Bull. Aust. Math. Soc. **106** (2022), 408–418
doi:10.1017/S0004972722000259

EXTREME VALUES OF THE RANKIN–SELBERG L-FUNCTIONS

CHI CUI and QIYU YANG

(Received 23 January 2022; accepted 9 February 2022; first published online 21 March 2022)

Abstract

In this paper, we study the extreme values of the Rankin–Selberg L-functions associated with holomorphic cusp forms in the vertical direction. Assuming the generalised Riemann hypothesis (GRH), we prove that

$$\max_{T^\delta \le t \le T} \left| L\left(\frac{1}{2} + it, f \times f\right) \right| \ge \exp\left(C \sqrt{\frac{\log T \log\log\log T}{\log\log T}} \right)$$

with $C \le \mathscr{X}\sqrt{1-\delta}$, where $\mathscr{X} := (2/\pi)\int_0^{\pi/3} \sin^2 \xi \, d\xi$ and $0 \le \delta < 1$.

2020 *Mathematics subject classification*: primary 11F11; secondary 11F66.

Keywords and phrases: extreme value, resonance method, Rankin–Selberg L-function.

1. Introduction

1.1. Background. The Lindelöf hypothesis (LH) asserts that for every $\varepsilon > 0$,

$$|\zeta(\tfrac{1}{2} + it)| = O(t^\varepsilon) \quad \text{as } t \to \infty.$$

In [9], Littlewood showed that a stronger form of LH follows from the Riemann hypothesis (RH): namely, for some positive constant $C_1 > 0$ and for all large $|t|$,

$$\left|\zeta\left(\frac{1}{2} + it\right)\right| = O\left(\exp\left(C_1 \frac{\log|t|}{\log\log|t|}\right)\right),$$

where the sharpest size of the implicit constant C_1 has been given by Chandee and Soundararajan [4].

In the opposite direction, Titchmarsh (see [12, Theorem 8.12]) proved that for any $\alpha < 1/2$ and large enough T,

$$\max_{t\in[0,T]} |\zeta(\tfrac{1}{2} + it)| \ge \exp((\log T)^\alpha).$$

This work is supported by the Science and Technology Development Fund, Macau SAR (File No. 0066/2020/A2).

Using the resonance method, Soundararajan [10] proved that there exists $t \in [T, 2T]$ such that

$$\left|\zeta\Big(\frac{1}{2} + it\Big)\right| \geq \exp\Big((1 + o(1))\frac{\sqrt{\log T}}{\sqrt{\log\log T}}\Big) \quad \text{as } T \to \infty.$$

Recently, Bondarenko and Seip [2] made a breakthrough by showing that for any constant $C_2 < 1/\sqrt{2}$,

$$\max_{t\in[\sqrt{T},T]}\left|\zeta\Big(\frac{1}{2} + it\Big)\right| \geq \exp\Big(C_2\frac{\sqrt{\log T \log\log\log T}}{\sqrt{\log\log T}}\Big).$$

Later, Bondarenko and Seip [3] improved their result by widening the allowable range of C_2, showing that the above bound holds for any $C_2 < 1$. Currently, the sharpest lower bound is due to De la Bretèche and Tenenbaum [5] who established

$$\max_{t\in[0,T]}\left|\zeta\Big(\frac{1}{2} + it\Big)\right| \geq \exp\Big((\sqrt{2} + o(1))\sqrt{\frac{\log T \log\log\log T}{\log\log T}}\Big) \quad \text{as } T \to \infty.$$

In this paper, we investigate the extreme values of the Rankin–Selberg L-functions associated with holomorphic cusp forms in the vertical direction. We begin with some definitions for these L-functions.

