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Tropical geometry and Newton–Okounkov cones for Grassmannian of planes from compactifications

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Abstract. We construct a family of compactifications of the affine cone of the Grassmannian variety of 2-planes. We show that both the tropical variety of the Plücker ideal and familiar valuations associated to the construction of Newton–Okounkov bodies for the Grassmannian variety can be recovered from these compactifications. In this way, we unite various perspectives for constructing toric degenerations of flag varieties.

1 Introduction

The study of toric degenerations of flag varieties is a meeting point for techniques from commutative algebra, algebraic geometry, and representation theory. Grassmannian varieties, in particular, being that they are often the most straightforward case to study after projective space, provide a testing ground for new constructions of toric degenerations, as well as a tractable class of examples for comparisons. A survey of recent activity in this area can be found in [FFL17c].

In this paper, we study the Grassmannian variety $Gr_2(\mathbb{C}^n)$ of 2-planes in $\mathbb{C}^n$. Let $I_{2,n}$ be the Plücker ideal that cuts out the affine cone $X \subset \mathbb{A}^{\binom{n}{2}}$ of $Gr_2(\mathbb{C}^n)$. Speyer and Sturmfels [SS04] provide a comprehensive understanding of the known toric degenerations of $Gr_2(\mathbb{C}^n)$, which are constructed from initial ideals of $I_{2,n}$ and organized by tropical geometry. In particular, in [SS04] it is shown that the cones of the tropical variety, $\mathrm{Trop}(I_{2,n})$, are in bijection with trees σ with n ordered leaves labeled by $\{1, \dots, n\}$ such that the valence of any nonleaf vertex is at least 3. In particular, the maximal cones of $\mathrm{Trop}(I_{2,n})$ are in bijection with trivalent trees σ, and the initial ideal associated with each of these cones is prime and binomial. We present a distinct construction of this class of well-known toric degenerations using the representation theory of SL_2 (Section 8), a quiver variety-style construction of X, and a family of compactifications X_σ (Section 4), one for each trivalent tree σ. Our main result is the following. See [KM19] and Section 2 for the notion of Khovanskii basis. Throughout the paper, we assume that no tree σ contains a vertex of valence 2.

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Theorem 1.1 *For each trivalent tree σ as above, and a total ordering $<$ on the edges of σ, we construct:*

1. *a simplicial cone C_σ of discrete, rank 1 valuations on $\mathbb{C}[X]$ with common Khovanskii basis given by the Plücker generators of $\mathbb{C}[X]$,*
2. *a rank $2n-3$ discrete valuation $\mathfrak{v}_{\sigma,<}$ on $\mathbb{C}[X]$ with Khovanskii basis given by the Plücker generators of $\mathbb{C}[X]$,*
3. *a compactification $X \subset X_\sigma$ by a combinatorial normal crossings divisor D_σ such that C_σ is spanned by the divisorial valuations associated to the components of D_σ, and $\mathfrak{v}_{\sigma,<}$ is a Parshin point valuation (see Section 8.2) built from a flag of subvarieties of X_σ obtained by intersection components of D_σ.*

Furthermore, the affine semigroup algebra $\mathbb{C}[S_\sigma]$ associated to the value semigroup S_σ of $\mathfrak{v}_{\sigma,<}$ is presented by the initial ideal corresponding to the cone in $\mathrm{Trop}(I_{2,n})$ *associated to σ.*

Remark 1.2 We observe that by [KM19, Lemma 3], the valuation $\mathfrak{v}_{\sigma,<}$ coincides with any homogeneous valuation with value semigroup S_σ constructed by one of the many methods used for constructing degenerations of flag varieties.

The compactification X_σ has a natural description in terms of the geometry of X. In Section 4, we construct X as a type of quiver variety coming from a choice of directed structure on the tree σ. In particular, each edge of σ is assigned a space, either SL_2 or $\mathbb{A}^2$. The compactification X_σ is then the space where these edge coordinates are allowed to take values in a compactification of SL_2 or $\mathbb{A}^2$. We also show that X_σ is always Fano (Proposition 6.9).

To explain our results, we recall the elements of two general theories underlying toric degeneration constructions. As a set, the Berkovich analytification X^{an} of an affine variety [Ber90] is the collection of all rank 1 valuations on the coordinate ring $\mathbb{C}[X]$, which restrict to the trivial valuation on $\mathbb{C}$. If $\mathcal{F} = \{f_1, \dots, f_n\}$ is a set of generators of $\mathbb{C}[X]$, it is well-known (see [Pay09]) that the evaluation map $ev_{\mathcal{F}}$, which sends $v \in X^{an}$ to $(v(f_1), \dots, v(f_n))$, maps onto the tropical variety $\mathrm{Trop}(I)$ of I, the ideal of forms that vanish on $\mathcal{F}$. It is difficult to find a section of this map. The main result of the work of Cueto et al. [CHW14] carries out such a construction in the projective setting for $Gr_2(\mathbb{C}^n)$. Part (1) of Theorem 1.1 extends to a version of this result on the affine cone X. We define a polyhedral complex of trees $\mathcal{T}(n)$ in Section 5, which is close (up to a lineality space) to the Biller–Holmes–Vogtmann space of phylogenetic trees [SS04, BHV01] .

Theorem 1.3 *There is a continuous map which identifies $\mathcal{T}(n)$ with a connected subcomplex of the analytification X^{an}. The evaluation map defined by the Plücker generators of $\mathbb{C}[X]$ takes $\mathcal{T}(n)$ isomorphically onto* $\mathrm{Trop}(I_{2,n})$.

Representation theory provides many constructions that are useful for construction of toric degenerations. These methods underlie constructions used by Alexeev and Brion [AB04], the Newton–Okounkov construction of Kaveh in [Kav15], and the birational sequence approach used in [FFL17a, FFL17b] (see also [MZ14]). The construction of $\mathfrak{v}_{\sigma,<}$ in Theorem 1.1 relies instead on properties of the tensor product in the category of SL_2 representations. These valuations are also used in [Man16].

The work on canonical bases in cluster algebras by Gross et al. [GHKK18], and then later used by Rietsch and Williams [RW19] and Bossinger et al. [BFF+18]on Grassmannians also provides a powerful organizing tool for toric degenerations. In [KM19, GHKK18, RW19] , compactifications of varieties by nice divisors are linked with the construction of a toric degeneration. We expect each compactification X_σ can be realized via a potential function construction in the manner of [GHKK18] and [RW19]. The compactification X_σ is also closely related to the compactification of the free group character variety $\mathcal{X}(F_g, \mathrm{SL}_2)$ by a combinatorial normal crossings divisor constructed in [Man18].

2 Background on valuations and tropical geometry

In this section, we introduce the necessary background on filtrations of commutative algebras and the functions associated to these filtrations, valuations, and quasi-valuations. We recall the critical notions of adapted basis and Khovanskii basis for a valuation, which enable computations with valuations. We also summarize the results of Kaveh and the first author [KM19], which directly relate higher-rank valuations to tropical geometry.

2.1 Quasi-valuations and filtrations

Let A be a commutative domain over $\mathbb{C}$, and let $\mathbb{Z}^r$ be the free Abelian group of rank r endowed with a total group ordering $\prec$ (e.g., the lexicographic ordering). A (decreasing) algebraic filtration F of A with values in $\mathbb{Z}^r$ is the data of a $\mathbb{C}$-subspace $F_\alpha \subset A$ for each $\alpha \in \mathbb{Z}^r$ such that $F_\alpha \supset F_\beta$ when $\alpha \prec \beta$, $F_\alpha F_\beta \subset F_{\alpha+\beta}$, $\forall \alpha, \beta \in \mathbb{Z}^r$, and $\bigcup_{\alpha\in\mathbb{Z}^r} F_\alpha = A$. We further assume that $1 \in F_0$ and $1 \notin F_\beta$ when $0 \prec \beta$. For any $\alpha \in \mathbb{Z}^r$, we let $F_{\succ\alpha}$ be $\bigcup_{\beta\succ\alpha} F_\beta$. For any such filtration, we can form the *associated graded* algebra:

$$gr_F(A) = \bigoplus_{\alpha\in\mathbb{Z}^r} F_\alpha / F_{\succ\alpha}. \tag{2.1}$$

Example 2.1 If A carries a $\mathbb{Z}^r$ grading, $A = \bigoplus_{\alpha\in\mathbb{Z}^r} A_\alpha$, then for any $\prec$ there is a filtration on A defined by setting $F_\alpha = \bigoplus_{\beta\succeq\alpha} A_\beta$. In this case, $gr_F(A)$ is canonically isomorphic to A.

