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Cantor spectrum for CMV and Jacobi matrices with coefficients arising from generalized skew-shifts

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Abstract. We consider continuous cocycles arising from CMV and Jacobi matrices. Assuming that the Verblunsky and Jacobi coefficients arise from generalized skew-shifts, we prove that uniform hyperbolicity of the associated cocycles is C^0-dense. This implies that the associated CMV and Jacobi matrices have a Cantor spectrum for a generic continuous sampling map.

Key words: CMV matrices, Jacobi matrices, spectra, cocycle, uniform hyperbolicity
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1. *Introduction*

Let X be a compact metric space and let $T : X \to X$ be a strictly ergodic homeomorphism (that is, T is minimal and uniquely ergodic), which fibers over an almost periodic dynamical system (generalized skew-shifts). This means that there exist an infinite compact abelian group $\mathbb{G}$ and an onto continuous map $f : X \to \mathbb{G}$ such that $T(f(x)) = T(x) + g$ for some $g \in \mathbb{G}$. We consider Cantero, Moral and Velazquez (CMV) matrices and Jacobi matrices whose Verblunsky coefficients and, respectively, Jacobi coefficients are obtained by a continuous sampling map along an orbit of T. Our interest is to investigate spectral properties.

By the nature of dynamically defined Verblunsky and Jacobi coefficients, our results rely on a connection between spectral properties and dynamics of linear cocycles, which was first established by Johnson [10], often called Johnson's theorem. Roughly speaking, Johnson's theorem provides a connection between the spectrum of self-adjoint linear differential operators and uniform hyperbolicity of the associated linear cocycles referred to as 'an exponential dichotomy' in Johnson [10].

Two similar results, which are directly connected to our work, are Damanik *et al* [6] for CMV matrices and Marx [11] for Jacobi matrices. In [6], the authors showed that the uniform spectrum of a CMV matrix consists of unimodular complex numbers whose associated cocycles are not uniformly hyperbolic. Likewise, in [11], the author proved that the uniform spectrum of a Jacobi matrix consists of energies whose associated cocycles are not uniformly hyperbolic.

In this paper, we consider the continuous cocycles arising from CMV and Jacobi matrices and show that uniform hyperbolicity is C^0-dense in both cases. Together with the results in [6, 11], this implies that the uniform spectrum of a CMV or Jacobi matrix is a Cantor set for a generic continuous sampling map.

Let us discuss a paper which is intimately related to our work. In Avila *et al* [1], the authors considered continuous $\mathrm{SL}(2, \mathbb{R})$-cocycles with the same base dynamics as in the present paper. The authors proved that if a cocycle is not uniformly hyperbolic and its homotopy class does not display a certain obstruction, it can be C^0-perturbed to become uniformly hyperbolic. Using this and 'a projection lemma', which was also proved in [1], the authors showed that uniform hyperbolicity is C^0-dense for the cocycles arising from Schrödinger operators. In turn, the C^0-denseness implies a Cantor spectrum for a generic continuous potential. In Bochi [2], the author extended the denseness of uniformly hyperbolicity to the denseness of dominated splitting in higher dimensions. Provided that T is a minimal diffeomorphism and the fiber dimension is at least three, a linear cocycle (or, more generally, a vector bundle automorphism) fibered over T can be approximated by another cocycle admitting a dominated splitting. The dominated splitting is equivalent to uniform hyperbolicity in $\mathrm{SL}(2, \mathbb{R})$-cocycles.

Our work fully utilizes their results on the general $\mathrm{SL}(2, \mathbb{R})$-cocycles. In addition, the proof of the Jacobi case is a direct application of the projection lemma. From our point of view, the applicability of the projection lemma is related to the solvability of a system of equations. We were unable to find a possible solvability for the case of CMV matrices. Thus, another constructive way of proof will be provided.

The spectral theory of Schrödinger operators and Jacobi matrices with dynamically defined potentials and coefficients, respectively, has been extensively studied for the past few decades in a variety of settings, e.g., random potentials, almost periodic potentials, subshift potentials etc (see [5, 12] for surveys). On the other hand, the case of CMV matrices is much less understood. Damanik and Lenz [7] considered ergodic families of Verblunsky coefficients generated by minimal aperiodic subshifts; thus, a sampling map in [7] may be regarded as a simple function taking finitely many values.

Conspicuously absent while interesting was the case of almost periodic Verblunsky coefficients; see [14, pp. 706–707]. Bourget *et al* [3] studied some almost periodic case but their model is modified so that it is distinguished from true CMV matrices. Recently, Wang and Damanik [16] considered quasiperiodic Verblunsky coefficients with analytic sampling maps and showed Anderson localization in the regime of positive Lyapunov exponents. Our work considers generalized skew-shifts, which include the almost periodic case with continuous sampling maps.

2. *Statement of results*

Let X be a compact metric space and let $T : X \to X$ be a homeomorphism. Given a continuous map $A : X \to \mathrm{SL}(2, \mathbb{R})$, a *continuous cocycle* $(T, A) : X \times \mathbb{R}^2 \to X \times \mathbb{R}^2$ is defined as $(x, v) \to (T(x), A(x)v)$. For $n \in \mathbb{Z}$, A^n is defined by $(T, A)^n = (T^n, A^n)$.

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Definition 2.1. A continuous cocycle

$$(T, A) : X \times \mathbb{R}^2 \to X \times \mathbb{R}^2, (x, v) \to (T(x), A(x)v)$$

is *uniformly hyperbolic* if there are $C > 0$ and $\lambda < 1$ and, for every $x \in X$, there exist one-dimensional subspaces E^s_x and E^u_x of $\mathbb{R}^2$ such that:

(1) $A(x)E^s_x = E^s_{T(x)}$ and $A(x)E^u_x = E^u_{T(x)}$;

(2) $\|A^n(x)v^s\| \le C\lambda^n\|v^s\|$ and $\|A^{-n}(x)v^u\| \le C\lambda^n\|v^u\|$

for every $v^s \in E^s_x$, $v^u \in E^u_x$, $x \in X$ and $n \ge 1$.

Equivalently, it is well known that (T, A) is uniformly hyperbolic if and only if there exist constants $c > 0$ and $\sigma > 1$ such that $\|A^n(x)\| > c\sigma^n$ for all $x \in X$ and $n \ge 1$. (See [15], for example.) This definition is equivalent to the usual hyperbolic splitting condition: see [17]. Thus, uniform hyperbolicity is an open condition in $C^0(X, \mathrm{SL}(2, \mathbb{R}))$.

In fact, E^s_x and E^u_x are unique if they exist and depend continuously on $x \in X$ (compare [15]). Thus, we may choose a continuous map u from X to $\mathbb{RP}^1$ such that $u(x) \in E^u_x$.

2.1. *CMV matrices.* Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Let μ be a non-trivial probability measure on $\partial\mathbb{D}$, that is, it is not supported on a finite set. Then we may define the nth monic orthogonal polynomial $\Phi_n := \Phi_n(z; d\mu)$ by $\Phi_n \perp z^l$ for $l = 0, 1, \ldots, n-1$. Thus, we have $\langle \Phi_n, \Phi_m \rangle = 0$ for all $m \ne n$ in $L^2(\partial\mathbb{D}, d\mu)$. Naturally, orthonormal polynomials ϕ_n are defined as $\phi_n(z) = \Phi(z)/\|\Phi(z)\|$.

It is well known that the monic orthogonal polynomials are generated by the Szegö recursion,

$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha_n}\Phi^*_n(z),$$

where $\{\alpha_0, \alpha_1, \alpha_2, \ldots\} \subset \mathbb{D}$ are suitably chosen parameters, called Verblunsky coefficients. Conversely, given a sequence $\{\alpha_0, \alpha_1, \alpha_2, \ldots\} \subset \mathbb{D}$, we may define monic orthogonal polynomials with respect to a non-trivial probability measure on $\partial\mathbb{D}$ by the Szegö recursion. In fact, Verblunsky's theorem says that there is a one-to-one correspondence between non-trivial probability measures and sequences in $\mathbb{D}$.