1.2. Rankin–Selberg *L*-functions. Let f be a primitive holomorphic cusp form of weight $k \geq 1$ for $SL_2(\mathbb{Z})$. Let

$$f(z) = \sum_{n\geq 1} \lambda_f(n) n^{(k-1)/2} e(nz)$$

be its normalised Fourier expansion at the cusp ∞, where $\lambda_f(n) \in \mathbb{R}$ ($n = 1, 2, \ldots$) are eigenvalues of Hecke operators $T(n)$ (that is, $T(n)f = \lambda_f(n)f$), normalised so that $\lambda_f(1) = 1$. By the work of Deligne, there exist $\alpha_f(p), \beta_f(p) \in \mathbb{C}$, satisfying

$$\alpha_f(p)\beta_f(p) = 1$$

and

$$\lambda_f(p^\nu) = \alpha_f(p)^\nu + \alpha_f(p)^{\nu-1}\beta_f(p) + \cdots + \beta_f(p)^\nu, \quad \text{for } \nu \geq 1.$$

The Ramanujan conjecture states that

$$|\alpha_f(p)| = |\beta_f(p)| = 1 \tag{1.1}$$

for all prime numbers p. For holomorphic f, this was proved by Deligne [6] in 1974. Thus for each prime number p, there is a unique $\xi_f(p) \in [0, \pi]$ such that

$$\lambda_f(p) = 2\cos \xi_f(p).$$

According to the Sato–Tate conjecture, the sequence $\{\xi_f(p)\}_p$ is equi-distributed on $[0, \pi]$ with respect to the measure $(2/\pi)\sin^2 \xi \, d\xi$, that is, for a given subinterval

https://doi.org/10.1017/S0004972722000259 Published online by Cambridge University Press

$[a, b] \subset [0, \pi]$,

$$|\{p \le x : \xi_f(p) \in [a, b]\}| \sim \frac{x}{\log x} \frac{2}{\pi} \int_a^b \sin^2 \xi \, d\xi \quad \text{as } x \to \infty. \tag{1.2}$$

This has been proved by Barnet-Lamb *et al.* [1]. (For Maass cusp forms, both (1.1) and (1.2) are still open.)

For $\mathrm{Re}\, s > 1$, the Rankin–Selberg L-function attached to f is

$$L(s, f \times f) := \prod_p \Big(1 - \frac{\alpha_f(p)^2}{p^s}\Big)^{-1} \Big(1 - \frac{\beta_f(p)^2}{p^s}\Big)^{-1} \Big(1 - \frac{1}{p^s}\Big)^{-2}.$$

It can be continued analytically to the whole complex plane with a simple pole at $s = 1$ and satisfies the functional equation

$$\Lambda(s, f \times f) = \Lambda(1 - s, f \times f),$$

for the complete L-function

$$\Lambda(s, f \times f) := \gamma(s, f \times f) L(s, f \times f)$$

and the gamma factor

$$\gamma(s, f \times f) := (2\pi)^{-2s} \Gamma(s) \Gamma(s + k - 1).$$

Following the argument of Bondarenko and Seip [3], we establish the following theorem.

THEOREM 1.1. *Assume the GRH. Let $0 \le \delta < 1$ be given. If T is sufficiently large, then there exists t with $T^\delta \le t \le T$ such that*

$$\Big|L\Big(\frac{1}{2} + it, f \times f\Big)\Big| \ge \exp\Big(C \sqrt{\frac{\log T \log\log\log T}{\log\log T}}\Big), \tag{1.3}$$

where C is a positive number depending on f, satisfying $C < \mathscr{X} \sqrt{1 - \delta}$, and where $\mathscr{X} := (2/\pi) \int_0^{\pi/3} \sin^2 \xi \, d\xi$.

The assumption of the GRH is only needed in the proof of Theorem 1.1 to handle the moments $\int_1^T |L(\frac{1}{2} + it, f \times f)|^{2r} \, dt$ (see Lemma 2.2). In fact, the convexity bound is sufficient for the proof of Lemma 2.1 and subconvexity bounds will not lead to any improvement of the result.

