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History-dependent decay rates for a logistic equation with infinite delay

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A logistic equation with infinite delay is considered under conditions that force its solution to approach a positive steady state at large times. It is shown that this rate of convergence depends on the initial history in some cases, and is independent of the history in others.

1. Introduction

The logistic equation can be derived from a simple model for the evolution of a single population in an environment with limited resources. However, its solution fails to reflect the observed evolution of many species, and models have been developed in which delay terms appear. Cushing [10,11], Gopalsamy and Lalli [12] and Miller [17], among others, studied the logistic equation with infinite delay

$$\dot{N}(t) = N(t)\left\{r - aN(t) - \int_0^\infty b(s)N(t-s)\,\mathrm{d}s\right\}, \quad t > 0, \tag{1.1 a}$$

with the initial condition

$$N(t) = \phi(t), \quad t \leqslant 0. \tag{1.1 b}$$

More recently it has been considered by Song and Baker [18] in work on the qualitative behaviour of numerical approximations to Volterra integro-differential equations.

It is assumed here that $a > 0$ and $r > 0$. Also, for simplicity, $b\colon [0,\infty) \to (0,\infty)$ is taken to be continuous and integrable. Equation (1.1 a) has two steady-state

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solutions, $N(t) \equiv 0$ and $N(t) \equiv K$, where

$$K = \frac{r}{a + \int_0^\infty b(s)\,\mathrm{d}s}. \tag{1.2}$$

Clearly, $K > 0$. If

$$a > \int_0^\infty b(s)\,\mathrm{d}s,$$

Miller [17] proved that the solution of (1.1) satisfies $N(t) \to K$ as $t \to \infty$, assuming that the initial history ϕ is bounded. In this paper we give an elementary proof which does not require ϕ to be bounded.

At what the rate does $N(t)$ converge to the steady state K as $t \to \infty$? In this paper the rate of convergence is examined, and shown to be dependent on the initial history ϕ if $b(t)\mathrm{e}^{\lambda t}$ decays subexponentially for some $\lambda \geqslant 0$. However, if $b(t) = \kappa \mathrm{e}^{-\lambda t}$, the rate of convergence is independent of ϕ. By linearizing about the steady state K, we obtain a weakly nonlinear equation.

This paper investigates the simple scalar nonlinear equation (1.1 a) in some special cases: no attempt is made to comprehensively characterize the different decay rates to equilibrium or add point delays to the equation. In his original pioneering work, Volterra considered a predator–prey system with infinite delay. We expect that the decay rates for such systems with matrix-valued kernels and vector-valued histories will exhibit more complicated behaviour.

Our method is to show that $N(t)$ satisfies a *linear* Volterra integral equation. This equation is then rescaled using appropriate weight functions. It is shown that this rescaled equation has asymptotically constant solutions by using admissibility properties of certain Volterra operators. The results on admissibility which are used in our proofs are summarized in Appendix A. For the kernels b and histories ϕ considered here, the appropriate weight functions involve subexponential functions, the relevant properties of which are given in Appendix B.

2. Existence, uniqueness and convergence to steady-state

We make assumptions on data that are natural in the context of population dynamics. It is supposed throughout the paper, without further repetition, that $a > 0$ *and* $r > 0$ *are constants, and* $b\colon [0, \infty) \to (0, \infty)$ *is a continuous function in* $L^1(0, \infty)$. We introduce a class of initial histories.

DEFINITION 2.1. Let Φ be the set of continuous $\phi\colon (-\infty, 0] \to [0, \infty)$ such that $\phi(0) > 0$, and for which $F(\cdot;\phi)\colon [0, \infty) \to [0, \infty)$ defined by

$$F(t;\phi) = \int_{-\infty}^{0} b(t-s)\phi(s)\,\mathrm{d}s, \quad t \geqslant 0, \tag{2.1}$$

is continuous and obeys $\lim_{t\to\infty} F(t;\phi) = 0$.

Next an answer is given to the question of which initial histories are in Φ. It is seen that there is a trade-off between the rate of decay of $b(t)$ as $t \to \infty$ and the rate of growth of $\phi(t)$ as $t \to -\infty$. It is clear that there can be unbounded ϕ in Φ if $b(t)$ decays quickly enough as $t \to \infty$.

PROPOSITION 2.2. *Let $\phi\colon(-\infty,0]\to[0,\infty)$ be continuous with $\phi(0)>0$, and $\lambda\geqslant 0$. If*

$$\int_0^\infty b(t)\mathrm{e}^{\lambda t}\,\mathrm{d}t<\infty$$

and there is $M\geqslant 0$ such that $\phi(t)\leqslant M\mathrm{e}^{-\lambda t}$ for all $t\leqslant 0$, then $\phi\in\Phi$.

Proof. We observe that

$$F(t;\phi)=\int_t^\infty b(u)\phi(t-u)\,\mathrm{d}u\leqslant M\int_t^\infty b(u)\mathrm{e}^{\lambda u}\,\mathrm{d}u\to 0$$

as $t\to\infty$. To prove the continuity of $F(\cdot;\phi)$, suppose that $t\geqslant 0$ is fixed. Let $0<h<T$. It is easy to show that

$$\begin{aligned}|F(t+h;\phi)-F(t;\phi)|&\leqslant\int_{-T}^{-h}b(t-s)|\phi(s+h)-\phi(s)|\,\mathrm{d}s+\int_{-h}^{0}b(t-s)\phi(s)\,\mathrm{d}s\\&\quad+\int_{-\infty}^{-T}b(t-s)|\phi(s+h)-\phi(s)|\,\mathrm{d}s\\&\leqslant\sup_{-T\leqslant s\leqslant -h}|\phi(s+h)-\phi(s)|\int_t^\infty b(u)\,\mathrm{d}u\\&\quad+\sup_{-h\leqslant s\leqslant 0}\phi(s)\int_t^{t+h}b(u)\,\mathrm{d}u+2M\int_{-\infty}^{-T}b(t-s)\mathrm{e}^{-\lambda s}\,\mathrm{d}s\\&\leqslant\sup_{-T\leqslant s\leqslant -h}|\phi(s+h)-\phi(s)|\int_t^\infty b(u)\,\mathrm{d}u\\&\quad+\sup_{-h\leqslant s\leqslant 0}\phi(s)\int_t^{t+h}b(u)\,\mathrm{d}u+2M\int_{T}^{\infty}b(u)\mathrm{e}^{\lambda u}\,\mathrm{d}s.\end{aligned}$$

The last term on right-hand side can be made arbitrarily small by taking T suitably large; having fixed $T>0$, the uniform continuity of ϕ on $[-T,0]$ ensures that the first term can be made arbitrarily small for small enough h. The other case, of $h<0$, can be dealt with similarly. □

The initial-history problem (1.1) is reformulated as an initial-value problem for a Volterra integro-differential equation. Indeed, with the aid of (1.1 *b*), (1.1 *a*) can be written as

$$\dot{N}(t)=N(t)\biggl\{r-aN(t)-\int_0^t b(t-s)N(s)\,\mathrm{d}s-F(t;\phi)\biggr\}.\tag{2.2}$$

THEOREM 2.3. *Suppose that ϕ is in Φ.*

(i) *The initial-value problem consisting of equation (2.2) and initial condition $N(0)=\phi(0)$ has a unique solution $N(\cdot;\phi)\colon[0,\infty)\to(0,\infty)$, which is bounded on in $[0,\infty)$.*

(ii) *If in addition*

$$a>\int_0^\infty b(s)\,\mathrm{d}s,\tag{2.3}$$

then $N(t;\phi)\to K$ as $t\to\infty$.

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The property that $\lim_{t\to\infty} F(t;\phi) = 0$ is not needed for part (i), but is required for part (ii). Standard results on existence, uniqueness and continuation of solutions of Volterra integral equations (see, for example, [7, 13]) yield most of part (i). For the proof of positivity, see, for example, [17].

Part (ii) is a global asymptotic stability result due to Miller [17, theorem 1]. He established the result under the assumption of *bounded* histories, but a careful examination of his proof shows that this boundedness assumption is not required if $\phi \in \Phi$. We provide an elementary proof, which uses an idea from [14]. Here and throughout the paper the explicit dependence of functions on the initial history ϕ is sometimes suppressed in our notation.

