is equivalent to $1 \leqslant d(P) < g(P)$. In the year of soft market, as $\kappa < 1$ or $\text{E}Y > P^M$, which is equivalent to the inequality $g(P) < d(P)$, the implication $P \in \mathcal{P}^s_{\text{CL-SW}}$ is equivalent to $g(P) < d(P) \leqslant 1$, the implication $P \in \mathcal{P}^s_{\text{CL-SL}}$ is equivalent to $g(P) < 1 < d(P)$, and the implication $P \in \mathcal{P}^s_{\text{CW-SL}}$ is equivalent to $1 \leqslant g(P) < d(P)$.

DEFINITION 2.8 (Premium loading). Call $\tau(P) = g(P) - 1$ *loading* on premium P.

Equivalent, but more familiar equality implying loading is $P = (1 + \tau(P))\,\text{E}Y$. Positive loading on P means an excess over $\text{E}Y$, while negative loading means deficiency of P with respect to $\text{E}Y$. Plainly, one has $\tau(P) \geqslant 0$, as $P \in \mathcal{P}^h_{\text{CW-SW}} \cup \mathcal{P}^h_{\text{CW-SL}} \cup \mathcal{P}^s_{\text{CW-SL}}$, and $\tau(P) \leqslant 0$, as $P \in \mathcal{P}^h_{\text{CL-SW}} \cup \mathcal{P}^s_{\text{CL-SW}} \cup \mathcal{P}^s_{\text{CL-SL}}$, with $\tau(P) = 0$ iff $P = \text{E}Y$.

2.6. Strategic planning and annual maneuvering

Important for further modeling is the strategic planning and annual maneuvering of the aggressive, actively defending and neutral or, competing with the market, companies. That matter crucially depends on the strategic goals (see Section 2.2) in which diversity is overwhelming. However, the companies are eager, forced or allowed to apply a limited number of controls for annual maneuvering.

TABLE 2.2

NEUTRAL COMPANY'S STRUGGLE "FOR JUST-SURVIVAL"

Quarters	Neutral $\mathfrak{N}$ (initial capital and initial share of middling size)	
DH	Profit-seeking control followed by a defensive evolution	CW-SW followed by CW-SL
DS	Passive defense: minimal solvency and subsistence constraints	CW-SL or CL-SL or ⊠
US	Passive defense: minimal solvency and subsistence constraints	CW-SL or CL-SL or ⊠
UH	If survived, $\mathfrak{N}$ recovers share and renews profitable operations	CL-SW or CW-SW (if ⊞)

Developing the outline of Section 2.4, consider competitors pursuing different strategic goals: a company $\mathfrak{N}$ seeking survival (Table 2.2), an aggressive company $\mathfrak{A}$ and an actively defending company $\mathfrak{D}$ which compete for co-existence (Table 2.3), and $\mathfrak{A}$ and $\mathfrak{D}$ which fight an all-or-none battle (Table 2.4).

TABLE 2.3

COMPETITION FOR CO-EXISTENCE

Quarters	Aggressive $\mathfrak{A}$ (large initial capital, small initial share)		Defensive $\mathfrak{D}$ (relatively small initial capital, large initial share)	
DH	Drastic reduction of prices and aggressive market share gain	CL-SW	Defensive evolution	CW-SL
DS	Close contest	CL-SW	Close contest	CW-SL
US	$\mathfrak{A}$ is contented and ceases aggression	CL-SW or CL-SL	$\mathfrak{D}$ agrees to end competition	CL-SL
UH	$\mathfrak{A}$ wins a share and starts profitable operations	CW-SW	$\mathfrak{D}$ loses a share and renews profitable operations	CW-SW

In Quarter DH, the aggressive company $\mathfrak{A}$ is already structured for competition. Eager to acquire a greater share from a demobilized adversary $\mathfrak{D}$ either by provoking its bankruptcy or by easier means, $\mathfrak{A}$ reduces premiums. Thus $\mathfrak{A}$ realizes a capital loss and market share win. The competitors remain either demobilized, i.e. non-resisting, or switch from Goal B to Goal A and make a *constrained* defensive evolution.

In Quarter DS, if the company $\mathfrak{A}$ remains mobilized and aggressive, it accelerates the premium decrease and keeps losing capital. Since $\mathfrak{D}$ and most contending companies became mobilized at this moment, a close contest starts.

TABLE 2.4

COMPETITION "OF ELIMINATION"

Quarters	Aggressive $\mathfrak{A}$ (large initial capital, small initial share)		Defensive $\mathfrak{D}$ (relatively small initial capital, large initial share)	
DH	Drastic reduction of prices and aggressive market share gain	CL-SW	Defensive evolution	CW-SL
DS	Close contest	CL-SW	Close contest	CW-SL
US	Struggle of attrition: $\mathfrak{A}$ is aggressive	CL-SW	Struggle of attrition: $\mathfrak{D}$ is defensive	CL-SL
	Struggle of attrition: $\mathfrak{A}$ is defensive	CL-SL	Struggle of attrition: $\mathfrak{D}$ is aggressive	CL-SW
	…	…	…	…
			$\mathfrak{D}$ loses share (capital) and agonizes,	CW-SL
	$\mathfrak{A}$ knocks $\mathfrak{D}$ out,	⊞		
	or		or	
	$\mathfrak{A}$ loses share (capital) and agonizes	CW-SL	$\mathfrak{D}$ knocks $\mathfrak{A}$ out	⊞
UH	$\mathfrak{A}$ wins spoils and starts profitable operations,	CW-SW	$\mathfrak{D}$ is eliminated,	⊠
	or		or	
	$\mathfrak{A}$ is eliminated	⊠	$\mathfrak{D}$ wins spoils and renews profitable operations	CW-SW

In Quarter US, the company $\mathfrak{A}$ is still losing its capital, and is *constrained* to wage a more careful price policy; $\mathfrak{A}$ becomes lenient and prices gradually increase. Its competitor $\mathfrak{D}$ is either broken in spirits and agrees to stop competing, or starts competing aggressively by launching spiral competition, when the rôle of aggressor and sufferer alternates. The issue may be ruin for either.

In Quarter UH, as there are no aggressive companies in the market, the switching from Goal A to Goal B is universal, and premium prices grow above the real costs of insurance. The successful companies are content with the

share won by action or by agreement, seek for profit, carry on demobilization and trade off the excessive capital for more market share.

Further discussion and quantification is deferred to Sections 5 and 6.

REMARK 2.3. (Open model). It is noteworthy that not only leading aggressive company $\mathfrak{A}$ (see Assumption 3) seeks to deprive $\mathfrak{D}$ of its market share, but so do all players on the insurance market which follow $\mathfrak{A}$ in the price reduction race. Company $\mathfrak{D}$ resists both the aggression of $\mathfrak{A}$ and the downfall of the market price produced by a concerted industry's action. In a sense, there are three interactive players: $\mathfrak{A}$, $\mathfrak{D}$ and the rest of the market. In the next section, the customers gained by $\mathfrak{A}$ are not quite the same as those who left $\mathfrak{D}$.

3. LUNDBERG-TYPE ANNUAL PROBABILITY MECHANISM OF INSURANCE

Models of the annual probability mechanism of insurance fit to be used in the multi-period model of competition-originated cycle must render information about at least the year-end risk reserve, year-end portfolio size and downward crossings of zero capital level, which is ruin, by the end of the year. The least set of the controlled variables is the initial capital, the initial portfolio size, and the premium intensity.

Develop the Lundberg-type model of the annual probability mechanism of insurance with varying portfolio size, with non-homogeneous income-outcome balance and probability of ruin set as a measure of insolvency.

3.1. Migration of insureds

Emigration of insureds, or insurer's market share losing, as time goes on, is induced by excess of P over P^M. In terms of price-to-market ratio $d(P) = P/P^M$ that is equivalent to $d(P) > 1$. Market share gain is induced by P below P^M, which is equivalent to $d(P) < 1$. In the sequel we often omit the argument in $d(P)$ for notation simplicity.

DEFINITION 3.1 (Migration rate functions). For $P \in \mathcal{P}^s (P \in \mathcal{P}^h)$ and price-to-market ratio $d(P) = P/P^M$, introduce a set $\mathcal{R} = \{r_d(s), s \geqslant 0\}$ of positive continuous functions, such that $r_d(0) = 1$ uniformly on $d \in \mathrm{R}^+$. Assume that

(i)   for $d > 1$, the function $r_d(s)$ is monotone decreasing in $s \in \mathrm{R}^+$, and $r_{d_1}(s) < r_{d_2}(s)$ for all $s \in \mathrm{R}^+$, as $d_1 > d_2 > 1$,

(ii) for $0 < d < 1$, the function $r_d(s)$ is monotone increasing in $s \in \mathrm{R}^+$, and $r_{d_1}(s) > r_{d_2}(s)$ for all $s \in \mathrm{R}^+$, as $1 > d_2 > d_1$,

(iii) for $d = 1$, the function $r_d(s)$, $s \in \mathrm{R}^+$, is identically unit.

The functions from the set $\mathcal{R}$ are called *migration rate* functions. The positive function $r_d = r_d(+\infty)$, $d > 0$, is called *ultimate migration rate* function.

One has from (i)-(iii), that $r_1 = 1$ and r_d decreases from $r_0 = \sup_{d < 1} r_d \geqslant 1$ to $r_{+\infty} = \inf_{d > 1} r_d \in [0, 1]$, as d increases from 0 to $+\infty$. The function r_d may be discontinuous. If $r_0 < \infty$, the ultimate immigration is bounded, and the insurer's capacity is finite. If $r_{+\infty} = 0$, the ultimate emigration is total. If $r_{+\infty} > 0$, it is not.

DEFINITION 3.2 (Share functions). For a set $\mathcal{R} = \{r_d(s),\ s \geqslant 0\}$ of migration rate functions and for the initial portfolio size $\lambda > 0$, introduce a set $\mathcal{R}_\lambda$ of continuous positive functions $\lambda_d(s)$ of time $s \in \mathbf{R}^+$, such that

$$\lambda_d(s) = \lambda r_d(s),\ \ s \geqslant 0. \tag{5}$$

Call[5] the functions from the set $\mathcal{R}_\lambda$ *market share* or *portfolio size* functions.
  Evidently, one has $\lambda r_{+\infty}(s) \leqslant \lambda_d(s) \leqslant \lambda r_0(s),\ s \geqslant 0$.

DEFINITION 3.3 (Intensity functions). Call

$$\Lambda_{d,\lambda}(t) = \int_0^t \lambda_d(s)ds,\ t \geqslant 0,$$

intensity function corresponding to the portfolio size function $\lambda_d(s),\ s \geqslant 0$.
  The following lemma is straightforward.

LEMMA 3.1. For $t \geqslant 0$, $\lambda > 0$, one has

$$\Lambda_{d_1,\lambda}(t) < \Lambda_{d_2,\lambda}(t) < \Lambda_{1,\lambda}(t) = \lambda t,\ \text{ as } 1 < d_2 < d_1,$$

$$\Lambda_{d_1,\lambda}(t) > \Lambda_{d_2,\lambda}(t) > \Lambda_{1,\lambda}(t) = \lambda t,\ \text{ as } 1 > d_2 > d_1 > 0.$$

PROOF. The proof is straightforward from Definitions 3.1-3.3.   □

Examples of the ultimate migration rate functions r_d abound. Each positive function $M(d)$, $d \geqslant 0$, which is monotone decreasing from $M(0)$ to $M(+\infty)$, as d increases from 0 to $+\infty$, yields

$$r_d = M(d)/M(1),\ \text{where}\ r_0 = M(0)/M(1) \geqslant 1,\ r_{+\infty} = M(+\infty)/M(1) \geqslant 0.$$

To be specific, pick up arbitrarily two constants $0 \leqslant c < 1 < C$ and construct r_d such that $C = r_0$, $c = r_{+\infty}$. Set

$$r_d = (e^{-\rho d^l} + \varrho)/(e^{-\rho} + \varrho),\ d \geqslant 0, \tag{6}$$

---

[5] For brevity, write $\mathcal{R}_1 = \mathcal{R}$.

where $l > 0$ and $\varrho = c/(C-c) > 0$, $\rho = -\ln((1-c)/(C-c)) > 0$. It is noteworthy that the function r_d is as closer to the step function

$$r_d = \begin{cases} C, & 0 < d < 1, \\ 1, & d = 1, \\ c, & d > 1, \end{cases}$$

as larger is taken the power l.

To set examples of migration rate functions $\mathcal{R}$, introduce

$$r_d(s) = r_d + (1-r_d)e^{-s} = 1 - (1-r_d)(1-e^{-s}), \quad s \in [0,t], \tag{7}$$

called *exponential* migration rate function. Since $e^{-s} \leqslant 1$, as $s \geqslant 0$, and $r_d(s)$ depends on r_d linearly,

$$r_{+\infty}(s) = r_{+\infty} + (1-r_{+\infty})e^{-s} \leqslant r_d(s) \leqslant r_0 + (1-r_0)e^{-s} = r_0(s), \quad s \in [0,t].$$

In the case $d = 1$, one has $r_d(s) \equiv 1$, and the portfolio all the time remains unchanged. In the case $d > 1$ (emigration) the value $1 - r_d > 0$ is the rate of ultimate emigrants and $(1-r_d)(1-e^{-s})$ is the portion of the ultimate emigrants who left the portfolio by time s. In the case $d < 1$ (immigration) rewrite (7) as

$$r_d(s) = r_d - (r_d - 1)e^{-s} = 1 + (r_d - 1)(1-e^{-s}), \quad s \in [0,t],$$

where $r_d - 1 > 0$ is the rate of ultimate immigrants and $(r_d - 1)(1-e^{-s})$ is the portion of the ultimate immigrants who joined the portfolio by time s. The migration rate functions (7) yields

$$\begin{aligned} \Lambda_{d,\lambda}(t) &= \int_0^t \lambda_d(s)ds = \lambda t r_d + \lambda(1-r_d)(1-e^{-t}) \\ &= \lambda t - \lambda(1-r_d)(e^{-t} + t - 1). \end{aligned} \tag{8}$$

Exponential migration may be exceedingly quick. Introduce $k > 0$ and call

$$r_d(s) = r_d + (1-r_d)(1+s)^{-k} = 1 - (1-r_d)(1-(1+s)^{-k}), \quad s \in [0,t], \tag{9}$$

power migration rate function. Since $(1+s)^{-k} \leqslant 1$, as $s \geqslant 0$, and since $r_d(s)$ depends on r_d linearly, for $s \in [0,t]$ one has

$$r_{+\infty}(s) = r_{+\infty} + (1-r_{+\infty})(1+s)^{-k} \leqslant r_d(s) \leqslant r_0 + (1-r_0)(1+s)^{-k} = r_0(s).$$

That yields

$$\Lambda_{d,\lambda}(t) = \begin{cases} \lambda t r_d + \lambda(1-r_d)((1+t)^{1-k} - 1)/(1-k), & k \neq 1, \\ \lambda t r_d + \lambda(1-r_d)\ln(1+t), & k = 1 \end{cases}$$

$$= \begin{cases} \lambda t - \lambda(1-r_d)[t - ((1+t)^{1-k} - 1)/(1-k)], & k \neq 1, \\ \lambda t - \lambda(1-r_d)(t - \ln(1+t)), & k = 1. \end{cases}$$

(10)

Note that $e^{-t} + t > 1$, $t > ((1+t)^{1-k} - 1)/(1-k)$, $t > \ln(1+t)$, as $t > 0$, in Eq. (8), (10). Bearing in mind that $1 - r_d > 0$ in the case of outgo (as $d > 1$) and $1 - r_d < 0$ in the case of inflow (as $d < 1$) of insureds, one has migration-originated adjustment terms for $\Lambda_{1,\lambda}(t) = \lambda t$ in $\Lambda_{d,\lambda}(t)$.