The positivity of the coefficients of the Rankin–Selberg L-functions is necessary for the method in the proof. Hence, a principal difference between our version of the resonance method and that used earlier by Bondarenko and Seip [3] is that we have to consider a suitable subcollection of the set of prime numbers in our resonator (see Section 2.3 for the details).

https://doi.org/10.1017/S0004972722000259 Published online by Cambridge University Press

2. Preparation for the proof

2.1. Convolution formula for $L(s, f \times f)$. We define the Fourier transform $\widehat{F}$ of F on $\mathbb{R}$ as

$$\widehat{F}(\xi) = \int_{-\infty}^{\infty} F(x) e^{-ix\xi}\, dx.$$

LEMMA 2.1. *Suppose that* $\frac{1}{2} \le \sigma < 1$ *and let* $F(x+iy)$ *be an analytic function in the horizontal strip* $\sigma - 2 \le y \le 0$ *satisfying the growth estimate*

$$\max_{\sigma-2\le y\le 0} |F(x+iy)| = O\Big(\frac{1}{|x|^2}\Big)$$

when $|x| \to \infty$. *Then for every real* t,

$$\int_{-\infty}^{\infty} L(\sigma + i(t+u), f\times f) F(u)\, du = \sum_{m=1}^{\infty} \frac{A(m)\widehat{F}(\log m)}{m^{\sigma+it}} - 2\pi i \rho_f F(-t + i(\sigma-1)),$$

where $L(s, f\times f) := \sum_{m=1}^{\infty} A(m) m^{-s}$ *and* ρ_f *is the residue of* $L(s, f\times f)$ *at* $s = 1$.

PROOF. Let T be a large positive number and let $\mathscr{R}(T)$ denote the contour consisting of the line segments connecting $\sigma - iT, 2 - iT, 2 + iT, \sigma + iT$.

By the residue theorem applied to $G(z) := L(z + it, f\times f) F(i\sigma - iz)$ in $\mathscr{R}(T)$,

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\mathscr{R}(T)} G(z)\, dz \\
&\quad = \frac{1}{2\pi i}\Big(\int_{\sigma+iT}^{\sigma-iT} + \int_{\sigma-iT}^{2-iT} + \int_{2-iT}^{2+iT} + \int_{2+iT}^{\sigma+iT}\Big) L(z+it, f\times f) F(i\sigma - iz)\, dz \\
&\quad = \operatorname*{Res}_{z=1-it} G(z).
\end{aligned}$$

Since $L(s, f\times f)$ is holomorphic in the s-plane except for a simple pole at $s = 1$,

$$\begin{aligned}
&\int_{\sigma-iT}^{\sigma+iT} L(z+it, f\times f) F(i\sigma - iz)\, dz \\
&\qquad = \int_{2-iT}^{2+iT} L(z+it, f\times f) F(i\sigma - iz)\, dz - 2\pi i \rho_f F(-t + i(\sigma-1)) \\
&\qquad\quad + \Big(\int_{\sigma-iT}^{2-iT} + \int_{2+iT}^{\sigma+iT}\Big) L(z+it, f\times f) F(i\sigma - iz)\, dz. \qquad (2.1)
\end{aligned}$$

Applying the upper bound for $F(x+iy)$ in Lemma 2.1, (2.1) and the convexity bound [8, (5.21)], we arrive at

$$\begin{aligned}
\int_{-T}^{T} L(\sigma + i(t+u), f\times f) F(u)\, du &= \int_{-T}^{T} L(2 + i(t+u), f\times f) F(u + i(\sigma-2))\, du \\
&\quad - 2\pi i \rho_f F(-t + i(\sigma-1)) + O(T^{-3/2+\varepsilon}).
\end{aligned}$$

https://doi.org/10.1017/S0004972722000259 Published online by Cambridge University Press

Also,

$$\begin{aligned}\int_{-\infty}^{\infty} & L(2+i(t+u), f\times f)F(u+i(\sigma-2))\,du \\ &= \sum_{m=1}^{\infty}\frac{A(m)}{m^{2+it}}\int_{-\infty}^{\infty}F(u+i(\sigma-2))e^{-iu\log m}e^{(\sigma-2)\log m}e^{-(\sigma-2)\log m}\,du \\ &= \sum_{m=1}^{\infty}\frac{A(m)}{m^{\sigma+it}}\widehat{F}(\log m).\end{aligned}$$

Combining these two formulae completes the proof of Lemma 2.1. □

2.2. Moments of *L*-functions. Define

$$I_r(T, f\times f) := \int_1^T \left|L\Big(\frac{1}{2}+it, f\times f\Big)\right|^{2r} dt. \tag{2.2}$$