The standard CMV matrix associated to the measures μ is a matrix representation discovered by Cantero *et al* [4] for multiplication by $z \in \partial\mathbb{D}$ in $L^2(\partial\mathbb{D}, d\mu)$. The matrix is given by

$$\mathcal{C} = \begin{bmatrix} \overline{\alpha_0} & \overline{\alpha_1}\rho_0 & \rho_1\rho_0 & & & & \\ \rho_0 & -\overline{\alpha_1}\alpha_0 & -\rho_1\alpha_0 & & & & \\ & \overline{\alpha_2}\rho_1 & -\overline{\alpha_2}\alpha_1 & \overline{\alpha_3}\rho_2 & \rho_3\rho_2 & & \\ & \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha_3}\alpha_2 & -\rho_3\rho_2 & & \\ & & & \overline{\alpha_4}\rho_3 & -\overline{\alpha_4}\alpha_3 & \overline{\alpha_5}\rho_4 & \\ & & & \rho_4\rho_3 & -\rho_4\alpha_3 & -\overline{\alpha_5}\rho_4 & \\ & & & & \ddots & \ddots & \ddots \end{bmatrix},$$

where $\rho_n = (1 - |\alpha_n|^2)^{1/2}$.

The basis for the representation is obtained by orthonormalizing $\{1, z, z^{-1}, z^2, z^{-2}, \ldots\}$. Note that the basis for the representation is not the orthonormal polynomials. The matrix

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representation based on the orthonormal polynomials is called the GGT (after Geronimus, Gragg and Teplyaev) representation; see [13, §4.1].

Let us briefly discuss why the CMV representation is a more suitable choice for spectral analysis. First of all, the set of orthonormal polynomials, $\{1, \phi_1, \phi_2, \ldots\}$, may not be a basis of $L^2(\partial\mathbb{D}, d\mu)$. Indeed, the orthonormal polynomials form a basis if and only if $\sum_{j=0}^{\infty} |\alpha_j|^2 = \infty$, where the α_j are the corresponding Verblunsky coefficients [13, Theorem 1.5.7]. Even for the case when the orthonormal polynomials form a basis, a row of its GGT representation has infinitely many non-zero terms. The five diagonal form of a CMV matrix allows us to connect the solution u of $\mathcal{C}u = zu$, $z \in \partial\mathbb{D}$, to 2×2 matrices and this provides very useful tools for spectral analysis. On the other hand, we do not have this connection for the GGT representation as its rows are not finite (see [13, §§4.1 and 4.2] for more discussion).

Now let us discuss CMV matrices over dynamical systems. Let (X, ν) be a probability measure space and let $T : X \to X$ be an invertible measure-preserving transformation. Under this setting, we may consider dynamically defined Verblunsky coefficients with a measurable function $f : X \to \mathbb{D}$. That is, our coefficients $\{\alpha_n\}_{n\in\mathbb{Z}_+}$ are defined by $\alpha_n = f(T^n x)$ for some $x \in X$. As T is an invertible map, we may also consider a bi-infinite sequence, $\{\alpha_n\}_{n\in\mathbb{Z}} = \{f(T^n x)\}_{n\in\mathbb{Z}}$. This leads to a bi-infinite CMV matrix, called an extended CMV matrix:

$$\mathcal{E}_x = \begin{bmatrix} \ddots & \ddots & \ddots & & & & \\ & -\overline{\alpha_0}\alpha_{-1} & \overline{\alpha_1}\rho_0 & \rho_1\rho_0 & & & \\ & -\rho_0\alpha_{-1} & -\overline{\alpha_1}\alpha_0 & -\rho_1\alpha_0 & & & \\ & & \overline{\alpha_2}\rho_1 & -\overline{\alpha_2}\alpha_1 & \overline{\alpha_3}\rho_2 & \rho_3\rho_2 & \\ & & \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha_3}\alpha_2 & -\rho_3\rho_2 & \\ & & & & \overline{\alpha_4}\rho_3 & -\overline{\alpha_4}\alpha_3 & \overline{\alpha_5}\rho_4 \\ & & & & \rho_4\rho_3 & -\rho_4\alpha_3 & -\overline{\alpha_5}\rho_4 \\ & & & & & \ddots & \ddots & \ddots \end{bmatrix},$$

where, again, $\rho_n = (1 - |\alpha_n|^2)^{1/2}$.

Extended CMV matrices are useful tools to study spectral properties. Let us now assume that $T : X \to X$ is an ergodic invertible measure-preserving transformation. With the associated extended CMV matrix, we have $\sigma(\mathcal{E}_x) = \sigma(\mathcal{E}_y)$ for ν-almost every $x, y \in X$. Moreover, the almost sure spectrum is purely essential. On the other hand, with the standard CMV matrix, what we obtain is that the essential spectrum coincides for ν-almost every $x \in X$ and the discrete spectrum may depend on $x \in X$ [14, Theorems 10.16.1 and 10.16.2]. Moreover, we may draw more conclusions from Kotani theory with the extended CMV matrix [14, Theorems 10.11.1–10.11.4].

The spectrum of extended CMV matrices associated to the dynamically defined Verblunsky coefficients is closely related to the Szegö cocycle defined as $(T, \overline{A}_z(x)) = (T, \overline{A}(f(x), z))$, $z \in \partial\mathbb{D}$, where

$$\overline{A}(f(x), z) := \frac{1}{z^{1/2}\sqrt{1 - |f(x)|^2}} \begin{bmatrix} z & -\bar{f}(x) \\ -f(x)z & 1 \end{bmatrix}.$$

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Here $\overline{A}_z(x)$ is an element of SU(1, 1), which may not be in SL(2, $\mathbb{R}$). However, there is an isomorphism between SU(1, 1) and SL(2, $\mathbb{R}$), which we will explain later in more detail. The branch $z^{-1/2}$ is chosen since the angle of $z^{-1/2}$ would describe the dependence on z in the SL(2, $\mathbb{R}$) representation.

Let X be a compact metric space. If $T : X \to X$ is a minimal homeomorphism and $f \in C^0(X, \mathbb{D})$, there is a uniform compact set $\Sigma \subset \partial\mathbb{D}$ with $\sigma(\mathcal{E}_x) = \Sigma$ for every $x \in X$. Damanik *et al* [6] showed that the uniform spectrum is given by $\Sigma = \partial\mathbb{D} \setminus U$, where

$$U = \{z \in \partial\mathbb{D} : (T, \overline{A}_z) \text{ is uniformly hyperbolic}\}.$$

Note that under the same hypothesis, the spectrum of standard CMV matrices may depend on $x \in X$.

Our strategy is to show that, given $\epsilon > 0$, if $(T, \overline{A}(f, z))$ is not uniformly hyperbolic, there exists $f' \in C^0(X, \mathbb{D})$ such that $\|\overline{A}(f, z) - \overline{A}(f', z)\|_{C^0} < \epsilon$ and $(T, \overline{A}(f', z))$ is uniformly hyperbolic. By combining with the result above, the following theorem holds.

THEOREM 2.2. *Let $T : X \to X$ be a strictly ergodic homeomorphism such that $h(T(x)) = h(x) + g$ for some $g \in \mathbb{G}$, where $h : X \to \mathbb{G}$ is an onto continuous map and $\mathbb{G}$ is an infinite compact abelian group. For a generic $f \in C^0(X, \mathbb{D})$, we have that $U = \partial\mathbb{D} \setminus \Sigma$ is dense and the associated CMV matrices have a Cantor spectrum.*

2.2. *Jacobi matrices.* Let μ be a non-trivial probability measure on $\mathbb{R}$ (not supported on a finite set) with a compact support. Then we may define the nth monic orthogonal polynomial $P_n(x)$ by $P_n \perp x^l$ for all $l = 0, 1, \ldots, n-1$. Naturally, the nth orthonormal polynomial is given as $p_n := P^n/\|P^n\|$. It is well known that the orthonormal polynomials obey the Jacobi recursion,

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_{n+1}p_n(x) + a_n p_{n-1}(x),$$

with suitably chosen real-valued sequences $a_n > 0$ and b_n, called Jacobi coefficients. Conversely, given real-valued bounded sequences $\{a_n\}$ and $\{b_n\}$ with $a_n > 0$ for all $n \in \mathbb{Z}_+$, the Jacobi recursion gives us a set of orthonormal polynomials with respect to a non-trivial probability measure with compact support.

The Jacobi matrix associated to the measure μ is the matrix representation for multiplication by x in $L^2(\mathbb{R}, d\mu)$ with respect to the basis $\{p_0, p_1, p_2, \ldots\}$:

$$\mathcal{J} = \begin{bmatrix} b_0 & a_0 & & & \\ a_0 & b_1 & a_1 & & \\ & a_1 & b_2 & a_2 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}.$$

Let (X, ν) be a probability measure space. Let $f_a, f_b : X \to \mathbb{R}$ be measurable maps with $f_a(x) > 0$ for all $x \in X$ and let $T : X \to X$ be an invertible ergodic transformation.