Let $f \in F_\alpha \subset A$ but $f \notin F_{\succ\alpha}$, then we have the initial form $\bar{f} \in F_\alpha/F_{\succ\alpha} \subset gr_F(A)$. It is straightforward to show that $\overline{fg} = \bar{f}\bar{g}$. We say that F only takes finite values if such an α exists for every $f \in A$. We assume from now on that F only takes finite values; this is the case for all the filtrations we consider in this paper.

Definition 2.1 Let F be a filtration as above. We define the associated quasi-valuation $\mathfrak{v}_F : A\backslash\{0\} \to \mathbb{Z}^r$ as follows:

$$\mathfrak{v}_F(f) = \alpha \textit{ such that } f \in F_\alpha, f \notin F_{\succ\alpha}. \tag{2.2}$$

The function $\mathfrak{v}_F$ always has the following properties:

(1) $\mathfrak{v}_F(fg) \succeq \mathfrak{v}_F(f) + \mathfrak{v}_F(g)$,
(2) $\mathfrak{v}_F(f+g) \succeq MIN\{\mathfrak{v}_F(f), \mathfrak{v}_F(g)\}$,
(3) $\mathfrak{v}_F(C) = 0$, $\forall C \in \mathbb{C}$.

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More generally, a function that satisfies (1)–(3) above is called a *quasi-valuation* on A. If $\mathfrak{w} : A\backslash\{0\} \to \mathbb{Z}^r$ is a quasi-valuation, we also get a corresponding filtration $F^{\mathfrak{w}}$ defined as follows:

$$F_\alpha^{\mathfrak{w}} = \{f \mid \mathfrak{w}(f) \geq \alpha\}. \tag{2.3}$$

One easily checks that the constructions, $F \to \mathfrak{v}_F$ and $\mathfrak{w} \to F^{\mathfrak{w}}$, are inverse to each other. Finally, a quasi-valuation $\mathfrak{v}$ is said to be a *valuation* if $\mathfrak{v}(fg) = \mathfrak{v}(f) + \mathfrak{v}(g)$, $\forall f, g \in A$.

2.2 Adapted bases

Now, we recall the notion of an *adapted basis* [KM19, Section 3]. Adapted bases facilitate computations and allow quasi-valuations to be treated as combinatorial objects. We continue to use quasi-valuations with values in $\mathbb{Z}^r$, but we observe that the results in this section work with any ordered group.

Definition 2.2 A $\mathbb{C}$-vector space basis $\mathbb{B} \subset A$ is said to be *adapted* to a filtration F if $F_\alpha \cap \mathbb{B}$ is a vector space basis for all $\alpha \in \mathbb{Z}^r$.

If $\{F_\alpha\}$ is a collection of vector subspaces of A (not necessarily forming a filtration) with the property that any intersection $F_\alpha \cap \mathbb{B}$ is a basis of F_α, then the same property holds for any vector subspace of A constructed by intersections and sums of the members of $\{F_\alpha\}$. It immediately follows then that if the F_α forms a filtration F, then $F_{>\alpha} \cap \mathbb{B}$ is a basis of $F_{>\alpha}$ and the equivalence classes $\bar{\mathbb{B}}_\alpha$ of basis members $\mathbb{B} \cap F_\alpha\backslash F_{>\alpha}$ form a basis of $F_\alpha/F_{>\alpha}$. We let $\bar{\mathbb{B}} \subset gr_F(A)$ be the disjoint union $\sqcup_{\alpha\in\mathbb{Z}^r}\bar{\mathbb{B}}_\alpha$; this is a basis of $gr_F(A)$, which is adapted to the grading by $\mathbb{Z}^r$. If a quasi-valuation $\mathfrak{v}$ corresponds to a filtration F with adapted basis $\mathbb{B}$, we say that $\mathbb{B}$ is adapted to $\mathfrak{v}$. The next proposition summarizes the basic properties of adapted bases.

Proposition 2.2 *Let $\mathfrak{v}$ be a quasi-valuation with adapted basis $\mathbb{B}$, then:*

(1) *for any $f \in A$ with $f = \sum_i C_i b_i$, $\mathfrak{v}(f) = MIN\{\mathfrak{v}(b_i) \mid C_i \neq 0\}$,*
(2) *if $\mathbb{B}'$ is another basis adapted to $\mathfrak{v}$, then any $b \in \mathbb{B}$ has a upper triangular expression in elements of $\mathbb{B}'$,*
(3) *if $\mathbb{B}$ is adapted to another quasi-valuation $\mathfrak{w}$ and $\mathfrak{v}(b) = \mathfrak{w}(b)$, $\forall b \in \mathbb{B}$, then $\mathfrak{v} = \mathfrak{w}$.*

Proof For part (1), let $\alpha = MIN\{\mathfrak{v}(b_i) \mid C_i \neq 0\}$ and note that $f \in F_\alpha$, $\{b_i \mid C_i \neq 0\} \subset F_\alpha$, and $\mathfrak{v}(f) \geq \alpha$. If $f \in F_\beta$ with $\beta > \alpha$, then all $b_i \in F_\beta$, which is a contradiction. For part (2), let $b \in \mathbb{B}$ have value $\mathfrak{v}(b) = \alpha$; then we can write $b = \sum C_i b_i'$ for $b_i' \in \mathbb{B}'$. From part (1) we have $\alpha = MIN\{\mathfrak{v}(b_i') \mid C_i \neq 0\}$. For part (3), note that for any $f \in A$ with $f = \sum C_i b_i$, we have $\mathfrak{v}(f) = MIN\{\mathfrak{v}(b_i) \mid C_i \neq 0\} = MIN\{\mathfrak{w}(b_i) \mid C_i \neq 0\} = \mathfrak{w}(f)$. ∎

More generally, if we are given a direct sum decomposition $A = \bigoplus_{i\in I} A_i$ of A as a vector space, we say that this decomposition is adapted to a filtration F if for any $\alpha \in \mathbb{Z}^r$ we have that F_α is a direct sum of a subset of the spaces A_i. Notice that, in this case, any selection of basis for each A_i gives a basis adapted to F. It is possible to define a sum operation on the set of all quasi-valuations adapted to a given basis $\mathbb{B} \subset A$.

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Definition 2.3 Let $\mathfrak{v}, \mathfrak{w} : A\backslash\{0\} \to \mathbb{Z}^r$ be quasi-valuations, which share a common adapted basis $\mathbb{B} \subset A$, then we define the *sum* $[\mathfrak{v} + \mathfrak{w}] : A\backslash\{0\} \to \mathbb{Z}^r$ to be $\mathfrak{v}(b) + \mathfrak{w}(b)$ for any $b \in \mathbb{B}$, and extend this to $f = \sum C_i b_i, b_i \in A$, by the setting $[\mathfrak{v} + \mathfrak{w}](f) = MIN\{[\mathfrak{v} + \mathfrak{w}](b_i) \mid C_i \neq 0\}$.