It is noteworthy that surveys of policy holders[6] accentuate migration rate functions with $r_d(t) > 0$, which agrees with our models.

REMARK 3.1. The families $\mathfrak{R}_\lambda$ may be further sophisticated. Firstly, one may deal with *random* migration rate functions rather than with *deterministic* ones. Secondly, the dependence structure may be set more complex. For example, if migration of insureds accelerates, as the initial risk reserve u decreases or increases, one may refine Eq. (6) and set $r_d(u) = (e^{-\rho(u)\,d^{l(u)}} + \varrho(u))/(e^{-\rho(u)} + \varrho(u))$ with appropriately chosen positive $l(u)$, $\rho(u)$ and $\varrho(u)$.

3.2. Migration and surplus process

Apply the classical Lundberg's approach to model the non-homogeneous risk reserve process under migration.

DEFINITION 3.4 (Claims arrival process). For $P \in \mathcal{P}^s$ ($P \in \mathcal{P}^h$), price-to-market ratio $d(P) = P/P^M$ and initial portfolio size λ, the *claims arrival* process is a non-homogeneous Poisson process $v_{d,\lambda}(s)$, $s \in [0, t]$, with intensity function $\Lambda_{d,\lambda}(s) = \int_0^s \lambda_d(z)\,dz$, $s \in [0, t]$, where $\lambda_d(s)$, $s \in [0, t]$, is a function from $\mathfrak{R}_\lambda$ corresponding to P, $d(P)$ and λ.

It is well known that $\Lambda_{d,\lambda}(s) = \mathrm{E}v_{d,\lambda}(s)$, $s \in [0, t]$.

---

[6] Quote from Subramanian (1998), p. 39: "Surveys of policyholders have consistently demonstrated some reluctance to switch insurers. In a survey of 2462 policyholders by Cummins et al. (see Cummins et al. (1974)), 54% of respondents confessed never to have shopped around for auto insurance prices. To the question "Which is the most important factor in your decision to buy insurance?", 40% responded the company, 29% the agent, and only 27% the premium. A similar survey of 2004 Germans (see Schlesinger et al. (1993)) indicated that, despite the fact that 67% of those responding knew that considerable price differences exist between automobile insurers, only 35% chose their carrier on the basis of their favorable premium. Therefore, we will assume that, given the opportunity to switch for a reduced premium, one-third of the policyholders will do so".

DEFINITION 3.5 (Total claim amount process). Assume that i.i.d. claim amounts Y_i, $i = 1, 2, \ldots$, are independent on the claims arrival process $v_{d,\lambda}(s)$, $s \in [0, t]$. The *total claim amount* process is the compound non-homogeneous Poisson process

$$Z_{d,\lambda}(s) = \sum_{i=1}^{v_{d,\lambda}(s)} Y_i, \qquad (11)$$

as $v_{d,\lambda}(s) > 0$, or zero, as $v_{d,\lambda}(s) = 0$, $s \in [0, t]$.

DEFINITION 3.6 (Premium income process). For $P \in \mathcal{P}^s$ ($P \in \mathcal{P}^h$), $d = P/P^M$ and initial portfolio size λ, the *premium income* process is a non-random process

$$P\Lambda_{d,\lambda}(s) = P \int_0^s \lambda_d(z)\,dz = P\lambda \int_0^s r_d(z)\,dz, \quad s \in [0, t], \qquad (12)$$

where $\lambda_d(s)$, $s \in [0, t]$, is a function from $\mathfrak{R}_\lambda$ corresponding to P, d and λ.

DEFINITION 3.7 (Risk reserve process). For $P \in \mathcal{P}^s$ ($P \in \mathcal{P}^h$), $d(P) = P/P^M$, for the initial portfolio size $\lambda > 0$ and for $u > 0$ called *initial risk reserve*, the random process

$$R_{u,\lambda,P}(s) = u + P\Lambda_{d(P),\lambda}(s) - Z_{d(P),\lambda}(s), \qquad (13)$$

as $v_{d,\lambda}(s) > 0$, or $u + P\Lambda_{d,\lambda}(s)$, as $v_{d,\lambda}(s) = 0$, $s \in [0, t]$, is called *risk reserve* process corresponding to total claim amount (11) and premium income (12) processes.

LEMMA 3.2. For the claims arrival process $v_{d,\lambda}(s)$, $s \in [0, t]$, one has

$$v_{d,\lambda}(s) = N_\lambda(\Lambda_{d,\lambda}(s)/\lambda), \ s \in [0, t],$$

where $N_\lambda(s)$, $s \in [0, t]$, is the homogeneous Poisson process with intensity $\lambda > 0$. Moreover, for the risk reserve process (13),

$$R_{u,\lambda,P}(s) = u + P\Lambda_{d(P),\lambda}(s) - Z_{d(P),\lambda}(s)$$

$$= u + [P\lambda](\Lambda_{d(P),\lambda}(s)/\lambda) - \sum_{i=1}^{N_\lambda(\Lambda_{d(P),\lambda}(s)/\lambda)} Y_i.$$

PROOF. See Bühlmann (1970), Theorem 1 on p. 38.   □

LEMMA 3.3. Introduce

$$\hat{R}_{u,\lambda,P}(s) = u + [P\lambda]s - \sum_{i=1}^{N_\lambda(s)} Y_i, \ \text{as } 0 \leqslant s \leqslant \Lambda_{d(P),\lambda}(t)/\lambda. \qquad (14)$$

For $\tau(s) = \Lambda_{d,\lambda}(s)/\lambda$, $0 \leqslant s \leqslant t$, one has

$$R_{u,\lambda,P}(s) = \hat{R}_{u,\lambda,P}(\tau(s)), \ 0 \leqslant s \leqslant t. \tag{15}$$

PROOF. The proof is standard (see Bühlmann (1970), p. 38-39 and Section 2.2.3, or Grandell (1991), Section 2.1 on p. 33, or Asmussen (2000), Remark 1.6 on p. 60). The time $\tau(s) = \Lambda_{d,\lambda}(s)/\lambda$, $0 \leqslant s \leqslant t$, is known under the name of *operational time*. Plainly, the passage of that time is no longer measured in calendar units, but in expected number of claims. □

REMARK 3.2 (Migration and expected surplus). The ratio $g(P) = P/EY \in \mathbb{R}^+$ is called (see Definition 2.7) price-to-cost. It may be noteworthy that

$$
\begin{aligned}
ER_{u,\lambda,P}(t) &= u + P\Lambda_{d,\lambda}(t) - EZ_d(s) = u + (P - EY)\Lambda_{d,\lambda}(t) \\
&= u + EY(g(P) - 1)\Lambda_{d,\lambda}(t).
\end{aligned}
\tag{16}
$$

If the inequality $g(P) > 1$ holds true, the averaged risk reserve is ascending, as time goes on, since $ER_{u,\lambda,P}(t) > u$. Conversely, if the inequality $g(P) < 1$ holds true, the averaged risk reserve is descending, as time goes on, since $ER_{u,\lambda,P}(t) < u$.

THEOREM 3.1 (Risk reserve distribution for exponential claims). Assume that Y_i, $i = 1,2,\ldots$, are i.i.d. exponential with intensity $\mu > 0$ (i.e., $1/\mu = EY$) and independent on the claims arrival process $v_{d,\lambda}(s)$, $s \in [0,t]$. For $P \in \mathcal{P}^s$ ($P \in \mathcal{P}^h$), $d = P/P^M$, for the initial portfolio size $\lambda > 0$ and initial risk reserve $u > 0$, and for $x \in \mathbb{R}$

$$P\{R_{u,\lambda,P}(t) \leqslant x\} =$$

$$
= \begin{cases}
1, & x > u + P\Lambda_{d,\lambda}(t), \\
1 - e^{-\Lambda_{d,\lambda}(t)} - e^{-\Lambda_{d,\lambda}(t)}\sqrt{\mu\Lambda_{d,\lambda}(t)} \\
\times \int_0^{u+P\Lambda_{d,\lambda}(t)-x} z^{-1/2} I_1\left(2\sqrt{\mu\Lambda_{d,\lambda}(t)z}\right) e^{-\mu z} dz, & x \leqslant u + P\Lambda_{d,\lambda}(t),
\end{cases}
$$

where $I_1(\cdot)$ is the modified Bessel function of unit order.

PROOF. It is straightforward from Lemma 3.3 and Theorem 2.1 in Malinovskii (2008a). □

3.3. Migration and probabilities of ruin

Introduce the probability of ruin in the Lundberg's model with migration, as the risk reserve process is non-homogeneous.

DEFINITION 3.8 (Probability of ruin). For all values of t, u and λ positive, for $P \in \mathcal{P}^s (P \in \mathcal{P}^h)$, the probability

$$\psi_{u,\lambda,P}(t) = P\left\{\inf_{0 < s \leqslant t} R_{u,\lambda,P}(s) < 0\right\} \tag{17}$$

is called *probability of ruin* within time t.

Direct corollary of Eq. (15) is

$$\inf_{0 < s \leqslant t} R_{u,\lambda,P}(s) = \inf_{0 < s \leqslant \Lambda_{d,\lambda}(t)/\lambda} \hat{R}_{u,\lambda,P}(s). \tag{18}$$

In a competitive business, the insurer may lose some of its insureds and market share. So, he receives less in premiums, which makes the probability of ruin greater, and pays fewer claims, which makes the probability of ruin less. Either of these counteracting factors may dominate. In the same way, the insurer may receive more in premiums and pay more claims, as he gains market share.

Examine that effect in the framework of Lundberg-type model, where simultaneous increase (decrease) of cumulative premiums and compound claims is introduced as a function of portfolio size, or market share, increase (decrease).

THEOREM 3.2. For all values of t, u and λ positive, the probability of ruin $\psi_{u,\lambda,P}(t)$ is monotone decreasing, as $P \in \mathcal{P}^s (P \in \mathcal{P}^h)$ monotone increases.

PROOF. Bearing in mind Eq. (18), one has

$$\psi_{u,\lambda,P}(t) = P\left\{\inf_{0 \leqslant s \leqslant \Lambda_{d(P),\lambda}(t)/\lambda} \hat{R}_{u,\lambda,P}(s) < 0\right\}$$

$$= P\left\{\inf_{0 \leqslant s \leqslant \Lambda_{d(P),\lambda}(t)/\lambda} \left(u + \underbrace{\left(P^M d(P)\right)}_{P} \lambda s - \sum_{i=1}^{N_\lambda(s)} Y_i\right) < 0\right\}.$$

Recall that $d(P) = P/P^M > 0$. By Lemma 3.1, $\Lambda_{d(P),\lambda}(t)$ monotone decreases, as $P \in \mathcal{P}^s$ monotone increases. The same holds true, as $P \in \mathcal{P}^h$. Both factors which depend on P in the expression for $\psi_{u,\lambda,P}(t)$ contribute to a monotone decrease of $\psi_{u,\lambda,P}(t)$, as P monotone increases, which completes the proof. $\square$

In the case of exponential claim size Y_i, $i = 1, 2, \ldots$, an explicit expression for the annual probability of ruin $\psi_{u,\lambda,P}(t)$ is available. Denote by $I_n(z)$ the modified Bessel function of n-th order, $z \in \mathbb{R}$ and $n = 0, 1, 2, \ldots$.

THEOREM 3.3. Assume that Y_i, $i = 1, 2, \ldots$, are i.i.d. exponential with intensity $\mu > 0$ (i.e., $1/\mu = EY$) and independent on the claims arrival process $v_{d,\lambda}(s)$,

$s \in (0, t]$. For $P \in \mathcal{P}^s (P \in \mathcal{P}^h)$, $d(P) = P/P^M$, $g(P) = P/EY$, for the initial portfolio size $\lambda > 0$ and for the initial risk reserve $u > 0$, one has[7]

$$\psi_{u, \lambda, P}(t) = e^{-u\mu} \sum_{n \geqslant 0} \frac{(u\mu)^n}{n!} g^{-(n+1)/2} \int_0^{\Lambda_{d,\lambda}(t)} \frac{n+1}{x} e^{-(1+g)x} I_{n+1}\left(2x\sqrt{g}\right) dx, \quad (19)$$

or, alternatively,

$$\psi_{u, \lambda, P}(t) = \psi_{u, \lambda, P}(+\infty) - \frac{1}{\pi} \int_0^\pi f_{u, \lambda, P}(x)\, dx, \quad (20)$$

where[8]

$$\psi_{u, \lambda, P}(+\infty) = \begin{cases} g^{-1} \exp\left\{-u\mu\left(1 - g^{-1}\right)\right\}, & g > 1, \\ 1, & g \leqslant 1 \end{cases} \quad (21)$$

and

$$\begin{aligned}
f_{u, \lambda, P}(x) = {}& g^{-1}\left(1 + g^{-1} - 2g^{-1/2}\cos x\right)^{-1} \\
& \times \exp\left\{u\mu\left(g^{-1/2}\cos x - 1\right) - \Lambda_{d,\lambda}(t)g\left(1 + g^{-1} - 2g^{-1/2}\cos x\right)\right\} \\
& \times \left[\cos\left(u\mu g^{-1/2}\sin x\right) - \cos\left(u\mu g^{-1/2}\sin x + 2x\right)\right].
\end{aligned}$$

PROOF. By Eq. (18), it is straightforward from Corollary 2.1 in Malinovskii (2008a). □

Examples of numerical evaluation, by means of Theorem 3.3, of the annual probabilities of ruin $\psi_{u, \lambda, P}(t)$ as functions of P for diverse initial portfolio sizes, and a reference level α, are shown on Fig. 1.

REMARK 3.3. Theorem 3.3 affords a most well know example of an explicit formula for the finite time probability of ruin $\psi_{u, \lambda, P}(t)$. A number of results allowing calculation of $\psi_{u, \lambda, P}(t)$ in a more complicated set-up (see Malinovskii (1998), Wang and Liu (2002), Sun (2007) and references therein) are known.

---

[7] For brevity sake, we omit the argument P in the notation for $d(P)$ and $g(P)$.
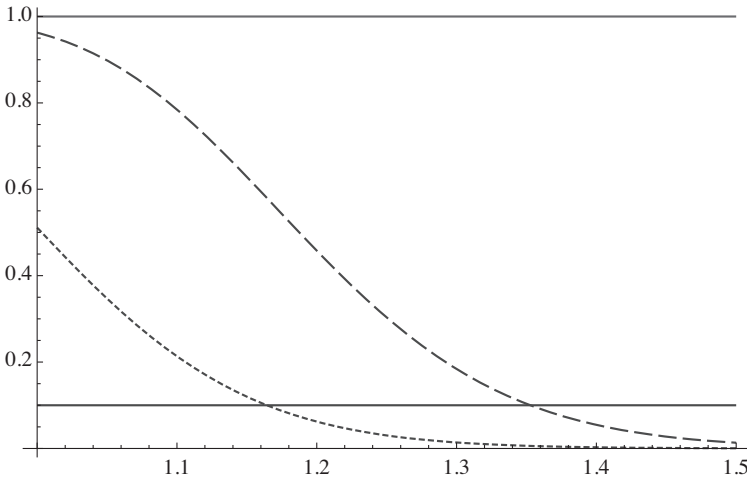[8] Bear in mind that the right hand side of Eq. (21) does not depend on λ.