As for the case of Verblunsky coefficients, we may consider dynamically defined Jacobi coefficients under this setting. Specifically, two-sided Jacobi coefficients $\{a_n\}_{n\in\mathbb{Z}}$ and $\{b_n\}_{n\in\mathbb{Z}}$ are defined by $a_n = f_a(T^n(x))$ and $b_n = f_b(T^n x)$, respectively. Then an associated bi-infinite Jacobi matrix naturally arises. As in the case of CMV matrices,

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there are many advantages of bi-infinite Jacobi matrices to study spectral properties. Given $x \in X$, the bi-infinite Jacobi matrix H_x is given by

$$H_x = \begin{bmatrix} \ddots & \ddots & \ddots & & & & & \\ & a_{-2} & b_{-1} & a_{-1} & & & & \\ & & a_{-1} & b_0 & a_0 & & & \\ & & & a_0 & b_1 & a_1 & & \\ & & & & a_1 & b_2 & a_2 & \\ & & & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

It is well known that the spectrum of the Jacobi matrix is closely related to the solutions of the difference equation, $(H_x - E)u = 0$, where $E \in \mathbb{R}$. Notice that a sequence $\{u_n\}$ is a solution of $(H_x - E)u = 0$ if and only if

$$a_n u_{n+1} + (b_n - E)u_n + a_{n-1}u_{n-1} = 0$$

for all $n \in \mathbb{Z}$.

Equivalently, $\{u_n\}$ obeys

$$\begin{bmatrix} u_n \\ a_{n-1}u_{n-1} \end{bmatrix} = A^n_{E,a,b}(x) \begin{bmatrix} u_0 \\ a_{-1}u_{-1} \end{bmatrix},$$

where

$$A_{E,a,b}(x) = \frac{1}{f_a(x)} \begin{bmatrix} E - f_b(x) & -1 \\ f_a(x)^2 & 0 \end{bmatrix}.$$

Let X be a compact metric space. If we assume that $f_a, f_b \in C^0(X, \mathbb{R})$ and $T : X \to X$ is a minimal homeomorphism, $\sigma(H_x)$ coincides for all $x \in X$.

Let Σ be the spectrum of H_x. Marx [11] showed that

$$\mathbb{R} \setminus \Sigma = \{E \in \mathbb{R} | (T, A_{E,a,b}) \text{ is uniformly hyperbolic}\}.$$

In fact, Marx [11] considered both singular and non-singular cocycles. We call a cocycle $(T, A) : X \times \mathbb{R}^2 \to X \times \mathbb{R}^2$ singular if $\det A(x_0) = 0$ for some $x_0 \in X$. For singular cocycles, uniform hyperbolicity is not applicable and, thus, uniform hyperbolicity is replaced by dominated splitting. For $\mathrm{SL}(2, \mathbb{R})$-cocycles, dominated splitting is equivalent to uniform hyperbolicity. (See [11].)

Later, given $\epsilon > 0$, we will prove that if $(T, A_{E,a,b})$ is not uniformly hyperbolic, there exists $f_{b'} \in C^0(X, \mathbb{R})$ such that $\|A_{E,a,b} - A_{E,a,b'}\|_{C^0} < \epsilon$ and $(T, A_{E,a,b'})$ is uniformly hyperbolic, where $b'_n = f_{b'}(T^n x)$. Together with the result in [11], this implies the following theorem.

THEOREM 2.3. *Let $T : X \to X$ be a strictly ergodic homeomorphism such that $h(T(x)) = h(x) + g$ for some $g \in \mathbb{G}$, where $h : X \to \mathbb{G}$ is an onto continuous map and $\mathbb{G}$ is an infinite compact abelian group. Let $f_a \in C^0(X, \mathbb{R})$ with $f_a(x) > 0$ for all $x \in X$. For generic $f_b \in C^0(X, \mathbb{R})$, we have that $\mathbb{R} \setminus \Sigma$ is dense and the associated Jacobi matrices have a Cantor spectrum.*

2.3. *Discussion of the results.* In addition to the results for general continuous $\mathrm{SL}(2, \mathbb{R})$-cocycles, the C^0-genericity of a Cantor spectrum for Schrödinger operators proved in Avila *et al* [1] was striking. Especially for the standard skew-shift with a sufficiently regular non-constant potential function $V : \mathbb{T}^2 \to \mathbb{R}$, it had been widely expected to have a pure point spectrum, which is not a Cantor set, with exponentially decaying eigenfunctions.

After the work, an obvious expectation for CMV and Jacobi matrices generated by the same base dynamics is to also have a generic Cantor spectrum. The present paper, to the best of our knowledge, first provides a proof for the statement.

We would like to mention that the spectrum of quasiperiodic Schrödinger operators with analytic potentials behaves in a very different way. For the case of shifts on the one-dimensional torus, Goldstein and Schlag [8] proved that a Cantor spectrum is obtained for analytic potentials in the regime of positive Lyapunov exponents with typical shifts, that is, with $x \in \mathbb{T}$ and $n \in \mathbb{Z}$, shifts $x + n\alpha$ for Lebesgue almost every $\alpha \in \mathbb{T}$. For the case of shifts on a multidimensional torus, it turned out to be harder to study and, thus, is much less understood. However, for a two-dimensional shift, Goldstein *et al* [9] showed that the spectrum consists of a single interval for large real analytic potentials satisfying certain restrictions.

3. *Results for* $\mathrm{SL}(2, \mathbb{R})$*-cocycles*

As noted in the introduction, our work is closely related to the results in Avila *et al* [1]. In this section, we discuss the results in [1] for general continuous $\mathrm{SL}(2, \mathbb{R})$-cocycles over the same base dynamics as in the present paper, that is, with a strictly ergodic homeomorphism $T : X \to X$ such that $h(T(x)) = h(x) + g$ for some $g \in \mathbb{G}$, where $h : X \to \mathbb{G}$ is an onto continuous map and $\mathbb{G}$ is an infinite compact abelian group.

We say that two cocycles (T, A) and $(T, \tilde{A})$ are conjugate (respectively, $\mathrm{PSL}(2, \mathbb{R})$-conjugate) if there exists a conjugacy $B \in C^0(X, \mathrm{SL}(2, \mathbb{R}))$ (respectively, $B \in C^0(X, \mathrm{PSL}(2, \mathbb{R}))$) such that $\tilde{A}(x) = B(T(x))A(x)B(x)^{-1}$.

We say that (T, A) is reducible if it is $\mathrm{PSL}(2, \mathbb{R})$-conjugate to a constant cocycle. We say that (T, A) is reducible up to homotopy if there exists a reducible cocycle $(T, \tilde{A})$ such that the maps A and $\tilde{A} : X \to \mathrm{SL}(2, \mathbb{R})$ are homotopic. Let *Ruth* be the set of all A such that (T, A) is reducible up to homotopy.

In Avila *et al* [1], the authors showed that if an $\mathrm{SL}(2, \mathbb{R})$-cocycle is not uniformly hyperbolic, it can be approximated by one that is conjugate to an $\mathrm{SO}(2, \mathbb{R})$-cocycle. Using this, it was proved that if a cocycle is in *Ruth*, then it can be approximated by a uniformly hyperbolic cocycle. As a uniformly hyperbolic cocycle is always reducible up to homotopy, this shows that uniform hyperbolicity is dense in *Ruth* [1, Theorem 2].

We will observe that a cocycle associated to a CMV matrix or Jacobi matrix is homotopic to a constant cocycle and hence in *Ruth*. Therefore, it can be C^0-perturbed so that it is a continuous $\mathrm{SL}(2, \mathbb{R})$-cocycle, which is uniformly hyperbolic. For the purpose of our work, a difficulty is that the perturbed cocycle need not be in the form of one associated with CMV matrices or Jacobi matrices.

The difficulty is nicely overcome for the case of Schrödinger operators in Avila *et al* [1] by using 'a projection lemma', which was also proved in their work. On the one hand, it makes the perturbed $\mathrm{SL}(2,\mathbb{R})$-cocycle conjugate to a cocycle associated to Schrödinger operators (one may say the perturbed $\mathrm{SL}(2,\mathbb{R})$-cocycle is projected). Of course, the conjugacy preserves the uniform hyperbolicity. On the other hand, the associated cocycle can be arbitrarily close to the original cocycle, which is not uniformly hyperbolic. In conclusion, it provides a uniformly hyperbolic cocycle associated to Schrödinger operators such that it is arbitrarily close to the original cocycle, which was not uniformly hyperbolic.