Proposition 2.3 *The following holds for the sum operation:*

(1) *the sum* $[\mathfrak{v} + \mathfrak{w}]$ *of two quasi-valuations is a quasi-valuation,*
(2) *the sum operation is commutative and associative,*
(3) *for any* $\mathfrak{v}$, $[\sum_{i=1}^n \mathfrak{v}] = n\mathfrak{v}$,
(4) *the sum has neutral element, the quasi-valuation* $\mathbf{o} : A\backslash\{0\} \to \mathbb{Z}^r$ *defined by* $\mathbf{o}(f) = 0$, $\forall f \in A\backslash\{0\}$,
(5) *the set of quasi-valuations adapted to* $\mathbb{B}$ *can be identified with the monoid of points in* $[\mathbb{Z}^r]^{\mathbb{B}}$ *which satisfy* $\mathfrak{v}(b_i) + \mathfrak{v}(b_j) \leq MIN\{\mathfrak{v}(b_k) \mid b_i b_j = \sum C_k b_k, C_k \neq 0\}$.

Proof For part (1), clearly $[\mathfrak{v} + \mathfrak{w}](C) = 0$, $\forall C \in \mathbb{C}$.

For $f, g \in A\backslash\{0\}$, we write $f = \sum C_i b_i$ and $g = \sum K_i b_i, \forall b_i \in \mathbb{B}$. The sum $[\mathfrak{v} + \mathfrak{w}](f + g)$ is then computed by $MIN\{\mathfrak{v}(b_i) + \mathfrak{w}(b_i) \mid C_i + K_i \neq 0\}$; this must be larger than $MIN\{\mathfrak{v}(b_i) + \mathfrak{w}(b_i) \mid C_i \neq 0\}$ or $MIN\{\mathfrak{v}(b_i) + \mathfrak{w}(b_i) \mid K_i \neq 0\}$. Thus, $[\mathfrak{v} + \mathfrak{w}](f + g) \geq MIN\{[\mathfrak{v} + \mathfrak{w}](f), [\mathfrak{v} + \mathfrak{w}](g)\}$.

Now, consider the product $fg = \sum C_i K_j b_i b_j$. First, we show that $[\mathfrak{v} + \mathfrak{w}](bb') \geq [\mathfrak{v} + \mathfrak{w}](b) + [\mathfrak{v} + \mathfrak{w}](b')$ for any $b, b' \in \mathbb{B}$: Let $bb' = \sum_k T_k b_k, b_k \in \mathbb{B}$. Then,

$$\begin{aligned}
[\mathfrak{v} + \mathfrak{w}](bb') &= MIN\{[\mathfrak{v} + \mathfrak{w}](b_k) \mid T_k \neq 0\} = MIN\{\mathfrak{v}(b_k) + \mathfrak{w}(b_k) \mid T_k \neq 0\} \\
&\geq MIN\{\mathfrak{v}(b_k) \mid T_k \neq 0\} + MIN\{\mathfrak{w}(b_k) \mid T_k \neq 0\} \\
&= \mathfrak{v}(bb') + \mathfrak{w}(bb') \text{ (by Proposition 2.2)} \\
&\geq \mathfrak{v}(b) + \mathfrak{v}(b') + \mathfrak{w}(b) + \mathfrak{w}(b') \\
&= [\mathfrak{v} + \mathfrak{w}](b) + [\mathfrak{v} + \mathfrak{w}](b').
\end{aligned}$$

Finally,

$$\begin{aligned}
[\mathfrak{v} + \mathfrak{w}](fg) &\geq MIN\{[\mathfrak{v} + \mathfrak{w}](b_i b_j) \mid b_i b_j \text{ appears in } fg\} \\
&\geq MIN\{[\mathfrak{v} + \mathfrak{w}](b_i) + [\mathfrak{v} + \mathfrak{w}](b_j) \mid b_i b_j \text{ appears in } fg\} \\
&= MIN\{\mathfrak{v}(b_i) + \mathfrak{w}(b_i) + \mathfrak{v}(b_j) + \mathfrak{w}(b_j) \mid b_i b_j \text{ appears in } fg\} \\
&\geq MIN\{\mathfrak{v}(b_i) + \mathfrak{w}(b_i) \mid C_i \neq 0\} + MIN\{\mathfrak{v}(b_j) + \mathfrak{w}(b_j) \mid K_j \neq 0\} \\
&= [\mathfrak{v} + \mathfrak{w}](f) + [\mathfrak{v} + \mathfrak{w}](g).
\end{aligned}$$

For parts (2)–(4), we observe that this operation is commutative and associative by definition. It is easy to check $[\sum_{i=1}^n \mathfrak{v}](b_i) = n\mathfrak{v}(b_i)$ for any $b_i \in A_i$; this implies that $[\sum_{i=1}^n \mathfrak{v}](f) = n\mathfrak{v}(f)$ for any $f \in A$. Similarly, $[\mathfrak{v} + \mathbf{o}](b_i) = \mathfrak{v}(b_i)$ for any $b_i \in A_i$; this implies that $\mathbf{o}$ is a neutral element. For part (5), we leave it to the reader to consider the map, which sends $\mathfrak{v}$ to the tuple $(\mathfrak{v}(b) \mid b \in \mathbb{B}) \in [\mathbb{Z}^r]^{\mathbb{B}}$. ■

Remark 2.4 For any non-negative real number $r \in \mathbb{R}_{\geq 0}$ and valuation $\mathfrak{v} : A\backslash\{0\} \to \mathbb{R}^d$ we obtain a new valuation $r\mathfrak{v} : A\backslash\{0\} \to \mathbb{R}^d$ by scaling the values of $\mathfrak{v}$ by r. In particular, Proposition 2.3 shows that scaling by a non-negative integer $n\mathfrak{v}$ coincides with the sum $[\mathfrak{v} + \cdots + \mathfrak{v}]$. If $\mathfrak{v}$ is a grading function, we can also scale $\mathfrak{v}$ by negative real numbers and obtain a valuation. Indeed, following Example 2.1, the filtration spaces $F_\alpha(r\mathfrak{v})$ differ from the spaces $F(\mathfrak{v})$ for $r \in \mathbb{R}_{<0}$, but both associated graded algebras are canonically isomorphic to the algebra A. If A is a domain, this means that both $\mathfrak{v}$ and $r\mathfrak{v}$ are valuations by definition. This will be important when we discuss the grading function $deg : \mathbb{C}[\mathbb{A}^2]\backslash\{0\} \to \mathbb{Z}$ in Section 5.

Proposition 2.3 illustrates how quasi-valuations with a common adapted basis tend to work well with each other. The following lemma shows a similar phenomenon.

Lemma 2.5 *If $\mathfrak{v}_1, \mathfrak{v}_2$ are quasi-valuations which are adapted to the same basis $\mathbb{B} \subset A$, then the function $\bar{\mathfrak{v}}_2$ on $gr_{\mathfrak{v}_1}(A)$, which assigns $\mathfrak{v}_2(b)$ to $\bar{b} \in \bar{\mathbb{B}} \subset gr_{\mathfrak{v}_1}(A)$, also defines a quasi-valuation.*

Proof We only have to check that we have $\bar{\mathfrak{v}}_2(\bar{b}_i\bar{b}_j) \geq \bar{\mathfrak{v}}_2(\bar{b}_i) + \bar{\mathfrak{v}}_2(\bar{b}_j)$. Now, $\bar{\mathfrak{v}}_2(\bar{b}_i\bar{b}_j) = MIN\{\bar{\mathfrak{v}}_2(\bar{b}_k) \mid \bar{b}_i\bar{b}_j = \sum C_k\bar{b}_k, C_k \neq 0\}$. But the equation $\bar{b}_i\bar{b}_j = \sum C_k\bar{b}_k$ is a truncation of the corresponding expansion of b_ib_j in A, where the associated inequality holds, that is, $\mathfrak{v}_2(b_i) + \mathfrak{v}_2(b_j) \leq MIN\{\mathfrak{v}_2(b_k) \mid b_ib_j = \sum C_kb_k, C_k \neq 0\}$. ■

The sum operation also works well with tensor products of algebras. Let $\mathfrak{v}_1$ be a quasi-valuation on A_1 and $\mathfrak{v}_2$ be a quasi-valuation on A_2, and let F^1, F^2 be the corresponding filtrations. We get two filtrations $\mathcal{F}^1, \mathcal{F}^2$ on $A_1 \otimes_{\mathbb{C}} A_2$ by setting $\mathcal{F}^1_\alpha = F^1_\alpha \otimes A_2$ and $\mathcal{F}^2_\alpha = A_1 \otimes F^2_\alpha$, with corresponding quasi-valuations $\mathfrak{v}_1, \mathfrak{v}_2$. By picking adapted bases (this is always possible for the algebras we consider in this paper) $\mathbb{B}_1$ and $\mathbb{B}_2$ we obtain a basis $\mathbb{B} = \{b_i \otimes b_j \mid b_i \in \mathbb{B}_1, b_j \in \mathbb{B}_2\} \subset A_1 \otimes_{\mathbb{C}} A_2$, which is simultaneously adapted to $\mathfrak{v}_1$ and $\mathfrak{v}_2$.