FIGURE 1: Level $\alpha = 0.1$ and probabilities of ruin (Y-axis) $\psi_{u, \lambda_1, P}(t) < \psi_{u, \lambda_2, P}(t)$, $\lambda_1 = 1.5 < \lambda_2 = 2.2$, as functions of P (X-axis); $P \in (1, 2)$ (i.e., $P \in \mathcal{P}_{\text{CL-SL}}^s$), when $t = 100$, $u = 150$, $EY = 2$, $P^M = 1$. The function $\Lambda_{d, \lambda}(t)$ is defined as in (10), with power $k = 1/2$ and r_d as in (6), where $l = 2$, $\rho = -\ln(2/5)$, $\varrho = 1/5$ (i.e., $C = 2$, $c = 1/3$).

3.4. Approximations for the probabilities of ruin

An alternative to diverse procedures of exact calculation of $\psi_{u, \lambda, P}(t)$ are approximations. Quite a handful of them is known. The following result is a counterpart (in the case $\tau(P) < 0$) of an approximation in Teugels (1982).

THEOREM 3.4. In the assumptions of Theorem 3.3, set $a(P) = (1 - \sqrt{1 + \tau(P)})^2$ and $b(P) = 1/\sqrt{1 + \tau(P)}$ and put $\Lambda_{d, \lambda}(t) \to \infty$, as $t \to \infty$. For $\tau(P) < 0$ and for $\phi_{u, \lambda, P}(t) = 1 - \psi_{u, \lambda, P}(t)$, one has

$$\phi_{u, \lambda, P}(t) = \frac{b^{3/2}(bu\mu + 1)}{2\sqrt{\pi} \, a(\Lambda_{d, \lambda}(t))^{3/2}} \, e^{-u\mu(1-b)} e^{-a\Lambda_{d, \lambda}(t)} \exp\left\{-\frac{b^3 (u\mu)^2}{4\Lambda_{d, \lambda}(t)}\right\} \left\{1 + \underline{O}\left(\Lambda_{d, \lambda}^{-1/2}(t)\right)\right\}$$

for $u \leqslant \underline{O}\left(\Lambda_d^{1/2}(t)\right)$, as $t \to \infty$.

PROOF. It is based on Theorem 3.3 where the explicit expression for $\phi_{u, \lambda, P}(t)$ is given. The result is yielded by the expansions technique developed in Section 3 of Malinovskii (2008b). □

Denote by $\Phi_{\{0,1\}}(\cdot)$ the standard normal distribution function. Of interest are the approximations fit to large u. The following result is suited for $u = \underline{O}(\Lambda_{d, \lambda}(t))$, as $t \to \infty$.

THEOREM 3.5. In the assumptions of Theorem 3.3, as $u \to \infty$, one has

(1) for $\tau(P) = P/\mathrm{E}Y - 1 < 0$,

$$\sup_{t \in \mathbf{R}^+} \left| \psi_{u,\lambda,P}(t) - \Phi_{\{0,1\}}\left(\left(\Lambda_{d,\lambda}(t) - M_\tau u\mu \right) / \left(S_\tau (u\mu)^{1/2} \right) \right) \right| = \underline{O}(u^{-1/2}), \quad (22)$$

where $M_\tau = -1/\tau$, $S_\tau^2 = -2/\tau^3$, and

(2) for $\tau(P) = P/\mathrm{E}Y - 1 > 0$,

$$\sup_{t \in \mathbf{R}^+} \left| \psi_{u,\lambda,P}(t) - C e^{-\kappa u\mu} \Phi_{\{0,1\}}\left(\left(\Lambda_{d,\lambda} - M_\tau u\mu \right) / \left(S_\tau (u\mu)^{1/2} \right) \right) \right| = \bar{o}\left(e^{-\kappa u} \right), \quad (23)$$

where $\kappa = \tau/(1+\tau)$, $C = 1/(1+\tau)$, $M_\tau = 1/(\tau(1+\tau))$, $S_\tau^2 = 2/\tau^3$.

PROOF. Bearing in mind Eq. (18), the proof of part (1) of Theorem 3.5 is straightforward from Theorem 5(I) in Malinovskii (1993), or from Section 3.1 in Malinovskii (2008a), and the proof of part (2) of Theorem 3.5 is straight-forward from Malinovskii (1994). □

REMARK 3.4. Though Theorems 3.4 and 3.5 are known to hold true in a quite more general set-up, we bounded ourselves by the Poisson-Exponential frame-work of this paper. Theorems 3.4 and 3.5 may be formulated in a considerably enhanced form. In Malinovskii (1993), the asymptotic expansions in Eq. (22), and in Malinovskii (1996a), the corresponding large deviations are obtained in a general framework of stopped random sequences which fits well Sparre Andersen's model.

In Malinovskii (1994), the asymptotic expansions in Eq. (23), and in Malinovskii (1996b), the corresponding large deviations are obtained in the framework of Sparre Andersen's model. The latter means non-trivial approx-imations for $\psi_{u,\lambda,P}(t)$, as $u, t \to \infty$ such that $u \ll \underline{O}(\Lambda_d(t))$ and $u \gg \underline{O}(\Lambda_d(t))$.

3.5. Probability mechanism of insurance

Since migration of insureds is modeled in Section 3.1 deterministically, concen-trate on the annual probability mechanisms with two random components: the year-end risk reserve and ruin. For the risk reserve process $R_{u,\lambda,P}(s)$, $s \in [0, t]$, defined in Eq. (13) and for a Borel set A introduce

$$\pi_{u,\lambda,P}(t; A, \{\text{ruin}\}) = \mathrm{P}\left\{ R_{u,\lambda,P}(t) \in A, \inf_{0 < s \leqslant t} R_{u,\lambda,P}(s) < 0 \right\},$$

$$\pi_{u,\lambda,P}(t; A, \{\text{no ruin}\}) = \mathrm{P}\left\{ R_{u,\lambda,P}(t) \in A, \inf_{0 < s \leqslant t} R_{u,\lambda,P}(s) > 0 \right\} \quad (24)$$

$$= \mathrm{P}\left\{ R_{u,\lambda,P}(t) \in A \right\} - \pi_{u,\lambda,P}(t; A, \{\text{ruin}\}),$$

where $u \in \mathrm{R}^+$, $d(P) = P/P^M$. Plainly,

$$0 \leqslant \pi_{u,\lambda,P}(t; A, \{\mathrm{ruin}\}) \leqslant \psi_{u,\lambda,P}(t),$$

$$0 \leqslant \pi_{u,\lambda,P}(t; A, \{\mathrm{no\ ruin}\}) \leqslant \mathrm{P}\{R_{u,\lambda,P}(t) \in A\}.$$

When the time-transformed process (14) is Poisson-Exponential, as in Theorem 3.3, more delicate analysis of (24) is possible. In this case one has

$$\pi_{u,\lambda,P}(t; A, \{\mathrm{ruin}\}) = \mathrm{P}\Big\{\widehat{R}_{u,\lambda,P}(\Lambda_{d,\lambda}(t)/\lambda) \in A, \inf_{0 \leqslant s \leqslant \Lambda_{d,\lambda}(t)/\lambda} \widehat{R}_{u,\lambda,P}(s) < 0\Big\}$$

$$= \int_0^\infty \int_0^{\Lambda_{d,\lambda}(t)/\lambda} \mathrm{P}\{\hat{\tau} \in ds, \hat{\delta} \in dy\}\, \mathrm{P}\Big\{\widehat{R}_{-y,\lambda,d}\big((\Lambda_{d,\lambda}(t)/\lambda) - s\big) \in A\Big\},$$

where $\hat{\tau} = \inf\{s > 0 : \hat{R}_{u,\lambda,P}(s) < 0\}$ is the time of ruin[9] and $\hat{\delta}$ is the corresponding deficit at ruin.

The joined distribution of $\hat{\tau}$ and $\hat{\delta}$ was considered by several authors (see Gerber Shiu (1997)). In the Poisson-Exponential case (see Asmussen (1984)) $\hat{\tau}$ is independent on $\hat{\delta}$; the latter is exponential with parameter μ. It yields

$$\pi_{u,\lambda,P}(t; A, \{\mathrm{ruin}\}) = \int_0^\infty \mathrm{P}\{\hat{\delta} \in dy\} \int_0^{\Lambda_{d,\lambda}(t)/\lambda} \mathrm{P}\{\hat{\tau} \in ds\}\, \mathrm{P}\Big\{\widehat{R}_{-y,\lambda,d}((\Lambda_{d,\lambda}(t)/\lambda) - s) \in A\Big\}$$

$$= \int_0^\infty \mu e^{-\mu y}\, dy \int_0^{\Lambda_{d,\lambda}(t)/\lambda} \mathrm{P}\{\hat{\tau} \in ds\}\, \mathrm{P}\Big\{\widehat{R}_{-y,\lambda,d}((\Lambda_{d,\lambda}(t)/\lambda) - s) \in A\Big\}.$$

The explicit expressions for $\pi_{u,\lambda,P}(t; A, \{\mathrm{ruin}\})$ are yielded therefore by the results like Theorems 3.1 and 3.3.

4. QUANTIFICATION OF ANNUAL CONTROLS

Annual controls are a few types of building blocks used to create an edifice of a competitive insurance business model. Seeking for transparency, we quantify the annul maneuvers and set control options of aggressive, actively defending and neutral companies under the regularity conditions of Theorem 3.3.

These conditions are restrictive and may be weakened considerably at the price of more complicated auxiliary results for probabilities of ruin within finite time (see Asmussen (2000), Malinovskii (1994), Malinovskii (1996b), Malinovskii (1998), Malinovskii (2000)).

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[9] Note that $\{\inf_{0 \leqslant s \leqslant t} R_{u,\lambda,P}(s) < 0\} = \{\inf_{0 < s \leqslant \Lambda_{d,\lambda}(t)/\lambda} \hat{R}_{u,\lambda,P}(s) < 0\} = \{\hat{\tau} \leqslant \Lambda_{d,\lambda}(t)/\lambda\}$.

We start with one more assumption which excludes such conventional control implements as borrowing or lending of capital, or selling or buying of portfolio shares. That simplifying assumption may be easily removed.

ASSUMPTION 4 (Capital and share's continuity). *Let each k-th year's initial capital (portfolio size) be equal to the $(k-1)$-st year-end risk reserve (portfolio size).*

4.1. Controls of three types

Controls may be classified with respect to many different factors: active or passive; aggressive or defensive; applied mainly in the years of soft, or hard market; adaptive, i.e. based on the previous year's observations, or rigid; interactive, or based on the information about the adversary, or non-interactive, and so on. Some examples will follow.

CONTROL 1. Let the initial capital be $u > 0$ and the initial portfolio size be $\lambda > 0$. Call (ε, α)-*control* the price $P \in \mathcal{P}^s (P \in \mathcal{P}^h)$ which guarantees ε-subsistence and α-solvency, i.e. such that[10]

$$\lambda_{d(P)}(t) \geqslant \varepsilon, \quad \psi_{u, \lambda, P}(t) \leqslant \alpha. \tag{25}$$

CONTROL 2. Let the initial capital be $u > 0$ and the initial portfolio size be $\lambda > 0$. Call (λ^T, α)-*control* the price $P \in \mathcal{P}^s (P \in \mathcal{P}^h)$ which makes year-end portfolio size equal to the target value $\lambda^T \gg \varepsilon > 0$ and guarantees α-solvency,

$$\lambda_{d(P)}(t) = \lambda^T, \quad \psi_{u, \lambda, P}(t) \leqslant \alpha. \tag{26}$$

CONTROL 3. Let the initial capital be $u > 0$ and the initial portfolio size be $\lambda > 0$. Call $(u^T, \varepsilon, \alpha)$-*control* the price $P \in \mathcal{P}^s (P \in \mathcal{P}^h)$ which makes year-end average capital equal to the target value $u^T > 0$ and guarantees ε-subsistence and α-solvency,

$$\mathrm{E} R_{u, \lambda, P}(t) = u^T, \quad \lambda_{d(P)}(t) \geqslant \varepsilon, \quad \psi_{u, \lambda, P}(t) \leqslant \alpha. \tag{27}$$

It is readily seen that Controls 2, 3 are active, aimed at the target values λ^T and u^T, while Control 1 is not. The constrains of Control 1 are rather vague. It may be applied (see Tables 2.2-2.4) by a neutral company $\mathfrak{N}$, or by $\mathfrak{A}$ and $\mathfrak{D}$, as they cease active operations.

   Control 2, as $\lambda^T \gg \lambda$, may be applied (see Tables 2.3 and 2.4) by an aggressive company $\mathfrak{A}$ which takes heeds about its α-solvency. That control suits an "offensive and invasive" share gain.

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[10] Convenient is notation with market price P^M and averaged losses $\mathrm{E}Y$ shown explicitly, e.g. $\lambda_{d(P)}(t \mid P^M)$, $\psi_{u, \lambda, P}(t \mid \mathrm{E}Y, P^M)$. In the sequel we will use it freely.
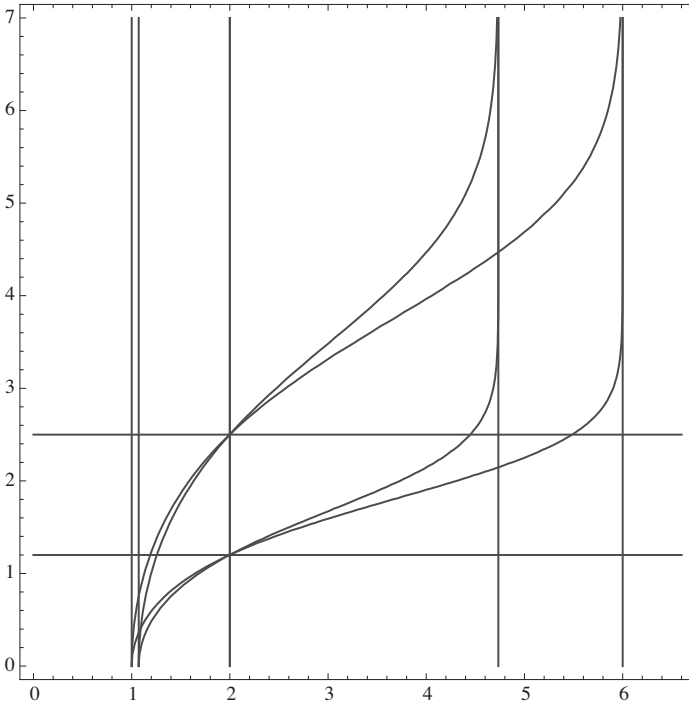
FIGURE 2: Control 1. Shown are functions $P_{\varepsilon,t}(\lambda \mid P_1^M) < P_{\varepsilon,t}(\lambda \mid P_2^M)$ (Y-axis; X-axis is λ) for $P_1^M = 1.2$, $P_2^M = 2.5$, $\varepsilon = 2.0$, $t = 55$ in the case of exponential (Example 4.1) and power (Example 4.2) migration rate functions. The ultimate migration rate is $r_d = (e^{-\rho d l} + \varrho)/(e^{-\rho} + \varrho)$, with $l = 2$, $\rho = -\ln(2/5)$, $\varrho = 1/5$ (i.e., $C = 2$, $c = 1/3$), and $\underline{\lambda}_{\varepsilon,t}^{[\exp]} = 1.0$, $\overline{\lambda}_{\varepsilon,t}^{[\exp]} = 6.0$ and $\underline{\lambda}_{\varepsilon,t}^{[\mathrm{pow}]} = 1.07$, $\overline{\lambda}_{\varepsilon,t}^{[\mathrm{pow}]} = 4.73$.