Our proof for the case of Jacobi matrices is a direct application of the above procedure. We were unable to find a possible way of application for the case of CMV matrices and we will explain in more detail this difficulty during the proof for the case of Jacobi matrices. However, we still use half of the projection lemma in [1]. We first need to introduce some notation. Given $A \in C^0(X, \mathrm{SL}(2,\mathbb{R}))$ and a non-empty subset $V \subset X$, let $C^0_{A,V}(X, \mathrm{SL}(2,\mathbb{R})) \subset C^0(X, \mathrm{SL}(2,\mathbb{R}))$ be the set of all $B \in C^0(X, \mathrm{SL}(2,\mathbb{R}))$ such that $B(x) = A(x)$ for $x \notin V$.

LEMMA 3.1. [1, Lemma 10] *Let $V \subset X$ be any non-empty open set and let $A \in C^0(X, \mathrm{SL}(2,\mathbb{R}))$ be arbitrary. Then there exist an open neighborhood $\mathcal{W}_{A,V} \subset C^0(X, \mathrm{SL}(2,\mathbb{R}))$ of A and continuous maps*

$$\Phi = \Phi_{A,V} : \mathcal{W}_{A,V} \to C^0_{A,\overline{V}}(X, \mathrm{SL}(2,\mathbb{R}))$$

and

$$\Psi = \Psi_{A,V} : \mathcal{W}_{A,V} \to C^0(X, \mathrm{SL}(2,\mathbb{R}))$$

satisfying

$$\Psi(B)(T(x)) \cdot B(x) \cdot [\Psi(B)(x)]^{-1} = \Phi(B)(x),$$

$$\Phi(A) = A \text{ and } \Psi(A) = \mathrm{id}.$$

4. *Proof of results*

4.1. *Proof for CMV matrices.* Define $\mathrm{SU}(1,1) := \{A \in U(1,1) : \det A = 1\}$. Let $\overline{J}$ be a matrix such that $\overline{J}^* = \overline{J} = \overline{J}^{-1}$ and $Tr(\overline{J}) = 0$. Then we may choose a unitary matrix W such that $W\mathrm{SU}(1,1;\overline{J})W^{-1} = \mathrm{SU}(1,1)$, where $\mathrm{SU}(1,1;\overline{J}) := \{A : A^*\overline{J}A = \overline{J}\}$.

Note that, with $J_r := [\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}]$, we have $\mathrm{SU}(1,1,J_r) = \mathrm{SL}(2,\mathbb{R})$: see [14, Proposition 10.4.1]. For our purpose, this may be read as

$$W^{-1}\mathrm{SU}(1,1)W = \mathrm{SU}(1,1,J_r) = \mathrm{SL}(2,\mathbb{R}),$$

where $W = 1/\sqrt{2}[\begin{smallmatrix} 1 & i \\ 1 & -i \end{smallmatrix}]$. Let $f \in C^0(X,\mathbb{D})$ and let $z \in \partial\mathbb{D}$ be given. Note that $\overline{A}(f,z) \in C^0(X, \mathrm{SU}(1,1))$. Thus, given $x \in X$, we have an $\mathrm{SL}(2,\mathbb{R})$ matrix

$$W^{-1}\overline{A}(f(x),z)W = \frac{1}{2z^{1/2}\sqrt{1-|f(x)|^2}} \begin{bmatrix} z - \bar{f}(x) - f(x)z + 1 & i(z + \bar{f}(x) - f(x)z - 1) \\ i(-z + \bar{f}(x) - f(x)z + 1) & z + \bar{f}(x) + f(x)z + 1 \end{bmatrix}.$$

Denote R_η as the 2×2 rotation matrix with the angle $\eta \in \mathbb{R}$. By a simple observation, we have following result.

LEMMA 4.1. *Let $z = e^{i\psi} \in \partial\mathbb{D}$ and let $f \in C^0(X, \mathbb{D})$ be given by $f(x) = r(x)e^{i\phi(x)}$. Then $W^{-1}\overline{A}(f(x), z)W$ is equal to*

$$\frac{1}{\sqrt{1 - r(x)^2}}\left(R_{\theta'} + r(x)\begin{bmatrix}-\cos\theta(x) & \sin\theta(x)\\ \sin\theta(x) & \cos\theta(x)\end{bmatrix}\right),$$

where $\theta' = \psi/2$ and $\theta(x) = (\psi/2) + \phi(x)$.

Given $z \in e^{i\psi} \in \partial\mathbb{D}$ with $\theta' := \psi/2$, define $S'_{\theta'} \subset \mathrm{SL}(2, \mathbb{R})$ as

$$S'_{\theta'} = \left\{\frac{1}{\sqrt{(1 - s^2)}}\left(R_{\theta'} + s\begin{bmatrix}-\cos\theta & \sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\right) : s \in [0, 1), \theta \in \mathbb{R}\right\}.$$

Let $A(x) := W^{-1}\overline{A}(f(x), z)W$. Then we have $A \in C^0(X, S'_{\theta'})$ and we may write it as in Lemma 4.1.

Given an $S'_{\theta'}$-valued cocycle (T, A), our goal is to construct another $S'_{\theta'}$-valued cocycle (T, B'') which is uniformly hyperbolic and arbitrarily C^0-close to (T, A). By the C^0-denseness result in [1], we may choose a uniformly hyperbolic $\mathrm{SL}(2, \mathbb{R})$-valued cocycle (T, B) which is arbitrarily C^0-close to A. From (T, B), we construct an $S'_{\theta'}$-valued cocycle (T, B') which is arbitrarily C^0-close to (T, A). From (T, B'), we finally construct an $S'_{\theta'}$-valued cocycle (T, B'') which is uniformly hyperbolic and arbitrarily C^0-close to A.

We assume that $f \in C^0(X, \mathbb{D})$ is not identically zero. Let $y \in X$ be an element such that $f(y) \neq 0$. We may choose a non-empty open set $V \subset X$ so that $y \in V$ and $f(x) \neq 0$ for all $x \in \overline{V}$. In fact, there exist $r_1, r_2 \in [0, 1)$ such that $r_1 \leq |f(x)| \leq r_2$ for all $x \in \overline{V}$. Thus, $r_1 \leq r(x) \leq r_2$ for all $x \in \overline{V}$.

By Lemma 3.1 and the C^0-denseness result of uniform hyperbolicity in [1], we may choose a uniformly hyperbolic cocycle (T, B) such that $B \in C^0_{A,\bar{V}}(X, \mathrm{SL}(2, \mathbb{R}))$ is arbitrarily C^0-close to A. Write B as

$$B(x) = \frac{1}{\sqrt{1 - r(x)^2}}\left(R_{\theta'} + r(x)\begin{bmatrix}b_{11}(x) & b_{12}(x)\\ b_{21}(x) & b_{22}(x)\end{bmatrix}\right).$$

Set $B'(x) := B(x)$ for $x \in X \setminus \overline{V}$. For $x \in \overline{V}$, we define $B' \in C^0(X, S'_{\theta'})$ as follows.

(1) Recall that since (T, B) is uniformly hyperbolic there exists a continuous map u from X to the projective space $\mathbb{RP}^1$ such that $u(x) \in E^u_x$. Let $R_{-\tau(x)} \cdot u(x) = (1, 0)$. Consider the matrix

$$\begin{bmatrix}y_{11}(x) & y_{12}(x)\\ y_{21}(x) & y_{22}(x)\end{bmatrix} := \begin{bmatrix}b_{11}(x) & b_{12}(x)\\ b_{21}(x) & b_{22}(x)\end{bmatrix} \cdot R_{\tau(x)}. \tag{1}$$

Note that we may choose a $\tau : X \to \mathbb{R}$ so that it is continuous since u is continuous.

(2) Normalize the vector $(y_{11}(x), y_{21}(x))$. Then we may write it as $(-\cos\tilde{\theta}, \sin\tilde{\theta})$ for some $\tilde{\theta} \in \mathbb{R}$.