Lemma 2.6 *For $\mathbb{B}, \mathfrak{v}_1$, and $\mathfrak{v}_2$ as above, we have $gr_{\mathfrak{v}_1+\mathfrak{v}_2}(A_1 \otimes_{\mathbb{C}} A_2) \cong gr_{\mathfrak{v}_1}(A_1) \otimes_{\mathbb{C}} gr_{\mathfrak{v}_2}(A_2)$. Moreover, $\mathfrak{v}_1 + \mathfrak{v}_2$ is independent of the choice of bases $\mathbb{B}_1$ and $\mathbb{B}_2$.*

Proof Clearly, as vector spaces, we have $gr_{\mathfrak{v}_1+\mathfrak{v}_2}(A_1 \otimes_{\mathbb{C}} A_2) \cong gr_{\mathfrak{v}_1}(A_1) \otimes_{\mathbb{C}} gr_{\mathfrak{v}_2}(A_2)$. So, it remains to show that the multiplication operations on both sides coincide. This follows from the fact that $\mathfrak{v}_1$ only sees the first tensor component, and $\mathfrak{v}_2$ only sees the second tensor component; this, in turn, implies that the lower terms of the product $[b_1 \otimes b_1'][b_2 \otimes b_2'] = [b_1b_2 \otimes b_1'b_2']$ are the same way on both sides. We leave the second statement to the reader. ■

Finally, we will need the following notion of *Khovanskii basis*.

Definition 2.4 (Khovanskii basis) We say $\mathcal{B} \subset A$ is a Khovanskii basis for a quasi-valuation $\mathfrak{v}$ if $gr_{\mathfrak{v}}(A)$ is generated by the equivalence classes $\bar{\mathcal{B}} \subset gr_{\mathfrak{v}}(A)$ as an algebra over $\mathbb{C}$.

2.3 Weight valuations and the tropical variety

For the following construction, see [KM19, Section 4]. We assume that A is presented as the image of a polynomial ring: $\pi : \mathbb{C}[\mathbf{x}] \to A$, with kernel $Ker(\pi) = I$. Here, $\mathbf{x} = \{x_1, \ldots, x_n\}$ is a system of parameters. We make the further assumption that A is a positively graded domain. Recall the notion of initial form $in_w(f)$ of a polynomial $f \in \mathbb{C}[\mathbf{x}]$ and initial ideal $in_w(I)$ associated to a weight vector $w \in \mathbb{Q}^n$. We will require the

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notion of the Gröbner fan $\mathcal{G}(I)$ associated to I; recall that this is a complete polyhedral fan in $\mathbb{Q}^n$ whose cones index the initial ideals of I. In particular, we have $in_w(I) = in_{w'}(I)$ for any w, w', which are members of the relative interior of the same cone in $\mathcal{G}(I)$. For this and other notions from Gröbner theory, see [Stu96] and [MS15].

The tropical variety $\mathrm{Trop}(I)$ can be identified with a subfan of $\mathcal{G}(I)$ given by those cones whose associated initial ideals contain no monomial (see [SS04, MS15]). The tropical variety $\mathrm{Trop}(I)$, and, more generally, the Gröbner fan $\mathcal{G}(I)$ of the ideal I help to organize the quasi-valuations on A with Khovanskii basis $\pi(\mathbf{x}) = \mathcal{B}$ by realizing all such functions as so-called weight quasi-valuations.

Definition 2.5 (Weight quasi-valuations) For $w \in \mathbb{Q}^n$, the weight quasi-valuation on $A = \mathbb{C}[\mathbf{x}]/I$ is defined on $f \in A$ as follows:

$$\mathfrak{v}_w(f) = MAX\{MIN\{\langle w, \alpha\rangle | \, p(\mathbf{x}) = \sum C_\alpha \mathbf{x}^\alpha, C_\alpha \neq 0\} \mid \pi(p) = f\}. \tag{2.4}$$

We summarize the properties of weight quasi-valuations that we will need in the following proposition (see [KM19, Section 4]). We let $gr_w(A)$ denote the associated graded algebra of $\mathfrak{v}_w$.

Proposition 2.7 *Let A be a positively graded algebra presented as $\mathbb{C}[\mathbf{x}]/I$ for a prime ideal I, then:*

(1) *for any $w \in \mathbb{Q}^n$, $gr_w(A) \cong \mathbb{C}[\mathbf{x}]/in_w(I)$,*
(2) *$\mathfrak{v}_w$ is adapted to any standard monomial basis of A associated to a monomial ordering on $I \subset \mathbb{C}[\mathbf{x}]$ which refines w,*
(3) *$\mathfrak{v} : A\backslash\{0\} \to \mathbb{Q}$ is a quasi-valuation with Khovanskii basis $\mathcal{B} = \pi(\mathbf{x})$ if and only if $\mathfrak{v} = \mathfrak{v}_w$ for some $w \in \mathbb{Q}^n$.*

If $in_w(I)$ is a prime ideal, then part (1) of Proposition 2.7 implies that $\mathfrak{v}_w$ is a valuation. In this case, we say that the cone C_w of the Gröbner fan containing w in its relative interior is a *prime cone*. With a mild assumption (each element of $\mathbf{x}$ is a standard monomial), we can conclude that $C_w \subset \mathrm{Trop}(I)$.

3 Constructions for SL_2 and $\mathbb{A}^2$

In this section, we define compactifications of SL_2 and $\mathbb{A}^2$, which are stable under the group actions on these spaces (respectively, by $\mathrm{SL}_2 \times \mathrm{SL}_2$ and SL_2). The divisorial valuations defined by the boundaries of these compactifications are used as building blocks in both the tropical and Newton–Okounkov constructions we give for the Grassmannian variety, and the compactifications themselves are key ingredients in the construction of the projective variety X_σ. Accordingly, the constructions presented here for SL_2 and $\mathbb{A}^2$ provide a reference point for the main results of the paper.

3.1 Representations of SL_2

Recall that SL_2 is a simple algebraic group over $\mathbb{C}$. This implies that any finite dimensional representation V of SL_2 decomposes uniquely into a direct sum of irreducible representations:

$$V \cong \bigoplus_{n\geq 0} Hom_{\mathrm{SL}_2}(V(n), V) \otimes V(n). \tag{3.1}$$

The representation $V(n)$ is the irreducible representation of SL_2 associated to the dominant weight $n \in \mathbb{Z}_{\geq 0}$. The representation $V(n)$ is isomorphic to the nth symmetric power $Sym^n(\mathbb{C}^2)$; in particular, $V(0)$ is isomorphic to $\mathbb{C}$ equipped with the trivial action by SL_2. The vector space $Hom_{\mathrm{SL}_2}(V(n), V)$ is the space of SL_2-maps from the irreducible $V(n)$ into V, which is called the multiplicity space of $V(n)$ in V. The space $Hom_{\mathrm{SL}_2}(V(0), V)$ is called the space of SL_2-invariants in V, which is also denoted by V^{SL_2}.