Control 2, as $0 < \lambda^{\mathrm{T}} \ll \lambda$, or Control 3, as $u^{\mathrm{T}} \gg u$, may be applied (see Tables 2.3 and 2.4) by a defending company $\mathfrak{D}$ to mobilize it for competition. That means to switch from a structure fit to fulfil profitable operations to a structure fit to defense. Such defensive evolution trades off a portion of the company's portfolio for extra capital, and $P \in \mathcal{P}_{\mathrm{CW\text{-}SL}}^h$ in Quarter DH, while $P \in \mathcal{P}_{\mathrm{CW\text{-}SL}}^s$ in Quarter DS.

Controls 2 and 3 are called interactive, as target values λ^{T} and u^{T} depend on the adversary's position. That is indispensable for "close contest" or "struggle of attrition" (see Tables 2.3 and 2.4). Both "close contest" or "struggle of attrition" refer exclusively to the case of soft market.

4.2. Quantification of Control 1

Consider prices which guarantee ε-subsistence as a function of the initial portfolio size λ. Recall from Section 3.1 that for $d(P) = P/P^M > 0$ and for migration rate function $r_d(s)$, $s \geqslant 0$, introduced in Definition 3.1, the market share function is $\lambda_{d(P)}(s) = \lambda r_{d(P)}(s)$, $s \geqslant 0$.

DEFINITION 4.1 (Subsistence price as function of share). For all values of ε, t positive, the solutions $P_{\varepsilon,t}(\lambda)$ of the *subsistence equation*

$$\lambda_{d(P)}(t) = \varepsilon \qquad (28)$$

with respect to $P \in \mathcal{P}^s (P \in \mathcal{P}^h)$, considered as a function of initial share λ, constitute exact upper bound of the set of *subsistence prices*, of level ε.

MONOTONY LEMMA 1. In the assumptions of Theorem 3.3, for all values of ε, t positive, for $\underline{\lambda}_{\varepsilon,t} = \varepsilon r_0^{-1}(t)$, $\bar{\lambda}_{\varepsilon,t} = \varepsilon r_{+\infty}^{-1}(t)$ and for any EY and P^M, the function $P_{\varepsilon,t}(\lambda)$ of the argument $\lambda \in (\underline{\lambda}_{\varepsilon,t}, \bar{\lambda}_{\varepsilon,t})$ is continuous and monotone increasing from 0 to $+\infty$, as λ increases from $\underline{\lambda}_{\varepsilon,t}$ to $\bar{\lambda}_{\varepsilon,t}$.

PROOF OF MONOTONY LEMMA 1. The proof is straightforward from Definitions 2.7, 3.1 and 4.1. □

EXAMPLE 4.1. For $\varepsilon > 0$, for the ultimate migration rate $r_d = (e^{-\rho d l} + \varrho) / (e^{-\rho} + \varrho)$, $d \geqslant 0$, set in Eq. (6), with $0 \leqslant c = r_{+\infty} < 1 < C = r_0 < \infty$, and for the exponential migration rate function $r_d(s) = r_d + (1 - r_d) e^{-s}$, $s \in [0, t]$, set in Eq. (7), the unique solution of the subsistence equation with respect $P \in \mathcal{P}^s$ $(P \in \mathcal{P}^h)$, writes as

$$\lambda\left(r_{d(P)} + \left(1 - r_{d(P)}\right) e^{-t}\right) = \varepsilon,$$

where $0 < \underline{\lambda}_{\varepsilon,t}^{[\exp]} \leqslant \lambda \leqslant \bar{\lambda}_{\varepsilon,t}^{[\exp]}$. In our particular case $\underline{\lambda}_{\varepsilon,t}^{[\exp]} = \varepsilon r_0^{-1}(t)$, $\bar{\lambda}_{\varepsilon,t}^{[\exp]} = \varepsilon r_{+\infty}^{-1}(t)$, where $r_0(t) = r_0 + (1 - r_0) e^{-t} = C + (1 - C) e^{-t}$ and $r_{+\infty}(t) = r_{+\infty} + (1 - r_{+\infty}) e^{-t} = c + (1 - c) e^{-t}$. The solution may be written explicitly, as

$$P_{\varepsilon,t}\left(\lambda \mid P^M\right) = P^M\left(-\ln\left\{\left(1 - (1 - \varepsilon/\lambda)(1 - e^{-t})^{-1}\right)\left(e^{-\rho} + \varrho\right) - \varrho\right\}/\rho\right)^{1/l}.$$

EXAMPLE 4.2. For $\varepsilon > 0$, for the ultimate migration rate $r_d = (e^{-\rho d l} + \varrho) / (e^{-\rho} + \varrho)$, $d \geqslant 0$, set in Eq. (6), with $0 \leqslant c = r_{+\infty} < 1 < C = r_0 < \infty$, and for the power migration rate function $r_d(s) = r_d + (1 - r_d)(1 + s)^{-k}$, $s \in [0, t]$, $k > 0$, set in Eq. (9), the subsistence equation with respect to P writes as

$$\lambda\left(r_{d(P)} + \left(1 - r_{d(P)}\right)(1 + t)^{-k}\right) = \varepsilon,$$

where $0 < \underline{\lambda}_{\varepsilon,t}^{[\mathrm{pow}]} \leqslant \lambda \leqslant \bar{\lambda}_{\varepsilon,t}^{[\mathrm{pow}]}$. In our particular case $\underline{\lambda}_{\varepsilon,t}^{[\mathrm{pow}]} = \varepsilon r_0^{-1}(t)$, $\bar{\lambda}_{\varepsilon,t}^{[\mathrm{pow}]} = \varepsilon r_{+\infty}^{-1}(t)$, where $r_0(t) = r_0 + (1 - r_0)(1 + t)^{-k} = C + (1 - C)(1 + t)^{-k}$ and $r_{+\infty}(t) = r_{+\infty} + (1 - r_{+\infty})(1 + t)^{-k} = c + (1 - c)(1 + t)^{-k}$.
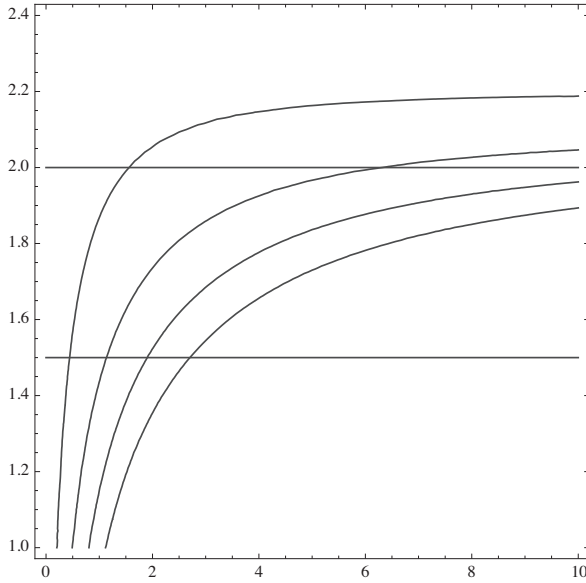
FIGURE 3: Control 1 in the year of soft market: $EY = 2$, $P^M = 1.5$. Shown are functions $P_{\alpha,t,u}(\lambda \mid EY, P^M)$ (Y-axis; X-axis is λ) for $\alpha = 0.1$, $t = 100$. From top downward: $u = 50, 100, 150, 200$. The function $\Lambda_{d,\lambda}(t)$ is defined as in (10), with power $k = 1/2$ and r_d as in (6), where $l = 2$, $\rho = -\ln(2/5)$, $\varrho = 1/5$ (i.e., $C = 2$, $c = 1/3$).
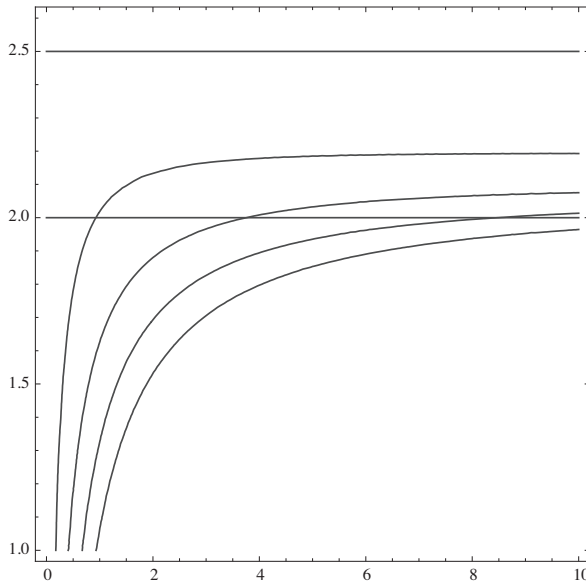

FIGURE 4: Control 1 in the year of hard market: $EY = 2$, $P^M = 2.5$. Shown are functions $P_{\alpha,t,u}(\lambda \mid EY, P^M)$ (Y-axis; X-axis is λ) for $\alpha = 0.1$, $t = 100$. From top downward: $u = 50, 100, 150, 200$. The function $\Lambda_{d,\lambda}(t)$ is defined as in (10), with power $k = 1/2$ and r_d as in (6), where $l = 2$, $\rho = -\ln(2/5)$, $\varrho = 1/5$ (i.e., $C = 2$, $c = 1/3$).

Investigate prices which guarantee α-solvency. Denote by $P_{\alpha,u}(+\infty)$ a solution of the equation $\exp\{u/P - u/\mathrm{E}Y\} = \alpha P/\mathrm{E}Y$ with respect to P. Mention it that $P_{\alpha,u}(+\infty)$ is positive and finite.

DEFINITION 4.2. (Solvency price as function of share). For all values of $\alpha \in (0,1)$ and t, u positive, the solutions $P_{\alpha,t,u}(\lambda)$ of the *solvency equation*

$$\psi_{u,\lambda,P}(t) = \alpha \qquad (29)$$

with respect to $P \in \mathcal{P}^s (P \in \mathcal{P}^h)$, considered as a function of initial share λ, constitute exact lower bound of the set of *solvency prices*, of level α.

Verify it that a solution of the solvency equation actually exists and is unique for each λ; analyze the solutions as a function of λ. To do so, apply the explicit formula for $\psi_{u,\lambda,P}(t)$ yielded by Theorem 3.3.

MONOTONY LEMMA 2. In the assumptions of Theorem 3.3, for all values of $\alpha \in (0,1)$ and t, u positive, and for any $\mathrm{E}Y$ and P^M, the function $P_{\alpha,t,u}(\lambda)$ of the argument $\lambda > 0$ is continuous, convex and monotone increasing from zero to $0 < P_{\alpha,u}(+\infty) < +\infty$, as λ increases from zero to $+\infty$.

PROOF OF MONOTONY LEMMA 2. Continuity, convexity and monotony follow from implicit function differentiation arguments analogous to those in the proof of Theorem 3.2 in Malinovskii (2008a), applied to the solvency equation (29) with the left hand side explicit from Eq. (19); that is a straightforward analogue of the proofs in Malinovskii (2008a) and Malinovskii (2008b) and requires merely some elementary calculus.

Exact upper bound $P_{\alpha,u}(+\infty)$ evidently is a solution of the equation $\lim_{\lambda \to +\infty} \psi_{u,\lambda,P}(t) = \alpha$ with respect to P. Since $\lim_{\lambda \to +\infty} \psi_{u,\lambda,P}(t) = \psi_{u,+\infty,P}(t) = \psi_{u,\lambda,P}(+\infty) = \psi_{u,+\infty,P}(+\infty)$ is given by Eq. (21), it rewrites as

$$g^{-1}(P)\exp\left\{-u\mu\left(1 - g^{-1}(P)\right)\right\} = \alpha.$$

It is noteworthy that $P_{\alpha,t,u}(+\infty) = P_{\alpha,+\infty,u}(+\infty) > \mathrm{E}Y$ for all values of $\alpha \in (0,1)$ and u positive. $\qquad\square$

In the case of power migration rate function, when $\Lambda_{d,\lambda}(t)$ is defined as in (10), the functions $P_{\varepsilon,t}(\lambda|P^M)$ and $P_{\alpha,t,u}(\lambda|\mathrm{E}Y, P^M)$ calculated numerically are shown on Fig. 5 and 6.

Quantifying aggressive actions and defensive reactions of interacting companies and using annual (corresponding to particular $\mathrm{E}Y$ and P^M) Control 1, one may use Fig. 5 and 6 to select admissible prices for diverse initial portfolio sizes λ. Those are "moves legally possible". The set of admissible prices lies between $P_{\varepsilon,t}(\lambda|P^M)$ and $P_{\alpha,t,u}(\lambda|\mathrm{E}Y, P^M)$; its shape for diverse λ is "a curved bird's beck" with the edge being the intersection point. The ordinate of the
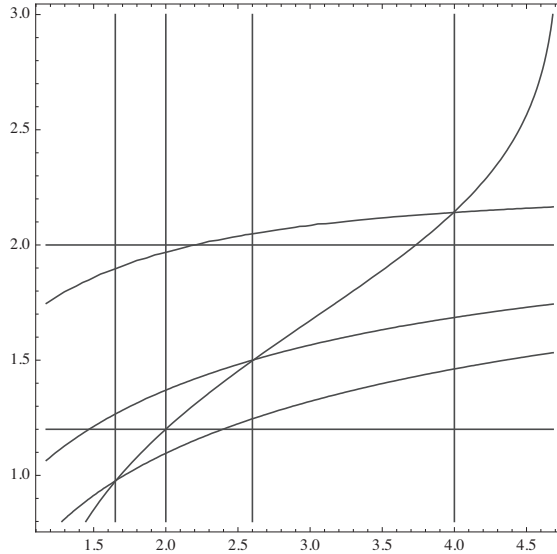
FIGURE 5: Control 1 in the year of soft market: $EY = 2$, $P^M = 1.2$. Shown (Y-axis; X-axis is λ)
are $P_{\varepsilon,t}(\lambda \mid P^M)$, $\lambda \in [\underline{\lambda}_{\varepsilon,t}, \overline{\lambda}_{\varepsilon,t}]$, with $\underline{\lambda}_{\varepsilon,t} = \varepsilon r_0^{-1}(t) = 1.07$, , $\overline{\lambda}_{\varepsilon,t} = \varepsilon r_{+\infty}^{-1}(t) = 4.73$, and $P_{\alpha,t,u}(\lambda \mid EY, P^M)$ for
$u = 40$, 100, 150 (from top downward), with $\alpha = 0.1$, $\varepsilon = 2$, $t = 55$. Vertical lines: $\lambda_{\alpha,\varepsilon,t}(150) = 1.65$,
$\lambda = \varepsilon = 2.0$, $\lambda_{\alpha,\varepsilon,t}(100) = 2.6$, $\lambda_{\alpha,\varepsilon,t}(40) = 4.0$. The function $\Lambda_{d,\lambda}(t)$ is defined as in (10), with power $k = 1/2$
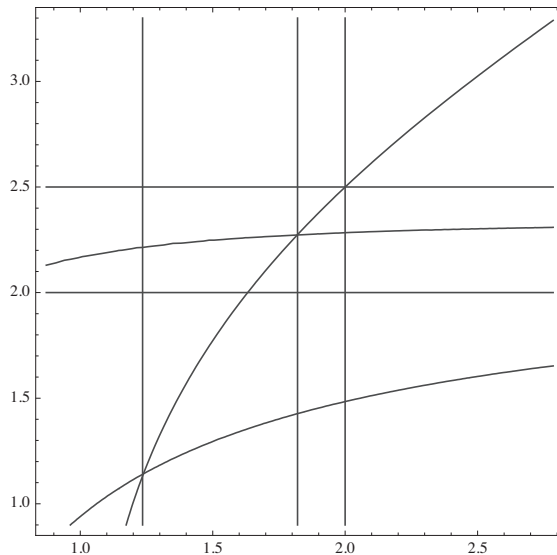and r_d as in (6), with $l = 2$, $\rho = -\ln(2/5)$, $\varrho = 1/5$ (i.e., $C = 2$, $c = 1/3$).

