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COMPARISONS OF SERIES AND PARALLEL SYSTEMS WITH HETEROGENEOUS COMPONENTS

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This paper carries out stochastic comparisons of series and parallel systems with independent and heterogeneous components in the sense of the hazard rate order, the reversed hazard rate order, and the likelihood ratio order. The main results extend and strengthen the corresponding ones by Misra and Misra [18] and by Ding, Zhang, and Zhao [8]. Meanwhile, the results on the hazard rate order of parallel systems and the reversed hazard order of series systems serve as nice supplements to Theorem 16.B.1 of Boland and Proschan [4] and Theorem 3.2 of Nanda and Shaked [20], respectively.

1. INTRODUCTION

As one popular fault-tolerant structure in reliability theory, a k-out-of-n system functions iff at least k of its components are working. In other words, k-out-of-n system fails only when there are at least $(n-k+1)$ failed ones among n components. In particular, the n-out-of-n system and the 1-out-of-n system are known as series system and parallel system, respectively. The lifetime of a k-out-of-n system can be represented as the $(n-k+1)$th order statistic of its component lifetimes. Specifically, lifetimes of series and parallel systems correspond to the smallest and largest order statistics, respectively. In the literature of the past three decades, there is a large of research articles on stochastic comparisons of k-out-of-n systems with independent and identical components. For a detailed summary, one may refer to the well-known monograph Shaked and Shanthikumar [24]. In contrast,

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for k-out-of-n systems with components having independent but not necessarily identically distributed lifetimes, going with complicated nature of distribution theory, only a few results are available. Among which, due to the simple structures, series and parallel systems with the underlying components' lifetimes having some specific distributions such as exponential and gamma received much attention. For example, Boland, El-Neweihi, and Proschan [3], Bon and Păltănea [5], Dykstra, Kochar, and Rojo [9], Joo and Mi [13], Khaledi, Farsinezhad, and Kochar [14], Khaledi and Kochar [15], Kochar and Rojo [16], Kochar and Xu [17], Navarro and Shaked [21], Pledger and Proschan [22], Proschan and Sethuraman [23], Zhao and Balakrishnan [27], Zhao, Li, and Balakrishnan [28] and the references therein.

The purpose of this paper is to further investigate the problem of comparing series and parallel systems with independent and heterogeneous components. We present some results on stochastic comparisons in terms of several stochastic orders. In Section 2, definitions of some stochastic orders are recalled, and some recent developments on ordering series and parallel systems with heterogenous components are reviewed. The hazard rate order of parallel systems and the reversed hazard rate order of series systems are given in Section 3, these results extend several known results in the literature. Section 4 presents some results on likelihood ratio order, which strengthen the recent results in the literature.

Throughout this paper, "increasing" and "decreasing" mean "non-decreasing" and "non-increasing", respectively. Unless otherwise stated, all the random variables considered in this paper will be assumed to be non-negative.

2. PRELIMINARIES

For the sake of handy reference, this section first recalls some most pertinent notations of stochastic orders to be used in the sequel, and then review the recent development on stochastic comparison of series and parallel systems.

2.1. Some Stochastic Orders

Let $F_X(t)$, $\overline{F}_X(t)$ and $f_X(t)$ be the distribution function, survival function and density function of random variable X, respectively. The hazard rate function and the reversed hazard rate function of X are, respectively, defined as $r_X(t) = f_X(t)/\overline{F}_X(t)$ and $\tilde{r}_X(t) = f_X(t)/F_X(t)$. Denote by $m_X(t) = \mathsf{E}(X - t|X > t)$ the mean residual lifetime function of X. Then, X is said to be smaller than Y in the

(i) *likelihood ratio order* (denoted by $X \le_{\rm lr} Y$ or $F_X \le_{\rm lr} F_Y$) if $f_Y(t)/f_X(t)$ is increasing in t;

(ii) *hazard rate order* (denoted by $X \le_{\rm hr} Y$ or $F_X \le_{\rm hr} F_Y$) if $r_X(t) \ge r_Y(t)$ for all $t \in \mathbb{R}$, or equivalently, $\overline{F}_Y(t)/\overline{F}_X(t)$ is increasing in t;

(iii) *reversed hazard rate order* (denoted by $X \le_{\rm rh} Y$ or $F_X \le_{\rm rh} F_Y$) if $\tilde{r}_X(t) \le \tilde{r}_Y(t)$ for all $t \in \mathbb{R}$, or equivalently, $F_Y(t)/F_X(t)$ is increasing in t;

(iv) *mean residual life order* (denoted by $X \le_{\rm mrl} Y$ or $F_X \le_{\rm mrl} F_Y$) if $m_Y(t) \ge m_X(t)$ for all $t \in \mathbb{R}_+$.

The following implications are well known:

$$X \le_{\rm rh} Y \Longleftarrow X \le_{\rm lr} Y \Longrightarrow X \le_{\rm hr} Y \Longrightarrow X \le_{\rm mrl} Y.$$

For more details of these stochastic orders and their inter-relationship, one may refer to Müller and Stoyan [19] and Shaked and Shanthikumar [24].

2.2. Recent Advances in the Literature

From now on, we assume that $X_1, \dots, X_n$ be independent and absolutely continuous random variables, which are not necessarily of identical probability distribution. Independent of the vector of $(X_1, \dots, X_n)$, let $Y_1, \dots, Y_m$ be another sequence of independent and absolutely continuous random variables. Denote by $T_k(X_1, \dots, X_n)$ the lifetime of a k-out-of-n system composed of components with lifetimes $X_1, \dots, X_n$. To avoid ambiguity, let us state the known results in the literature in two separate groups.