Proof of theorem 2.3(ii). The first step in proving (ii) is to establish that the solution N of (2.2) and $N(0) = \phi(0)$, which is known to be bounded and positive, also obeys

$$\int_0^\infty N(t)\,\mathrm{d}t = \infty. \tag{2.4}$$

The solution of (2.2) satisfies

$$N(t) = \phi(0)\exp\biggl\{\int_0^t [r - F(s;\phi)]\,\mathrm{d}s - a\int_0^t N(s)\,\mathrm{d}s - \int_0^t\int_0^s b(s-u)N(u)\,\mathrm{d}u\,\mathrm{d}s\biggr\}. \tag{2.5}$$

The first term in curly brackets diverges to ∞ as $t \to \infty$, since $r > 0$ and $F(t;\phi) \to 0$ as $t \to \infty$. If N is in $L^1(0,\infty)$, the second and third terms converge to a finite value, and (2.5) then implies that $N(t) \to \infty$ as $t \to \infty$, contradicting the boundedness of N. Hence, (2.4) must be satisfied.

The next step is to observe that $x(\cdot;\phi)\colon [0,\infty) \to \mathbb{R}$ defined by

$$x(t;\phi) = N(t;\phi) - K, \quad t \geqslant 0, \tag{2.6}$$

is bounded and satisfies the linear equation

$$\dot{x}(t) = -aN(t)x(t) - N(t)\biggl[\int_0^t b(t-s)x(s)\,\mathrm{d}s + f(t;\phi)\biggr],$$

where

$$f(t;\phi) = \int_{-\infty}^0 b(t-s)(\phi(s) - K)\,\mathrm{d}s, \quad t \geqslant 0. \tag{2.7}$$

Use of an integrating factor leads to

$$x(t) = \exp\biggl\{-a\int_0^t N(u)\,\mathrm{d}u\biggr\}[\phi(0) - K] - \int_0^t N(\tau)\exp\biggl\{-a\int_\tau^t N(u)\,\mathrm{d}u\biggr\}\biggl(\int_0^\tau b(\tau-s)x(s)\,\mathrm{d}s + f(\tau)\biggr)\mathrm{d}\tau.$$

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Because x is bounded, $\bar{x} = \limsup_{t\to\infty} |x(t)|$ is finite. It is easily deduced using (2.4) that

$$\bar{x} \leqslant \limsup_{t\to\infty} \frac{\int_0^t N(\tau)\exp\{a\int_0^\tau N(u)\,\mathrm{d}u\}(\int_0^\tau b(\tau-s)|x(s)|\,\mathrm{d}s + |f(\tau)|)\,\mathrm{d}\tau}{\exp\{a\int_0^t N(u)\,\mathrm{d}u\}}. \tag{2.8}$$

To shorten displayed equations, we set

$$g(\tau) = \int_0^\tau b(\tau-s)|x(s)|\,\mathrm{d}s + |f(\tau)|, \quad \tau \geqslant 0.$$

It is a consequence of $\phi \in \varPhi$ and $b \in L^1(0,\infty)$ that

$$|f(t)| \leqslant \int_{-\infty}^0 b(t-s)|\phi(s)-K|\,\mathrm{d}s \leqslant F(t) + K\int_{-\infty}^0 b(t-s)\,\mathrm{d}s \to 0$$

as $t\to\infty$. Hence,

$$\limsup_{\tau\to\infty} g(\tau) \leqslant \biggl(\int_0^\infty b(s)\,\mathrm{d}s\biggr)\bar{x}.$$

Let $T > 0$. We deduce from (2.4) and (2.8) that

$$\begin{aligned}\bar{x} &\leqslant \limsup_{t\to\infty} \frac{\int_T^t N(\tau)\exp\{a\int_0^\tau N(u)\,\mathrm{d}u\}\,\mathrm{d}\tau}{\exp\{a\int_0^t N(u)\,\mathrm{d}u\}} \sup_{\tau\geqslant T} g(\tau)\\ &= \frac{1}{a}\limsup_{t\to\infty} \frac{\exp\{a\int_0^t N(s)\,\mathrm{d}s\} - \exp\{a\int_0^T N(s)\,\mathrm{d}s\}}{\exp\{a\int_0^t N(u)\,\mathrm{d}u\}} \sup_{\tau\geqslant T} g(\tau)\\ &= \frac{1}{a}\sup_{\tau\geqslant T} g(\tau).\end{aligned}$$

Letting $T\to\infty$,

$$\bar{x} \leqslant \biggl(\frac{1}{a}\int_0^\infty b(u)\,\mathrm{d}u\biggr)\bar{x}.$$

We conclude from (2.3) that $\bar{x} = 0$. □

3. Main result and discussion

3.1. Main result

Our main result shows that the decay rate of $N(t;\phi)$ to the steady state K can depend on the initial history ϕ.

THEOREM 3.1. *Suppose that ϕ is in $\varPhi$, $t \mapsto b(t)\mathrm{e}^{\lambda t}$ is subexponential with $\lambda \geqslant 0$ and*

$$K\int_0^\infty \mathrm{e}^{\lambda s} b(s)\,\mathrm{d}s < aK - \lambda. \tag{3.1}$$

(i) *If there is a decreasing $\delta\colon (0,\infty) \to (0,\infty)$ such that*

$$\frac{b(t)}{\mathrm{e}^{-\lambda t}\int_t^\infty b(s)\mathrm{e}^{\lambda s}\,\mathrm{d}s} \sim \delta(t) \quad \text{as } t\to\infty \tag{3.2}$$

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and

$$\lim_{t\to-\infty} \mathrm{e}^{\lambda t}[\phi(t)-K] =: L(\phi,K) \text{ exists,} \tag{3.3}$$

then

$$N(t;\phi)-K = \mathrm{e}^{-\lambda t}\int_t^\infty \mathrm{e}^{\lambda s} b(s)\,\mathrm{d}s[c_1(\phi)+o(1)] \quad \text{as } t\to\infty, \tag{3.4}$$

where $c_1(\phi)\neq 0$ *if and only if* $L(\phi,K)\neq 0$.

(ii) *If* $t\mapsto b(t)\mathrm{e}^{\lambda t}$ *is decreasing, and*

$$\int_{-\infty}^0 |\phi(t)-K|\mathrm{e}^{\lambda t}\,\mathrm{d}t<\infty, \tag{3.5}$$

then

$$N(t;\phi)-K = b(t)[c_2(\phi)+o(1)] \quad \text{as } t\to\infty. \tag{3.6}$$

3.2. Discussion and examples

The statement of the theorem refers to a kind of slowly varying function, called subexponential: these are discussed in Appendix B. The relative decay rates of the functions in (3.4) and (3.6) are related by

$$\lim_{t\to\infty}\frac{b(t)}{\mathrm{e}^{-\lambda t}\int_t^\infty \mathrm{e}^{\lambda s}b(s)\,\mathrm{d}s}=0, \tag{3.7}$$

so that

$$\mathrm{e}^{-\lambda t}\int_t^\infty \mathrm{e}^{\lambda s}b(s)\to 0$$

more slowly than $b(t)\to 0$ as $t\mapsto\infty$. In practice, (3.2) is not much stronger than (3.7). For instance, if $t\mapsto \mathrm{e}^{\lambda t}b(t)$ is in the space of regularly varying functions at infinity with index $-\alpha<-1$, it is known that (cf. [6, theorem 1.5.11(ii)])

$$\frac{b(t)}{\mathrm{e}^{-\lambda t}\int_t^\infty \mathrm{e}^{\lambda s}b(s)\,\mathrm{d}s}\sim\frac{\alpha-1}{t} \quad \text{as } t\to\infty,$$

so that (3.2) holds with $\delta(t)=(\alpha-1)/(1+t)$.

Remark 3.2. It is not assumed in theorem 3.1 that (2.3) holds, since this is implied by (3.1), the two conditions coinciding if $\lambda=0$.