FIGURE 6: Control 1 in the year of hard market: $EY = 2$, $P^M = 2.5$. Shown (Y-axis; X-axis is λ)
are $P_{\varepsilon,t}(\lambda \mid P^M)$, $\lambda \in [\underline{\lambda}_{\varepsilon,t}, \overline{\lambda}_{\varepsilon,t}]$, with $\underline{\lambda}_{\varepsilon,t} = \varepsilon r_0^{-1}(t) = 1.07$, $\overline{\lambda}_{\varepsilon,t} = \varepsilon r_{+\infty}^{-1}(t) = 4.73$, and $P_{\alpha,t,u}(\lambda \mid EY, P^M)$
for $u = 30$, 130 (from top downward), with $\alpha = 0.1$, $\varepsilon = 2$, $t = 55$. Vertical lines: $\lambda_{\alpha,\varepsilon,t}(130) = 1.24$,
$\lambda_{\alpha,\varepsilon,t}(30) = 1.82$, $\lambda = \varepsilon = 2.0$. The function $\Lambda_{d,\lambda}(t)$ is defined as in (10), with power $k = 1/2$ and r_d as in (6),
with $l = 2$, $\rho = -\ln(2/5)$, $\varrho = 1/5$ (i.e., $C = 2$, $c = 1/3$).

beck's edge point yields the lowest price among all admissible prices. For λ less than the abscissa of this point, admissible choices of Control 1 there are none.

4.3. Quantification of Control 2

Quantification of Control 2 is similar to quantification of Control 1 up to replacement of subsistence equation (28) by the *targeting equation*

$$\lambda_{d(P)}(t) = \lambda^{\mathrm{T}}, \tag{30}$$

where λ^{T} is set larger than the least allowed level ε. Denote by $P_{\lambda^{\mathrm{T}},t}(\lambda)$ the solution of (30) with respect to $P \in \mathcal{P}^s (P \in \mathcal{P}^h)$, considered as a function of initial share λ.

MONOTONY LEMMA 3. In the assumptions of Theorem 3.3, for all values of λ^{T}, t positive, for $\underline{\lambda}_{\lambda^{\mathrm{T}},t} = \lambda^{\mathrm{T}} r_0^{-1}(t)$, $\overline{\lambda}_{\lambda^{\mathrm{T}},t} = \lambda^{\mathrm{T}} r_{+\infty}^{-1}(t)$ and for any EY and P^M, the function $P_{\lambda^{\mathrm{T}},t}(\lambda)$ of the argument $\lambda \in (\underline{\lambda}_{\lambda^{\mathrm{T}},t}, \overline{\lambda}_{\lambda^{\mathrm{T}},t})$ is continuous and monotone increasing from 0 to $+\infty$, as λ increases from $\underline{\lambda}_{\lambda^{\mathrm{T}},t}$ to $\overline{\lambda}_{\lambda^{\mathrm{T}},t}$.

PROOF OF MONOTONY LEMMA 3. The same as the proof of Monotony Lemma 1. □

EXAMPLE 4.3. For $d(P) = P/P^M > 0$, by Eq. (9) and (6), one has

$$r_{d(P)}(t) = 1 - \left(1 - r_{d(P)}\right)\left(1 - (1 + t)^{-k}\right),$$

where $r_{d(P)} = (e^{-\rho d(P)^l} + \varrho)/(e^{-\rho} + \varrho)$. Eq. (30) rewrites as $r_{d(P)} = 1 - (1 - \lambda^{\mathrm{T}}/\lambda)/(1 - (1 + t)^{-k})$, and the explicit solution of (30) is

$$P_{\lambda^{\mathrm{T}},t}(\lambda \mid P^M) =$$
$$P^M \left(-(1/\rho) \ln\left(\left(1 - (1 - \lambda^{\mathrm{T}}/\lambda) / \left(1 - (1 + t)^{-k}\right)\right)\left(e^{-\rho} + \varrho\right) - \varrho\right)\right)^{1/l}.$$

In the case of power migration rate function, when $\Lambda_{d,\lambda}(t)$ is defined as in (10), the functions $P_{\lambda_i^{\mathrm{T}},t}(\lambda \mid P^M)$ and $P_{\alpha,t,u}(\lambda \mid EY, P^M)$ calculated numerically are shown on Fig. 7 and 8.

Quantifying aggressive actions and defensive reactions of interacting companies and using annual (corresponding to particular EY and P^M) Control 2, one may use Fig. 7 and 8 to select admissible prices for diverse initial portfolio sizes λ. For λ less than the abscissa of the point of intersection of $P_{\lambda_i^{\mathrm{T}},t}(\lambda \mid P^M)$ and $P_{\alpha,t,u}(\lambda \mid EY, P^M)$, admissible choices of Control 2 there are none.
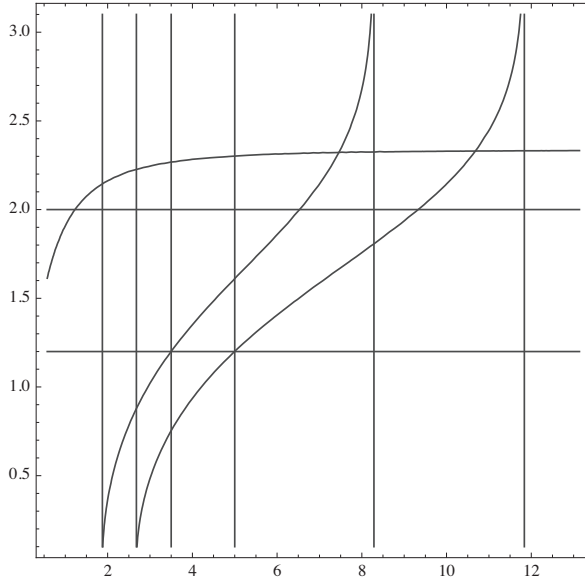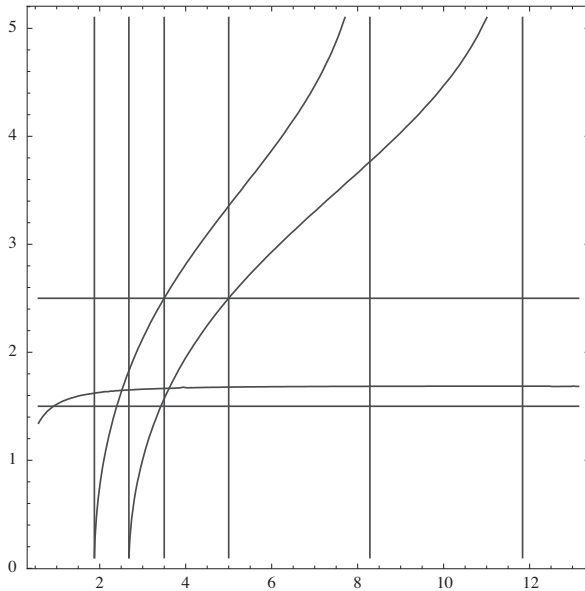
FIGURE 7: Control 2 in the year of soft market: $P^M = 1.2$, $EY = 2.0$. Shown (Y-axis; X-axis is λ) are $P_{\lambda_i^T, t}(\lambda | P^M)$, $i = 1, 2$, for $3.5 = \lambda_1^T < \lambda_2^T = 5.0$, and $P_{\alpha, t, u}(\lambda | EY, P^M)$ for $\alpha = 0.1$, $t = 55$, $u = 30$. The function $\Lambda_{d, \lambda}(t)$ is defined as in (10), with power $k = 1/2$ and r_d as in (6), with $l = 2$, $\rho = -\ln(2/5)$, $\varrho = 1/5$ (i.e., $C = 2$, $c = 1/3$).

FIGURE 8: Control 2 in the year of hard market: $P^M = 2.5$, $EY = 1.5$. Shown (Y-axis; X-axis is λ) are $P_{\lambda_i^T, t}(\lambda | P^M)$, $i = 1, 2$, for $3.5 = \lambda_1^T < \lambda_2^T = 5.0$, and $P_{\alpha, t, u}(\lambda | EY, P^M)$ for $\alpha = 0.1$, $t = 55$, $u = 30$. The function $\Lambda_{d, \lambda}(t)$ is defined as in (10), with power $k = 1/2$ and r_d as in (6), with $l = 2$, $\rho = -\ln(2/5)$, $\varrho = 1/5$ (i.e., $C = 2$, $c = 1/3$).

4.4. Quantification of Control 3

Quantification of Control 3 goes along the lines of quantification of Control 1. The additional requirement is the *targeting equation*

$$\mathrm{E}\,R_{u,\lambda,P}(t) = u^{\mathrm{T}}, \tag{31}$$

with respect to $P \in \mathcal{P}^s (P \in \mathcal{P}^h)$, where u^{T} is the target average capital. By Eq. (16), the targeting equation (31) rewrites as

$$u + \mathrm{E}\,Y(g(P) - 1)\,\Lambda_{d(P),\lambda}(t) = u^{\mathrm{T}}. \tag{32}$$

DEFINITION 4.3 (Capital targeting price as function of share). For all values of t, u positive, the solutions $P_{u \to u^{\mathrm{T}}, t}(\lambda)$ of the *targeting equation* $\mathrm{E}\,R_{u,\lambda,P}(t) = u^{\mathrm{T}}$ with respect to $P \in \mathcal{P}^s (P \in \mathcal{P}^h)$, constitute a function of initial share λ, called price *targeting average capital* at point u^{T}.

MONOTONY LEMMA 4. In the assumptions of Theorem 3.3, for all values of u^{T}, t, u positive, and for any $\mathrm{E}Y$ and P^M, the function $P_{u \to u^{\mathrm{T}}, t}(\lambda)$ of the argument $\lambda > 0$ is a constant equal to $\mathrm{E}Y$, as $u = u^{\mathrm{T}}$, is continuous and monotone increasing to $\mathrm{E}Y$ from below, as $u > u^{\mathrm{T}}$, and is continuous and monotone decreasing from above to $\mathrm{E}Y$, as $u < u^{\mathrm{T}}$.

PROOF OF MONOTONY LEMMA 4. The assertion in the case $u = u^{\mathrm{T}}$ is evident from (32). The monotony, as $u > u^{\mathrm{T}}$ or $u < u^{\mathrm{T}}$, is straightforward from implicit function differentiation arguments analogous to those in the proof of Theorem 3.2 in Malinovskii (2008a), and requires merely some elementary calculus.   □

In the case of power migration rate function, when $\Lambda_{d,\lambda}(t)$ is defined as in (10), the functions $P_{u \to u^{\mathrm{T}}, t}(\lambda \,|\, \mathrm{E}Y, P^M)$ calculated numerically are shown on Fig. 9 and 10. Quantifying aggressive actions and defensive reactions of interacting companies and using annual (corresponding to particular $\mathrm{E}Y$ and P^M) Control 3, one must make allowance for the admissible regions shown on Fig. 5 and 6.

5. STRATEGIC MODELING FOR A NEUTRAL COMPANY

For a neutral company $\mathfrak{N}$, starting at the top of a prospering phase and eager to survive until Quarter UH (see Table 2.2), the competitor is the market as a whole[11]. Consider a cycle directed by the market prices

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[11] Or the sequence (33) of the market prices, at which the company can not influence. In that sense it is referred to as *neutral*.
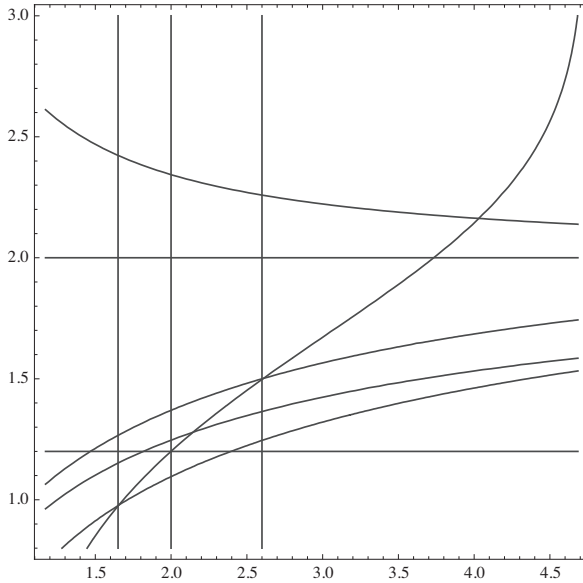
FIGURE 9: Control 3 in the year of soft market: $EY = 2$, $P^M = 1.2$. Additionally (cf. Fig. 5) shown are (Y-axis; X-axis is λ) functions $P_{u \to u^T, t}(\lambda | EY, P^M)$ with $u = 150$ and $u^T = 70$ (increasing curve) and $u = 100$ and $u^T = 120$ (decreasing curve).
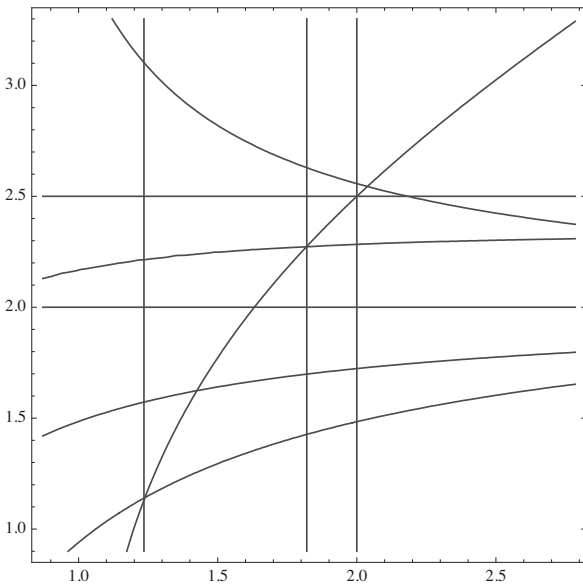

FIGURE 10: Control 3 in the year of hard market: $EY = 2$, $P^M = 2.5$. Additionally (cf. Fig. 6) shown are (Y-axis; X-axis is λ) functions $P_{u \to u^T, t}(\lambda | EY, P^M)$ with $u = 130$ and $u^T = 90$ (increasing curve) and $u = 30$ and $u^T = 90$ (decreasing curve).

$$P_1^M > P_2^M > \cdots > P_{\vartheta_{DH}}^M > \mathrm{E}\,Y > P_{\vartheta_{DH}+1}^M > P_{\vartheta_{DH}+2}^M > \cdots > P_{\vartheta_{DH}+\vartheta_{DS}}^M$$

$$< P_{\vartheta_{DH}+\vartheta_{DS}+1}^M < P_{\vartheta_{DH}+\vartheta_{DS}+2}^M < \cdots < P_{\vartheta_{DH}+\vartheta_{DS}+\vartheta_{US}}^M < \mathrm{E}\,Y. \qquad (33)$$

Assume that the main strategic goal of $\mathfrak{N}$ is "just-survival" (see Section 2.2). More formally, that means "to keep the portfolio size above $\varepsilon > 0$, while the annual probability of ruin is kept below $\alpha > 0$".

5.1. A multi-period controlled risk model

Model for $\mathfrak{N}$ the insurance process matching diagram (1). That requires (see Malinovskii (2007) – Malinovskii (2009a)) a definition over the elementary state space $(\Omega, \mathscr{F})$ of a controlled random sequence $(\mathbf{W}_k, \mathbf{U}_k)$, $k = 0, 1, 2, \ldots$, assuming values from a state space W and a control space U. That is (see Gihman and Skorokhod (1979)) a standard procedure based on the annual mechanisms of insurance π_k, $k = 1, 2, \ldots$, and annual controls γ_k, $k = 0, 1, 2, \ldots$.