2.2.1. Systems with two components. Suppose X_i and Y_i have the exponential lifetimes with respective hazard rates λ_i and γ_i, $i = 1, 2$. Joo and Mi [13] showed that

$$\gamma_1 \le \min\{\lambda_1, \lambda_2\} \Longrightarrow T_1(X_1, X_2) \le_{\text{hr}} T_1(X_1, Y_1). \tag{2.1}$$

Afterwards, Zhao and Balakrishnan [27] further strengthened the hazard rate order in (2.1) to the likelihood ratio order. On the other hand, Da, Ding, and Li [6] generalized the exponential distribution in (2.1) to any life distribution as below,

$$X_1 \le_{\text{hr}} (\le_{\text{mrl}}) X_2 \le_{\text{hr}} (\le_{\text{mrl}}) Y_1 \Longrightarrow T_1(X_1, X_2) \le_{\text{hr}} (\le_{\text{mrl}}) T_1(X_1, Y_1). \tag{2.2}$$

Also, a similar result for the likelihood ratio order can be found in Example 1.C.36 of Shaked and Shanthikumar [24] as follows:

$$X_1 \le_{\text{lr}} X_2 \le_{\text{lr}} Y_1 \Longrightarrow T_1(X_1, X_2) \le_{\text{lr}} T_1(X_1, Y_1). \tag{2.3}$$

Further, Misra and Misra [18] extended (2.2) and (2.3) to

$$X_i \le_{\text{hr}} (\le_{\text{mrl}}, \le_{\text{lr}}) Y_1,\ i = 1, 2 \Longrightarrow T_1(X_1, X_2) \le_{\text{hr}} (\le_{\text{mrl}}, \le_{\text{lr}}) T_1(X_1, Y_1). \tag{2.4}$$

Meanwhile, they also proved that

$$X_i \ge_{\text{rh}} (\ge_{\text{lr}}) Y_1,\ i = 1, 2 \Longrightarrow T_2(X_1, X_2) \ge_{\text{rh}} (\ge_{\text{lr}}) T_2(X_1, Y_1). \tag{2.5}$$

2.2.2. Systems with $n \ge 2$ components. Recently, Ding, Zhang, and Zhao [8] turned their attention to comparing systems of n component rather than systems of only two components. They showed that

$$X_i \le_{\text{hr}} (\ge_{\text{rh}}) Y_1,\ i = 1, \dots, n \Longrightarrow T_k(X_1, \dots, X_n) \le_{\text{hr}} (\ge_{\text{rh}}) T_k(Y_1, X_2, \dots, X_n), \tag{2.6}$$

for $k = 1, \dots, n$, and

$$X_i \le_{\text{mrl}} Y_1,\ i = 1, \dots, n \Longrightarrow T_1(X_1, \dots, X_{n-1}, X_n) \le_{\text{mrl}} T_1(Y_1, X_2, \dots, X_n). \tag{2.7}$$

Ding et al. [8] also considered the likelihood ratio order of series and parallel systems under the framework of multiple-outlier model. Suppose $X_i \overset{\text{st}}{=} X_1$ for $i = 1, \dots, m$, $X_i \overset{\text{st}}{=} X_n$ for $i = m+1, \dots, n$, and $Y_i \overset{\text{st}}{=} Y_m$ for $i = 1, \dots, m$. They built

$$X_1 \le_{\text{lr}} Y_m \quad \text{and} \quad X_n \le_{\text{lr}} Y_m \Longrightarrow T_1(X_1, \dots, X_n) \le_{\text{lr}} T_1(Y_1, \dots, Y_m, X_{m+1}, \dots, X_n) \tag{2.8}$$

and

$$X_1 \ge_{\text{lr}} Y_m \quad \text{and} \quad X_n \ge_{\text{lr}} Y_m \Longrightarrow T_n(X_1, \dots, X_n) \ge_{\text{lr}} T_n(Y_1, \dots, Y_m, X_{m+1}, \dots, X_n). \tag{2.9}$$

Apparently, results in (2.6)–(2.9) strengthen and generalize those in (2.4) and (2.5).

This paper further studies the series and parallel systems with n components. For any $m = 1, \ldots, n-1$, it is proved that if $X_i \le_{\rm hr} Y_j$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$, then

$$T_1(X_1, \ldots, X_n) \le_{\rm hr} T_1(Y_1, \ldots, Y_m, X_{m+1}, \ldots, X_n), \tag{2.10}$$

and if $X_i \ge_{\rm rh} Y_j$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$, then

$$T_n(X_1, \ldots, X_n) \ge_{\rm rh} T_n(Y_1, \ldots, Y_m, X_{m+1}, \ldots, X_n). \tag{2.11}$$

Clearly, the results in (2.10) and (2.11) essentially generalize the corresponding ones in (2.6) and (2.7) through considering more than one spare for series and parallel systems, respectively. In addition, for $m = 1$, we also obtain the following results: if $X_i \le_{\rm lr} Y_1$ for $i = 1, \ldots, n$, then

$$T_1(X_1, \ldots, X_n) \le_{\rm lr} T_1(Y_1, X_2, \ldots, X_n), \tag{2.12}$$

and if $X_i \ge_{\rm lr} Y_1$ for $i = 1, \ldots, n$, then

$$T_n(X_1, \ldots, X_n) \ge_{\rm lr} T_n(Y_1, X_2, \ldots, X_n). \tag{2.13}$$

Apparently, (2.12) and (2.13) improve the ones in (2.8) and (2.9), respectively.

3. HAZARD RATE ORDER AND REVERSED HAZARD RATE ORDER

Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be two sets of independent and absolutely continuous random variables, for every i, X_i and Y_i have survival functions $\overline{F}_i$ and $\overline{G}_i$, and distribution functions F_i and G_i, respectively. Recall that $T_n(X_1, \ldots, X_n)$ and $T_1(X_1, \ldots, X_n)$ represent the lifetimes of series and parallel systems with vector of lifetimes of components $(X_1, \ldots, X_n)$, respectively. Likewise, denote by $T_n(Y_1, \ldots, Y_m, X_{m+1}, \ldots, X_n)$ and $T_1(Y_1, \ldots, Y_m, X_{m+1}, \ldots, X_n)$ the lifetimes of resulting series and parallel systems by replacing $(X_1, \ldots, X_m)$ with $(Y_1, \ldots, Y_m)$, respectively.

In the situation with $m = n$, according to Theorem 3.1 below, a k-out-of-n system with heterogeneous components could attain smaller hazard rate through decreasing the hazard rates of all components to the level no larger than the lowest hazard rate.