For any two SL_2-representations V and W, we can consider the tensor product $V \otimes W$ equipped with the diagonal action $g \circ (v \otimes w) = g \circ v \otimes g \circ w$. Similarly, the vector space of homomorphisms $Hom(V, W)$ is naturally equipped with a representation structure; in particular, the dual vector space $V^* = Hom(V, V(0))$ is called the dual representation. For any $n \in \mathbb{Z}_{\geq 0}$, we have $V(n)^* \cong V(n)$. These operations endow the category $Rep(\mathrm{SL}_2)$ of finite dimensional SL_2-representations with the structure of a symmetric, monoidal, semi-simple category with dualizing object $V(0)$. It is an important problem for any such category to determine the rule for decomposition of a tensor product of irreducible representations into irreducibles:

$$V(j) \otimes V(k) = \bigoplus_{i\geq 0} Hom_{\mathrm{SL}_2}(V(i), V(j) \otimes V(k)) \otimes V(i). \tag{3.2}$$

We have $Hom_{\mathrm{SL}_2}(V(i), V(j) \otimes V(k)) \cong Hom_{\mathrm{SL}_2}(V(0), V(i)^* \otimes V(j) \otimes V(k)) \cong [V(i) \otimes V(j) \otimes V(k)]^{\mathrm{SL}_2}$ using the properties of tensor product and duals, so this problem can be reduced to computing the invariant spaces $[V(i) \otimes V(j) \otimes V(k)]^{\mathrm{SL}_2}$. The following formula can be derived from the *Pieri rule* [FH91, 6.1]:

$$[V(i) \otimes V(j) \otimes V(k)]^{\mathrm{SL}_2} \cong \begin{cases} \mathbb{C} & \text{if } i+j+k \in 2\mathbb{Z},\ \ |i-j| \leq k \leq i+j, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

We refer to $i + j + k \in 2\mathbb{Z}$ as the *parity condition* on a triple of integers. We say that (i, j, k) satisfy the *triangle inequalities* if $0 \leq i, j, k$ and $|i - j| \leq k \leq i + j$; this is because these are precisely the conditions needed to guarantee that i, j, k can be the sides of a Euclidean triangle.

3.2 Coordinate algebras of SL_2 and $\mathbb{A}^2$

Recall the isotypical decomposition of the coordinate ring of SL_2 as an $\mathrm{SL}_2 \times \mathrm{SL}_2$-representation:

$$\mathbb{C}[\mathrm{SL}_2] = \bigoplus_{n\geq 0} V(n) \otimes V(n). \tag{3.4}$$

The multiplication operation $m : \mathbb{C}[\mathrm{SL}_2] \otimes \mathbb{C}[\mathrm{SL}_2] \to \mathbb{C}[\mathrm{SL}_2]$ is not graded by dominant weight, but the dominant weights still define a filtration. For any n and $m \in \mathbb{Z}_{\geq 0}$, we have:

$$m\Big([V(m)\otimes V(m)]\otimes[V(n)\otimes V(n)]\Big)\subset\bigoplus_{k\le n+m}V(k)\otimes V(k). \tag{3.5}$$

In particular, the projection of $m\Big([V(m)\otimes V(m)]\otimes[V(n)\otimes V(n)]\Big)$ onto $V(n+m)\otimes V(n+m)$ is an instance of the so-called *Cartan multiplication* operation on tensor products of irreducible representations, and is never 0 (see [HMM17, Section 3]). There is an algebraic filtration of $\mathbb{C}[\mathrm{SL}_2]$ by the spaces:

$$F_m=\bigoplus_{n\le m}V(n)\otimes V(n). \tag{3.6}$$

Using equation (3.5), it is straightforward to check that $m(F_m\otimes F_n)\subset F_{m+n}$.

Let $U\subset\mathrm{SL}_2$ be the group of upper triangular 2×2 matrices with 1's along the diagonal. Using right multiplication by elements of U, any element of SL_2 can be taken to a matrix whose entries depend only on the two entries in the first column. Since both of these entries cannot be zero, we find that $\mathrm{SL}_2/U\cong\mathbb{A}^2\backslash\{0\}$. Since the origin is a codimension-2 subvariety of $\mathbb{A}^2$, we have an isomorphism of the algebra of U-invariants $\mathbb{C}[\mathrm{SL}_2]^U$ with the coordinate ring of $\mathbb{A}^2$; namely a polynomial ring on two variables.

The group U acts on the right hand component of each tensor product $V(n)\otimes V(n)\subset\mathbb{C}[\mathrm{SL}_2]$. As each $V(n)$ is irreducible, with a one-dimensional subspace of highest weight vectors, the space $V(n)^U$ has dimension 1, so $V(n)\otimes V(n)^U\cong V(n)$. It follows that $\mathbb{C}[\mathbb{A}^2]=\mathbb{C}[SL_2]^U$ has the following isotypical decomposition:

$$\mathbb{C}[\mathbb{A}^2]=\bigoplus_{n\ge 0}V(n). \tag{3.7}$$

Indeed, $V(n)\cong Sym^n(\mathbb{C}^2)$, so equation (3.7) is the direct sum decomposition of the polynomial ring on two variables into its homogeneous components. The multiplication operation on $\mathbb{C}[\mathbb{A}^2]$, just normal polynomial multiplication, is accordingly the Cartan multiplication operation for SL_2: $V(n)\otimes V(m)\to V(n+m)$. This grading endows $\mathbb{A}^2$ with an action by $\mathbb{G}_m$ on the right in addition to its natural action by SL_2 on the left. In particular, $t\in\mathbb{G}_m$ acts on $f\in V(n)$ by the rule $f\circ t=ft^n$.

The associated graded algebra $gr_F(\mathbb{C}[\mathrm{SL}_2])$ of the filtration F has an identical isotypical decomposition to $\mathbb{C}[\mathrm{SL}_2]$,

$$gr_F(\mathbb{C}[\mathrm{SL}_2])=\bigoplus_{n\ge 0}V(n)\otimes V(n). \tag{3.8}$$

The difference between these two algebras is found in their multiplication operations, where the multiplication in $gr_F(\mathbb{C}[\mathrm{SL}_2])$ is computed by the Cartan multiplication operation. Following [HMM17, Section 3] and [Pop87], we say that $gr_F(\mathbb{C}[\mathrm{SL}_2])$ is the coordinate algebra of the horospherical contraction SL_2^c of SL_2. The coordinate ring $\mathbb{C}[\mathrm{SL}_2^c]$ can also be constructed by means of invariant theory. We have $\mathbb{G}_m$ act antidiagonally through the right actions on two copies of the coordinate ring of $\mathbb{A}^2$. In particular, for $t\in\mathbb{G}_m$ and $f\in V(n)\otimes V(m)\subset\mathbb{C}[\mathbb{A}^2]\otimes\mathbb{C}[\mathbb{A}^2]$ we have $f\circ t=ft^{m-n}$. The only components that are invariant under this action are those with $m=n$. The coordinate ring of the horospherical contraction SL_2^c can be constructed

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by taking invariants with respect to this action:

$$\mathbb{C}[\mathrm{SL}_2^c] = \left[\mathbb{C}[\mathbb{A}^2] \otimes \mathbb{C}[\mathbb{A}^2]\right]^{\mathbb{G}_m}. \tag{3.9}$$

3.3 Valuations on $\mathbb{C}[\mathrm{SL}_2]$ and $\mathbb{C}[\mathbb{A}^2]$

The algebra $\mathbb{C}[\mathrm{SL}_2^c]$ is a domain, so it follows that the filtration F defines a valuation $v : \mathbb{C}[\mathrm{SL}_2]\backslash\{0\} \to \mathbb{Z}$. This valuation is computed on a regular function $f \in \mathbb{C}[\mathrm{SL}_2]$ with $f = \sum f_n$, $f_n \in V(n) \otimes V(n)$, by the rule:

$$v(f) = MIN\{-n \mid f_n \neq 0\}. \tag{3.10}$$

Abusing notation, we say that $\mathbb{C}[\mathrm{SL}_2^c]$ is the associated graded algebra of v. Likewise, the algebra $\mathbb{C}[\mathbb{A}^2]$ is equipped with its degree valuation $deg : \mathbb{C}[\mathbb{A}^2]\backslash\{0\} \to \mathbb{Z}$, which is computed using almost the same formula; for $f \in \mathbb{C}[\mathbb{A}^2]$ with $f = \sum f_n$, $f_n \in V(n)$, we have $deg(f) = MIN\{-n \mid f_n \neq 0\}$. Notice that this is the *negative* of the homogeneous degree function on $\mathbb{C}[\mathbb{A}^2]$. Where v is an $\mathrm{SL}_2 \times \mathrm{SL}_2$-invariant valuation on $\mathbb{C}[\mathrm{SL}_2]$, deg is invariant with respect to the action of $\mathrm{SL}_2 \times \mathbb{G}_m$ on $\mathbb{A}^2$. This will feature prominently in our constructions involving the Plücker algebra.