LEMMA 2.2 [11, Theorem 1.1]. *Assume the GRH. Let f be a primitive holomorphic cusp form of weight $k \geq 1$ for $SL_2(\mathbb{Z})$. Let $I_r(T, f\times f)$ be defined as in (2.2). Then for all real numbers $r > 0$ and sufficiently large T,*

$$I_r(T, f\times f) \ll_{r,f} T(\log T)^{r^2}.$$

2.3. Construction of the resonator. The resonance method can be traced back to a paper of Voronin [13]. It was developed independently and significantly refined by Hilberdink [7] and by Soundararajan [10].

A resonator (in the spirit of [3, Section 3]) is a function of the form $R(t)$, where

$$R(t) = \sum_{m\in\mathscr{M}'}\frac{r(m)}{m^{it}}, \tag{2.3}$$

and $\mathscr{M}'$ is a suitable finite set of positive integers whose construction is given below.

Let $0 < \delta < 1$ be a fixed real number, and let κ and γ with $0 < \kappa < 1$ and $0 < \gamma < 1$ be parameters still to be chosen. Define $N = [T^\kappa]$, where $[x]$ denotes the integer part of x. Let P be the set of prime numbers p such that

$$e\log N\log_2 N < p < \exp((\log_2 N)^\gamma)\log N\log_2 N \quad \text{and} \quad 1 \leq \lambda_f(p) \leq 2. \tag{2.4}$$

We define $h(n)$ to be the multiplicative function supported on the set of square-free numbers such that

$$h(p) := \sqrt{\frac{\log N\log_2 N}{\log_3 N}}\,\frac{1}{\sqrt{p}(\log p - \log_2 N - \log_3 N)} \quad \text{for } p \in P,$$

https://doi.org/10.1017/S0004972722000259 Published online by Cambridge University Press

and $h(p) = 0$ otherwise. Fix Δ with $1 < \Delta < 1/\gamma$. For each $\ell \in \{1, \dots, [(\log_2 N)^\gamma]\}$, we define the sets

$$P_\ell := \{p : e^\ell \log N \log_2 N < p \le e^{\ell+1} \log N \log_2 N,\ 1 \le \lambda_f(p) \le 2\},$$
$$M_\ell := \Big\{n \in \mathrm{supp}(h) : n \text{ has at least } \frac{\Delta \log N}{\ell^2 \log_3 N} \text{ prime divisors in } P_\ell\Big\}.$$

Next, we define the set

$$\mathscr{M} := \mathrm{supp}(h) \setminus \bigcup_{\ell=1}^{[(\log_2 N)^\gamma]} M_\ell.$$

Then $\mathscr{M}$ is the set of square-free numbers n that have at most $\Delta \log N/(\ell^2 \log_3 N)$ divisors in P_ℓ.

Now, let $\mathscr{J}$ be the set of integers j such that

$$[(1+T^{-1})^j, (1+T^{-1})^{j+1}] \bigcap \mathscr{M} \neq \emptyset,$$

and let m_j to be the minimum of $[(1+T^{-1})^j, (1+T^{-1})^{j+1}] \bigcap \mathscr{M}$ for j in $\mathscr{J}$. Consider the set

$$\mathscr{M}' := \{m_j : j \in \mathscr{J}\},$$

and define

$$r(m_j) := \Big(\sum_{n \in \mathscr{M},\, (1+T^{-1})^{j-1} \le n \le (1+T^{-1})^{j+2}} h(n)^2\Big)^{1/2} \quad \text{for } m_j \in \mathscr{M}'.$$

Finally, we set $\Phi(t) = e^{-t^2/2}$.

LEMMA 2.3. *We have*

(i) $|R(t)|^2 \le R(0)^2 \ll N \sum_{l \in \mathscr{M}} h(l)^2$;
(ii) $\int_{-\infty}^{\infty} |R(t)|^2 \Phi(t/T)\, dt \ll T \sum_{l \in \mathscr{M}} h(l)^2$.