Example 3.3. A simple example of a subexponential function is $b(t)=(1+t)^{-\alpha}$ for some $\alpha>1$. In this case $\lambda=0$ and

$$\frac{b(t)}{\int_t^\infty b(s)\,\mathrm{d}s}=\frac{\alpha-1}{1+t}=\delta(t).$$

If the history tends to a constant unequal to K as $t\to-\infty$, then (3.3) is true and

$$N(t;\phi)-K=\frac{1+t}{(1+t)^\alpha}\frac{1}{\alpha-1}[c_1+o(1)] \quad \text{as } t\to\infty.$$

If $\phi(t) \to K$ as $t \to -\infty$, $L(\phi, K) = 0$, $c_1 = 0$ and the decay rate is no longer $(1+t)^{1-\alpha}$. But if the history tends to K as $t \to -\infty$ rapidly enough for (3.5) to hold, then

$$N(t;\phi) - K = \frac{1}{(1+t)^{\alpha}}[c_2 + o(1)] \quad \text{as } t \to \infty.$$

Thus, the rate is exactly $(1+t)^{-\alpha}$ if $c_2 \neq 0$, and faster than $(1+t)^{-\alpha}$ if $c_2 = 0$.

Example 3.4. An example of a subexponential function which does not decay polynomially is $b(t) = \mathrm{e}^{-(1+t)^{\alpha}}$ for some $\alpha \in (0,1)$. In this case $\lambda = 0$. By L'Hôpital's rule,

$$\lim_{t\to\infty} \frac{b(t)(1+t)^{1-\alpha}}{\int_t^{\infty} b(s)\,\mathrm{d}s} = \alpha,$$

and $\delta(t) = \alpha/(1+t)^{1-\alpha}$. If the history tends to a constant unequal to K as $t \to -\infty$, then (3.3) is true and

$$N(t;\phi) - K = \frac{1}{\alpha}(1+t)^{1-\alpha}\mathrm{e}^{-(1+t)^{\alpha}}[c_1 + o(1)] \quad \text{as } t \to \infty.$$

If $\phi(t) \to K$ as $t \to -\infty$, $L(\phi, K) = 0$, $c_1 = 0$ and the decay rate is no longer $(1+t)^{1-\alpha}\mathrm{e}^{-(1+t)^{\alpha}}$. But if the history tends to K as $t \to -\infty$ rapidly enough for (3.5) to hold, then

$$N(t;\phi) - K = \mathrm{e}^{-(1+t)^{\alpha}}[c_2 + o(1)] \quad \text{as } t \to \infty.$$

Thus, the rate is exactly $\mathrm{e}^{-(1+t)^{\alpha}}$ if $c_2 \neq 0$, and faster than $\mathrm{e}^{-(1+t)^{\alpha}}$ if $c_2 = 0$.

3.3. The purely exponential case

For the sake of comparison, we discuss the case where $b(t) = \kappa \mathrm{e}^{-\lambda t}$ for all $t \geqslant 0$. Then (1.1) can be reduced to an initial-value problem for a planar system of ordinary differential equations of the form

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x}(0) = \boldsymbol{\alpha}(\phi),$$

where the new variables are

$$x_1(t) = N(t) - K, \qquad x_2(t) = \kappa \int_{-\infty}^{t} \mathrm{e}^{-\lambda(t-s)}(N(s) - K)\,\mathrm{d}s,$$

and

$$A = \begin{pmatrix} -aK & -K \\ \kappa & -\lambda \end{pmatrix}, \qquad \boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} -ax_1^2 - x_1x_2 \\ 0 \end{pmatrix},$$

$$\boldsymbol{\alpha}(\phi) = \begin{pmatrix} \phi(0) - K \\ \displaystyle\int_0^{\infty} \mathrm{e}^{\lambda s}(\phi(s) - K)\,\mathrm{d}s \end{pmatrix}.$$

We observe that A depends on a, b and r, and that the initial history ϕ affects only the initial value $\boldsymbol{\alpha}(\phi)$.

The eigenvalues of the matrix A always have negative real parts. Indeed it can be shown that they are real and unequal if $K < K_1(a,\lambda,\kappa)$ or $K > K_2(a,\lambda,\kappa)$, and complex if $K_1(a,\lambda,\kappa) < K < K_2(a,\lambda,\kappa)$, where $0 < K_1(a,\lambda,\kappa) < K_2(a,\lambda,\kappa)$ are the roots of

$$0 = a^2K^2 - 2(a\lambda + 2\kappa)K + \lambda^2.$$

To illustrate what happens, we confine our attention to $K < K_1(a,\lambda,\kappa)$, which is equivalent to

$$r < r_1 := \left(a + \frac{\kappa}{\lambda}\right)K_1(a,\lambda,\kappa).$$

Let the eigenvalues of A be $-\mu_1$ and $-\mu_2$, with $0 < \mu_1 < \mu_2$. Then [16, theorem 3.5, ch. VIII] implies that $x(t;\boldsymbol{\alpha}) = \mathrm{e}^{-\mu_1 t}(c_1(\boldsymbol{\alpha}) + o(1))$ as $t \to \infty$ for $\boldsymbol{\alpha}$ in a neighbourhood of $\mathbf{0}$ in $\mathbb{R}^2$. The constant $c_1(\boldsymbol{\alpha})$ can vanish, but if that holds and $\boldsymbol{\alpha} \neq \mathbf{0}$, then $x(t;\boldsymbol{\alpha}) = \mathrm{e}^{-\mu_2 t}(c_2(\boldsymbol{\alpha}) + o(1))$ with $c_2(\boldsymbol{\alpha}) \neq 0$, so that the solution decays more quickly than $\mathrm{e}^{-\mu_1 t}$.

4. Proof of theorem 3.1

4.1. Transformation of (2.2) to another integral equation

In order to prove theorem 3.1, we express $x(\cdot;\phi)$ as the solution of a *linear* Volterra integral equation, about whose kernel and forcing function qualitative properties can be inferred from the fact that $x(t) \to 0$ as $t \to \infty$.

It is easily shown that $x(\cdot;\phi)$ is the unique solution of the initial-value problem

$$\dot{x}(t) = -aKx(t) - K\int_0^t b(s)x(t-s)\,\mathrm{d}s + c(t)x(t) - Kf(t), \quad t > 0, \tag{4.1 a}$$

$$x(0) = \phi(0) - K, \tag{4.1 b}$$

where f is given by (2.7) and

$$c(t;\phi) = -ax(t;\phi) - \int_0^t b(t-u)x(u;\phi)\,\mathrm{d}u - f(t;\phi), \quad t \geqslant 0. \tag{4.2}$$

The original nonlinearity has been subsumed in the non-autonomous term $c(t)x(t)$.

Now we show that x solves an integral equation. To this end, we introduce the differential resolvent $z\colon [0,\infty) \to \mathbb{R}$ by

$$\dot{z}(t) = -aKz(t) - K\int_0^t b(t-s)z(s)\,\mathrm{d}s, \quad t > 0, \qquad z(0) = 1, \tag{4.3}$$

and

$$h(t;\phi) = z(t)(\phi(0) - K) - K\int_0^t z(t-s)f(s;\phi)\,\mathrm{d}s, \quad t \geqslant 0. \tag{4.4}$$

By taking the convolution of z with each term in (4.1 a), we obtain the *variation of constants* formula

$$x(t) = \int_0^t z(t-s)c(s)x(s)\,\mathrm{d}s + h(t), \quad t \geqslant 0. \tag{4.5}$$

We view both h and c as known, since the first depends on the resolvent z and the initial history ϕ while the latter depends on the initial history through the known solution $x(\cdot;\phi)$.

It can be shown that (2.3) implies that $\lim_{t\to\infty} z(t) = 0$ and that $z \in L^1(0,\infty)$. As already observed, $\lim_{t\to\infty} f(t;\phi) = 0$ because $\phi \in \Phi$, and therefore $h(t;\phi) \to 0$ as $t \to \infty$ by the dominated convergence theorem. From (4.5), the rate at which $x(t;\phi) \to 0$ as $t \to \infty$ should depend on the rates at which both $z(t) \to 0$ as $t \to \infty$ and $f(t;\phi) \to 0$ as $t \to \infty$.