Now, we define the *Rees algebra* of the valuation v:

$$R = \bigoplus_{m\geq 0} F_m t^m = \bigoplus_{m\geq n\geq 0} V(n) \otimes V(n) t^m. \tag{3.11}$$

The parameter $t \in V(0) \otimes V(0)t \subset F_1 t$ acts by "shifting" the copy $V(n) \otimes V(n)t^m$ of the space $V(n) \otimes V(n) \subset F_m$ to the copy of the same space $V(n) \otimes V(n)t^{m+1} \subset F_{m+1}t^{m+1}$. Since t is not a 0-divisor, this action makes R into a flat $\mathbb{C}[t]$-module. For the following see [HMM17, Section 3].

Lemma 3.1 *The following hold for the $\mathbb{C}[t]$ action on R.*

(1) $\frac{1}{t}R \cong \mathbb{C}[\mathrm{SL}_2] \otimes \mathbb{C}[t, t^{-1}]$,
(2) $R/tR \cong gr_F(\mathbb{C}[\mathrm{SL}_2]) \cong \mathbb{C}[\mathrm{SL}_2^c]$.

Part (1) of Lemma 3.1 says that away from the origin we have $R/(t-a)R \cong \mathbb{C}[SL_2]$, whereas part (2) says at the special fiber R/tR, we obtain $\mathbb{C}[SL_2^c]$.

In coordinates $\mathbb{C}[\mathrm{SL}_2] \cong \mathbb{C}[a, b, c, d]/\langle ad - bc - 1\rangle$ for $a, b, c, d \in V(1) \otimes V(1)$. Cartan multiplication must be surjective (the image is irreducible), so it follows that $a, b, c, d \in V(1) \otimes V(1)$ generate $\mathbb{C}[\mathrm{SL}_2^c]$ as well. Picking coordinates $V(1) \cong \mathbb{C}\{x, y\}$ we can set $a = x \otimes x$, $b = x \otimes y$, $c = y \otimes x$, and $d = y \otimes y$ (i.e., these are the "matrix entries" of a 2×2 matrix). Computing in $\mathbb{C}[\mathrm{SL}_2^c]$, we see that $ad - bc = (x \otimes x)(y \otimes y) - (x \otimes y)(y \otimes x)$. In the coordinate ring $\mathbb{C}[\mathbb{A}^2] \otimes \mathbb{C}[\mathbb{A}^2]$ this is $(xy - xy) \otimes (xy - yx) = 0$. It follows that we can identify SL_2^c with the singular 2×2 matrices. If we set $A = at, B = bt, C = ct, D = dt \in R$ we can compute $AD - BC - t^2 = 0$; this defines a presentation of R. Passing from a general point ($t \neq 0$) to the origin ($t = 0$) degenerates SL_2 to SL_2^c: the singular 2×2 matrices.

Remark 3.2 In [HMM17] and [Man18], a different Rees family is used. Instead of $AD - BC - t^2$, the family is presented by $AD - BC - s$, where s is a parameter of

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homogeneous degree 2. In this way, the family we consider, $Spec(R)$, is a double cover of the family cut out by $AD - BC - s$ considered in *loc. cit.*

3.4 Compactifications

Now, we define a compactification of SL_2 by setting $\overline{\mathrm{SL}}_2 = Proj(R)$.

Proposition 3.3 *The following are true of the projective scheme $\overline{\mathrm{SL}}_2$:*

(1) *$\overline{\mathrm{SL}}_2$ has an algebraic action by $\mathrm{SL}_2 \times \mathrm{SL}_2$,*
(2) *$\overline{\mathrm{SL}}_2$ can be identified with the closed subscheme of $\mathbb{P}^4$ cut out by $AD - BC - t^2 = 0$,*
(3) *the $\mathrm{SL}_2 \times \mathrm{SL}_2$-stable irreducible divisor $D \subset \overline{\mathrm{SL}}_2$ defined by setting $t = 0$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$,*
(4) *SL_2 is isomorphic to the Zariski-open complement of D,*
(5) *the line bundle $\mathcal{O}(1)$ defined by the divisor D satisfies $H^0(\overline{\mathrm{SL}}_2, \mathcal{O}(m)) \cong F_m$. Furthermore, this line bundle induces $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$ on $D \cong \mathbb{P}^1 \times \mathbb{P}^1$,*
(6) *the valuation $ord_D : \mathbb{C}[\mathrm{SL}_2]\backslash\{0\} \to \mathbb{Z}$ is equal to v.*

Proof This is essentially contained in [Man18], but we will also give a proof here. Part (1) follows from the definition of $\overline{\mathrm{SL}}_2$ as $Proj$ of an $\mathrm{SL}_2 \times \mathrm{SL}_2$-algebra. Similarly, parts (2), (3), (4), and (5) follow from the presentation of R given above. For part (6), we identify SL_2 with the open subset $Spec([\frac{1}{t}R]_0) \subset \overline{\mathrm{SL}}_2$. The role of t as a placeholder in the direct sum decomposition of the Rees algebra makes the use of "t" in this description of SL_2 misleading; to be precise, we refer to the regular function $1t \subset V(0) \otimes V(0)t^1$. Taking ord_D of a regular function measures divisibility by $1t$, so we will determine what degree of $1t$ divides an element $f \in V(n) \otimes V(n)$. In order to be in the degree-0 part of $\frac{1}{t}R$, we must divide $V(n) \otimes V(n)t^m$ by $(1t)^m$ to obtain $\frac{1}{(1t)^m}[V(n) \otimes V(n)t^m]$. However, every function in this component is already divisible by $(1t)^{m-n}$, so we obtain $\frac{1}{(1t)^n}[V(n) \otimes V(n)t^n]$; this is the component that maps to $V(n) \otimes V(n)$ under the isomorphism $[\frac{1}{t}R]_0 \cong \mathbb{C}[\mathrm{SL}_2(\mathbb{C})]$. It follows that $ord_D(f) = -n$ for any $f \in V(n) \otimes V(n) \subset \mathbb{C}[\mathrm{SL}_2]$. Since D is $\mathrm{SL}_2 \times \mathrm{SL}_2$-invariant, the valuation ord_D is as well; as a consequence (see [Tim11, Chapter 4]), we compute $ord_D(f)$ for $f = \sum f_n$, $f_n \in V(n) \otimes V(n)$, by taking $MIN\{ord_D(f_n) \mid f_n \neq 0\}$. ∎

A similar statement holds for $\mathbb{A}^2$. We form the Rees algebra $S = \bigoplus_{m \geq n \geq 0} V(n)t^m$ with respect to the valuation deg, and take $Proj(S)$ to obtain the $\mathrm{SL}_2 \times \mathbb{G}_m$-stable compactification $\mathbb{A}^2 \subset \mathbb{P}^2$. The divisor at infinity in this compactification is $Proj(\mathbb{C}[\mathbb{A}^2]) \cong \mathbb{P}^1$. The sections of this divisor recover $\mathcal{O}(1)$ on both $\mathbb{P}^2$ and the boundary $\mathbb{P}^1$. The valuation computed by taking order along the boundary recovers the degree valuation $deg : \mathbb{C}[\mathbb{A}^2]\backslash\{0\} \to \mathbb{Z}$.