PROOF. The proof for (i) follows from the definition of $R(t)$ in (2.3) and the Cauchy–Schwarz inequality. The proof for (ii) follows the same outline as in [3, Lemma 5]. □

Define

$$\mathcal{A}(N) := \frac{1}{\sum_{i \in \mathbb{N}} h(i)^2} \sum_{n \in \mathbb{N}} \frac{h(n)}{\sqrt{n}} \sum_{d \mid n} h(d)\sqrt{d} \quad \text{and} \quad \mathscr{X} := \frac{2}{\pi} \int_0^{\pi/3} \sin^2 \xi \, d\xi. \tag{2.5}$$

We make use of the following four lemmas.

LEMMA 2.4. *We have*

$$\mathcal{A}(N) \ge \exp\Big((\gamma \mathscr{X} + o(1)) \sqrt{\frac{\log N \log_3 N}{\log_2 N}}\Big) \quad \textit{as } N \to \infty.$$

https://doi.org/10.1017/S0004972722000259 Published online by Cambridge University Press

PROOF. From the construction of h as a multiplicative function,

$$\frac{1}{\sum_{i\in\mathbb{N}} h(i)^2}\sum_{n\in\mathbb{N}}\frac{h(n)}{\sqrt{n}}\sum_{d|n}h(d)\sqrt{d} = \frac{\prod_{p\in P}(1+h(p)p^{-1/2}\prod_{d|p}h(d)\sqrt{d})}{\prod_{p\in P}(1+h(p)^2)}$$
$$= \prod_{p\in P}\frac{1+h(p)^2+h(p)p^{-1/2}}{1+h(p)^2}$$
$$= \exp\Big((1+o(1))\sum_{p\in P}\frac{h(p)}{\sqrt{p}}\Big). \tag{2.6}$$

Following the proof of [2, Lemma 1], (1.2) and the definitions of P in (2.4) and $\mathscr{X}$ in (2.5) lead to

$$\sum_{p\in P}\frac{h(p)}{\sqrt{p}} = \sqrt{\frac{\log N\log_2 N}{\log_3 N}}\sum_{p\in P}\frac{1}{p(\log p-\log_2 N-\log_3 N)}$$
$$= \sqrt{\frac{\log N\log_2 N}{\log_3 N}}\int_{e\log N\log_2 N}^{\exp((\log_2 N)^\gamma)\log N\log_2 N}\frac{d\,(\frac{(1+o(1))x}{\log x}\frac{2}{\pi}\int_0^{\pi/3}\sin^2\xi\,d\xi)}{x(\log x-\log_2 N-\log_3 N)}$$
$$= (\gamma\mathscr{X}+o(1))\sqrt{\frac{\log N\log_3 N}{\log_2 N}}. \tag{2.7}$$

Inserting (2.7) into (2.6) completes the proof of the lemma. □

LEMMA 2.5 [2, Lemma 3]. *We have*

$$\frac{1}{\sum_{i\in\mathbb{N}} h(i)^2}\sum_{n\in\mathscr{M}}\frac{h(n)}{\sqrt{n}}\sum_{\substack{d|n\\ d\le n/N^\varepsilon}}h(d)\sqrt{d} = o(\mathcal{A}(N)) \quad \text{as } N\to\infty,$$

where the implicit constant depends only on ε.

LEMMA 2.6 [2, Lemma 2]. *We have*

$$\frac{1}{\sum_{i\in\mathbb{N}} h(i)^2}\sum_{\substack{n\in\mathbb{N}\\ n\notin\mathscr{M}}}\frac{h(n)}{\sqrt{n}}\sum_{d|n}h(d)\sqrt{d} = o(\mathcal{A}(N)) \quad \text{as } N\to\infty.$$