Lemma 4.1. *Suppose that $b(t) = \beta(t)e^{-\lambda t}$ for all $t \geqslant 0$, where β is a subexponential function with $\lambda \geqslant 0$. Assume also that (3.1) holds. Then*

$$\lim_{t\to\infty} \frac{z(t)}{b(t)} =: L_b z = \frac{-K}{(aK - \lambda + K\int_0^\infty e^{\lambda s} b(s)\,ds)^2} < 0 \tag{4.6}$$

and

$$\int_0^\infty |z(t)| e^{\lambda t}\,dt < \infty. \tag{4.7}$$

Proof. Firstly, we define $p(t) = e^{\lambda t} z(t)$, so that $z(t) = e^{-\lambda t} p(t)$. Substituting this and its derivative into (4.3) yields

$$\dot{p}(t) = -(aK - \lambda)p(t) - K\int_0^t \beta(t-s)p(s)\,ds, \quad t \geqslant 0,$$

and $p(0) = 1$. By multiplying by the integrating factor $e^{-(aK-\lambda)t}$, integrating and using Fubini's theorem,

$$\begin{aligned} p(t) &= e^{-(aK-\lambda)t} - K\int_0^t \int_0^{t-s} e^{-(aK-\lambda)(t-s-v)}\beta(v)\,dv\,p(s)\,ds \\ &= e^{-(aK-\lambda)t} + \int_0^t \Gamma(t-s)p(s)\,ds, \quad t \geqslant 0, \end{aligned} \tag{4.8}$$

where Γ denotes the convolution

$$\Gamma(t) = -K\int_0^t e^{-(aK-\lambda)(t-v)}\beta(v)\,dv, \quad t \geqslant 0. \tag{4.9}$$

By dividing (4.8) by $\beta(t)$ and introducing

$$\eta(t) = \frac{p(t)}{\beta(t)}, \qquad \xi(t) = \frac{e^{-(aK-\lambda)t}}{\beta(t)},$$
$$H(t,s) = \frac{\Gamma(t-s)\beta(s)}{\beta(t)}, \quad 0 \leqslant s \leqslant t,$$

we obtain the integral equation (A 6).

The proof is completed by applying theorem A.2 to show that the solution η of (A 6) tends to a limit as $t \to \infty$. It is a consequence of (3.1) that $0 \leqslant \lambda < aK$. Due to (B 8) we observe that $\xi(t) \to 0$ as $t \to \infty$. A simple consequence of (4.9) is that

$$\int_0^\infty |\Gamma(t)|\,dt = -\int_0^\infty \Gamma(t)\,dt = \frac{K}{aK - \lambda}\int_0^\infty \beta(u)\,du.$$

Hypothesis (3.1) therefore says that

$$\int_0^\infty |\Gamma(t)|\,\mathrm{d}t < 1. \tag{4.10}$$

Applying (B 9) with $\varepsilon = aK - \lambda$ gives

$$\lim_{t\to\infty} \frac{\Gamma(t)}{\beta(t)} = \frac{-K}{aK-\lambda}. \tag{4.11}$$

Therefore, this and (B 2) imply that

$$\begin{aligned}\lim_{t\to\infty} H(t,s) &= \lim_{t\to\infty} \frac{\beta(t-s)}{\beta(t)} \lim_{t\to\infty} \frac{\Gamma(t-s)}{\beta(t-s)}\beta(s) \\ &= \frac{-K}{aK-\lambda}\beta(s) =: H_\infty(s)\end{aligned}$$

uniformly with respect to $s \in [0,T]$ for all $T > 0$. We have proved that (A 1) holds. Since β is integrable, we have that H_∞ is in $L^1(0,\infty)$.

Let $t \geqslant 2T$. Splitting the range of integration, we observe that

$$\int_T^t H(t,s)\,\mathrm{d}s = \int_T^{t-T} H(t,s)\,\mathrm{d}s + \int_{t-T}^t H(t,s)\,\mathrm{d}s. \tag{4.12}$$

By the definition of H,

$$\int_T^{t-T} |H(t,s)|\,\mathrm{d}s = \int_T^{t-T} |\Gamma(t-s)| \frac{\beta(s)}{\beta(t)}\,\mathrm{d}s \leqslant M \int_T^{t-T} \beta(t-s)\frac{\beta(s)}{\beta(t)}\,\mathrm{d}s,$$

where

$$M = \sup_{t\geqslant 0} \frac{|\Gamma(t)|}{\beta(t)}$$

is finite because of (4.11). It follows from (B 3) that

$$\lim_{T\to\infty} \limsup_{t\to\infty} \int_T^{t-T} |H(t,s)|\,\mathrm{d}s = 0.$$

Changing variables in the last integral in (4.12), we get

$$\begin{aligned}\int_{t-T}^t H(t,s)\,\mathrm{d}s &= \int_{t-T}^t \Gamma(t-s)\frac{\beta(s)}{\beta(t)}\,\mathrm{d}s \\ &= \int_0^T \Gamma(u)\frac{\beta(t-u)}{\beta(t)}\,\mathrm{d}u \\ &= \int_0^T \Gamma(u)\left(\frac{\beta(t-u)}{\beta(t)} - 1\right)\mathrm{d}u + \int_0^T \Gamma(u)\,\mathrm{d}u.\end{aligned}$$

Therefore,

$$\begin{aligned}&\left|\int_T^t H(t,s)\,\mathrm{d}s - \int_0^\infty \Gamma(u)\,\mathrm{d}u\right| \\ &\quad\leqslant \int_T^{t-T} |H(t,s)|\,\mathrm{d}s + \int_0^T |\Gamma(u)|\,\mathrm{d}u \max_{0\leqslant u\leqslant T}\left|\frac{\beta(t-u)}{\beta(t)} - 1\right| + \int_T^\infty |\Gamma(u)|\,\mathrm{d}u.\end{aligned}$$

Since β obeys (B 2), we see that

$$\limsup_{t\to\infty}\left|\int_T^t H(t,s)\,\mathrm{d}s - \int_0^\infty \Gamma(u)\,\mathrm{d}u\right| \leqslant \limsup_{t\to\infty}\int_T^{t-T}|H(t,s)|\,\mathrm{d}s + \int_T^\infty |\Gamma(u)|\,\mathrm{d}u,$$

and hence that (A 3) holds with

$$V = \int_0^\infty \Gamma(u)\,\mathrm{d}u.$$

A similar argument shows that

$$W := \lim_{T\to\infty}\limsup_{t\to\infty}\int_T^t |H(t,s)|\,\mathrm{d}s \leqslant \int_0^\infty |\Gamma(u)|\,\mathrm{d}u. \tag{4.13}$$

By (4.10), $0 \leqslant W < 1$ and (A 2) is true.

All the conditions of theorem A.2 apply to the solution η of (A 6), and therefore

$$\begin{aligned}\eta(\infty) &= \frac{1}{1-V}\int_0^\infty H_\infty(s)\eta(s)\,\mathrm{d}s \\ &= \frac{-K}{aK-\lambda+K\int_0^\infty \beta(s)\,\mathrm{d}s}\int_0^\infty p(s)\,\mathrm{d}s. \end{aligned}\tag{4.14}$$

Integrating both sides of (4.8) over $(0,\infty)$ yields

$$\int_0^\infty p(t)\,\mathrm{d}t = \frac{1}{aK-\lambda} - \frac{K}{aK-\lambda}\int_0^\infty \beta(s)\,\mathrm{d}s\int_0^\infty p(s)\,\mathrm{d}s,$$

which leads to

$$\int_0^\infty p(t)\,\mathrm{d}t = \frac{1}{aK-\lambda+K\int_0^\infty \beta(s)\,\mathrm{d}s}. \tag{4.15}$$

Since

$$\eta(\infty) = \lim_{t\to\infty}\frac{p(t)}{\beta(t)} = \lim_{t\to\infty}\frac{z(t)}{b(t)},$$

equation (4.6) is obtained if we put (4.15) into (4.14). Because $\lim_{t\to\infty} z(t)/b(t)$ exists and

$$\int_0^\infty \mathrm{e}^{\lambda s} b(s)\,\mathrm{d}s$$

is finite,

$$\int_0^\infty \mathrm{e}^{\lambda s}|z(s)|\,\mathrm{d}s$$

is also finite, which is (4.7). □

4.2. Proof of part (i) of theorem 3.1

Let $\gamma\colon [0,\infty) \to (0,\infty)$ be defined by

$$\gamma(t) = \mathrm{e}^{-\lambda t}\int_t^\infty b(s)\mathrm{e}^{\lambda s}\,\mathrm{d}s = \mathrm{e}^{-\lambda t}\int_t^\infty \beta(s)\,\mathrm{d}s, \quad t \geqslant 0. \tag{4.16}$$

Since subexponential functions satisfy (B 5),

$$\lim_{t\to\infty}\frac{b(t)}{\gamma(t)}=0, \tag{4.17}$$

so that $\gamma(t)\to 0$ more slowly than $b(t)\to 0$ as $t\to\infty$. This is just (3.7).