THEOREM 3.1 (Boland and Proschan [1], Theorem 16.B.1): *Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be two groups of independent and (not necessarily identically distributed) absolutely continuous random variables, all with support (a, b). If $X_i \le_{\rm hr} Y_j$ for all i and j, then,*

$$T_k(X_1, \ldots, X_n) \le_{\rm hr} T_k(Y_1, \ldots, Y_n), \quad \textit{for } k = 1, 2, \ldots, n.$$

Naturally, one may wonder whether the k-out-of-n system may also attain smaller hazard rate by only decreasing the hazard rates of a portion of working components in the system. This prompts us to build Theorem 3.2, which serves as a positive answer for parallel systems.

The main theorems of this section rely on the following lemma.

LEMMA 3.1 (Ding and Li [7], Lemma 3): *Let g_1, g_2 and g be non-negative continuous functions such that both*

$$\frac{g_1(t)}{g_2(t)} \quad \textit{and} \quad \frac{g(t)}{g_2(t)-g_1(t)}$$

are increasing in t, and g_1 and g_2 have no crossing. Then,

$$\frac{g(t)+g_1(t)}{g(t)+g_2(t)}$$

is also increasing in t.

We are now ready to state the main results.

THEOREM 3.2: *For $n > m \geq 0$, if $X_i \leq_{\rm hr} Y_j$ for all $i = 1, 2, \ldots, n$ and $j = 1, \ldots, m$, then,*

$$T_1(X_1, \ldots, X_n) \leq_{\rm hr} T_1(Y_1, \ldots, Y_m, X_{m+1}, \ldots, X_n).$$

PROOF: Denote, for $t \geq 0$ and $i = 1, 2, \ldots, n$,

$$X_i(t) = \begin{cases} 1, & \text{if } X_i > t, \\ 0, & \text{otherwise.} \end{cases}$$

Then, we only need to prove the increasing property in $t \in \mathbb{R}_+$ of the following function:

$$\begin{aligned} &\frac{\mathsf{P}(T_1(Y_1, \ldots, Y_m, X_{m+1}, \ldots, X_n) > t)}{\mathsf{P}(T_1(X_1, \ldots, X_n) > t)} \\ &\quad = \frac{\mathsf{P}\left(\sum_{i=m+1}^n X_i(t) \geq 1\right) + \mathsf{P}\left(\sum_{i=m+1}^n X_i(t) = 0\right) \mathsf{P}\left(\sum_{i=1}^m Y_i(t) \geq 1\right)}{\mathsf{P}\left(\sum_{i=m+1}^n X_i(t) \geq 1\right) + \mathsf{P}\left(\sum_{i=m+1}^n X_i(t) = 0\right) \mathsf{P}\left(\sum_{i=1}^m X_i(t) \geq 1\right)} \\ &\quad = \frac{\mathsf{P}\left(\sum_{i=m+1}^n X_i(t) \geq 1\right) + \mathsf{P}\left(\sum_{i=1}^m Y_i(t) \geq 1\right) \prod_{i=m+1}^n F_i(t)}{\mathsf{P}\left(\sum_{i=m+1}^n X_i(t) \geq 1\right) + \mathsf{P}\left(\sum_{i=1}^m X_i(t) \geq 1\right) \prod_{i=m+1}^n F_i(t)}. \end{aligned}$$

By Theorem 3.1, it holds that

$$\frac{\mathsf{P}\left(\sum_{i=1}^m Y_i(t) \geq 1\right) \prod_{i=m+1}^n F_i(t)}{\mathsf{P}\left(\sum_{i=1}^m X_i(t) \geq 1\right) \prod_{i=m+1}^n F_i(t)} = \frac{\mathsf{P}\left(\sum_{i=1}^m Y_i(t) \geq 1\right)}{\mathsf{P}\left(\sum_{i=1}^m X_i(t) \geq 1\right)} = \frac{\mathsf{P}(T_1(Y_1, \ldots, Y_m) > t)}{\mathsf{P}(T_1(X_1, \ldots, X_m) > t)}$$

is increasing in $t \in \mathbb{R}_+$. Moreover, since $T_1(Y_1, \ldots, Y_m) \geq_{\rm hr} T_1(X_1, \ldots, X_m)$ implies

$$T_1(Y_1, \ldots, Y_m) \geq_{\rm st} T_1(X_1, \ldots, X_m),$$

we then have the following inequality, for all $t \geq 0$,

$$\mathsf{P}\left(\sum_{i=1}^m Y_i(t) \geq 1\right) \prod_{i=m+1}^n F_i(t) \geq \mathsf{P}\left(\sum_{i=1}^m X_i(t) \geq 1\right) \prod_{i=m+1}^n F_i(t).$$

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Thus, according to Lemma 3.1, it suffices to show that

$$\frac{\mathsf{P}\left(\sum_{i=m+1}^{n} X_i(t) \geq 1\right)}{\left[\mathsf{P}\left(\sum_{i=1}^{m} Y_i(t) \geq 1\right) - \mathsf{P}\left(\sum_{i=1}^{m} X_i(t) \geq 1\right)\right] \prod_{i=m+1}^{n} F_i(t)}$$
$$= \frac{1}{1 - \frac{\mathsf{P}\left(\sum_{i=1}^{m} X_i(t)\geq 1\right)}{\mathsf{P}\left(\sum_{i=1}^{m} Y_i(t)\geq 1\right)}} \frac{\mathsf{P}\left(\sum_{i=m+1}^{n} X_i(t) \geq 1\right)}{\mathsf{P}\left(\sum_{i=1}^{m} Y_i(t) \geq 1\right) \prod_{i=m+1}^{n} F_i(t)}$$
$$= \frac{1}{1 - \frac{\mathsf{P}\left(\sum_{i=1}^{m} X_i(t)\geq 1\right)}{\mathsf{P}\left(\sum_{i=1}^{m} Y_i(t)\geq 1\right)}} \frac{\sum_{i=m+1}^{n} \overline{F}_i(t) \prod_{j=m+1}^{i-1} F_j(t)}{\left[\sum_{l=1}^{m} \overline{G}_l(t) \prod_{j=1}^{l-1} G_j(t)\right] \prod_{i=m+1}^{n} F_i(t)}$$
$$= \frac{1}{1 - \frac{\mathsf{P}\left(\sum_{i=1}^{m} X_i(t)\geq 1\right)}{\mathsf{P}\left(\sum_{i=1}^{m} Y_i(t)\geq 1\right)}} \sum_{i=m+1}^{n} \frac{\overline{F}_i(t)}{\sum_{l=1}^{m} \overline{G}_l(t) \prod_{j=1}^{l-1} G_j(t)} \cdot \frac{1}{\prod_{j=i}^{n} F_j(t)}$$

is decreasing in $t \in \mathbb{R}_+$, here, conventionally, we denote

$$\prod_{i=1}^{0} G_i(t) = \prod_{i=m}^{m-1} F_i(t) = 1, \quad \text{for any } t \geq 0.$$