LEMMA 2.7. *Suppose that*

$$L(t) := \sum_{m=1}^{\infty}\frac{A(m)a_m}{m^{1/2+it}}$$

https://doi.org/10.1017/S0004972722000259 Published online by Cambridge University Press

is absolutely convergent and that $a_n \geq 0$ *for every* n. *Let* ε *be a positive number and* γ *be the parameter defining the set* P. *Then*

$$\int_{-\infty}^{\infty} L(t)|R(t)|^2\Phi\Big(\frac{t}{T}\Big)\,dt \geq T\min_{m\leq T^{\varepsilon}} a_m \cdot \exp\bigg((\gamma\mathscr{X} + o(1))\sqrt{\frac{\kappa\log T\log_3 T}{\log_2 T}}\bigg)\sum_{l\in\mathscr{M}} h(l)^2.$$

PROOF. It follows from the explicit expression for $R(t)$ that

$$\begin{aligned}\int_{-\infty}^{\infty} L(t)|R(t)|^2\Phi\Big(\frac{t}{T}\Big)\,dt &= \sqrt{2\pi}T\sum_{m,n\in\mathscr{M}'}\sum_{k=1}^{\infty}\frac{A(k)a_k r(m)r(n)}{\sqrt{k}}\Phi\Big(T\log\frac{km}{n}\Big)\\ &\geq \sqrt{2\pi}T\min_{j\leq T^{\varepsilon}} a_j\sum_{m,n\in\mathscr{M}'}\sum_{\substack{k\in\mathscr{M}\\ k\leq T^{\varepsilon}}}\frac{r(m)r(n)}{\sqrt{k}}\Phi\Big(T\log\frac{km}{n}\Big). \end{aligned}\tag{2.8}$$

Here we used the fact that $A(k) \geq 1$ for $k \in \mathscr{M}$ by the construction of $\mathscr{M}$.

For a given k in $\mathscr{M}$, consider all pairs m', n' in $\mathscr{M}'$ such that $|km'/n' - 1| \leq 3/T$. We use the notation

$$J(m') := [(1 + T^{-1})^j, (1 + T^{-1})^{j+1}),$$

where j is the unique integer such that $(1 + T^{-1})^j \leq m' \leq (1 + T^{-1})^{j+1}$. Using the Cauchy–Schwarz inequality and the definition of $r(m')$, we find

$$r(m')r(n') \geq \sum_{\substack{m,n\in\mathscr{M},\, mk=n\\ m\in J(m'),\, n\in J(n')}} h(m)h(n),$$

and hence, by the definition of $\mathscr{M}'$,

$$\sum_{\substack{m',n'\in\mathscr{M}'\\ |km'/n'-1|\leq 3/T}} r(m')r(n') \geq \sum_{m,n\in\mathscr{M},\, mk=n} h(m)h(n).$$

Now dividing this inequality by $\sqrt{k}$ and summing over all k in $\mathscr{M}\cap[1, T^{\varepsilon}]$ and combining the result with (2.8), and Lemmas 2.5 and 2.6, we get

$$\begin{aligned}\int_{-\infty}^{\infty} L(t)|R(t)|^2\Phi\Big(\frac{t}{T}\Big)\,dt &\gg T\min_{j\leq T^{\varepsilon}} a_j\sum_{n\in\mathscr{M}}\frac{h(n)}{\sqrt{n}}\sum_{d|n,\, d\geq n/T^{\varepsilon}} h(d)\sqrt{d}\\ &\gg T\min_{j\leq T^{\varepsilon}} a_j\cdot\mathcal{A}(N)\sum_{n\in\mathscr{M}} h(n)^2. \end{aligned}\tag{2.9}$$

Combining Lemma 2.4 and (2.9) completes the proof of the lemma. □

3. Proof of Theorem 1.1

We choose

$$F(t) := \frac{\sin^2((\varepsilon\log T)t)}{(\varepsilon\log T)t^2} \tag{3.1}$$

https://doi.org/10.1017/S0004972722000259 Published online by Cambridge University Press

and note that

$$\widehat{F}(\xi) = \frac{\pi}{2} \max\Big(1 - \frac{|\xi|}{2\varepsilon \log T}, 0\Big). \tag{3.2}$$