(3) Set B' as

$$B'(x) = \frac{1}{\sqrt{1 - r(x)^2}}\left(R_{\theta'} + r(x)\begin{bmatrix}-\cos\tilde{\theta}(x) & \sin\tilde{\theta}(x)\\ \sin\tilde{\theta}(x) & \cos\tilde{\theta}(x)\end{bmatrix} R_{-\tau(x)}\right).$$

Observe that, as an element in the projective line of $\mathbb{R}^2$, we have

$$\begin{bmatrix} -\cos\tilde{\theta}(x) & \sin\tilde{\theta}(x) \\ \sin\tilde{\theta}(x) & \cos\tilde{\theta}(x) \end{bmatrix} R_{-\tau(x)} \cdot u(x) = \begin{bmatrix} -\cos\tilde{\theta}(x) & \sin\tilde{\theta}(x) \\ \sin\tilde{\theta}(x) & \cos\tilde{\theta}(x) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} y_{11}(x) & y_{12}(x) \\ y_{21}(x) & y_{22}(x) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{bmatrix} u(x).$$

LEMMA 4.2. *Given* $\epsilon > 0$, *there exists* $\delta > 0$ *so that* $\|A - B\|_{C^0} < \delta$ *implies that* $\|B - B'\|_{C^0} < \epsilon$.

Proof. It suffices to show that given $\epsilon > 0$, there exists $\delta > 0$ so that

$$\left\| \begin{bmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{bmatrix} - \begin{bmatrix} -\cos\theta(x) & \sin\theta(x) \\ \sin\theta(x) & \cos\theta(x) \end{bmatrix} \right\|_{C^0} < \delta$$

implies that

$$\left\| \begin{bmatrix} -\cos\tilde{\theta}(x) & \sin\tilde{\theta}(x) \\ \sin\tilde{\theta}(x) & \cos\tilde{\theta}(x) \end{bmatrix} R_{-\tau(x)} - \begin{bmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{bmatrix} \right\|_{C^0} < \epsilon.$$

Set $\delta = \epsilon/4$. Then

$$\left\| \begin{bmatrix} y_{11}(x) & y_{12}(x) \\ y_{21}(x) & y_{22}(x) \end{bmatrix} - \begin{bmatrix} -\cos\theta(x) & \sin\theta(x) \\ \sin\theta(x) & \cos\theta(x) \end{bmatrix} R_{\tau(x)} \right\|_{C^0} < \epsilon/4.$$

In particular, we have

$$\left\| \begin{bmatrix} y_{11}(x) \\ y_{21}(x) \end{bmatrix} - \begin{bmatrix} -\cos(\theta(x)+\tau(x)) \\ \sin(\theta(x)+\tau(x)) \end{bmatrix} \right\| < \epsilon/4$$

and

$$\left\| \begin{bmatrix} y_{12}(x) \\ y_{22}(x) \end{bmatrix} - \begin{bmatrix} \sin(\theta(x)+\tau(x)) \\ \cos(\theta(x)+\tau(x)) \end{bmatrix} \right\| < \epsilon/4.$$

This implies that

$$\left\| \begin{bmatrix} y_{21}(x) \\ -y_{11}(x) \end{bmatrix} - \begin{bmatrix} y_{12}(x) \\ y_{22}(x) \end{bmatrix} \right\| < \epsilon/2.$$

Therefore, we have

$$\left\| \begin{bmatrix} y_{11}(x) & y_{21}(x) \\ y_{21}(x) & -y_{11}(x) \end{bmatrix} - \begin{bmatrix} y_{11}(x) & y_{12}(x) \\ y_{21}(x) & y_{22}(x) \end{bmatrix} \right\|_{C^0} < \epsilon/2.$$

If necessary, choose a C^0-close B to A (so, smaller δ) so that

$$\left\| \frac{1}{\sqrt{y_{11}(x)^2 + y_{21}(x)^2}} \begin{bmatrix} y_{11}(x) & y_{21}(x) \\ y_{21}(x) & -y_{11}(x) \end{bmatrix} - \begin{bmatrix} y_{11}(x) & y_{21}(x) \\ y_{21}(x) & -y_{11}(x) \end{bmatrix} \right\|_{C^0} < \epsilon/2$$

for all $x \in X$.

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Then, by the triangle inequality,

$$\left\| \frac{1}{\sqrt{y_{11}(x)^2 + y_{21}(x)^2}} \begin{bmatrix} y_{11}(x) & y_{21}(x) \\ y_{21}(x) & -y_{11}(x) \end{bmatrix} - \begin{bmatrix} y_{11}(x) & y_{12}(x) \\ y_{21}(x) & y_{22}(x) \end{bmatrix} \right\|_{C^0} < \epsilon.$$

In conclusion,

$$\left\| \begin{bmatrix} -\cos\tilde{\theta}(x) & \sin\tilde{\theta}(x) \\ \sin\tilde{\theta}(x) & \cos\tilde{\theta}(x) \end{bmatrix} - \begin{bmatrix} y_{11}(x) & y_{12}(x) \\ y_{21}(x) & y_{22}(x) \end{bmatrix} \right\|_{C^0} < \epsilon,$$

which implies that

$$\left\| \begin{bmatrix} -\cos\tilde{\theta}(x) & \sin\tilde{\theta}(x) \\ \sin\tilde{\theta}(x) & \cos\tilde{\theta}(x) \end{bmatrix} R_{-\tau(x)} - \begin{bmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{bmatrix} \right\|_{C^0} < \epsilon. \qquad \square$$

Let N be the closed annulus on the $\mathbb{R}^2$-plane centered at the origin with radius $r_1/\sqrt{1-r_1^2} \le \rho \le r_2/\sqrt{1-r_2^2}$. That is,

$$N := \left\{ \frac{r}{\sqrt{1-r^2}}(\cos\eta, \sin\eta) \in \mathbb{R}^2 : r_1 \le r \le r_2, \eta \in \mathbb{R} \right\}.$$

Note that

$$\frac{1}{\sqrt{1-r^2}} - \frac{r}{\sqrt{1-r^2}} < 1.$$

Let $\underline{\epsilon} < 1$ be a number such that

$$\frac{1}{\sqrt{1-r_1^2}} - \frac{\underline{\epsilon} \cdot r_1}{\sqrt{1-r_1^2}} < 1$$

and let $\overline{\epsilon} > 1$ be a number such that

$$\frac{1}{\sqrt{1-r_2^2}} - \frac{\overline{\epsilon} \cdot r_2}{\sqrt{1-r_2^2}} > 0.$$

Given $\underline{\epsilon} < \epsilon < \overline{\epsilon}$, we define $h_\epsilon : N \to [0, 1)$ and $g_\epsilon : N \to [-\pi, \pi]$ as follows. Given $t \in N$ with $|t| = r/\sqrt{1-r^2}$, we may choose $\eta \in \mathbb{R}$ such that

$$t = \frac{r}{\sqrt{1-r^2}} \begin{bmatrix} -\cos\eta & \sin\eta \\ \sin\eta & \cos\eta \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Consider the vector

$$\frac{1}{\sqrt{1-r^2}} \left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} + r \begin{bmatrix} -\cos\eta & \sin\eta \\ \sin\eta & \cos\eta \end{bmatrix} \cdot \begin{bmatrix} \epsilon \\ 0 \end{bmatrix} \right). \tag{2}$$

Then there exist unique $s \in [0, 1)$ and $\beta \in [-\pi, \pi]$ so that

$$\frac{1}{\sqrt{1-s^2}} \left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -\cos\eta & \sin\eta \\ \sin\eta & \cos\eta \end{bmatrix} \cdot R_\beta \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

coincides with (2). We define $h_\epsilon(t) = s$ and $g_\epsilon(t) = \beta$. Here are some properties of h_ϵ.

LEMMA 4.3.

(a) *h_ϵ is continuous.*
(b) *Given $t \in N$ with $|t| = r/\sqrt{1-r^2}$, we have $h_\epsilon(t) \to r$ as $\epsilon \to 1$.*
(c) *Let $t \in N$ with $|t| = r/\sqrt{1-r^2}$ and let $\epsilon' \leq \epsilon \leq 1$. Then we have $h_{\epsilon'}(t) \leq h_\epsilon(t) \leq r$.*
(d) *Let $t \in N$ with $|t| = r/\sqrt{1-r^2}$ and let $\epsilon' \geq \epsilon \geq 1$. Then we have $h_{\epsilon'}(t) \geq h_\epsilon(t) \geq r$.*
(e) *Let $h : N \to [0, 1)$ be the function defined by $h(t) = r$ if $|t| = r/\sqrt{1-r^2}$. Let $\{\epsilon_n\}$ be a sequence such that $\epsilon_n \to 1$ and $\epsilon_n \leq \epsilon_{n+1}$ (or $\epsilon_n \geq \epsilon_{n+1}$) for all n. Then $\{h_{\epsilon_n}\}$ converges uniformly to h.*

Proof. Parts (a) to (d) are easy to check. For part (e), suppose that $\{\epsilon_n\}$ is a sequence such that $\epsilon_n \to 1$ and $\epsilon_n \leq \epsilon_{n+1} \leq 1$ for all n. By part (b), $\{h_{\epsilon_n}\}$ pointwise converges to h. By part (c), $h_{\epsilon_n}(t) \leq h_{\epsilon_{n+1}}(t)$ for all n and for all $t \in T$.