Since $X_i \leq_{\text{hr}} Y_j$ for all $i = m+1, \ldots, n$ and $j = 1, \ldots, m$, it holds that

$$\frac{\overline{F}_i(t)}{\sum_{l=1}^{m} \overline{G}_l(t) \prod_{j=1}^{l-1} G_j(t)} = \frac{1}{\sum_{l=1}^{m} \frac{\overline{G}_l(t)}{\overline{F}_i(t)} \prod_{j=1}^{l-1} G_j(t)}$$

is decreasing in $t \in \mathbb{R}_+$. This completes the proof. ∎

Theorem 3.2 extends both (2.4) by Misra and Misra [18] and (2.6) by Ding et al. [8] in the sense of enhancing a parallel system through improving more than one component. It is still an open problem whether a general k-out-of-n system may be enhanced in such a manner.

As a consequence of Theorems 3.1 and 3.2, Corollary 3.1 follows immediately.

Corollary 3.1: *If $X_i \leq_{\text{hr}} Y_1 \leq_{\text{hr}} \ldots \leq_{\text{hr}} Y_n$ for all $i = 1, \ldots, n$, then, for $k = 1, \ldots, n$,*

$$T_1(X_1, \ldots, X_n) \leq_{\text{hr}} \ldots \leq_{\text{hr}} T_1(Y_1, \ldots, Y_k, X_{k+1}, \ldots, X_n) \leq_{\text{hr}} \ldots \leq_{\text{hr}} T_1(Y_1, \ldots, Y_n).$$

Inspired by Theorems 3.1 and 3.2, we shall consider to improve a k-out-of-n system through replacing the m $(0 \leq m \leq n)$ "weakest" components. It should be remarked here that the reversed hazard rate order had been built for the case with $m = n$ as below.

Theorem 3.3 (Nanda and Shaked [20], Thm. 3.2): *Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be two set of independent (not necessarily identically distributed) random variables, all are absolutely continuous with support (a, b). If $X_i \geq_{\text{rh}} Y_j$ for all i and j, then*

$$T_k(X_1, \ldots, X_n) \geq_{\text{rh}} T_k(Y_1, \ldots, Y_n) \quad \text{for } k = 1, 2, \ldots, n.$$

Note that $\min\{X_1, X_2, \ldots, X_n\} = (\max\{(1/X_1), \ldots, (1/X_n)\})^{-1}$, for positive random variables $X_1, \ldots, X_n$, by Theorem 1.B.41 in Shaked and Shanthikumar [24], next theorem follows from Theorem 3.2, dealing with series systems in the situation that $n > m \geq 1$.

Theorem 3.4: *For $n > m \geq 0$, if $X_i \geq_{\rm rh} Y_j$ for all $i = 1, 2, \ldots, n$ and $j = 1, \ldots, m$, then,*

$$T_n(X_1, \ldots, X_n) \geq_{\rm rh} T_n(Y_1, \ldots, Y_m, X_{m+1}, \ldots, X_n)$$

The reversed hazard rate orders of series systems in (2.5) and (2.6), due to Misra and Misra [18] and Ding et al. [8], respectively, may be viewed as special cases of Theorem 3.4 in the sense that they actually concern only one spare.

Analogously, the following corollary is an immediate consequence of Theorems 3.3 and 3.4.

Corollary 3.2: *If $Y_1 \leq_{\rm rh} \ldots \leq_{\rm rh} Y_n \leq_{\rm rh} X_i$ for all $i = 1, \ldots, n$, then,*

$$T_n(X_1, \ldots, X_n) \geq_{\rm rh} \ldots \geq_{\rm rh} T_n(X_1, \ldots, X_k, Y_{k+1}, \ldots, Y_n) \geq_{\rm rh} \ldots \geq_{\rm rh} T_n(Y_1, \ldots, Y_n),$$

for $k = 1, \ldots, n$.

It is still an open problem whether Theorems 3.2 and 3.4 may be extended to a general k-out-of-n system. The following two examples seem to drop a hint of the positive answer.

Example 3.1: Set, for $t \geq 0$,

$$r_{X_1}(t) = 0.2t + 0.8, \quad r_{X_2}(t) = 0.4t + 0.6, \quad r_{X_3}(t) = 0.6t + 0.5, \quad r_{X_4}(t) = t + 0.4,$$

and

$$r_{Y_1}(t) = 0.1t + 0.4, \quad r_{Y_2}(t) = 0.15t + 0.35, \quad r_{Y_3}(t) = 0.2t + 0.3.$$

Evidently, $r_{Y_i}(t) \leq r_{X_j}(t)$, that is, $Y_i \geq_{\rm hr} X_j$ for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$. While there exists once change of sigh of difference of hazard rate functions $r_{X_i}(t) - r_{X_j}(t)$ and $r_{Y_i}(t) - r_{Y_j}(t)$ for $i < j$ from '$+$' to '$-$'.

As is seen in Figure 1(a), the hazard rate of the lifetime $T_2(X_1, X_2, X_3, X_4)$ of a 2-out-of-4 system lies above that of $T_2(Y_1, Y_2, Y_3, X_4)$. That is, $T_2(X_1, X_2, X_3, X_4) \leq_{\rm hr} T_2(Y_1, Y_2, Y_3, X_4)$.