By the convexity bound [8, (5.21)] and the growth estimate for $F(x+iy)$,

$$\begin{aligned} &\int_{-T^\delta}^{T^\delta} \int_{-\infty}^{\infty} \Big|L\Big(\frac{1}{2} + i(t+u), f \times f\Big)\Big| F(u)\, du\, dt \\ &\quad \ll T^\delta + \int_{-T^\delta}^{T^\delta} \int_{|u| \le T^\delta} \Big|L\Big(\frac{1}{2} + i(t+u), f \times f\Big)\Big| F(u)\, du\, dt \\ &\quad \ll T^\delta + \int_{-2T^\delta}^{2T^\delta} \Big|L\Big(\frac{1}{2} + it, f \times f\Big)\Big|\, dt. \end{aligned} \tag{3.3}$$

By Lemma 2.2,

$$\int_{-2T^\delta}^{2T^\delta} \Big|L\Big(\frac{1}{2} + it, f \times f\Big)\Big|\, dt \ll_f T^{\delta+\varepsilon}. \tag{3.4}$$

Combining (3.3) and (3.4),

$$\int_{-T^\delta}^{T^\delta} \int_{-\infty}^{\infty} \Big|L\Big(\frac{1}{2} + i(t+u), f \times f\Big)\Big| F(u) |R(t)|^2 \Phi\Big(\frac{t}{T}\Big)\, du\, dt \ll_f T^{\delta+\kappa+\varepsilon} \sum_{l \in \mathcal{M}} h(l)^2 \tag{3.5}$$

by a trival estimation of $R(0)^2$ in Lemma 2.3. Because of the rapid decay of $\Phi(t)$,

$$\int_{|t| > T \log T} \int_{-\infty}^{\infty} \Big|L\Big(\frac{1}{2} + i(t+u), f \times f\Big)\Big| F(u) |R(t)|^2 \Phi\Big(\frac{t}{T}\Big)\, du\, dt = o\Big(\sum_{l \in \mathcal{M}} h(l)^2\Big). \tag{3.6}$$

Combining (3.5) and (3.6), we deduce that

$$\begin{aligned} &\int_{T^\delta \le |t| \le T \log T} \int_{-\infty}^{\infty} \Big|L\Big(\frac{1}{2} + i(t+u), f \times f\Big)\Big| F(u) |R(t)|^2 \Phi\Big(\frac{t}{T}\Big)\, du\, dt \\ &\quad = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Big|L\Big(\frac{1}{2} + i(t+u), f \times f\Big)\Big| F(u) |R(t)|^2 \Phi\Big(\frac{t}{T}\Big)\, du\, dt + O_f\Big(T^{\delta+\kappa+\varepsilon} \sum_{l \in \mathcal{M}} h(l)^2\Big). \end{aligned} \tag{3.7}$$

We now require $\delta + \kappa < 1$ and apply Lemma 2.3(ii) to the left-hand side of (3.7). We obtain

$$\begin{aligned} &\max_{T^\delta/2 \le t \le 2T \log T} \Big|L\Big(\frac{1}{2} + it, f \times f\Big)\Big| T \sum_{l \in \mathcal{M}} h(l)^2 \\ &\quad \gg \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Big|L\Big(\frac{1}{2} + i(t+u), f \times f\Big)\Big| F(u) |R(t)|^2 \Phi\Big(\frac{t}{T}\Big)\, du\, dt + O(T) \sum_{l \in \mathcal{M}} h(l)^2. \end{aligned} \tag{3.8}$$

https://doi.org/10.1017/S0004972722000259 Published online by Cambridge University Press

By Lemma 2.1,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} L\Big(\frac{1}{2}+i(t+u), f\times f\Big)F(u)|R(t)|^2\Phi\Big(\frac{t}{T}\Big)\,du\,dt$$
$$= \int_{-\infty}^{\infty}\sum_{m=1}^{\infty}\frac{A(m)\widehat{F}(\log m)}{m^{1/2+it}}|R(t)|^2\Phi\Big(\frac{t}{T}\Big)\,dt - 2\pi i\rho_f\int_{-\infty}^{\infty}F(-t-i/2)|R(t)|^2\Phi\Big(\frac{t}{T}\Big)\,dt. \tag{3.9}$$