To be particular, introduce the state vectors

$$\mathsf{w}_k^{\langle \mathfrak{N} \rangle} = \left(\mathsf{w}_k^{\langle 1, \mathfrak{N} \rangle}, \mathsf{w}_k^{\langle 2, \mathfrak{N} \rangle}, \mathsf{w}_k^{\langle 3, \mathfrak{N} \rangle} \right) \in \mathbf{W} = R \times R^+ \times \{0, 1\}, \quad k = 0, 1, 2, \ldots,$$

whose components are: k-th *year-end capital* of $\mathfrak{N}$, k-th *year-end portfolio size* of $\mathfrak{N}$ and *indicator* whether $\mathfrak{N}$ suffered a ruin within k-th year, or not[12]. The components of the control vectors

$$\mathsf{u}_{k-1}^{\langle \mathfrak{N} \rangle} = \left(\mathsf{u}_{k-1}^{\langle 1, \mathfrak{N} \rangle}, \mathsf{u}_{k-1}^{\langle 2, \mathfrak{N} \rangle}, \mathsf{u}_{k-1}^{\langle 3, \mathfrak{N} \rangle} \right) \in \mathbf{U} = \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^+, \quad k = 1, 2, \ldots,$$

are: k-th year *starting capital* of $\mathfrak{N}$, k-th year *initial portfolio size* of $\mathfrak{N}$ and k-th year *premium intensity* of $\mathfrak{N}$.

For $k = 1, 2, \ldots$, the annual Markov controls are yielded by

$$\mathsf{u}_{k-1}^{\langle \mathfrak{N} \rangle} = \gamma_{k-1} \left(\mathsf{w}_{k-1}^{\langle \mathfrak{N} \rangle} \right). \qquad (34)$$

The corresponding annual transition function of the probability mechanism of insurance (see Section 3.5; bear in mind deterministic migration assumed in this paper and independence of the annual interclaim times and claim amounts which is a common *risk scenario* assumption) is

$$\pi_k \left(\mathsf{w}_{k-1}^{\langle \mathfrak{N} \rangle}, \mathsf{u}_{k-1}^{\langle \mathfrak{N} \rangle}; d\mathsf{w}_k^{\langle \mathfrak{N} \rangle} \right)$$
$$= \mathrm{P} \left\{ R_{\mathsf{u}_{k-1}}^{\mathfrak{N}}(t) \in d\mathsf{w}_k^{\langle 1, \mathfrak{N} \rangle}, \lambda_{\mathsf{u}_{k-1}}^{\mathfrak{N}}(t) \in d\mathsf{w}_k^{\langle 2, \mathfrak{N} \rangle}, \mathbf{1}_{\{M_{\mathsf{u}_{k-1}}^{\mathfrak{N}}(t) < 0\}} \in d\mathsf{w}_k^{\langle 3, \mathfrak{N} \rangle} \right\},$$

---

[12] Let $\mathsf{w}_k^{\langle 3, \mathfrak{N} \rangle} = 1$ means ruin within k-th year, $\mathsf{w}_k^{\langle 3, \mathfrak{N} \rangle} \neq 1$ means survival.

where $M_{\mathsf{u}_{k-1}}^{\mathfrak{N}}(t) = \inf_{0 < s \leqslant t} R_{\mathsf{u}_{k-1}}^{\mathfrak{N}}(s)$.

The controlled random sequence $(\mathbf{W}_k, \mathbf{U}_k)$, $k = 0, 1, 2, \ldots$, with the Markov control (34) is equivalent to the Markov chain with the transition probabilities

$$P\left(\mathsf{w}_{k-1}^{\langle \mathfrak{N} \rangle}; d\mathsf{w}_k^{\langle \mathfrak{N} \rangle}\right) = \pi_k\left(\mathsf{w}_{k-1}^{\langle \mathfrak{N} \rangle}, \gamma_{k-1}\left(\mathsf{w}_{k-1}^{\langle \mathfrak{N} \rangle}\right); d\mathsf{w}_k^{\langle \mathfrak{N} \rangle}\right), \quad k = 1, 2, \ldots,$$

on the state space $(\mathsf{W}, \mathcal{W})$.

Write $\mathbf{P}^{\pi, \gamma}$ for the probabilities on the elementary state space $(\Omega, \mathcal{F})$ corresponding to the Markov chains with the initial state $\mathsf{w}_0^{\langle \mathfrak{N} \rangle}$ and the transition probability P, and denote by $\mathbf{E}^{\pi, \gamma}$ the expectation with respect to that measure.

5.2. An example of a "just-survival" adaptive strategy

Introduce the following scenario for a company $\mathfrak{N}$ whose strategic goal is "just-survival".

SCENARIO 1. The evolution of the market prices agrees with Assumption 3. Though insurer $\mathfrak{N}$ can not predict competition intensity over the cycle, i.e. neither the whole sequence (33), nor the values ϑ_{DH}, ϑ_{DS}, ϑ_{US} are known to him in advance. But watching over $\mathfrak{A}$, insurer $\mathfrak{N}$ can infer correctly about each next-year market price.

For $k = 1, 2, \ldots$, and for the initial state vector with components

$$\mathsf{w}_0^{\langle 1, \mathfrak{N} \rangle} > 0, \ \mathsf{w}_0^{\langle 2, \mathfrak{N} \rangle} > 0, \ \mathsf{w}_0^{\langle 3, \mathfrak{N} \rangle} = 0,$$

let the Markov strategy γ for $\mathfrak{N}$ be composed of the following annual controls. If $\mathsf{w}_{k-1}^{\langle 3, \mathfrak{N} \rangle} \neq 1$, one has[13]

$$\begin{aligned}
\mathsf{u}_{k-1}^{\langle 1, \mathfrak{N} \rangle} &= \mathsf{w}_{k-1}^{\langle 1, \mathfrak{N} \rangle}, \\
\mathsf{u}_{k-1}^{\langle 2, \mathfrak{N} \rangle} &= \mathsf{w}_{k-1}^{\langle 2, \mathfrak{N} \rangle}, \\
\mathsf{u}_{k-1}^{\langle 3, \mathfrak{N} \rangle} &\in \left(P_{\varepsilon, t}\left(\mathsf{w}_{k-1}^{\langle 2, \mathfrak{N} \rangle} \,\middle|\, P_k^M\right), P_{\alpha, t, \mathsf{w}_{k-1}^{\langle 1, \mathfrak{N} \rangle}}\left(\mathsf{w}_{k-1}^{\langle 2, \mathfrak{N} \rangle} \,\middle|\, \mathrm{E}\,Y, P_k^M\right) \right),
\end{aligned} \quad (35)$$

provided (call that k-th year *non-voidness* condition)

$$P_{\varepsilon, t}\left(\mathsf{w}_{k-1}^{\langle 2, \mathfrak{N} \rangle} \,\middle|\, P_k^M\right) \geqslant P_{\alpha, t, \mathsf{w}_{k-1}^{\langle 1, \mathfrak{N} \rangle}}\left(\mathsf{w}_{k-1}^{\langle 2, \mathfrak{N} \rangle} \,\middle|\, \mathrm{E}\,Y, P_k^M\right). \quad (36)$$

---

[13] The market price P_k^M is known by Scenario 1.

It is noteworthy that, by Assumption 3, $P_k^M = P_{k-1}^{\mathfrak{A}}$, $k = 2, 3, \ldots$, until $\mathfrak{A}$ remains aggressive. In that sense adaptive control (35) is *interactive*.

Both solvency and subsistence properties of the strategy γ composed of the annual controls (35) are straightforward from the definition.

THEOREM 5.1. *In the framework of the multi-period controlled risk model for $\mathfrak{R}$, for the strategy γ defined in Eq. (35),*

$$\sup_{\mathsf{w}_0^{\langle\mathfrak{R}\rangle} \in R^+ \times R^+ \times \{0\}} \mathbf{P}^{\pi,\gamma}\left\{ \left\{\begin{array}{c}\text{first ruin occurs} \\ \text{in } k\text{-th year}\end{array}\right\} \cup \left\{\begin{array}{c}\text{first fall of portfolio size} \\ \text{below } \varepsilon \text{ occurs in } k\text{-th year}\end{array}\right\} \right\} \leqslant \alpha$$

for $k = 1, 2, \ldots$.

COROLLARY 5.1. *In the framework of Theorem 5.1, for $n = 1, 2, \ldots$*

$$\sup_{\mathsf{w}_0^{\langle\mathfrak{R}\rangle} \in R^+ \times R^+ \times \{0\}} \mathbf{P}^{\pi,\gamma}\left\{ \left\{\begin{array}{c}\text{ruin within } n \\ \text{years occur}\end{array}\right\} \cup \left\{\begin{array}{c}\text{portfolio size at the end} \\ \text{of } n\text{-th year is below } \varepsilon\end{array}\right\} \right\}$$

$$\leqslant \sum_{k=1}^{n} \sup_{\mathsf{w}_0^{\langle\mathfrak{R}\rangle} \in R^+ \times R^+ \times \{0\}} \mathbf{P}^{\pi,\gamma}\left\{ \left\{\begin{array}{c}\text{first ruin occurs} \\ \text{in } k\text{-th year}\end{array}\right\} \cup \left\{\begin{array}{c}\text{first fall of portfolio size} \\ \text{below } \varepsilon \text{ occurs in } k\text{-th year}\end{array}\right\} \right\}$$

$$\leqslant n\alpha.$$

Further quantification of the strategy γ composed of the annual controls (35), including verification of non-voidness condition (36), is a straightforward application of results of Section 4.2.

5.3. Optimized adaptive strategy

Optimize the strategy γ composed of annual controls (35) to enable survival for $\mathfrak{R}$ during as lengthy soft market, as possible.

The optimal behavior of $\mathfrak{R}$ is to earn as much capital as possible in the prospering phase, to reduce the portfolio size to the least allowed size $\varepsilon > 0$ at the edge of the soft market phase, and to keep it minimal until the beginning of the next prospering phase. The market share traded off for capital timely means reduce of the losses and secure of the capital needed for a longest possible — whichever may be the length of the soft market phase — struggle for survival.

So, in Eq. (35) advisable is to choose

$$\mathsf{u}_{k-1}^{\langle 3, \mathfrak{R}\rangle} \in \left(P_{\varepsilon, t}\left(\mathsf{w}_{k-1}^{\langle 2, \mathfrak{R}\rangle} \,\middle|\, P_k^M\right),\ P_{\alpha, t, \mathsf{w}_{k-1}^{\langle 1, \mathfrak{R}\rangle}}\left(\mathsf{w}_{k-1}^{\langle 2, \mathfrak{R}\rangle} \,\middle|\, \mathbf{E}Y, P_k^M\right) \right) \cap \mathcal{P}_{\text{CW-SW}}^{h},$$

$$\text{as} \quad k = 1, 2, \ldots, \vartheta_{DH} - 1,$$

$$u^{\langle 3, \mathfrak{N}\rangle}_{\vartheta_{DH}-1} = P_{\varepsilon, t}\left(w^{\langle 2, \mathfrak{N}\rangle}_{\vartheta_{DH}-1} \,\Big|\, P^M_{\vartheta_{DH}}\right),$$

$$\text{provided } P_{\varepsilon, t}\left(w^{\langle 2, \mathfrak{N}\rangle}_{\vartheta_{DH}-1} \,\Big|\, P^M_{\vartheta_{DH}}\right) \geqslant P_{\alpha, t, w^{\langle 1, \mathfrak{N}\rangle}_{\vartheta_{DH}-1}}\left(w^{\langle 2, \mathfrak{N}\rangle}_{\vartheta_{DH}-1} \,\Big|\, EY, P^M_{\vartheta_{DH}}\right),$$

$$u^{\langle 3, \mathfrak{N}\rangle}_{k-1} = P_{\varepsilon, t}\left(w^{\langle 2, \mathfrak{N}\rangle}_{k-1} \,\Big|\, P^M_k\right) = P_{\varepsilon, t}\left(\varepsilon \,\Big|\, P^M_k\right) = P^M_k,$$

$$\text{provided } P^M_k = P_{\varepsilon, t}\left(\varepsilon \,\Big|\, P^M_k\right) \geqslant P_{\alpha, t, w^{\langle 1, \mathfrak{N}\rangle}_{k-1}}\left(\varepsilon \,\Big|\, EY, P^M_k\right),$$

$$\text{as } k = \vartheta_{DH} + 1, ..., \vartheta_{DH} + \vartheta_{DS} + \vartheta_{US}.$$

Quantification of the optimized strategy γ, including verification of non-voidness conditions, is a straightforward application of results of Section 4.3 or 4.4.

REMARK 5.1. If additional *smoothness* constraints descending from practical requirements are applied, defensive evolution may be lasting for several years. If the soft market phase is expected soon, advantageous may be to start reducing the portfolio size in good time, in a year $k < \vartheta_{DH}$ sufficiently close to ϑ_{DH}, but such that $P^M_k / EY > 1$ is still large.

6. STRATEGIC MODELING FOR COMPETITIVE COMPANIES

Assume that at the height of a prospering phase, which remains the starting point for our considerations, there are two dominating companies on the market: one aggressive $\mathfrak{A}$, another defending $\mathfrak{D}$. The former means that $\mathfrak{A}$ initiated the competition-originated cycle by applying prices

$$P^M_1 > P^{\mathfrak{A}}_1 = P^M_2 > P^{\mathfrak{A}}_2 = P^M_3 > \cdots > EY,$$

which triggered the concerted industry's response (see Section 2.3). The latter means that the assaulted company $\mathfrak{D}$ is able to actively compete, and is able to initiate its own concerted industry's response.

Assume first (see Section 2.2) that the strategic goal of aggressive $\mathfrak{A}$ is just "to win a share", or "to win a share and not to be ruined whichever the response of $\mathfrak{D}$ may be", while the goal of $\mathfrak{D}$ is "not to be ruined and to ruin the competitor $\mathfrak{A}$". Ruin of $\mathfrak{A}$ would re-gain for $\mathfrak{D}$ the formerly surrendered market share, and the goal "not to lose a share" is an offshoot of its main strategic goal.

6.1. A multi-period interactive controlled risk model

The multi-period controlled interactive risk model for $\mathfrak{A}$ and $\mathfrak{D}$ over the elementary state space $(\Omega, \mathcal{F})$ with evolution matching the diagram (2), is based on the annual mechanisms of insurance $\pi_k = (\pi^{\mathfrak{A}}_k, \pi^{\mathfrak{D}}_k)$, $k = 1, 2, \ldots$, and annual

controls $\gamma_k = (\gamma_k^{\mathfrak{A}}, \gamma_k^{\mathfrak{D}})$, $k = 0, 1, 2, \ldots$. It is known (see Malinovskii (2007) – Malinovskii (2009a)) that a corresponding controlled random sequence $(\mathbf{W}_k, \mathbf{U}_k)$, $k = 0, 1, 2, \ldots$, assuming values from a state space W and a control space U, exists.