Example 3.2: Set, for $t \geq 0$,

$$\tilde{r}_{X_1}(t) = \frac{10.4}{e^{0.8t} - 1}, \quad \tilde{r}_{X_2}(t) = \frac{7.2}{e^{0.6t} - 1}, \quad \tilde{r}_{X_3}(t) = \frac{2.2}{e^{0.2t} - 1}, \quad \tilde{r}_{X_4}(t) = \frac{1}{e^{0.1t} - 1},$$

and

$$\tilde{r}_{Y_1}(t) = \frac{10}{e^{t} - 1}, \quad \tilde{r}_{Y_2}(t) = \frac{4}{e^{0.8t} - 1}.$$

One may easily verify that there must be once change of sign of difference of reversed hazard rate functions $\tilde{r}_{X_i}(t) - \tilde{r}_{X_j}(t)$ and $\tilde{r}_{Y_i}(t) - \tilde{r}_{Y_j}(t)$ for $i < j$ from "$+$" to "$-$". Moreover, $\tilde{r}_{X_i}(t) \geq \tilde{r}_{Y_j}(t)$ for $t \geq 0$, i.e, $X_i \geq_{rh} Y_j$ for $i = 1, 2, 3, 4$ and $j = 1, 2$.

As shown in Figure 1(b), the reversed hazard rate of the lifetime $T_3(X_1, X_2, X_3, X_4)$ of one 3-out-of-4 system lies above that of $T_3(Y_1, Y_2, Y_3, X_4)$, implying $T_3(X_1, X_2, X_3, X_4) \geq_{\rm rh} T_3(Y_1, Y_2, X_3, X_4)$.

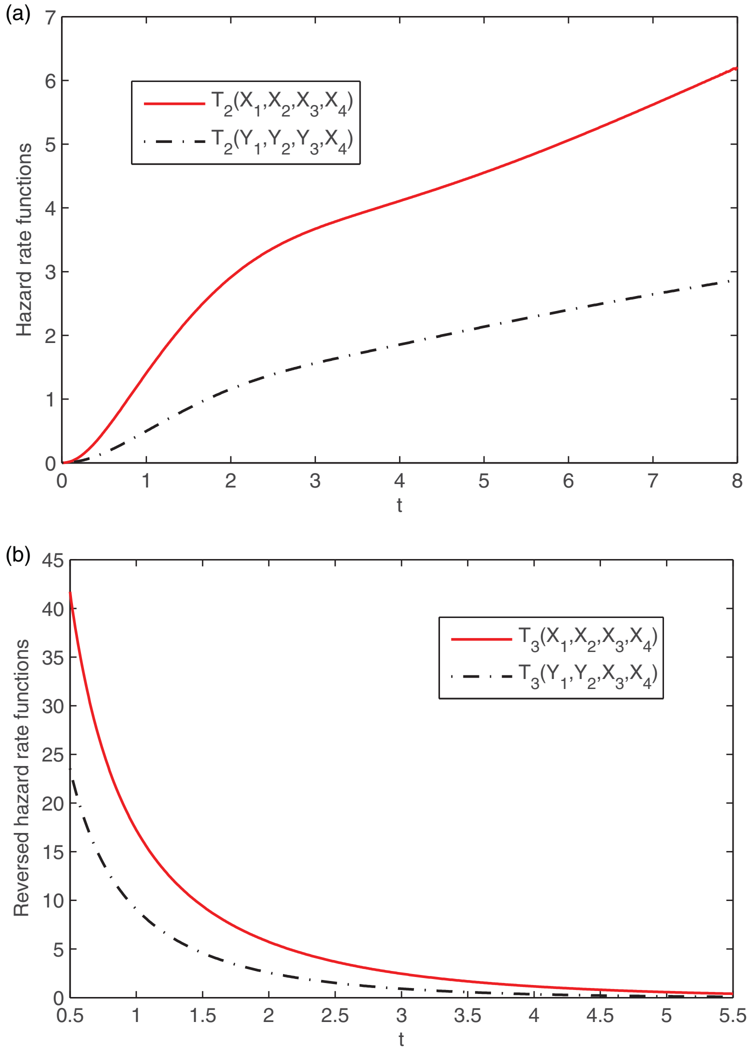

FIGURE 1. (Color online) Hazard rate and reversed hazard rate of k-out-of-n systems (a) $T_2(X_1, X_2, X_3, X_4)$ and $T_2(Y_1, Y_2, Y_3, X_4)$ and (b) $T_3(X_1, X_2, X_3, X_4)$ and $T_3(Y_1, Y_2, X_3, X_4)$.

4. LIKELIHOOD RATIO ORDER

Along the same line in the preceding section, we pursue the same problem in terms of the likelihood ratio order in this section, and the concerned k-out-of-n systems have only one spare. For convenience, let Y be lifetime of the spare with probability density $g(t)$, distribution function $G(t)$ and survival function $\overline{G}(t)$, and is independent of $X_1, \ldots, X_n$.

A key tool to be used in deriving the main results in this section is the theory of permanent. One may refer to Bapat and Beg [1], Bapat and Kochar [2], Hu, Lu, and Wen [10], Hu, Wang, and Zhu [11], and Wen, Lu, and Hu [26] for more related topics on permanent.

Let $\boldsymbol{A} = (a_{ij})$ is an $n \times n$ matrix, then the *permanent* of $\boldsymbol{A}$ is defined as $\sum_{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)}$, where the summation is taken over all permutations $\sigma = (\sigma(1), \ldots, \sigma(n))$ of $(1, \ldots, n)$. If

$\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ are vectors in $\mathbb{R}^n$, denote by $[\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n]$ the permanent of the $n \times n$ matrix $(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$, and by $[\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-1}]^{(i)}$ the permanent of the matrix by deleting row i of $[\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-1}]$. The permanent

$$\Big[\underbrace{\boldsymbol{a}_1}_{r_1}, \underbrace{\boldsymbol{a}_n}_{r_2}, \ldots\Big]$$

is obtained by taking r_1 copies of $\boldsymbol{a}_1$, r_2 copies of $\boldsymbol{a}_2$, and so on. If r_i equals 1, then it is omitted in the notation above. If $r_i = 0$, then it is understood that $\boldsymbol{a}_i$ does not appear in the permanent. If $r_i < 0$, for some i, the permanent is defined to be zero.