Thus, by Dini's theorem, $\{h_{\epsilon_n}\}$ uniformly converges to h. A similar argument shows the uniform convergence of $\{h_{\epsilon_n}\}$ with $\epsilon_n \geq \epsilon_{n+1} \geq 1$ for all n. □

Here are some properties of g_ϵ.

LEMMA 4.4.

(a) *g_ϵ is continuous.*
(b) *Given $t \in N$, we have $g_\epsilon(t) \to 0$ as $\epsilon \to 1$.*
(c) *Let $t = \rho(\cos\eta, \sin\eta) \in N$ with $\eta \in [0, \pi]$. If $\epsilon' \leq \epsilon \leq 1$, we have $g_{\epsilon'}(t) \leq g_\epsilon(t) \leq 0$. If $\epsilon' \geq \epsilon \geq 1$, we have $g_{\epsilon'}(t) \geq g_\epsilon(t) \geq 0$.*
(d) *Let $t = \rho(\cos\eta, \sin\eta) \in N$ with $\eta \in [\pi, 2\pi]$. If $\epsilon' \leq \epsilon \leq 1$, we have $g_{\epsilon'}(t) \geq g_\epsilon(t) \geq 0$. If $\epsilon' \geq \epsilon \geq 1$, we have $g_{\epsilon'}(t) \leq g_\epsilon(t) \leq 0$.*
(e) *Let $\{\epsilon_n\}$ be a sequence such that $\epsilon_n \to 1$ and $\epsilon_n \leq \epsilon_{n+1}$ (or $\epsilon_n \geq \epsilon_{n+1}$) for all n. Then $\{g_{\epsilon_n}\}$ uniformly converges to $g = 0$.*

Proof. Parts (a) to (d) are easy to check. For part (e), suppose that $\{\epsilon_n\}$ is a sequence such that $\epsilon_n \to 1$ and $\epsilon_n \leq \epsilon_{n+1} \leq 1$ for all n. By part (b), $\{g_{\epsilon_n}\}$ pointwise converges to $g = 0$. By part (c), $g_{\epsilon_n}(t) \leq g_{\epsilon_{n+1}}(t)$ for all n if $t = \rho(\cos\eta, \sin\eta)$ with $\eta \in [0, \pi]$. By part (d), $g_{\epsilon_n}(t) \geq g_{\epsilon_{n+1}}(t)$ for all n if $t = \rho(\cos\eta, \sin\eta)$ with $\eta \in [\pi, 2\pi]$.

Thus, by Dini's theorem, $\{g_{\epsilon_n}\}$ uniformly converges to $g = 0$. A similar argument shows the uniform convergence of $\{g_{\epsilon_n}\}$ with $\epsilon_n \geq \epsilon_{n+1} \geq 1$ for all n. □

Note that by setting $h(\epsilon, t) := h_\epsilon(t)$ and $g(\epsilon, t) = g_\epsilon(t)$, it is easy to see that $h : (\underline{\epsilon}, \overline{\epsilon}) \times N \to [0, 1)$ and $g : (\underline{\epsilon}, \overline{\epsilon}) \times N \to [-\pi, \pi]$ are continuous functions.

Now set $B''(x) = B(x)$ for $x \in X \setminus \overline{V}$. For $x \in \overline{V}$, define $B'' \in C^0(X, S'_{\theta'})$ as follows.

(1) Consider

$$B'(x) = \frac{1}{\sqrt{1 - r(x)^2}}\left(R_{\theta'} + r(x)\begin{bmatrix} -\cos\tilde{\theta}(x) & \sin\tilde{\theta}(x) \\ \sin\tilde{\theta}(x) & \cos\tilde{\theta}(x) \end{bmatrix} R_{-\tau(x)}\right)$$

and $\epsilon(x) = \sqrt{y_{11}(x)^2 + y_{21}(x)^2}$ (see equation (1)). We may assume that $\underline{\epsilon} < \epsilon(x) < \overline{\epsilon}$ for all $x \in X$ by taking a C^0-close enough B to A.

(2) Let $\omega(x) \in [0, 2\pi]$ be such that $R_{\omega(x)} \cdot R_{\theta'} \cdot u(x) = (-1, 0)$. Define $t(x) \in N$ as

$$t(x) = \rho(x) \cdot R_{\omega(x)} \cdot \begin{bmatrix} -\cos\tilde{\theta}(x) & \sin\tilde{\theta}(x) \\ \sin\tilde{\theta}(x) & \cos\tilde{\theta}(x) \end{bmatrix} R_{-\tau(x)} u(x),$$

where $\rho(x) = r(x)/\sqrt{1 - r(x)^2}$.

(3) Define $B''(x)$ as

$$B''(x) = \frac{1}{\sqrt{1-s^2}} \left(R_{\theta'} + s \begin{bmatrix} -\cos\tilde{\theta}(x) & \sin\tilde{\theta}(x) \\ \sin\tilde{\theta}(x) & \cos\tilde{\theta}(x) \end{bmatrix} R_{-\tau(x)} R_\beta \right),$$

where $s = h_{\epsilon(x)}(t(x))$, and let $\beta = g_{\epsilon(x)}(t(x))$.

Note that, by construction, we have $B'' \in C^0(X, S'_{\theta'})$ and

$$B(x)u(x) = B''(x)u(x) \tag{3}$$

for all $x \in X$. Since (T, B) is uniformly hyperbolic, there exist a constant C and $\sigma > 1$ such that

$$\|(B'')^n(x)\| \geq \|(B'')^n(x)u(x)\| = \|B^n(x)u(x)\| \geq C\sigma^n$$

for all $x \in X$ and $n \geq 1$, where the equality holds by the equation (3). This gives us the following lemma.

LEMMA 4.5. *The cocycle (T, B'') is uniformly hyperbolic.*

PROPOSITION 4.6. *Let $f \in C^0(X, \mathbb{D})$ with $f(x) \neq 0$ for some $x \in X$ and let $A = W^{-1}\overline{A}(f, z)W \in C^0(X, S'_{\theta'})$, where $\theta' \in \mathbb{R}$. Suppose that (T, A) is not uniformly hyperbolic. Given $\epsilon' > 0$, there exists $B'' \in C^0(X, S'_{\theta'})$ such that $\|A - B''\|_{C^0} < \epsilon'$ and (T, B'') is uniformly hyperbolic.*

Proof. Write A as in Lemma 4.1. Let $\epsilon' > 0$ be given. Let $y \in X$ be such that $r(y) := r \neq 0$. Choose an open set $V \subset X$ such that $y \in V$ and $r(x) \neq 0$ for all $x \in \overline{V}$. We may assume that $r_1 \leq r(x) \leq r_2$ for all $x \in \overline{V}$, where $r_1, r_2 \in [0, 1)$.

Choose $\delta_1, \delta_2 > 0$ so that $|s - r_2| < \delta_1$ and $|\beta| < \delta_2$ imply that

$$\begin{aligned} &\left\| \frac{1}{\sqrt{1-r_2^2}} R_{\theta'} - \frac{1}{\sqrt{1-s^2}} R_{\theta'} \right\| \\ &+ \left\| \frac{r_2}{\sqrt{1-r_2^2}} \begin{bmatrix} -\cos\eta & \sin\eta \\ \sin\eta & \cos\eta \end{bmatrix} - \frac{s}{\sqrt{1-s^2}} \begin{bmatrix} -\cos\eta & \sin\eta \\ \sin\eta & \cos\eta \end{bmatrix} \right\| \\ &+ \left\| \frac{s}{\sqrt{1-s^2}} \begin{bmatrix} -\cos\eta & \sin\eta \\ \sin\eta & \cos\eta \end{bmatrix} - \frac{s}{\sqrt{1-s^2}} \begin{bmatrix} -\cos\eta & \sin\eta \\ \sin\eta & \cos\eta \end{bmatrix} R_\beta \right\| < \frac{\epsilon'}{3} \end{aligned}$$

for all $\eta \in \mathbb{R}$.