LEMMA 4.2. *Suppose that $b(t)=\beta(t)\mathrm{e}^{-\lambda t}$ for all $t\geqslant 0$, where $\lambda\geqslant 0$ and β is a subexponential function. Then, for each $T>0$,*

$$\frac{\gamma(t-s)\mathrm{e}^{-\lambda s}}{\gamma(t)}\to 1 \quad \textit{as } t\to\infty \textit{ uniformly for } s\in[0,T] \tag{4.18}$$

Proof. Firstly, we note that

$$\frac{\gamma(t-s)\mathrm{e}^{-\lambda s}}{\gamma(t)}-1=\frac{\int_{t-s}^{\infty}\beta(u)\,\mathrm{d}u}{\int_{t}^{\infty}\beta(u)\,\mathrm{d}u}-1=\frac{\int_{t-s}^{t}\beta(u)\,\mathrm{d}u}{\int_{t}^{\infty}\beta(u)\,\mathrm{d}u}.$$

But we can deduce from L'Hôpital's rule and (B 5) that

$$\lim_{t\to\infty}\frac{\int_{t-s}^{t}\beta(u)\,\mathrm{d}u}{\int_{t}^{\infty}\beta(u)\,\mathrm{d}u}=\lim_{t\to\infty}\left(\frac{\beta(t-s)}{\beta(t)}-1\right)=0$$

uniformly with respect to $s\in[0,T]$. Hence, (4.18) holds. □

LEMMA 4.3. *Suppose that $b(t)=\beta(t)\mathrm{e}^{-\lambda t}$ for all $t\geqslant 0$, where $\lambda\geqslant 0$ and β is a subexponential function. If $\phi\in\Phi$ satisfies (3.3),*

$$\lim_{t\to\infty}\frac{f(t;\phi)}{\gamma(t)}=L(\phi,K),$$

where $L(\phi,K)$ is given by (3.3).

Proof. For brevity we write $\psi(t)=\mathrm{e}^{\lambda t}[\phi(t)-K]-L(\phi,K)$, $t\leqslant 0$. A short computation using the definitions of $f(\cdot;\phi)$ and γ leads to

$$\frac{f(t;\phi)}{\gamma(t)}-L(\phi,K)=\frac{\mathrm{e}^{-\lambda t}}{\gamma(t)}\int_{t}^{\infty}\beta(s)\psi(t-s)\,\mathrm{d}s=:J(t).$$

Therefore, to prove the result, it is enough to show that $J(t)\to 0$ as $t\to\infty$. To do so, the integral over $[t,\infty)$ is split into integrals over $[t,t+T]$ and $[t+T,\infty)$. Thus, for $T\geqslant 0$,

$$\begin{aligned}|J(t)|&\leqslant\frac{\mathrm{e}^{-\lambda t}}{\gamma(t)}\int_{t}^{t+T}\beta(s)|\psi(t-s)|\,\mathrm{d}s+\frac{\mathrm{e}^{-\lambda t}}{\gamma(t)}\int_{t+T}^{\infty}\beta(s)|\psi(t-s)|\,\mathrm{d}s\\&\leqslant\int_{0}^{T}\frac{\mathrm{e}^{-\lambda(t+u)}\beta(t+u)}{\gamma(t+u)}\frac{\gamma(t+u)\mathrm{e}^{\lambda u}}{\gamma(t)}|\psi(-u)|\,\mathrm{d}u\\&\qquad+\sup_{u\leqslant -T}|\psi(u)|\frac{\int_{t+T}^{\infty}\beta(s)\,\mathrm{d}s}{\int_{t}^{\infty}\beta(s)\,\mathrm{d}s}.\end{aligned}$$

By lemma 4.2 and (4.17), the integrand of the first integral tends to zero as $t \to \infty$ uniformly with respect to $u \in [0, T]$, and hence the integral tends to zero. By L'Hôpital's rule and (B 2),

$$\lim_{t\to\infty} \frac{\int_{t+T}^{\infty} \beta(s)\,\mathrm{d}s}{\int_t^{\infty} \beta(s)\,\mathrm{d}s} = \lim_{t\to\infty} \frac{\beta(t+T)}{\beta(t)} = 1.$$

It follows that

$$\limsup_{t\to\infty} |J(t)| \leqslant \sup_{u\leqslant -T} |\psi(u)|.$$

Since $\psi(u) \to 0$ as $u \to -\infty$, we conclude that $\lim_{t\to\infty} |J(t)| = 0$. □

Lemma 4.4. *Suppose that $b(t) = \beta(t)\mathrm{e}^{-\lambda t}$ for all $t \geqslant 0$, where $\lambda \geqslant 0$ and β is a subexponential function. Assume also that (3.1) holds. If $\phi \in \Phi$ satisfies (3.3), then*

$$\lim_{t\to\infty} \frac{h(t;\phi)}{\gamma(t)} = -L(\phi, K)K \int_0^{\infty} z(u)\mathrm{e}^{\lambda u}\,\mathrm{d}u, \tag{4.19}$$

where $L(\phi, K)$ is given by (3.3).

Proof. From (4.6) and (4.17),

$$\frac{z(t)}{\gamma(t)} = \frac{z(t)}{b(t)} \frac{b(t)}{\gamma(t)} \to 0 \quad \text{as } t \to \infty. \tag{4.20}$$

It follows then from (4.4) that it suffices to prove that

$$\frac{1}{\gamma(t)} \int_0^t z(t-s)f(s)\,\mathrm{d}s = \int_0^t H(t,s)\xi(s)\,\mathrm{d}s$$

tends to a limit as $t \to \infty$, where in this proof

$$H(t,s) = \frac{z(t-s)\gamma(s)}{\gamma(t)}, \quad 0 \leqslant s \leqslant t; \qquad \xi(t) = \frac{f(t)}{\gamma(t)}, \quad t \geqslant 0.$$

This is done by applying theorem A.1. Lemma 4.3 says that $\lim_{t\to\infty} \xi(t)$ exists. It remains to verify that the hypotheses concerning H hold.

Note that lemma 4.2 and (4.20) imply that

$$H(t,s) = \frac{z(t-s)}{\gamma(t-s)} \frac{\gamma(t-s)\mathrm{e}^{-\lambda s}}{\gamma(t)} \gamma(s)\mathrm{e}^{\lambda s} \to 0$$

as $t \to \infty$, so that $H_\infty(s) = 0$. Moreover, lemma 4.2 assures us that this convergence is uniform for $0 \leqslant s \leqslant T$.