Denote

$$\boldsymbol{f}(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_{n-1}(t) \end{pmatrix}, \quad \boldsymbol{f}'(t) = \begin{pmatrix} f_1'(t) \\ \vdots \\ f_{n-1}'(t) \end{pmatrix}, \quad \boldsymbol{F}(t) = \begin{pmatrix} F_1(t) \\ \vdots \\ F_{n-1}(t) \end{pmatrix}, \quad \overline{\boldsymbol{F}}(t) = \begin{pmatrix} \overline{F}_1(t) \\ \vdots \\ \overline{F}_{n-1}(t) \end{pmatrix}.$$

For the sake of convenience, we also adopt the following notations from Hu, Zhu, and Wei [12]:

$$[p, q, l] = \Big[\underbrace{\boldsymbol{f}(t)}_{p}, \underbrace{\boldsymbol{F}(t)}_{q}, \underbrace{\overline{\boldsymbol{F}}(t)}_{l}\Big], \quad [p', q, l] = \Big[\underbrace{\boldsymbol{f}'(t)}_{p}, \underbrace{\boldsymbol{F}(t)}_{q}, \underbrace{\overline{\boldsymbol{F}}(t)}_{l}\Big], \tag{4.1}$$

$$[p, q, l]_f = \Big[\underbrace{\begin{pmatrix} \boldsymbol{f}(t) \\ f_n(t) \end{pmatrix}}_{p}, \underbrace{\begin{pmatrix} \boldsymbol{F}(t) \\ F_n(t) \end{pmatrix}}_{q}, \underbrace{\begin{pmatrix} \overline{\boldsymbol{F}}(t) \\ \overline{F}_n(t) \end{pmatrix}}_{l}\Big], \quad [p', q, l]_f = \Big[\underbrace{\begin{pmatrix} \boldsymbol{f}'(t) \\ f_n'(t) \end{pmatrix}}_{p}, \underbrace{\begin{pmatrix} \boldsymbol{F}(t) \\ F_n(t) \end{pmatrix}}_{q}, \underbrace{\begin{pmatrix} \overline{\boldsymbol{F}}(t) \\ \overline{F}_n(t) \end{pmatrix}}_{l}\Big], \tag{4.2}$$

and

$$[p, q, l]_g = \Big[\underbrace{\begin{pmatrix} \boldsymbol{f}(t) \\ g(t) \end{pmatrix}}_{p}, \underbrace{\begin{pmatrix} \boldsymbol{F}(t) \\ G(t) \end{pmatrix}}_{q}, \underbrace{\begin{pmatrix} \overline{\boldsymbol{F}}(t) \\ \overline{G}(t) \end{pmatrix}}_{l}\Big], \quad [p', q, l]_g = \Big[\underbrace{\begin{pmatrix} \boldsymbol{f}'(t) \\ g'(t) \end{pmatrix}}_{p}, \underbrace{\begin{pmatrix} \boldsymbol{F}(t) \\ G(t) \end{pmatrix}}_{q}, \underbrace{\begin{pmatrix} \overline{\boldsymbol{F}}(t) \\ \overline{G}(t) \end{pmatrix}}_{l}\Big], \tag{4.3}$$

where p, q and l are integers such that $p + q + l = n - 1$ for (4.1), and $p + q + l = n$ for (4.2) and (4.3).

It is useful to represent the distribution function of order statistics in terms of permanent when the random variables are not identical. The density functions of $T_1(X_1, \ldots, X_n)$ and $T_n(X_1, \ldots, X_n)$ may be respectively represented as

$$f_{T_1(X_1,\ldots,X_{n-1},X_n)}(t) = \frac{1}{(n-1)!}[1, n-1, 0]_f, \quad \text{for } t \in \mathbb{R}_+, \tag{4.4}$$

and

$$f_{T_n(X_1,\ldots,X_{n-1},X_n)}(t) = \frac{1}{(n-1)!}[1, 0, n-1]_f, \quad \text{for } t \in \mathbb{R}_+. \tag{4.5}$$

Before stating the main results, let us recall the well-known Alexandroff's inequality for permanent.

Lemma 4.1 (van Lint [25]): *Let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-1}$ and $\boldsymbol{b}$ be non-negative vectors in n-dimension real space $\mathbb{R}^n$, $n \geq 2$. Then,*

$$[\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-1}, \boldsymbol{b}]^2 \geq [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-1}, \boldsymbol{a}_{n-1}] \cdot [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-2}, \boldsymbol{b}, \boldsymbol{b}].$$

Theorem 4.1: *If* $X_i \leq_{\mathrm{lr}} Y$ *for* $i = 1, 2, \ldots, n$, *then,*

$$T_1(X_1, \ldots, X_{n-1}, X_n) \leq_{\mathrm{lr}} T_1(X_1, \ldots, X_{n-1}, Y).$$

Proof: Misra and Misra [18, (Corollary 2.3)] proved the case with $n = 2$. Let us only consider the case with $n > 2$.

In virtue of (4.4), we need to prove the following rational function

$$\phi(t) = \frac{f_{T_1(X_1,\ldots,X_{n-1},Y)}(t)}{f_{T_1(X_1,\ldots,X_{n-1},X_n)}(t)} = \frac{[1, n-1, 0]_g}{[1, n-1, 0]_f}$$

is increasing in $t \in \mathbb{R}_+$.

Taking the derivative of $\phi(t)$ with respect to t, we have

$$\begin{aligned}\phi'(t) &\overset{\text{sgn}}{=} [1, n-1, 0]_f \cdot [1', n-1, 0]_g + (n-1)[1, n-1, 0]_f \cdot [2, n-2, 0]_g \\ &\quad - [1', n-1, 0]_f \cdot [1, n-1, 0]_g - (n-1)[2, n-2, 0]_f \cdot [1, n-1, 0]_g \\ &\overset{\text{def}}{=} \varphi(t).\end{aligned}$$