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Then, given $r \in [r_1, r_2]$, $\bar{s}$ with $|r - \bar{s}| < \delta_1$ and $|\beta| < \delta_2$, we have

$$\left\| \frac{1}{\sqrt{1-r^2}}\left(R_{\theta'} + r\begin{bmatrix} -\cos\eta & \sin\eta \\ \sin\eta & \cos\eta \end{bmatrix}\right) - \frac{1}{\sqrt{1-\bar{s}^2}}\left(R_{\theta'} + \bar{s}\begin{bmatrix} -\cos\eta & \sin\eta \\ \sin\eta & \cos\eta \end{bmatrix} R_\beta\right)\right\| < \frac{\epsilon'}{3}$$

for all $\eta \in \mathbb{R}$.

Let $N \subset \mathbb{R}^2$ be the annulus of radius $r_1/\sqrt{1-r_1^2} \le \rho \le r_2/\sqrt{1-r_2^2}$. Define $l : N \to [0, 1)$ as $l(t) := r$ when $|t| = r/\sqrt{1-r^2}$ and define $h_\epsilon : N \to [0, 1)$ and $g_\epsilon : N \to [-\pi, \pi]$ as above. By parts (c), (d) and (e) of Lemmas 4.3 and 4.4, we may choose $\delta > 0$ so that $|1 - \epsilon| < \delta$ implies that $|h_\epsilon(t) - l(t)| < \delta_1$ and $|g_\epsilon(t)| < \delta_2$ for all $t \in N$.

By Lemma 3.1, we may choose $B \in C^0_{A,\overline{V}}(X, \mathrm{SL}(2, \mathbb{R}))$, which is arbitrarily close to A, and (T, B) is uniformly hyperbolic. Let $B \in C^0_{A,\overline{V}}(X, \mathrm{SL}(2, \mathbb{R}))$ be close enough to A so that $|\epsilon(x) - 1| < \delta$ for all $x \in X$, where $\epsilon(x) := \sqrt{y_{11}(x)^2 + y_{21}(x)^2}$ (see equation (1)). Define $B' \in C^0(X, S'_{\theta'})$ and $B'' \in C^0(X, S'_{\theta'})$ as previously. Then we have $|h_{\epsilon(x)}(t(x)) - r(x)| < \delta_1$ and $|g_{\epsilon(x)}(t(x))| < \delta_2$ for all $x \in X$. (Note that $l(t(x)) = r(x)$.) With such B, we have $\|B' - B''\|_{C^0} < \epsilon'/3$.

From here, choose a C^0-close B (if necessary) so that $\|A - B\|_{C^0} < \epsilon'/3$ and $\|B - B'\|_{C^0} < \epsilon'/3$. In conclusion, we have

$$\|A - B''\|_{C^0} \le \|A - B\|_{C^0} + \|B - B'\|_{C^0} + \|B' - B''\|_{C^0} < \epsilon'$$

while $B'' \in C^0(X, S'_{\theta'})$ and (T, B'') is uniformly hyperbolic. □

Proof of Theorem 2.2. Let $f \in C^0(X, \mathbb{D})$ and suppose that f is not identically zero. Let $A := W^{-1}\overline{A}(f, z)W \in C^0(X, S'_{\theta'})$ and suppose that (T, A) is not uniformly hyperbolic. Choose $B \in C^0(X, \mathrm{SL}(2, \mathbb{R}))$, $B' \in C^0(X, S'_{\theta'})$ and $B'' \in C^0(X, S'_{\theta'})$ as in the proof of Proposition 4.6.

Define $\beta : X \to \mathbb{D}$ as $\beta(x) = h_{\epsilon(x)}(t(x))e^{i\eta(x)}$, where

$$\eta(x) = \tilde{\theta}(x) - \tau(x) + g_{\epsilon(x)}(t(x)) - \theta'.$$

Note that we have $\beta \in C^0(X, \mathbb{D})$ and

$$W^{-1}\overline{A}(\beta(x), z)W = W^{-1}\frac{1}{z^{1/2}\sqrt{1-|\beta(x)|^2}}\begin{bmatrix} z & -\bar{\beta}(x) \\ -\beta(x)z & 1 \end{bmatrix} W = B''(x).$$

Since W is unitary and $\|A - B''\|_{C^0} < \epsilon'$, we have

$$\|WAW^{-1} - WB''W^{-1}\|_{C^0} = \|\overline{A}(f, z) - \overline{A}(\beta, z)\|_{C^0} < \epsilon'.$$

This implies that we may choose $\beta \in C^0(X, \mathbb{D})$, which is arbitrarily C^0-close to f, and $(T, \overline{A}(\beta, z))$ is uniformly hyperbolic.

Now suppose that $f \in C^0(X, \mathbb{D})$ is identically zero. Then we may choose $f' \in C^0(X, \mathbb{D})$ such that it is not identically zero and arbitrarily C^0-close to f. With f', we may repeat a similar procedure as above.

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Now, for $z \in \partial\mathbb{D}$, consider the set

$$UH_z = \{f \in C^0(X, \mathbb{D}) : (T, \overline{A}(f, z)) \text{ is uniformly hyperbolic}\}.$$

Then UH_z is open and dense by the previous argument. Thus, we may choose a countable dense subset $\{z_n\}$ of $\partial\mathbb{D}$ to conclude that for $f \in \bigcap_n UH_{z_n}$, the set $\partial\mathbb{D} \setminus \Sigma$ is dense. Note that Σ is a compact set and contains no isolated points by ergodicity of T. Together with the result in [6], the associated CMV matrices have a Cantor spectrum for a generic $f \in C^0(X, \mathbb{D})$. $\square$

5. *Proof for Jacobi matrices*

Let $f_a, f_b \in C^0(X, \mathbb{R})$ with $f_a(x) > 0$ for all $x \in X$. Fix $x \in X$ and let H_x be a two-sided Jacobi matrix with $a_n = f_a(T^n x)$, $b_n = f_b(T^n x)$. Define $J \subset \mathrm{SL}(2, \mathbb{R})$ as

$$J = \left\{ \begin{bmatrix} t & -\dfrac{1}{a} \\ a & 0 \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R}) | t, a \in \mathbb{R}, a > 0 \right\}.$$

Then we have $A_{E,a,b} \in C^0(X, J)$. Recall that given a non-empty subset $K \subset X$, $C^0_{A,K}(X, \mathrm{SL}(2, \mathbb{R}))$ denotes the set of all $B \in C^0(X, \mathrm{SL}(2, \mathbb{R}))$ such that $A(x) = B(x)$ for $x \notin K$.

LEMMA 5.1. *Let $K \subset X$ be a compact set such that $K \cap T(K) = \emptyset$ and $K \cap T^2(K) = \emptyset$. Let $A \in C^0(X, J)$ be such that for every $x \in K$, we have $\mathrm{tr}A(x) \neq 0$. Then there exist an open neighborhood $\mathcal{W}_{A,K} \subset C^0_{A,K}(X, \mathrm{SL}(2, \mathbb{R}))$ of A and continuous maps*

$$\Phi = \Phi_{A,K} : \mathcal{W}_{A,K} \to C^0(X, J)$$

and

$$\Psi = \Psi_{A,K} : \mathcal{W}_{A,K} \to C^0(X, \mathrm{SL}(2, \mathbb{R}))$$

satisfying

$$\Psi(B)(T(x)) \cdot B(x) \cdot [\Psi(B)(x)]^{-1} = \Phi(B)(x),$$

$$\Phi(A) = A \text{ and } \Psi(A) = \mathrm{id}.$$

Proof. Let $B \in C^0_{A,K}(X, \mathrm{SL}(2, \mathbb{R}))$. Let $\Phi(B)(x) = A(x)$ if $x \notin \bigcup_{i=-1}^{1} T^i(K)$. Fix $x \in K$. Let

$$B(T(x))B(x)B(T^{-1}(x)) = \begin{bmatrix} t_1 & -\dfrac{1}{a_1} \\ a_1 & 0 \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} t_3 & -\dfrac{1}{a_3} \\ a_3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} t_1(pt_3 + qa_3) - \dfrac{rt_3 + sa_3}{a_1} & -\dfrac{pt_1}{a_3} + \dfrac{r}{a_1 a_3} \\ a_1(pt_3 + qa_3) & -\dfrac{pa_1}{a_3} \end{bmatrix}$$

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and let

$$A(x) = \begin{bmatrix} t_2 & -\dfrac{1}{a_2} \\ a_2 & 0 \end{bmatrix}.$$

For $x \in \bigcup_{i=-1}^{1} T^i(K)$, our goal is to define $\Phi(B)(x)$ so that

$$\Phi(B)(T(x))\Phi(B)(x)\Phi(B)(T^{-1}(x)) = \begin{bmatrix} t_1(pt_3 + qa_3) - \dfrac{rt_3 + sa_3}{a_1} & -\dfrac{pt_1}{a_3} + \dfrac{r}{a_1 a_3} \\ a_1(pt_3 + qa_3) & -\dfrac{pa_1}{a_3} \end{bmatrix}$$

while we have $\Phi(B)(x) \in J$.