Setting $a_m := \widehat{F}(\log m)$ and applying Lemma 2.7,

$$\int_{-\infty}^{\infty}\sum_{m=1}^{\infty}\frac{A(m)\widehat{F}(\log m)}{m^{1/2+it}}|R(t)|^2\Phi\Big(\frac{t}{T}\Big)\,dt$$
$$\geq T\min_{m\leq T^{\varepsilon}}\widehat{F}(\log m)\cdot\exp\bigg((\gamma\mathscr{X}+o(1))\sqrt{\frac{\kappa\log T\log_3 T}{\log_2 T}}\bigg)\sum_{l\in\mathscr{M}}h(l)^2. \tag{3.10}$$

Applying (3.1) and Lemma 2.3(i),

$$\int_{-\infty}^{\infty}F(-t-i/2)|R(t)|^2\Phi\Big(\frac{t}{T}\Big)\,dt \ll T^{\kappa+\varepsilon}\sum_{l\in\mathscr{M}}h(l)^2. \tag{3.11}$$

In view of (3.2), we note that $\min_{m\leq T^{\varepsilon}}\widehat{F}(\log m)\geq\pi/4$. Hence, choosing ε small enough and combining (3.8)–(3.11), we find that the asserted bound (1.3) holds for some t satisfying $T^{\delta}/2\leq t\leq 2T\log T$. We obtain the desired restriction $T^{\delta}\leq t\leq T$ after a trivial adjustment, changing T to $T/(2\log T)$ and making δ slightly smaller.

Acknowledgement

The authors are grateful to Xuanxuan Xiao for drawing their attention to the problem considered in this paper.

References

[1] T. Barnet-Lamb, D. Geraghty, M. Harris and R. Taylor, 'A family of Calabi–Yau varieties and potential automorphy II', *Publ. Res. Inst. Math. Sci.* **47** (2011), 29–98.

[2] A. Bondarenko and K. Seip, 'Large greatest common divisor sums and extreme values of the Riemann zeta function', *Duke Math. J.* **166** (2017), 1685–1701.

[3] A. Bondarenko and K. Seip, 'Extreme values of the Riemann zeta function and its argument', *Math. Ann.* **372** (2018), 999–1015.

[4] V. Chandee and K. Soundararajan, 'Bounding $|\zeta(\frac{1}{2}+it)|$ on the Riemann hypothesis', *Bull. Lond. Math. Soc.* **43** (2011), 243–250.

[5] R. De la Bretèche and G. Tenenbaum, 'Sommes de Gál et applications', *Proc. Lond. Math. Soc.* **119** (2019), 104–134.

[6] P. Deligne, 'La Conjecture de Weil I', *Publ. Math. Inst. Hautes Études Sci.* **43** (1974), 273–307.

[7] T. Hilberdink, 'An arithmetical mapping and applications to Ω-results for the Riemann zeta function', *Acta Arith.* **139** (2009), 341–367.

https://doi.org/10.1017/S0004972722000259 Published online by Cambridge University Press

[8] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, American Mathematical Society Colloquium Publications, 53 (American Mathematical Society, Providence, RI, 2004).
[9] J. E. Littlewood, 'On the zeros of the Riemann zeta-function', *Math. Proc. Cambridge Philos. Soc.* **22** (1924), 295–318.
[10] K. Soundararajan, 'Extreme values of zeta and *L*-functions', *Math. Ann.* **342** (2008), 467–486.
[11] H. Tang and X. Xiao, 'Integral moments of automorphic *L*-functions', *Int. J. Number Theory* **12** (2016), 1827–1843.
[12] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd edn (Oxford University Press, New York, 1986), edited and with a preface by D. R. Heath-Brown.
[13] S. M. Voronin, 'Lower bounds in Riemann zeta-function theory', *Izv. Akad. Nauk SSSR Ser. Mat.* **52** (1988), 882–892.

CHI CUI, Faculty of Information Technology,
Macau University of Science and Technology, Macau, PR China
e-mail: ccuicynthia@gmail.com

QIYU YANG, Faculty of Information Technology,
Macau University of Science and Technology, Macau, PR China
e-mail: qyyang.must@gmail.com

https://doi.org/10.1017/S0004972722000259 Published online by Cambridge University Press