Since $Y \geq_{\mathrm{lr}} X_n$ implies the increasing property of $g(t)/f_n(t)$ and $Y \geq_{\mathrm{rh}} X_n$, it holds that

$$g'(t) f_n(t) - g(t) f_n'(t) \geq 0, \quad \text{for all } t,$$

and

$$g(t) F_n(t) \geq G(t) f_n(t), \quad \text{for all } t.$$

By applying Laplace's expansion along the last row of all above permanent, we have

$$\begin{aligned}\varphi(t) &= \big(f_n(t)[0, n-1, 0] + (n-1)F_n(t)[1, n-2, 0]\big) \\ &\quad \times \big(g'(t)[0, n-1, 0] + (n-1)G(t)[1', n-2, 0]\big) \\ &\quad + \big(f_n(t)[0, n-1, 0] + (n-1)F_n(t)[1, n-2, 0]\big) \\ &\quad \times \big(2g(t)[1, n-2, 0] + (n-2)G(t)[2, n-3, 0]\big) \\ &\quad - \big(f_n'(t)[0, n-1, 0] + (n-1)F_n(t)[1', n-2, 0]\big) \\ &\quad \times \big(g(t)[0, n-1, 0] + (n-1)G(t)[1, n-2, 0]\big) \\ &\quad - \big(2f_n(t)[1, n-2, 0] + (n-2)F_n(t)[2, n-3, 0]\big) \\ &\quad \times \big(g(t)[0, n-1, 0] + (n-1)G(t)[1, n-2, 0]\big) \\ &= \big(g'(t)f_n(t) - g(t)f_n'(t)\big)[0, n-1, 0]^2 \\ &\quad + (n-1)\big(g'(t)F_n(t) - f_n'(t)G(t)\big)[0, n-1, 0][1, n-2, 0] \\ &\quad - (n-1)\big(g(t)F_n(t) - f_n(t)G(t)\big)[0, n-1, 0][1', n-2, 0] \\ &\quad + (n-1)\big(g(t)F_n(t) - f_n(t)G(t)\big) \\ &\quad \times \big(2(n-1)[1, n-2, 0]^2 - (n-2)[0, n-1, 0][2, n-3, 0]\big) \\ &\geq (n-1)\left(g'(t)F_n(t) - f_n'(t)G(t)\right)[0, n-1, 0][1, n-2, 0] \\ &\quad - (n-1)\big(g(t)F_n(t) - f_n(t)G(t)\big)[0, n-1, 0][1', n-2, 0] \\ &\quad + (n-1)\big(g(t)F_n(t) - f_n(t)G(t)\big) \\ &\quad \times \big(2(n-1)[1, n-2, 0]^2 - (n-2)[0, n-1, 0][2, n-3, 0]\big)\end{aligned}$$

$$
\begin{aligned}
&\geq (n-1)\big(g(t)F_n(t) - f_n(t)G(t)\big)[0, n-1, 0]\left(\frac{g'(t)}{g(t)}[1, n-2, 0] - [1', n-2, 0]\right)\\
&\quad + (n-1)\big(2(n-1)[1, n-2, 0]^2 - (n-2)[0, n-1, 0][2, n-3, 0]\big)\\
&\quad \times \big(g(t)F_n(t) - f_n(t)G(t)\big)\\
&\overset{\text{sgn}}{=} [0, n-1, 0]\varphi_1(t) + \varphi_2(t),
\end{aligned}
$$

where

$$
\varphi_1(t) = \frac{g'(t)}{g(t)}[1, n-2, 0] - [1', n-2, 0],
$$

and

$$
\varphi_2(t) = 2(n-1)[1, n-2, 0]^2 - (n-2)[0, n-1, 0][2, n-3, 0].
$$

By Lemma 4.1, it holds that

$$
\varphi_2(t) \geq 0, \quad \text{for all } t \geq 0.
$$

Note that, for $i = 1, \ldots,\ n-1$, $Y \geq_{\text{lr}} X_i$ implies

$$
\frac{g'(t)}{g(t)} - \frac{f_i'(t)}{f_i(t)} \geq 0, \quad \text{for all } t,
$$

by the Laplace's expansion along the first column of the two permanents, we also have

$$
\begin{aligned}
\varphi_1(t) &= \frac{g'(t)}{g(t)}[1, n-2, 0] - [1', n-2, 0]\\
&= \sum_{i=1}^{n-1} \frac{g'(t)}{g(t)} f_i(t)[0, n-2, 0]^{(i)} - \sum_{i=1}^{n-1} f_i'(t)[0, n-2, 0]^{(i)}\\
&= \sum_{i=1}^{n-1} f_i(t)[0, n-2, 0]^{(i)} \left(\frac{g'(t)}{g(t)} - \frac{f_i'(t)}{f_i(t)}\right)\\
&\geq 0.
\end{aligned}
$$

Now, it may be concluded that

$$
\phi'(t) = \varphi(t) = \varphi_1(t) + \varphi_2(t) \geq 0,
$$

for all $t \geq 0$. And thus, we complete the proof. ■

To end this section, we present the likelihood ratio order of series systems, which is analogous to Theorem 4.1.

Theorem 4.2: *If $X_i \geq_{\text{lr}} Y$ for $i = 1, 2, \ldots, n$, then*

$$
T_n(X_1, \ldots, X_{n-1}, X_n) \geq_{\text{lr}} T_n(X_1, \ldots, X_{n-1}, Y).
$$

Proof: The case with $n = 2$ was accomplished by Misra and Misra [18, (Corollary 3.2)] . We only need to prove the case with $n > 2$.

In virtue of (4.5), it is enough to prove that

$$\omega(t) = \frac{f_{T_n(X_1,\ldots,X_{n-1},X_n)}(t)}{f_{T_n(X_1,\ldots,X_{n-1},Y)}(t)} = \frac{[1,0,n-1]_f}{[1,0,n-1]_g}$$

is increasing in $t \in \mathbb{R}_+$.