It remains to establish (A 2) and (A 3). Let $T \geqslant 0$. For $t \geqslant 2T$, we can write

$$\int_T^t |H(t,s)|\,\mathrm{d}s = \int_T^{t-T} |H(t,s)|\,\mathrm{d}s + \int_{t-T}^t |H(t,s)|\,\mathrm{d}s.$$

Using the notation

$$I(t,T) := \int_T^{t-T} \beta(t-s) \frac{\int_s^{\infty} \beta(u)\,\mathrm{d}u}{\int_t^{\infty} \beta(u)\,\mathrm{d}u}\,\mathrm{d}s, \quad t \geqslant 2T, \tag{4.21}$$

we observe that

$$\int_T^{t-T} |H(t,s)|\,\mathrm{d}s = \int_T^{t-T} \frac{|z(t-s)|}{b(t-s)}\beta(t-s)\frac{\int_s^\infty \beta(u)\,\mathrm{d}u}{\int_t^\infty \beta(u)\,\mathrm{d}u}\,\mathrm{d}s$$
$$\leqslant \sup_{u\geqslant 0}\frac{|z(u)|}{b(u)} I(t,T), \tag{4.22}$$

because $|z(t)|/b(t)$ tends to a limit as $t\to\infty$. By lemma 4.2,

$$\int_{t-T}^{t} |H(t,s)|\,\mathrm{d}s = \int_0^T |z(u)|\mathrm{e}^{\lambda u}\frac{\gamma(t-u)\mathrm{e}^{-\lambda u}}{\gamma(t)}\,\mathrm{d}u \to \int_0^T |z(u)|\mathrm{e}^{\lambda u}\,\mathrm{d}u \tag{4.23}$$

as $t\to\infty$. Therefore,

$$\limsup_{t\to\infty}\int_T^t |H(t,s)|\,\mathrm{d}s \leqslant \sup_{u\geqslant 0}\frac{|z(u)|}{b(u)}\limsup_{t\to\infty} I(t,T) + \int_0^T |z(u)|\mathrm{e}^{\lambda u}\,\mathrm{d}u, \tag{4.24}$$

and it is a consequence of proposition B.2 that

$$\lim_{T\to\infty}\limsup_{t\to\infty}\int_T^t |H(t,s)|\,\mathrm{d}s \leqslant \int_0^\infty |z(u)|\mathrm{e}^{\lambda u}\,\mathrm{d}u.$$

Hence, (A 2) is satisfied.

Next it is proved that (A 3) holds with

$$V = \int_0^\infty z(u)\mathrm{e}^{\lambda u}\,\mathrm{d}u. \tag{4.25}$$

Let $T\geqslant 0$. For $t\geqslant 2T$,

$$\left|\int_T^t H(t,s)\,\mathrm{d}s - \int_0^\infty z(u)\mathrm{e}^{\lambda u}\,\mathrm{d}u\right|$$
$$\leqslant \int_T^{t-T} |H(t,s)|\,\mathrm{d}s + \left|\int_{t-T}^t H(t,s)\,\mathrm{d}s - \int_0^T z(u)\mathrm{e}^{\lambda u}\,\mathrm{d}u\right| + \int_T^\infty |z(u)|\mathrm{e}^{\lambda u}\,\mathrm{d}u. \tag{4.26}$$

By the same argument that led to (4.23),

$$\int_{t-T}^t H(t,s)\,\mathrm{d}s \to \int_0^T z(u)\mathrm{e}^{\lambda u}\,\mathrm{d}u \quad \text{as } t\to\infty.$$

This and (4.22) imply that

$$\limsup_{t\to\infty}\left|\int_T^t H(t,s)\,\mathrm{d}s - \int_0^\infty z(u)\mathrm{e}^{\lambda u}\,\mathrm{d}u\right|$$
$$\leqslant \sup_{u\geqslant 0}\frac{|z(u)|}{b(u)}\limsup_{t\to\infty} I(t,T) + \int_T^\infty |z(u)|\mathrm{e}^{\lambda u}\,\mathrm{d}u.$$

Clearly, lemma 4.1 and (B 6) imply (A 3) with V given by (4.25).

By substituting the expressions found for $\lim_{t\to\infty} z(t)/\gamma(t)$, $H_\infty(s)$ and V into (A 4), we obtain (4.19). □

Proof of part (i) of theorem 3.1. We see from (4.5) that

$$\frac{x(t)}{\gamma(t)} = \int_0^t \frac{z(t-s)c(s)\gamma(s)}{\gamma(t)} \frac{x(s)}{\gamma(s)}\,\mathrm{d}s + \frac{h(t)}{\gamma(t)},$$

which is in the form of (A 6) if

$$\eta(t) = \frac{x(t)}{\gamma(t)}, \quad H(t,s) = \frac{z(t-s)c(s)\gamma(s)}{\gamma(t)}, \quad \xi(t) = \frac{h(t)}{\gamma(t)}.$$

The kernel $H(t,s)$ used here differs by a factor of $c(s)$ from that employed in the proof of lemma 4.4. theorem A.2 is now used to show that $\lim_{t\to\infty}\eta(t)$ exists.

Lemma 4.4 asserts that $\lim_{t\to\infty}\xi(t)$ exists. The remaining assumptions of theorem A.2 are now verified. It is clear from (4.20) that

$$H(t,s) = \frac{z(t-s)}{\gamma(t-s)} \frac{\gamma(t-s)\mathrm{e}^{-\lambda s}}{\gamma(t)} \gamma(s)\mathrm{e}^{\lambda s} c(s) \to 0$$

as $t\to\infty$. With the aid of lemma 4.2 it can be shown that the convergence is uniform with respect to s in compact $[0,T]$. To establish (A 5), observe that, for $t \geqslant T$,

$$\begin{aligned}\limsup_{t\to\infty}\int_T^t |H(t,s)|\,\mathrm{d}s &= \limsup_{t\to\infty}\int_T^t |z(t-s)|\frac{\gamma(s)}{\gamma(t)}|c(s)|\,\mathrm{d}s \\ &\leqslant \sup_{s\geqslant T}|c(s)|\biggl(\sup_{u\geqslant 0}\frac{|z(u)|}{b(u)}\limsup_{t\to\infty} I(t,T) + \int_0^T |z(u)|\mathrm{e}^{\lambda u}\,\mathrm{d}u\biggr),\end{aligned}$$

where I is given by (4.21). Because $c(t)\to 0$ as $t\to\infty$,

$$\limsup_{T\to\infty}\limsup_{t\to\infty}\int_T^t |H(t,s)|\,\mathrm{d}s = 0,$$

and (A 5) holds with $W=0$. We deduce from

$$\biggl|\int_T^t H(t,s)\,\mathrm{d}s\biggr| \leqslant \int_T^t |H(t,s)|\,\mathrm{d}s \tag{4.27}$$

that (A 3) is satisfied with $V=0$. Then theorem A.2 asserts that $\eta(t) = x(t)/\gamma(t)$ converges as $t\to\infty$. Indeed, by substituting for $H_\infty(s)$ and V into (A 7), we observe that

$$\eta(\infty) = \xi(\infty) = -L(\phi,K)K\int_0^\infty z(u)\mathrm{e}^{\lambda u}\,\mathrm{d}u,$$

which clearly vanishes if and only if $L(\phi,K)=0$. □

4.3. Proof of part (ii) of theorem 3.1

Lemma 4.5. *Suppose that $b(t) = \beta(t)\mathrm{e}^{-\lambda t}$ for all $t \geqslant 0$, where $\lambda \geqslant 0$ and β is a decreasing subexponential function. If $\phi \in \Phi$ satisfies (3.5), then*

$$\lim_{t\to\infty}\frac{f(t;\phi)}{b(t)} = \int_{-\infty}^0 \mathrm{e}^{\lambda u}(\phi(u)-K)\,\mathrm{d}u.$$

Proof. We use the notation $\chi(t) = \phi(t) - K$ for $t \leqslant 0$. Let $T > 0$. Then

$$\frac{f(t)}{b(t)} - \int_{-\infty}^{0} \chi(s)\mathrm{e}^{\lambda s}\,\mathrm{d}s = \int_{-T}^{0} \left\{\frac{\beta(t-s)}{\beta(t)} - 1\right\}\chi(s)\mathrm{e}^{\lambda s}\,\mathrm{d}s + \int_{-\infty}^{-T} \frac{\beta(t-s)}{\beta(t)}\chi(s)\mathrm{e}^{\lambda s}\,\mathrm{d}s - \int_{-\infty}^{-T} \chi(s)\mathrm{e}^{\lambda s}\,\mathrm{d}s. \quad (4.28)$$

The first term in (4.28) can be estimated according to

$$\left|\int_{-T}^{0} \left\{\frac{\beta(t-s)}{\beta(t)} - 1\right\}\chi(s)\mathrm{e}^{\lambda s}\,\mathrm{d}s\right| \leqslant \max_{-T\leqslant s\leqslant 0}\left|\frac{\beta(t-s)}{\beta(t)} - 1\right| \int_{-T}^{0} |\chi(s)|\mathrm{e}^{\lambda s}\,\mathrm{d}s. \quad (4.29)$$

It follows from (B 2) that

$$\lim_{t\to\infty} \int_{-T}^{0} \left\{\frac{\beta(t-s)}{\beta(t)} - 1\right\}\chi(s)\mathrm{e}^{\lambda s}\,\mathrm{d}s = 0. \quad (4.30)$$

Because of the monotonicity, the second term in (4.28) can be estimated by

$$\left|\int_{-\infty}^{-T} \frac{\beta(t-s)}{\beta(t)}\chi(s)\mathrm{e}^{\lambda s}\,\mathrm{d}s\right| \leqslant \int_{-\infty}^{-T} \frac{\beta(t-s)}{\beta(t)}|\chi(s)|\mathrm{e}^{\lambda s}\,\mathrm{d}s \leqslant \int_{-\infty}^{-T} |\chi(s)|\mathrm{e}^{\lambda s}\,\mathrm{d}s.$$

We deduce from (4.28)–(4.30) that

$$\limsup_{t\to\infty} \left|\frac{f(t)}{b(t)} - \int_{-\infty}^{0} \chi(s)\mathrm{e}^{\lambda s}\,\mathrm{d}s\right| \leqslant 2\int_{-\infty}^{-T} |\chi(s)|\mathrm{e}^{\lambda s}\,\mathrm{d}s.$$

Letting $T \to \infty$ gives the required result. □

The following result is a consequence of [5, Proposition 4]. It can also be derived directly from theorem A.1, just as lemma 4.4 was.