4 Construction of X and X_σ

In this section, we describe a construction of the affine cone X over the Plücker embedding of the Grassmannian variety $\mathrm{Gr}_2(\mathbb{C}^n)$ which depends on the choice of a tree σ with n labeled leaves. This construction uses aspects of the geometry of SL_2 and $\mathbb{A}^2$ described in Section 3. We obtain a compactification $X_\sigma \supset X$ by performing the same construction with the compactifications $\overline{\mathrm{SL}}_2 \supset \mathrm{SL}_2$ and $\mathbb{P}^2 \supset \mathbb{A}^2$. In this section,

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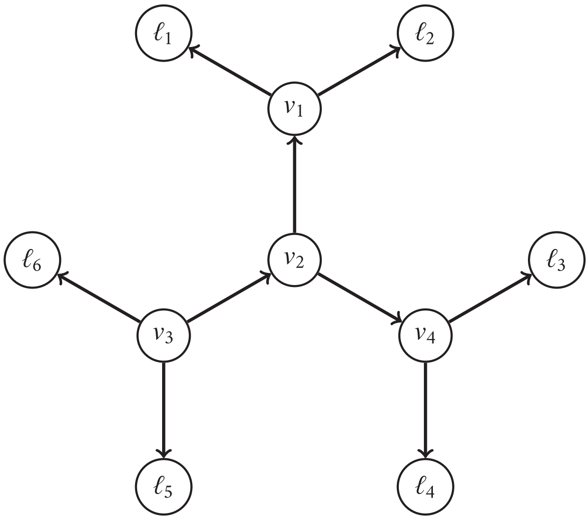

Figure 1: An oriented tree σ.

we make frequent use of the language of Geometric Invariant Theory (GIT). For background on this subject, see the book of Dolgachev [Dol03].

4.1 Constructing the affine cone X from a tree σ

We fix a tree σ with n leaves labeled by $i \in [n]$ with a cyclic ordering $i_1 \to \cdots \to i_n \to i_1$. Let $V(\sigma)$ be the set of nonleaf vertices of σ, and $E(\sigma)$ be the set of edges of σ. We further define $L(\sigma)$ to be the set of leaf-edges of σ, i.e., those edges that connect to a leaf, and $E^\circ(\sigma)$ to be the set of nonleaf edges. In particular, we have $E(\sigma) = E^\circ(\sigma) \sqcup L(\sigma)$. We let $\ell_i \in L(\sigma)$ denote the leaf-edge, which is connected to the leaf labeled i.

We select an orientation on σ; in particular, we choose a direction on each $e \in E(\sigma)$ so that the head of $\ell_i \in L(\sigma)$ points toward the leaf i. This information is necessary to construct X and X_σ, but ultimately the construction is independent of this choice (Figure 1).

We define a space $M(\sigma)$ and an algebraic group $G(\sigma)$ using elements of the tree σ. The space $M(\sigma)$ is a product of copies of SL_2 and $\mathbb{A}^2$, with one copy of SL_2 for each nonleaf edge, and one copy of $\mathbb{A}^2$ for each leaf-edge:

$$M(\sigma) = \prod_{e \in E^\circ(\sigma)} \mathrm{SL}_2 \times \prod_{\ell \in L(\sigma)} \mathbb{A}^2. \tag{4.1}$$

Similarly, the group $G(\sigma)$ is a product of copies of SL_2, with one copy of SL_2 for each nonleaf vertex:

$$G(\sigma) = \prod_{v \in V(\sigma)} \mathrm{SL}_2. \tag{4.2}$$

Now, we define an action of $G(\sigma)$ on $M(\sigma)$. For a nonleaf vertex $v \in V(\sigma)$, we have the corresponding copy of $\mathrm{SL}_2 \subset G(\sigma)$ act on the left-hand side of the space assigned

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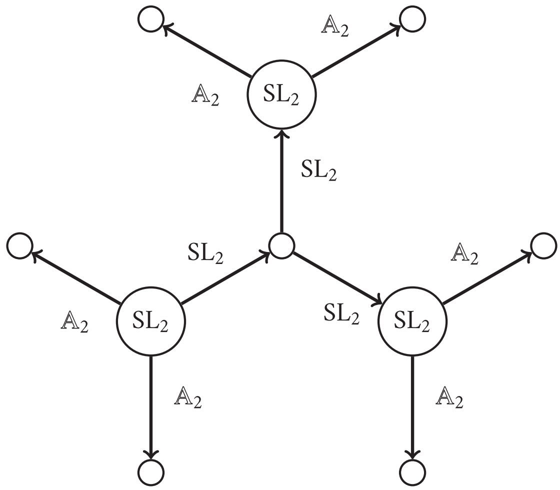

Figure 2: The space $M(\sigma)$ with action by $G(\sigma)$.

to an outgoing edge, and on the right-hand side of any incoming edge. Notice that leaf-edges are always assigned a copy of $\mathbb{A}^2$, which comes with an action by $SL_2 \times \mathbb{G}_m$ as described in Section 3; so for any vertex v connected to a leaf-edge the corresponding copy of SL_2 acts on the left-hand side of $\mathbb{A}^2$ by our conventions (Figure 2).

For the argument below, it will be convenient to work with a closely related space, $\tilde{M}(\sigma) = \prod_{e \in E(\sigma)} SL_2$, also equipped with an action of $G(\sigma)$. There is also a right action of the unipotent group $U^{L(\sigma)}$ on this space, and taking the GIT quotient by this action results in an isomorphism of $G(\sigma)$-spaces: $\tilde{M}(\sigma)//U^{L(\sigma)} \cong M(\sigma)$.

Proposition 4.1 *For any tree σ with n labeled leaves, the GIT quotient $M(\sigma)//G(\sigma)$ is isomorphic to X.*

Proof It is well-known (see [Dol03]) that X can be constructed as the GIT quotient $SL_2 \backslash\backslash [\mathbb{A}^2 \times \cdots \times \mathbb{A}^2]$; this is equivalent to the fact that the Plücker algebra is generated by the 2×2 minors of a $2 \times n$ matrix of parameters. This quotient can be recovered from the GIT construction above as the case of the tree σ_n with n labeled leaves, one nonleaf vertex, and the natural cyclic ordering $1 \to \cdots \to n$. Therefore, to prove the proposition, it suffices to show that all of the GIT constructions are isomorphic to $M(\sigma_n)//G(\sigma_n)$. The cyclic ordering does not affect the isomorphism type, so the problem can be reduced to showing the following statement: for any tree σ as above, and a tree σ' obtained from σ by contracting an edge $e \in E^\circ(\sigma)$, we have $M(\sigma)//G(\sigma) \cong M(\sigma')//G(\sigma')$.

Geometric invariant theory quotients can be performed in stages, so we can further reduce to the case of the trees σ with only one nonleaf edge e, and σ' with no nonleaf edges. Moreover, as $\tilde{M}(\sigma)//U^{L(\sigma)} \cong M(\sigma)$, we may work with the spaces $\tilde{M}(\sigma)$ and $\tilde{M}(\sigma')$. Let $e \in E^\circ(\sigma)$ have vertices v_1, v_2, with the orientation along e pointing $v_1 \to v_2$. Let v_1 have leaf edges $\ell_1, \ldots, \ell_s$ and v_2 have leaf edges $k_1, \ldots, k_r$. We orient $\ell_1, \ldots, \ell_s$ and $k_1, \ldots, k_r$ away from v_1, v_2. We make this choice without a loss of generality as we have the involution $g \to g^{-1}$, which is an isomorphism on the scheme SL_2, which interchanges the left and right actions. We let $v \in V(\sigma')$ be the lone nonleaf vertex of σ', and by abuse of notation we let $\ell_1, \ldots, \ell_s$ and $k_1, \ldots, k_r$ be its leaf-edges, oriented in the same fashion.