Set $\mathrm{W} = R \times R^+ \times \{0,1\} \times R \times R^+ \times \{0,1\}$ and $\mathrm{U} = R^+ \times R^+ \times R^+ \times R^+ \times R^+ \times R^+$. Introduce the state vectors

$$\mathsf{w}_k = \left(\mathsf{w}_k^{\langle 1, \mathfrak{A} \rangle}, \mathsf{w}_k^{\langle 2, \mathfrak{A} \rangle}, \mathsf{w}_k^{\langle 3, \mathfrak{A} \rangle}, \mathsf{w}_k^{\langle 1, \mathfrak{D} \rangle}, \mathsf{w}_k^{\langle 2, \mathfrak{D} \rangle}, \mathsf{w}_k^{\langle 3, \mathfrak{D} \rangle} \right) \in \mathrm{W}, \quad k = 1, 2, \ldots,$$

whose components are: k-th *year-end capital*, k-th *year-end portfolio size* and *indicator* whether ruin has occurred in the k-th year, or not, for $\mathfrak{A}$ and $\mathfrak{D}$ respectively. The components of the control vectors

$$\mathsf{u}_{k-1} = \left(\mathsf{u}_{k-1}^{\langle 1, \mathfrak{A} \rangle}, \mathsf{u}_{k-1}^{\langle 2, \mathfrak{A} \rangle}, \mathsf{u}_{k-1}^{\langle 3, \mathfrak{A} \rangle}, \mathsf{u}_{k-1}^{\langle 1, \mathfrak{D} \rangle}, \mathsf{u}_{k-1}^{\langle 2, \mathfrak{D} \rangle}, \mathsf{u}_{k-1}^{\langle 3, \mathfrak{D} \rangle} \right) \in \mathrm{U}, \quad k = 0, 1, 2, \ldots,$$

are: k-th year *starting capital*, k-th year *initial portfolio size* and k-th year *premium intensity*, for $\mathfrak{A}$ and $\mathfrak{D}$ respectively.

For $k = 1, 2, \ldots$, consider the annual Markov controls

$$\mathsf{u}_{k-1} = \gamma_{k-1}(\mathsf{w}_{k-1}). \tag{37}$$

The corresponding annual transition function of the probability mechanism of insurance (see Section 3.5; bear in mind independence of the annual interclaim times and claim amounts which is a common *risk scenario* assumption) is

$$\pi_k\left(\mathsf{w}_{k-1}, \mathsf{u}_{k-1}; d\mathsf{w}_k\right) = \mathbf{P}\Big\{ R_{\mathsf{u}_{k-1}}^{\mathfrak{A}}(t) \in d\mathsf{w}_k^{\langle 1, \mathfrak{A} \rangle}, \lambda_{\mathsf{u}_{k-1}}^{\mathfrak{A}}(t) \in d\mathsf{w}_k^{\langle 2, \mathfrak{A} \rangle}, \mathbf{1}_{\{M_{\mathsf{u}_{k-1}}^{\mathfrak{A}}(t) < 0\}} \in d\mathsf{w}_k^{\langle 3, \mathfrak{A} \rangle},$$

$$R_{\mathsf{u}_{k-1}}^{\mathfrak{D}}(t) \in d\mathsf{w}_k^{\langle 1, \mathfrak{D} \rangle}, \lambda_{\mathsf{u}_{k-1}}^{\mathfrak{D}}(t) \in d\mathsf{w}_k^{\langle 2, \mathfrak{D} \rangle}, \mathbf{1}_{\{M_{\mathsf{u}_{k-1}}^{\mathfrak{D}}(t) < 0\}} \in d\mathsf{w}_k^{\langle 3, \mathfrak{D} \rangle} \Big\},$$

where $M_{\mathsf{u}_{k-1}}^{\mathfrak{A}}(t) = \inf_{0 < s \leqslant t} R_{\mathsf{u}_{k-1}}^{\mathfrak{A}}(s)$, $M_{\mathsf{u}_{k-1}}^{\mathfrak{D}}(t) = \inf_{0 < s \leqslant t} R_{\mathsf{u}_{k-1}}^{\mathfrak{D}}(s)$.

The controlled random sequence $(\mathbf{W}_k, \mathbf{U}_k)$, $k = 0, 1, 2, \ldots$, with the Markov control (37), becomes equivalent to the Markov chain with the transition probabilities

$$P\left(\mathsf{w}_{k-1}; d\mathsf{w}_k\right) = \pi_k\left(\mathsf{w}_{k-1}, \gamma_{k-1}(\mathsf{w}_{k-1}); d\mathsf{w}_k\right), \quad k = 1, 2, \ldots,$$

on the state space $(\mathrm{W}, \mathcal{W})$.

Write $\mathbf{P}^{\pi, \gamma}$ for the probabilities on the elementary state space $(\Omega, \mathcal{F})$ corresponding to the Markov chains with transition probability P, and denote by $\mathbf{E}^{\pi, \gamma}$ the expectation with respect to that measure.

6.2. A glance at competition "of elimination"

Introduce the following scenario (see Section 2.6, Table 2.4) for two companies $\mathfrak{A}$ and $\mathfrak{D}$, whose strategic goals are highly antagonistic.

SCENARIO 2. The evolution of the market prices agrees with Assumption 3. The initial strategic goal of the aggressive insurer $\mathfrak{A}$ starting with portfolio of small size $\mathsf{w}_0^{\langle 2, \mathfrak{A} \rangle}$, but possessing large initial capital $\mathsf{w}_0^{\langle 1, \mathfrak{A} \rangle}$, is "to win a share" according to the *business plan*

$$0 < \lambda_1^{\mathrm{T}, \mathfrak{A}} < \lambda_2^{\mathrm{T}, \mathfrak{A}} < \lambda_3^{\mathrm{T}, \mathfrak{A}} < ...,$$

where $\lambda_k^{\mathrm{T}, \mathfrak{A}}$ is a *directive target value* for k-th year. Defense of $\mathfrak{D}$ starts with portfolio of large size $\mathsf{w}_0^{\langle 2, \mathfrak{D} \rangle}$, but relatively small capital $\mathsf{w}_0^{\langle 1, \mathfrak{D} \rangle}$. The strategic goal of both players, as soon as aggression of $\mathfrak{A}$ is triggered off and $\mathfrak{D}$ is forced to defend, is to eliminate the adversary.

The competition "of elimination" consists of the following periods.

I. For $\mathfrak{A}$: aggressive market share gain. For $k = 1, 2, \ldots$, and for the initial state vector with components

$$\mathsf{w}_0^{\langle 1, \mathfrak{A} \rangle} > 0, \ \mathsf{w}_0^{\langle 2, \mathfrak{A} \rangle} > 0, \ \mathsf{w}_0^{\langle 3, \mathfrak{A} \rangle} = 0,$$

the adaptive Markov strategy of $\mathfrak{A}$ is governed by an *increasing* directive target values $\lambda_k^{\mathrm{T}, \mathfrak{A}}$, $k = 1, 2, \ldots$, and consists of the following annual controls: if $\mathsf{w}_{k-1}^{\langle 3, \mathfrak{A} \rangle} \neq 1$, set[14]

$$\mathsf{u}_{k-1}^{\langle 1, \mathfrak{A} \rangle} = \mathsf{w}_{k-1}^{\langle 1, \mathfrak{A} \rangle},$$

$$\mathsf{u}_{k-1}^{\langle 2, \mathfrak{A} \rangle} = \mathsf{w}_{k-1}^{\langle 2, \mathfrak{A} \rangle},$$

$$\mathsf{u}_{k-1}^{\langle 3, \mathfrak{A} \rangle} = P_{\lambda_k^{\mathrm{T}, \mathfrak{A}}, t} \left(\mathsf{w}_{k-1}^{\langle 2, \mathfrak{A} \rangle} \,\middle|\, P_k^M \right),$$

provided $P_{\lambda_k^{\mathrm{T}, \mathfrak{A}}, t} \left(\mathsf{w}_{k-1}^{\langle 2, \mathfrak{A} \rangle} \,\middle|\, P_k^M \right) \geqslant P_{\alpha, t, \mathsf{w}_{k-1}^{\langle 1, \mathfrak{A} \rangle}} \left(\mathsf{w}_{k-1}^{\langle 2, \mathfrak{A} \rangle} \,\middle|\, \mathrm{E} Y, P_k^M \right)$. Quantification of that segment of the strategy γ straightforwardly applies the results of Section 4.3.

I. For $\mathfrak{D}$: defensive evolution. For $k = 1, 2, \ldots$, and for the initial state vector with components

$$\mathsf{w}_0^{\langle 1, \mathfrak{D} \rangle} > 0, \ \mathsf{w}_0^{\langle 2, \mathfrak{D} \rangle} > 0, \ \mathsf{w}_0^{\langle 3, \mathfrak{D} \rangle} = 0$$

---

[14] The market price P_k^M is known by Scenario 1.

the adaptive Markov strategy of $\mathfrak{D}$ is governed by the sequence

$$\boldsymbol{\psi}_k^{\langle\mathfrak{A}\rangle} = \boldsymbol{\psi}_{\mathsf{w}_{k-1}^{\langle\mathfrak{A}\rangle}}(t) \leqslant \alpha, \quad k = 1, 2, ...,$$

and consists of the following annual controls: if $\mathsf{w}_{k-1}^{\langle 3, \mathfrak{D}\rangle} \neq 1$, set

$$
\begin{aligned}
\mathsf{u}_{k-1}^{\langle 1, \mathfrak{D}\rangle} &= \mathsf{w}_{k-1}^{\langle 1, \mathfrak{D}\rangle}, \\
\mathsf{u}_{k-1}^{\langle 2, \mathfrak{D}\rangle} &= \mathsf{w}_{k-1}^{\langle 2, \mathfrak{D}\rangle}, \\
\mathsf{u}_{k-1}^{\langle 3, \mathfrak{D}\rangle} &= P_{\boldsymbol{\psi}_k^{\langle\mathfrak{A}\rangle}, t, \mathsf{w}_{k-1}^{\langle 1, \mathfrak{D}\rangle}}\left(\mathsf{w}_{k-1}^{\langle 2, \mathfrak{D}\rangle}\middle| P_k^M\right),
\end{aligned}
\tag{38}
$$

provided $P_{\varepsilon, t}\left(\mathsf{w}_{k-1}^{\langle 2, \mathfrak{D}\rangle}\middle| P_k^M\right) \geqslant P_{\boldsymbol{\psi}_k^{\langle\mathfrak{A}\rangle}, t, \mathsf{w}_{k-1}^{\langle 1, \mathfrak{D}\rangle}}\left(\mathsf{w}_{k-1}^{\langle 2, \mathfrak{D}\rangle}\middle| P_k^M\right)$. Quantification of that segment of the strategy γ straightforwardly applies the results of Section 4.3 or 4.4.

REMARK 6.1. Evidently, the control (38) is *interactive* since $\mathsf{u}_{k-1}^{\langle 3, \mathfrak{D}\rangle}$ explicitly depends on $\mathsf{w}_{k-1}^{\langle 2, \mathfrak{A}\rangle}$ through $\boldsymbol{\psi}_k^{\langle\mathfrak{A}\rangle}$, and *adaptive* since it depends on $\mathsf{w}_{k-1}^{\langle 2, \mathfrak{D}\rangle}$.

II. For $\mathfrak{A}$ and $\mathfrak{D}$: close contest. As the portfolio sizes become even

$$\mathsf{w}_{k-1}^{\langle 2, \mathfrak{A}\rangle} \approx \mathsf{w}_{k-1}^{\langle 2, \mathfrak{D}\rangle},$$

the prices of $\mathfrak{A}$ and $\mathfrak{D}$ may be chosen evenly,

$$P_{\lambda_k^{\mathrm{T}, \mathfrak{A}}, t}\left(\mathsf{w}_{k-1}^{\langle 2, \mathfrak{A}\rangle}\middle| P_k^M\right) \approx P_{\boldsymbol{\psi}_k^{\langle\mathfrak{A}\rangle}, t, \mathsf{w}_{k-1}^{\langle 1, \mathfrak{D}\rangle}}\left(\mathsf{w}_{k-1}^{\langle 2, \mathfrak{D}\rangle}\middle| P_k^M\right),$$

and decreasing year-by-ear, falling eventually below $\mathrm{E}Y$. That company has an advantage whose capital, $\mathsf{w}_{k-1}^{\langle 1, \mathfrak{A}\rangle}$ or $\mathsf{w}_{k-1}^{\langle 1, \mathfrak{D}\rangle}$, is somewhat larger, and whose probability of ruin is therefore smaller. That phase may embrace the prosperous phase, but is pertinent to the soft market one. Quantification of that strategy γ's segment applies the results of Section 4.2 or 4.3.

III. For $\mathfrak{A}$ and $\mathfrak{D}$: struggle of attrition. That phase is relevant to soft market. As company $\mathfrak{A}$ decides to cease the annual premiums reduction below the past-year market price, the company $\mathfrak{D}$ wages aggression instead, by reducing its prices. The company $\mathfrak{A}$ becomes defending, while $\mathfrak{D}$ aggressive. On that spiral competition the capital of both $\mathfrak{A}$ and $\mathfrak{D}$ is expending over each convolution, and the probability of ruin increases for both. That company has an advantage, whose capital is larger, and whose probability of ruin is therefore smaller.

IV. For 𝔄 or 𝔇: victory or defeat. The struggle of attrition, if not stopped with a truce, comes to a shortage of capital and to victory or defeat of 𝔄 or 𝔇.

7. FURTHER RESULTS ON QUANTIFICATION OF ANNUAL CONTROLS

This section contains a few among a range of results useful to quantify the annual controls. An inventory of such results would help to construct the strategies fit to pursue competitive goals, whichever they may be. Remarkable are closed-form analytical expressions, if available.

7.1. Solvency prices as function of initial capital

Introduce the following definition.

DEFINITION 7.1. (Solvency prices as function of capital). For all values of $\alpha \in (0,1)$ and t, λ positive, the solutions $P_{\alpha,t,\lambda}(u)$ of the *solvency equation* $\psi_{u,\lambda,P}(t) = \alpha$ with respect to $P \in \mathcal{P}^s (P \in \mathcal{P}^h)$, considered as a function of capital u, constitute a function called *solvency prices*, of level α.

MONOTONY LEMMA 5. In the assumptions of Theorem 3.3, for $\alpha \in (0,1)$, $t, \lambda \in R^+$ fixed and for any EY and P^M, the price function $P_{\alpha,t,\lambda}(u)$ is continuous, concave and monotone decreasing, as u increases.

PROOF OF MONOTONY LEMMA 5. It is straightforward from implicit function differentiation arguments analogous to those in the proof of Theorem 3.2 in Malinovskii (2008a), and requires some elementary calculus.   □
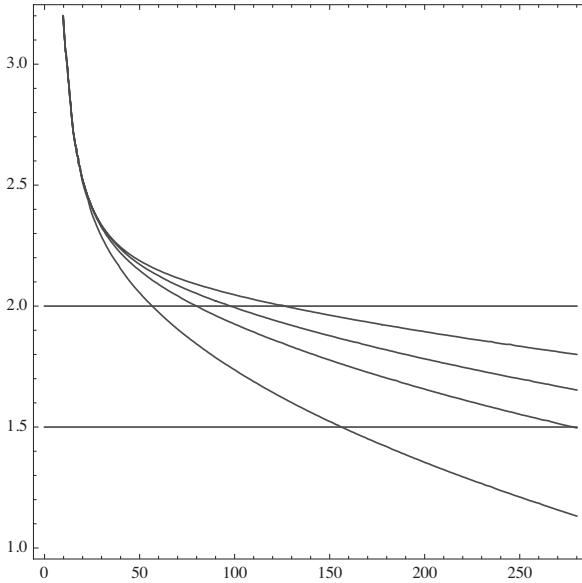
The functions $P_{\alpha,t,\lambda}(u \,|\, EY, P^M)$ in the years of soft and hard market calculated numerically are shown on Fig. 11 and 12.

7.2. Solvency capital as function of initial share

Useful is the following definition.

DEFINITION 7.2 (Solvency capital as function of share). For all values of $\alpha \in (0,1)$, t positive and $P \in \mathcal{P}^s (P \in \mathcal{P}^h)$, the solutions $u_{\alpha,t,P}(\lambda)$ of the *solvency equation* $\psi_{u,\lambda,P}(t) = \alpha$ with respect to u, considered as a function of initial share λ, constitute a function called *solvency capital*, of level α.