LEMMA 4.6. *Suppose that $b(t) = \beta(t)\mathrm{e}^{-\lambda t}$ for all $t \geqslant 0$, where $\lambda \geqslant 0$ and β is a decreasing subexponential function. Assume also that (3.1) holds. If $\phi \in \Phi$ satisfies (3.5), then*

$$\lim_{t\to\infty} \frac{h(t)}{b(t)} =: L_b h \text{ exists.}$$

Proof of part (ii) of theorem 3.1. Dividing equation (4.5) by $b(t)$ yields

$$\frac{x(t)}{b(t)} = \int_{0}^{t} \frac{z(t-s)c(s)b(s)}{b(t)}\frac{x(s)}{b(s)}\,\mathrm{d}s + \frac{h(t)}{b(t)},$$

which can be written as (A 6) if

$$\eta(t) = \frac{x(t)}{b(t)}, \qquad H(t,s) = \frac{z(t-s)c(s)b(s)}{b(t)}, \qquad \xi(t) = \frac{h(t)}{b(t)}.$$

Note that the kernel $H(t,s)$ used here is different to those employed previously. Theorem A.2 is now used to show that $\lim_{t\to\infty}\eta(t)$ exists.

Lemma 4.6 asserts that $\lim_{t\to\infty}\xi(t)$ exists. The remaining assumptions of theorem A.2 are now verified. It is clear from

$$H(t,s)=\frac{z(t-s)}{b(t-s)}\frac{\beta(t-s)}{\beta(t)}\beta(s)c(s),$$

and (B 2) that $H(t,s)\to(L_bz)\beta(s)c(s)=:H_\infty(s)$ as $t\to\infty$, uniformly with respect to s in compact $[0,T]$.

To establish (A 2), observe that for $t\geqslant T$,

$$\begin{aligned}\int_T^t|H(t,s)|\,\mathrm{d}s&=\int_T^t\frac{|z(t-s)|}{b(t-s)}\frac{\beta(t-s)\beta(s)}{\beta(t)}|c(s)|\,\mathrm{d}s\\&\leqslant\sup_{s\geqslant T}|c(s)|\sup_{0\leqslant u\leqslant t-T}\frac{|z(u)|}{b(u)}\int_T^t\frac{\beta(t-s)\beta(s)}{\beta(t)}\,\mathrm{d}s,\end{aligned}$$

which implies by (B 4) that

$$\limsup_{t\to\infty}\int_T^t|H(t,s)|\,\mathrm{d}s\leqslant\sup_{s\geqslant T}|c(s)|\sup_{u\geqslant 0}\frac{|z(u)|}{b(u)}\biggl(2\int_0^\infty\beta(s)\,\mathrm{d}s-\int_0^T\beta(s)\,\mathrm{d}s\biggr).$$

Because $c(t)\to 0$ as $t\to\infty$,

$$\limsup_{T\to\infty}\limsup_{t\to\infty}\int_T^t|H(t,s)|\,\mathrm{d}s=0,$$

and (A 2) holds with $W=0$. We deduce from

$$\biggl|\int_T^t H(t,s)\,\mathrm{d}s\biggr|\leqslant\int_T^t|H(t,s)|\,\mathrm{d}s \tag{4.31}$$

that (A 3) is satisfied with $V=0$. The conclusions of theorem A.2 are therefore true; in particular a limit formula can be obtained from (A 7). □

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Appendix A. Admissibility results

In this appendix, $H\colon\Delta\to\mathbb{R}$ is a continuous function on

$$\Delta:=\{(t,s)\in\mathbb{R}^2\colon 0\leqslant s\leqslant t\}.$$

Associated with H is the operator $\mathcal{H}\colon C[0,\infty)\to C[0,\infty)$ defined by

$$(\mathcal{H}\xi)(t)=\int_0^t H(t,s)\xi(s)\,\mathrm{d}s,\quad t\geqslant 0.$$

Firstly, we state a theorem, which is a variant of a part of an exercise in [9, Exercise 3, p. 74].

Theorem A.1. *Suppose that, for all $T > 0$,*

$$H(t,s) \to H_\infty(s) \quad \text{as } t \to \infty \text{ uniformly with respect to } s \in [0,T]. \tag{A 1}$$

Further assume that

$$W := \lim_{T\to\infty} \limsup_{t\to\infty} \int_T^t |H(t,s)|\,\mathrm{d}s < \infty, \tag{A 2}$$

$$\lim_{T\to\infty} \limsup_{t\to\infty} \left| \int_T^t H(t,s)\,\mathrm{d}s - V \right| = 0 \quad \text{for some } V \in \mathbb{R}. \tag{A 3}$$

Then $\lim_{t\to\infty}(\mathcal{H}\xi)(t)$ *exists for all* ξ *for which* $\lim_{t\to\infty}\xi(t) =: \xi(\infty)$ *exists, and*

$$\lim_{t\to\infty} (\mathcal{H}\xi)(t) = \int_0^\infty H_\infty(s)\xi(s)\,\mathrm{d}s + V\xi(\infty). \tag{A 4}$$

Proof. Firstly, it is demonstrated that $s \mapsto H_\infty(s)$ is in $L^1(0,\infty)$, and hence that the first term in (A 4) is well defined for all bounded ξ, for if (A 2) is satisfied, there is $T_0 \geqslant 0$ such that

$$\limsup_{t\to\infty} \int_{T_0}^t |H(t,s)|\,\mathrm{d}s$$

is finite. Then (A 1) implies that

$$\limsup_{t\to\infty} \int_0^t |H(t,s)|\,\mathrm{d}s = \int_0^{T_0} |H_\infty(s)|\,\mathrm{d}s + \limsup_{t\to\infty} \int_{T_0}^t |H(t,s)|\,\mathrm{d}s < \infty.$$

For all $T > 0$, it follows that

$$\int_0^T |H_\infty(s)|\,\mathrm{d}s = \lim_{t\to\infty} \int_0^T |H(t,s)|\,\mathrm{d}s \leqslant \limsup_{t\to\infty} \int_0^t |H(t,s)|\,\mathrm{d}s < \infty.$$

Let $T > 0$. Then, for all $t \geqslant T$,

$$\begin{aligned}
\int_0^t H(t,s)\xi(s)\,\mathrm{d}s &- \int_0^\infty H_\infty(s)\xi(s)\,\mathrm{d}s - V\xi(\infty) \\
&= \int_0^T [H(t,s) - H_\infty(s)]\xi(s) - \int_T^\infty H_\infty(s)\xi(s)\,\mathrm{d}s \\
&\quad + \left(\int_T^t H(t,s)\,\mathrm{d}s - V \right)\xi(\infty) + \int_T^t H(t,s)[\xi(s) - \xi(\infty)]\,\mathrm{d}s.
\end{aligned}$$