By a simple calculation,

$$\begin{bmatrix} t_1' & -\dfrac{1}{a_1'} \\ a_1' & 0 \end{bmatrix} \begin{bmatrix} t_2' & -\dfrac{1}{a_2'} \\ a_2' & 0 \end{bmatrix} \begin{bmatrix} t_3' & -\dfrac{1}{a_3'} \\ a_3' & 0 \end{bmatrix} = \begin{bmatrix} t_1'\left(t_2't_3' - \dfrac{a_3'}{a_2'}\right) - \dfrac{a_2'a_3'}{a_1'} & -\dfrac{t_1't_2'}{a_3'} + \dfrac{a_2'}{a_1'a_3'} \\ a_1'\left(t_2't_3' - \dfrac{a_3'}{a_2'}\right) & -\dfrac{t_2'a_1'}{a_3'} \end{bmatrix}.$$

Set $a_1' = a_1$, $a_2' = a_2$ and $a_3' = a_3$. We may write t_i', $i = 1, 2, 3$, as

$$t_i' = \frac{E - b_i'}{a_i'},$$

where $E, b_i' \in \mathbb{R}$. Set

$$b_2' = E - pa_2,$$

$$b_3' = E - a_3' \frac{pt_3 + qa_3 + \dfrac{a_3'}{a_2'}}{t_2'}$$

and

$$b_1' = E - \frac{a_1'a_3'}{t_2'}\left(\frac{a_2'}{a_1'a_3'} + \frac{pt_1}{a_3} - \frac{r}{a_1a_3}\right).$$

Note that we have $p \neq 0$ by choosing a proper neighborhood $\mathcal{W}_{A,K}$ of A since we assume that $\operatorname{tr}A(x) \neq 0$ for all $x \in K$. This, in turn, implies that $t_2' \neq 0$.

By setting

$$\Phi(B)(T(x)) = \begin{bmatrix} t_1' & -\dfrac{1}{a_1'} \\ a_1' & 0 \end{bmatrix}, \quad \Phi(B)(x) = \begin{bmatrix} t_2' & -\dfrac{1}{a_2'} \\ a_2' & 0 \end{bmatrix}$$

and

$$\Phi(B)(T^{-1}(x)) = \begin{bmatrix} t_3' & -\dfrac{1}{a_3'} \\ a_3' & 0 \end{bmatrix},$$

we have

$$\Phi(B)(T(x))\Phi(B)(x)\Phi(B)(T^{-1}(x)) = B(T(x))B(x)B(T^{-1}(x)).$$

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Let $\Psi(B)(x) = \text{id}$ for $x \notin K \cup T(K)$ and let

$$\Psi(B)(x) = \Phi(B)(T^{-1}(x)) \cdot [B(T^{-1}(x)]^{-1}$$

for $x \in K$. Let

$$\Psi(B)(x) = \Phi(B)(T^{-1}(x)) \cdot \Phi(B)(T^{-2}(x)) \cdot [B(T^{-2}(x)]^{-1} \cdot [B(T^{-1}(x)]^{-1}$$

for $x \in T(K)$. All properties are easy to check. □

By combining Lemmas 3.1 and 5.1, we obtain the following result.

PROPOSITION 5.2. *Let $A \in C^0(X, J)$ be a map whose trace is not identically zero. Then there exist an open neighborhood $\mathcal{W} \subset C^0(X, \mathrm{SL}(2, \mathbb{R}))$ of A and continuous maps*

$$\Phi = \Phi_A : \mathcal{W} \to C^0(X, J) \textit{ and } \Psi = \Psi_A : \mathcal{W} \to C^0(X, \mathrm{SL}(2, \mathbb{R}))$$

satisfying

$$\Psi(B)(T(x)) \cdot B(x) \cdot [\Psi(B)(x)]^{-1} = \Phi(B)(x),$$

$$\Phi(A) = A \textit{ and } \Psi(A) = \text{id}.$$

Proof. Let $x \in X$ be such that $\mathrm{tr}A(x) \neq 0$. Let V be an open neighborhood of x such that with $K = \bar{V}$, we have $\mathrm{tr}A(x) \neq 0$ for $x \in K$, $K \cap T(X) = \emptyset$ and $K \cap T^2(X) = \emptyset$.

Let $\Phi_{A,V} : \mathcal{W}_{A,V} \to C^0_{A,\bar{V}}(X, \mathrm{SL}(2, \mathbb{R})$ and $\Psi_{A,V} : \mathcal{W}_{A,V} \to C^0(X, \mathrm{SL}(2, \mathbb{R}))$ be given by Lemma 3.1. Let $\Phi_{A,K} : \mathcal{W}_{A,K} \to C^0(X, J)$ and $\Psi_{A,K} : \mathcal{W}_{A,K} \to C^0(X, \mathrm{SL}(2, \mathbb{R}))$ be given by Lemma 5.1. Let $\mathcal{W}$ be the domain of $\Phi := \Phi_{A,K} \circ \Phi_{A,V}$ and let $\Psi = (\Psi_{A,K} \circ \Phi_{A,V}) \cdot \Psi_{A,V}$. With Φ and Ψ, the result follows. □

Remark. Let us discuss why we do not apply the above procedure in the case of CMV matrices. To make a similar argument as in the proof of Lemma 5.1, one essential part is to construct $\Phi(B) \in C^0(X, S'_{\theta'})$ such that

$$\begin{aligned} &B(T^m(x)) \cdots B(x) \cdots B(T^{-n}(x)) \\ &\quad = \Phi(B)(T^m(x)) \cdots \Phi(B)(x) \cdots \Phi(B)(T^{-n}(x)) \end{aligned}$$

for some $m, n \in \mathbb{Z}_+$. The conjugation property almost automatically follows then. As we observed in the proof of Lemma 5.1, the construction is related to the solvability of a system of equations.

If we write $\Phi(B)$ as in Lemma 4.1, the product of matrices,

$$\Phi(B)(T^m(x)) \cdots \Phi(B)(x) \cdots \Phi(B)(T^{-n}(x)),$$

may be very complicated. In addition, we have many constraints since a matrix as in Lemma 4.1 has a very particular form while $0 \leq r(x) < 1$.

Alternatively, we may consider a product of matrices as in the form of $\mathrm{SU}(1, 1)$. In this case, the product involves complex numbers and conjugations of those while solutions must be in the complex unit disk. Moreover, we anticipate that it is unlikely that solutions are written as linear forms.

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Now we are ready to prove Theorem 2.3. Let $A \in C^0(X, J)$. Recall that if A is not uniformly hyperbolic, we may choose $B' \in C^0(X, \mathrm{SL}(2, \mathbb{R}))$ such that B' is arbitrarily C^0-close to A and (T, B') is uniformly hyperbolic.

Proof of Theorem 2.3. For $E \in \mathbb{R}$, define the set

$$UH_E := \{f_b \in C^0(X, \mathbb{R}) | (T, A_{E,a,b}) \text{ is uniformly hyperbolic}\}.$$

Suppose that a cocycle $(T, A_{E,a,b})$ is not uniformly hyperbolic. If $f_b(x) \neq E$ for some $x \in X$, $(T, A_{E,a,b})$ can be approximated by a uniformly hyperbolic cocycle $(T, A_{E,a,b'})$ for some $f_{b'} \in C^0(X, \mathbb{R})$ by Proposition 5.2.

Suppose that $f_b(x) = E$ for all $x \in X$. Then we may find $b' \in C^0(X, \mathbb{R})$ such that $f_{b'}$ is arbitrarily C^0-close to f_b and $f_{b'}(x) \neq E$ for some $x \in X$. From $(T, A_{E,a,b'})$, we may choose a uniformly hyperbolic cocycle $(T, A_{E,a,b''})$ for some $f_{b''} \in C^0(X, \mathbb{R})$, which is arbitrarily C^0-close by Proposition 5.2. This shows that the set UH_E is open and dense. Thus, there exists a countable dense subset $\{E_n\} \subset \mathbb{R}$ so that for all $f_b \in \bigcap_n UH_{E_n}$, the set $\mathbb{R} \setminus \Sigma$ is dense. Note that Σ is a compact set and contains no isolated points by ergodicity of T. Together with the result in [11], the associated Jacobi matrices have a Cantor spectrum for a generic $f \in C^0(X, \mathbb{D})$. □

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