MONOTONY LEMMA 6. In the assumptions of Theorem 3.3, for $\alpha \in (0,1)$, $t \in R^+$ and $P \in \mathcal{P}$ fixed, and for any EY and P^M, the function $u_{\alpha,t,P}(\lambda)$ increases, as λ increases.

PROOF OF MONOTONY LEMMA 6. That proof applies implicit function differentiation arguments analogous to those in the proof of Theorem 3.2 in Malinovskii (2008a) and requires some elementary calculus.   □

FIGURE 11: Functions $P_{\alpha,t,\lambda}(u\,|\,EY, P^M)$ (Y-axis; X-axis is u) in the year of soft market: $EY = 2$, $P^M = 1.5$. Here $\alpha = 0.1$, $t = 100$ and (from top downward) $\lambda = 10, 6, 4, 2$. The function $\Lambda_{d,\lambda}(t)$ is defined as in (10), with power $k = 1/2$ and r_d as in (6), where $l = 2$, $\rho = -\ln(2/5)$, $\varrho = 1/5$ (i.e., $C = 2$, $c = 1/3$).
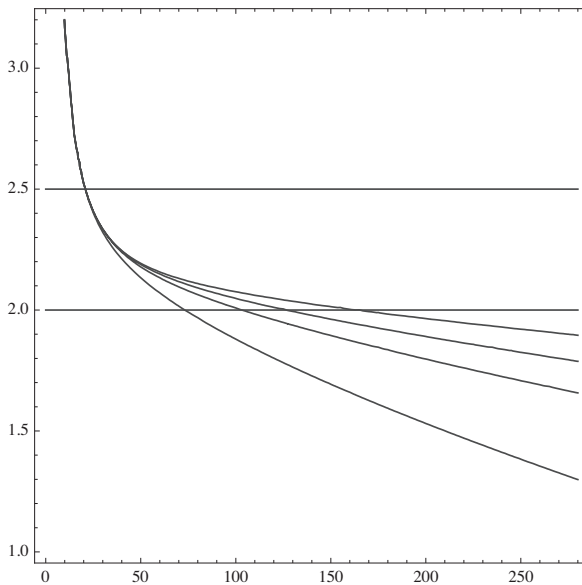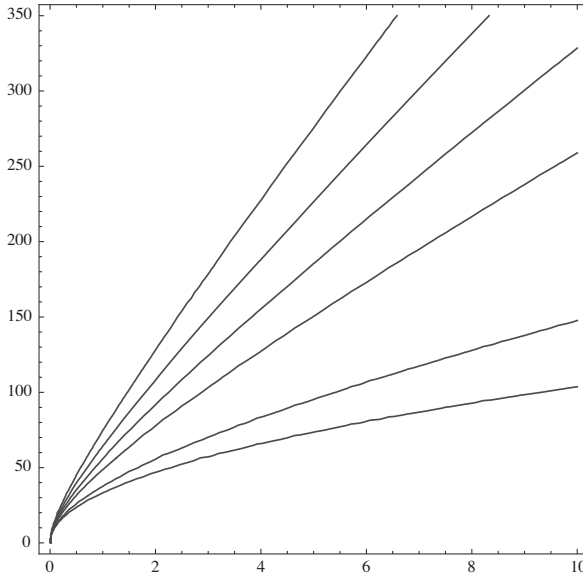
FIGURE 12: Functions $P_{\alpha,t,\lambda}(u\,|\,EY, P^M)$ (Y-axis; X-axis is u) in the year of hard market: $EY = 2$, $P^M = 2.5$. Here $\alpha = 0.1$, $t = 100$ and (from top downward) $\lambda = 10, 6, 4, 2$. The function $\Lambda_{d,\lambda}(t)$ is defined as in (10), with power $k = 1/2$ and r_d as in (6), where $l = 2$, $\rho = -\ln(2/5)$, $\varrho = 1/5$ (i.e., $C = 2$, $c = 1/3$).
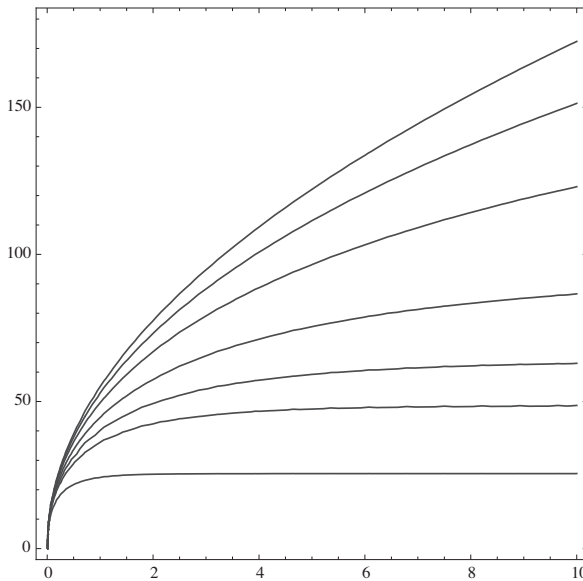
FIGURE 13: Functions $u_{\alpha,t,P}(\lambda \mid EY, P^M)$ (Y-axis; X-axis is λ) in the year of soft market:
$EY = 2$, $P^M = 1$. Here $\alpha = 0.1$, $t = 100$, $P \in \mathcal{P}_{\text{CL-SL}}^s$ and (from top downward) $P = 1.4$, 1.5, 1.6, 1.7, 1.9, 2.0.
The function $\Lambda_{d,\lambda}(t)$ is defined as in (10), with power $k = 1/2$ and r_d as in (6), where $l = 2$,
$\rho = -\ln(2/5)$, $\varrho = 1/5$ (i.e., $C = 2$, $c = 1/3$).

FIGURE 14: Functions $u_{\alpha,t,P}(\lambda \mid EY, P^M)$ (Y-axis; X-axis is λ) in the year of hard market:
$EY = 2$, $P^M = 3$. Here $\alpha = 0.1$, $t = 100$, $P \in \mathcal{P}_{\text{CW-SW}}^h$ and (from top downward) $P = 2.00$, 2.02, 2.05, 2.10,
2.15, 2.20, 2.40. The function $\Lambda_{d,\lambda}(t)$ is defined as in (10), with power $k = 1/2$ and r_d as in (6), where $l = 2$,
$\rho = -\ln(2/5)$, $\varrho = 1/5$ (i.e., $C = 2$, $c = 1/3$).

The functions $u_{\alpha,t,P}(\lambda \mid EY, P^M)$ in the years of soft and hard market calculated numerically are shown on Fig. 13 and 14.

Address a closed-form approximation for $u_{\alpha,t,P}(\lambda)$ which holds true whichever the loading $\tau(P) = P/EY - 1$ on premium P may be, and irrespectively of soft (i.e., $P^M < EY$) or hard (i.e., $P^M > EY$) market.

THEOREM 7.1. In the assumptions of Theorem 3.3, for all values of $\alpha \in (0, 1/2)$, t, μ, λ and P positive and such that $\Lambda_{d(P),\lambda}(t) \to \infty$, as $t \to \infty$, one has

(1) for $P < EY$ (negative loading on premium P) and a constant[15] $0 < c_{\alpha, P} < \Phi_{\{0,1\}}^{-1}(1 - \alpha)$,

$$u_{\alpha,t,P}(\lambda)\mu = (EY - P)\Lambda_{d(P),\lambda} + c_{\alpha,P} EY(2\Lambda_{d(P),\lambda}(t))^{1/2} + \underline{O}(1), \text{ as } t \to \infty,$$

(2) for $P = EY$ (zero loading on premium P) and $c_\alpha = \Phi_{\{0,1\}}^{-1}(1 - \alpha/2) > 0$,

$$u_{\alpha,t,EY}(\lambda)\mu = c_\alpha EY(2\Lambda_{d(EY),\lambda}(t))^{1/2} + \underline{O}(1), \text{ as } t \to \infty,$$

(3) for $P > EY$ (positive loading on premium P),

$$\lim_{t \to \infty} u_{\alpha,t,P}(\lambda)\mu = -\frac{1 + \tau(P)}{\tau(P)} \ln\big(\alpha(1 + \tau(P))\big).$$

PROOF. The proof of part (2) of Theorem 7.1 is straightforward from Theorem 4.1 in Malinovskii (2008b). The proof of part (1) and (3) of Theorem 7.1 are straightforward from parts (1) and (2) of Theorem 3.5. $\square$

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[15] The constant $c_{\alpha, P}$ may be specified as a solution of a non-linear equation. To save space, we omit this result.

References

ASMUSSEN, S. (1984; 1985) Approximations for the probability of ruin within finite time, *Scandinavian Actuarial Journal*, **64**, 31-57.

ASMUSSEN, S. (2000) *Ruin Probabilities.* World Scientific Publishers: Singapore.

ASMUSSEN, S. and ROLSKI, T. (1994) Risk theory in a periodic environment: the Cramér-Lundberg approximation and Lundbergs inequality. *Math. Oper. Res.*, **19(2)**, 410-433.

Best's Insolvency Study: Property/Casualty Insurers 1969-1990 (Oldwick, NJ: A.M. Best Company, June 1991).

BÜHLMANN, H. (1970) *Mathematical Methods in Risk Theory.* Springer Verlag, New York.

CUMMINS, J.D. and OUTREVILLE, J.F. (1987) An international analysis of underwriting cycles in property-liability insurance, *Journal of Risk and Insurance*, June 1987, 246-262.

CUMMINS, D., MCGILL, D., WINKLEVOSS, H. and ZELTEN, H. (1974) *Consumer Attitudes Toward Auto and Homeowners Insurance.* Philadelphia, Pa.: Department of Insurance, Wharton School.

D'ARCY, S.P., GORVETT, R.W., HERBERS, J.A., HETTINGER, T.E., LEHMANN, S.G. and MILLER, M.J. (1997) Building a public access PC-based DFA model. *CAS Forum*, 1997.

DAYKIN, C.D., PENTIKÄINEN, T. and PESONEN, M. (1996) *Practical Risk Theory for Actuaries.* Chapman and Hall, London, etc.

DOHERTY, N.A. and GARVEN, J.R. (1995) Insurance cycles: interest rates and the capacity constraint model. *The Journal of Business*, **68(3)**, 383-404.

DOHERTY, N.A. and KANG, H.B. (1988) Interest rates and insurance price cycles, *Journal of Banking and Finance*, **12(2)**, 199-214.

FELDBLUM, S. (2001) Underwriting cycles and business strategies, *Proceedings of the Casualty Actuarial Society*, **LXXXVIII**, 175-235.

FELDBLUM, S. (2007a) Underwriting cycles and insurance solvency. Casualty Actuarial Society Discussion Paper Program Casualty Actuarial Society – Arlington, Virginia 1992: May, Vol. 1, 383-438, http://www.casact.org/pubs/dpp/dpp92/92dpp383.pdf

FELDBLUM, S. (2007b) Underwriting cycles and ruin probability, Manuscript.

GERBER, H.U. and SHIU, E.S.W. (1997) The joined distribution of the time of ruin, the surplus immediately before ruin, and the deficit at ruin, *Insurance: Mathematics and Economics*, **21**, 129-137.

GIHMAN, I.I. and SKOROKHOD, A.V. (1979) *Controlled Stochastic Processes.* Springer-Verlag, New York etc.

GRANDELL, J. (1991) *Aspects of Risk Theory.* Springer-Verlag, New York etc.

JOHNSON, G. and SCHOLES, K. (1988) *Exploring Corporate Strategy. Text and Cases* (2nd edn). Prentice-Hall, Hemel Hempstead.

JOSKOW, P. (1973) Cartels, Competition and Regulation in the Property-Liability Insurance Industry, *Bell Journal of Economics* **4**, 375-427.

KAUFMANN, R., GADMER, A. and KLATT, R. (2001) Introduction to dynamic financial analysis. *RiskLab Switzerland*, April 26, 2001.

HARRINGTON, S.E. and DANZON, P.M. (1994) Pricing cutting in liability insurance markets, *The Journal of Business*, **67(4)**, 511-538.

MALINOVSKII, V.K. (1993) Limit theorems for stopped random sequences. I: rates of convergence and asymptotic expansions, *Theory Probab. Appl.*, **38**, 673-693.

MALINOVSKII, V.K. (1994) Corrected normal approximation for the probability of ruin within finite time, S*candinavian Actuarial Journal*, 161-174.

MALINOVSKII, V.K. (1996a) Limit theorems for stopped random sequences. II: probabilities of large deviations, *Theory Probab. Appl.*, **41**, 70-90.

MALINOVSKII, V.K. (1996b) Approximations and upper bounds on probabilities of large deviations in the problem of ruin within finite time, *Scandinavian Actuarial Journal*, 124-147.

MALINOVSKII, V.K. (1998) Non-poissonian claims' arrivals and calculation of the probability of ruin, *Insurance: Mathematics and Economics*, **22**, 123-138.

MALINOVSKII, V.K. (2000) Probabilities of ruin when the safety loading tends to zero, *Advances in Applied Probability*, **32**, 885-923.

MALINOVSKII, V.K. (2007) Zone-adaptive control strategy for a multiperiodic model of risk. *Annals of Actuarial Science*, **2(II)**, 391-409.

MALINOVSKII, V.K. (2008a) Adaptive control strategies and dependence of finite time ruin on the premium loading, *Insurance: Mathematics and Economics*, **42**, 81-94.

MALINOVSKII, V.K. (2008b) Risk theory insight into a zone-adaptive control strategy. *Insurance: Mathematics and Economics*, **42**, 656-667.

MALINOVSKII, V.K. (2009a) Scenario analysis for a multi-period diffusion model of risk, *ASTIN Bulletin*, **39**, 649-676.

MALINOVSKII, V.K. (2009b) *Survive a downswing phase of the underwriting cycle*, 39th ASTIN Colloquium in Helsinki, Finland, 1-4 June 2009.

OWEN, G. (1982) *Game theory*. 2nd Ed., Academic Press.

PENTIKÄINEN, T. (1988) On the solvency of insurers. In: Classical Insurance Solvency Theory, Ed. by: Cummins, J.D. and Derring, R.A., Boston etc., Kluwer, 1988, 1-48.

RANTALA, J. (1988) Fluctuations in insurance business results: some control theoretical aspects. In book: *Transactions of the 23-rd International Congress of Actuaries*, Helsinki 1988, **1**, 43-78.

SCHLEZINGER, H. and VON DER SCHULENBURG, J.M. (1993) Consumer information and decisions to switch insurers, *Journal of Risk and Insurance*, **LV(4)**, 591-615.

SOROS, G. (1994) *The Alchemy of Finance: Reading the Mind of the Market*. John Wiley & Sons, Inc. New York, etc.

SUBRAMANIAN, K. (1998) Bonus-Malus systems in a competitive environment, *North American Actuarial Journal*, **2**, 38-44.

SUN, Ch. (2007) Laplace transform of the survival probability under Sparre Andersen model, *Appl. Math. J. Chinese Univ. Ser. B*, **22(1)**, 109-118.

TEUGELS, J.L. (1982) Estimation of ruin probabilities, *Insurance: Mathematics and Economics*, **1**, 163-175.

VENEZIAN, E.C. (1985) Ratemaking methods and profit cycles in property and liability insurance, *Journal of Risk and Insurance*, September 1985, 477-500.

VON NEUMANN, J. and MORGENSTERN, O. (1944) *Theory of games and economic behavior*. Princeton University Press.

WANG, R. and LIU, H. (2002) On the ruin probability under a class of risk processes, *ASTIN Bulletin*, **32(1)**, 81-90.

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