The absolute value of the right-hand side is less than or equal to

$$\begin{aligned}
\int_0^T |H(t,s) - H_\infty(s)|\,\mathrm{d}s \sup_{s\geqslant 0}|\xi(s)| &+ \int_T^\infty |H_\infty(s)|\,\mathrm{d}s \sup_{s\geqslant 0}|\xi(s)| \\
&+ \left| \int_T^t H(t,s)\,\mathrm{d}s - V \right| |\xi(\infty)| + \int_T^t |H(t,s)|\,\mathrm{d}s \sup_{s\geqslant T}|\xi(s) - \xi(\infty)|.
\end{aligned}$$

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The limit superior of this expression as $t \to \infty$ is equal to

$$\int_T^\infty |H_\infty(s)|\,\mathrm{d}s \sup_{s\geqslant 0}|\xi(s)| + \limsup_{t\to\infty}\left|\int_T^t H(t,s)\,\mathrm{d}s - V\right||\xi(\infty)| \\ + \limsup_{t\to\infty}\int_T^t |H(t,s)|\,\mathrm{d}s \sup_{s\geqslant T}|\xi(s)-\xi(\infty)|.$$

But the limit superior of this expression as $T \to \infty$ is zero, due to H_∞ being in L^1, the property of V in (A 3), the finiteness of W in (A 2) and the fact that $\lim_{s\to\infty}\xi(s) = \xi(\infty)$. □

The next result extends [5, theorem 5] to non-convolution integral equations (cf. [5, theorem A.1]); it is also the counterpart of [4, theorems 3.1 and 5.1] and [15, theorem 3.1], which concern linear non-convolution difference equations.

Theorem A.2. *Suppose that (A 1) and (A 3) hold, and that (A 2) is satisfied with*

$$W < 1. \tag{A 5}$$

Assume that ξ is in $C[0,\infty)$ and $\lim_{t\to\infty}\xi(t) =: \xi(\infty)$ exists. If $\eta\colon [0,\infty) \to \mathbb{R}$ is the continuous solution of

$$\eta(t) = \xi(t) + \int_0^t H(t,s)\eta(s)\,\mathrm{d}s, \quad t \geqslant 0, \tag{A 6}$$

then $\lim_{t\to\infty}\eta(t) =: \eta(\infty)$ exists, and satisfies the limit formula

$$\eta(\infty) = (1-V)^{-1}\left[\xi(\infty) + \int_0^\infty H_\infty(s)\eta(s)\,\mathrm{d}s\right]. \tag{A 7}$$

Remark A.3. $(1-V)^{-1}$ in (A 7) is finite because $|V| < 1$: this is due to (A 5) and

$$\left|\limsup_{t\to\infty}\int_T^t H(t,s)\,\mathrm{d}s\right| \leqslant \limsup_{t\to\infty}\int_T^t |H(t,s)|\,\mathrm{d}s.$$

Appendix B. Subexponential functions

The definition of subexponential function is based on the hypotheses of [8, theorem 3]. Variants have been used in several papers, including [1,2,5]. We employ the definition used in [5].

Definition B.1. Let $\beta\colon [0,\infty) \to (0,\infty)$ be a continuous function. Then β is *subexponential* if

$$\int_0^\infty \beta(t)\,\mathrm{d}t < \infty, \tag{B 1}$$

$$\lim_{t\to\infty}\frac{\beta(t-s)}{\beta(t)} = 1 \quad \text{uniformly for } 0 \leqslant s \leqslant T, \text{ for all } T > 0, \tag{B 2}$$

$$\lim_{T\to\infty}\limsup_{t\to\infty}\int_T^{t-T}\frac{\beta(t-s)\beta(s)}{\beta(t)}\,\mathrm{d}s = 0. \tag{B 3}$$

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Condition (B 3) is convenient to use in proofs. However, if (B 1) and (B 2) are both true, (B 3) is equivalent to

$$\lim_{t\to\infty}\int_0^t \frac{\beta(t-s)\beta(s)}{\beta(t)}\,\mathrm{d}s = 2\int_0^\infty \beta(s)\,\mathrm{d}s. \tag{B 4}$$

Simple examples of subexponential functions are $\beta(t) = (1+t)^{-\alpha}$ for $\alpha > 1$, $\beta(t) = \mathrm{e}^{-(1+t)^\alpha}$ for $0 < \alpha < 1$ and $\beta(t) = \mathrm{e}^{-t/\log(t+2)}$. Thus, subexponential functions need not decay polynomially.

It was shown in [1, p. 20] that a subexponential function β has the property that

$$\lim_{t\to\infty}\frac{\beta(t)}{\int_t^\infty \beta(u)\,\mathrm{d}u} = 0. \tag{B 5}$$

The argument can be given briefly. If $S > 0$,

$$\begin{aligned}\beta(t)\biggl(\int_t^\infty \beta(u)\,\mathrm{d}u\biggr)^{-1} &\leqslant \frac{\beta(t)}{\beta(t+S)}\biggl(\beta(t+S)\biggl(\int_t^{S+t}\beta(u)\,\mathrm{d}u\biggr)^{-1}\biggr)\\ &= \frac{\beta(t)}{\beta(t+S)}\biggl(\int_0^S \frac{\beta(t+S-s)}{\beta(t+S)}\,\mathrm{d}s\biggr)^{-1} \to \frac{1}{S}\end{aligned}$$

as $t\to\infty$. Since $S > 0$ can be made arbitrarily large, the result is true. The proof of part (i) of theorem 3.1 relies on

$$\lim_{T\to\infty}\limsup_{t\to\infty}\int_T^{t-T}\beta(t-s)\frac{\int_s^\infty \beta(u)\,\mathrm{d}u}{\int_t^\infty \beta(u)\,\mathrm{d}u}\,\mathrm{d}s = 0 \tag{B 6}$$

being true. To ensure this, we impose a condition on the rate of convergence in (B 5): this condition is just (3.2) rewritten in terms of β.

Proposition B.2. *If β is a subexponential function, and there is a decreasing $\delta\colon(0,\infty)\to(0,\infty)$ such that*

$$\frac{\beta(t)}{\int_t^\infty \beta(s)\,\mathrm{d}s} \sim \delta(t) \quad \text{as } t\to\infty, \tag{B 7}$$

then (B 6) holds.

Proof. Let $\varepsilon > 0$. By (B 7) there is an $S(\varepsilon) > 0$ such that

$$\frac{\delta(t)}{1+\varepsilon}\int_t^\infty \beta(s)\,\mathrm{d}s < \beta(t) < (1+\varepsilon)\delta(t)\int_t^\infty \beta(s)\,\mathrm{d}s,$$

whenever $t > S(\varepsilon)$. Let $T > S(\varepsilon)$. If $t \geqslant 2T$,

$$\begin{aligned}\int_T^{t-T}\beta(t-s)\frac{\int_s^\infty \beta(u)\,\mathrm{d}u}{\int_t^\infty \beta(u)\,\mathrm{d}u}\,\mathrm{d}s &< (1+\varepsilon)^2\int_T^{t-T}\frac{\beta(t-s)\beta(s)}{\beta(t)}\frac{\delta(t)}{\delta(s)}\,\mathrm{d}s\\ &\leqslant (1+\varepsilon)^2\int_T^{t-T}\frac{\beta(t-s)\beta(s)}{\beta(t)}\,\mathrm{d}s,\end{aligned}$$

since δ is decreasing. The result is now a consequence of (B 3). □

In our proofs we also use the fact that subexponential functions have the property that, for every $\varepsilon > 0$,

$$\beta(t)\mathrm{e}^{\varepsilon t} \to \infty \quad \text{as } t \to \infty, \tag{B 8}$$

(cf. [3]). Also we employ the formula

$$\lim_{t\to\infty} \frac{1}{\beta(t)} \int_0^t \beta(t-s)\mathrm{e}^{-\varepsilon s}\,\mathrm{d}s = \frac{1}{\varepsilon}. \tag{B 9}$$

This can be deduced from [5, proposition 4], or by writing

$$\int_0^t \frac{\beta(t-s)}{\beta(t)}\mathrm{e}^{-\varepsilon s}\,\mathrm{d}s = \int_0^t \underbrace{\frac{\beta(t-s)\beta(s)}{\beta(t)}}_{H(t,s)} \underbrace{\frac{\mathrm{e}^{-\varepsilon s}}{\beta(s)}}_{\xi(s)}\,\mathrm{d}s,$$

and applying theorem A.1.

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