In a similar manner to that in the proof of Theorem 4.1, we have

$$\begin{aligned}
\omega'(t) &\overset{\text{sgn}}{=} [1',0,n-1]_f[1,0,n-1]_g - (n-1)[2,0,n-2]_f[1,0,n-1]_g \\
&\quad - [1,0,n-1]_f[1',0,n-1]_g + (n-1)[1,0,n-1]_f[2,0,n-2]_g \\
&= (g(t)f_n'(t) - g'(t)f_n(t))\,[0,0,n-1]^2 \\
&\quad - (n-1)\left(g'(t)\overline{F}_n(t) - f_n'(t)\overline{G}(t)\right)[0,0,n-1][1,0,n-2] \\
&\quad + (n-1)\left(g(t)\overline{F}_n(t) - f_n(t)\overline{G}(t)\right)[0,0,n-1][1',0,n-2] \\
&\quad + (n-1)\left(g(t)\overline{F}_n(t) - f_n(t)\overline{G}(t)\right) \\
&\quad \times \left(2(n-1)[1,0,n-2]^2 - (n-2)[0,0,n-1][2,0,n-3]\right) \\
&\geq (n-1)\big(g(t)\overline{F}_n(t) - f_n(t)\overline{G}(t)\big)[0,0,n-1] \\
&\quad \times \left([1',0,n-2] - \frac{g'(t)}{g(t)}[1,0,n-2]\right) \\
&\quad + (n-1)\big(g(t)\overline{F}_n(t) - f_n(t)\overline{G}(t)\big) \\
&\quad \times \left(2(n-1)[1,0,n-2]^2 - (n-2)[0,0,n-1][2,0,n-3]\right) \\
&\overset{\text{sgn}}{=} [0,0,n-1]\omega_1(t) + \omega_2(t),
\end{aligned}$$

where

$$\omega_1(t) = [1',0,n-2] - \frac{g'(t)}{g(t)}[1,0,n-2],$$

and

$$\omega_2(t) = 2(n-1)[1,0,n-2]^2 - (n-2)[0,0,n-1][2,0,n-3].$$

By Lemma 4.1, we have $\omega_2(t) \geq 0$ for all $t \geq 0$. On the other hand, due to $Y \geq_{\text{lr}} X_i$, $i = 1,\ldots,n-1$, it holds that, for all $t \geq 0$,

$$\begin{aligned}
\omega_1(t) &= [1',0,n-2] - \frac{g'(t)}{g(t)}[1,0,n-2] \\
&= \sum_{i=1}^{n-1} f_i(t)[0,0,n-2]^{(i)} \left(\frac{f_i'(t)}{f_i(t)} - \frac{g'(t)}{g(t)}\right) \\
&\geq 0.
\end{aligned}$$

So, $\omega(t)$ is increasing in $t \geq 0$. The proof is completed. ■

It is clear that, the results (2.8) and (2.9) due to Ding et al. [8] on the likelihood ratio order in the multiple-outlier models can be directly obtained from Theorems 4.1 and 4.2, respectively.

By the end, we present the following two examples, which tell that Theorems 4.1 and 4.2 can not be generalized to k-out-of-n systems.

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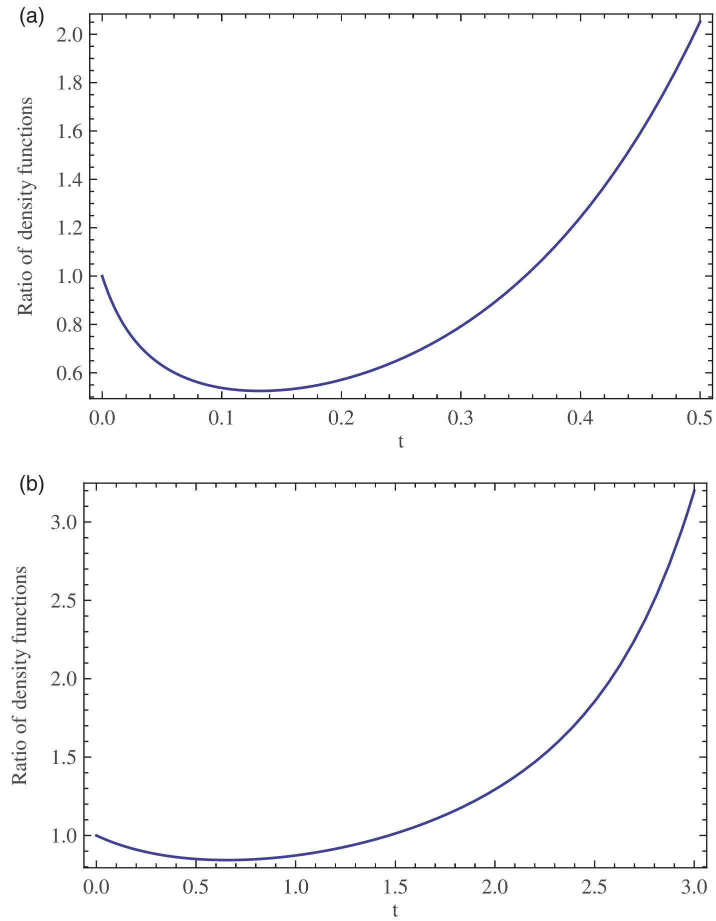

FIGURE 2. (Color online) Ratio of the two probability densities. (a) Series systems with two components and (b) Parallel systems with two components.

Example 4.1: Set, for $t \geq 0$,

$$F_1(t) = \left(1 - e^{-5t}\right)^4, \quad F_2(t) = \left(1 - e^{-6t}\right)^5, \quad G(t) = \left(1 - e^{-2t}\right)^5.$$

One may easily verify $Y \geq_{\rm lr} X_1$ and $Y \geq_{\rm lr} X_2$.

Consider the two series systems with respective lifetimes $T_2(X_1, X_2)$ and $T_2(X_1, Y)$. As can be seen in Figure 2(a), the ratio of density function of $T_2(X_1, X_2)$ to that of $T_2(X_1, Y)$ is not monotone at all. That is, there is no likelihood ratio order between $T_2(X_1, X_2)$ and $T_2(X_1, Y)$.

Example 4.2: Set, for $t \geq 0$,

$$\overline{F}_1(t) = \exp\{-0.5t^2 - 0.4t\}, \quad \overline{F}_2(t) = \exp\{-0.1t^2 - t\}, \quad \overline{G}(t) = \exp\{-0.5t^2 - t\}.$$

It is easy to verify that $Y \leq_{\rm lr} X_1$ and $Y \leq_{\rm lr} X_2$.

For two parallel systems with respective lifetimes $T_1(X_1, X_2)$ and $T_1(X_1, Y)$, as illustrated in Figure 2(b), the ratio of density function of $T_1(X_1, X_2)$ to that of $T_1(X_1, Y)$ does not have any monotone property, this implies that there is no likelihood ratio order between $T_1(X_1, X_2)$ and $T_1(X_1, Y)$.

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