We claim that there is an isomorphism $\tilde{M}(\sigma)//(SL_2 \times SL_2) \cong \tilde{M}(\sigma')//SL_2$. Once more, we appeal to GIT-in-stages and show that $\left(\prod_{\ell_i} SL_2\right) \times SL_2//SL_2 \cong \prod_{\ell_i} SL_2$ as spaces with an action by SL_2. Here, SL_2 acts on the right-hand side of the second component of $\left(\prod_{\ell_i} SL_2\right) \times SL_2$, and on the left-hand sides of the components of $\prod_{\ell_i} SL_2$.

To prove this, we show something more general. Let X be a G-variety for a reductive group G, and let G act on $X \times G$ diagonally on X and the left-hand side of G, then $X \times G//G$ retains an action of G through the right-hand side of G in $X \times G$. As G-varieties we have $X \times G//G \cong X$. To show this, map $(x, g) \in X \times G$ to $g^{-1}x \in X$; this is a map of G-spaces, which intertwines the right action on G in $X \times G$ with the action on X. This map is constant on the orbits of $X \times G$ under the diagonal action, which are, in turn, all closed; and furthermore there is an algebraic section $X \to X \times G$ sending x to (x, Id) for $Id \in G$ the identity. This proves the result. ∎

By Proposition 4.1, each tree σ defines a different realization of $X = SL_2 \backslash\backslash [\mathbb{A}^2 \times \cdots \times \mathbb{A}^2]$ with added "hidden variables" given by the SL_2 components along the nonleaf edges. The combinatorial and geometric constructions we make for X are then derived from this new information.

4.2 The compactification X_σ

We define a projective variety $\overline{M}(\sigma)$ using the same recipe used to define $M(\sigma)$:

$$\overline{M}(\sigma) = \prod_{e \in E^\circ(\sigma)} \overline{SL}_2 \times \prod_{\ell \in L(\sigma)} \mathbb{P}^2. \tag{4.3}$$

The $SL_2 \times SL_2$ and $SL_2 \times \mathbb{G}_m$ actions on SL_2 and $\mathbb{A}^2$, respectively, both extend to their compactifications $\overline{SL}_2$ and $\mathbb{P}^2$. It follows that there is an action of $G(\sigma)$ on $\overline{M}(\sigma)$. The line bundles defined in Proposition 3.3 on $\overline{SL}_2$ and $\mathbb{P}^2$ (both denoted $\mathcal{O}(1)$ by abuse of notation) are linearized with respect to the actions on these spaces; it follows that the outer tensor product bundle $\mathcal{L} = \boxtimes_{e \in E(\sigma)} \mathcal{O}(1)$ is $G(\sigma)$-linearized as well. With these observations in mind we define X_σ as the corresponding GIT quotient:

$$X_\sigma = \overline{M}(\sigma)//_{\mathcal{L}} G(\sigma). \tag{4.4}$$

Before we show that X_σ is a compactification of X (see Proposition 4.3), we describe the coordinate ring $\mathbb{C}[X] = \mathbb{C}[M(\sigma)]^{G(\sigma)}$ and the projective coordinate ring $\mathbb{C}[X_\sigma] = \bigoplus_{n \geq 0} H^0(\overline{M}(\sigma), \mathcal{L}^{\otimes n})^{G(\sigma)}$ in terms of the tree σ. In the sequel, we will refer to a σ-weight $\mathbf{s} \in \mathbb{Z}_{\geq 0}^{E(\sigma)}$, which is an assignment of non-negative integers to the edges of σ. The following decompositions of the coordinate ring of $\mathbb{C}[M(\sigma)]$ and the projective coordinate ring $\mathbb{C}[\overline{M}(\sigma)] = \bigoplus_{n \geq 0} H^0(\overline{M}(\sigma), \mathcal{L}^{\otimes n})$ can be computed from the isotypical decompositions of $\mathbb{C}[SL_2]$, $\mathbb{C}[\mathbb{A}^2]$, $\mathbb{C}[\mathbb{P}^2] = \bigoplus_{n \geq 0} H^0(\mathbb{P}^2, \mathcal{O}(n))$ and

$\mathbb{C}[\overline{\mathrm{SL}}_2] = \bigoplus_{n\geq 0} H^0(\overline{\mathrm{SL}}_2, \mathcal{O}(n))$:

$$\mathbb{C}[X] = \mathbb{C}[M(\sigma)]^{G(\sigma)} = \Big[\bigotimes_{e\in E^\circ(\sigma)} \mathbb{C}[\mathrm{SL}_2] \otimes \bigotimes_{\ell\in L(\sigma)} \mathbb{C}[\mathbb{A}^2]\Big]^{G(\sigma)} = \tag{4.5}$$

$$\bigoplus_{\mathbf{s}\in\mathbb{Z}_{\geq 0}^{E(\sigma)}} \Big[\bigotimes_{e\in E^\circ(\sigma)} V(\mathbf{s}(e)) \otimes V(\mathbf{s}(e)) \otimes \bigotimes_{\ell\in L(\sigma)} V(\mathbf{s}(\ell))\Big]^{G(\sigma)}.$$

To ease notation, we let $W_\sigma(\mathbf{s}) = \big[\bigotimes_{e\in E^\circ(\sigma)} V(\mathbf{s}(e)) \otimes V(\mathbf{s}(e)) \otimes \bigotimes_{\ell\in L(\sigma)} V(\mathbf{s}(\ell))\big]^{G(\sigma)}$, so that $\mathbb{C}[X] = \bigoplus_{\mathbf{s}\in\mathbb{Z}_{\geq 0}^{E(\sigma)}} W_\sigma(\mathbf{s})$. Roughly speaking, each $\mathbf{s}\in\mathbb{Z}_{\geq 0}^{E(\sigma)}$ assigns two irreducible representations $V(\mathbf{s}(e)) \otimes V(\mathbf{s}(e))$ to each nonleaf edge $e\in E^\circ(\sigma)$, one for the head of e and one for the tail of e. This pair is acted on through the right and left actions of SL_2 on the copy of SL_2 assigned to e. Similarly, $\mathbf{s}$ assigns one representation $V(\mathbf{s}(\ell))$ to each leaf-edge $\ell\in L(\sigma)$; this space is acted on by SL_2 through the left action on $\mathbb{A}^2$. Note that for any $\mathbf{s}\in\mathbb{Z}_{\geq 0}^{E(\sigma)}$, the space $W_\sigma(\mathbf{s})$ can be written as the following tensor product:

$$W_\sigma(\mathbf{s}) = \bigotimes_{v\in V(\sigma)} \big[V(\mathbf{s}(e_1(v))) \otimes \cdots \otimes V(\mathbf{s}(e_k(v)))\big]^{\mathrm{SL}_2}, \tag{4.6}$$

where $e_1(v),\dots,e_k(v)$ are the edges of σ containing v, see Figure 3.

Lemma 4.2 *Let σ' be a tree with n leaves, which are obtained from σ by contracting an edge $e\in E(\sigma)$, then there is a corresponding direct sum decomposition:*

$$W_{\sigma'}(\mathbf{s}') = \bigoplus_{\{\mathbf{s}\in\mathbb{Z}_{\geq 0}^{E(\sigma)} \mid \forall e'\in E(\sigma'),\ \mathbf{s}(e')=\mathbf{s}'(e')\}} W_\sigma(\mathbf{s}). \tag{4.7}$$

Proof Let v_1, v_2 be the endpoints of e, and let $v\in V(\sigma')$ be the vertex created by bringing v_1 and v_2 together. We prove this lemma by considering the link of v in σ'. Everything we do is compatible with the geometric arguments given in Proposition 4.1. Let $e_1,\dots,e_k$ be the edges of σ' which contain v, and let $e_1,\dots,e_s, e$ and $e, e_{s+1},\dots,e_k$ be the edges of σ which contain v_1 and v_2, respectively. For the sake of simplicity, we orient all edges e_i away from the vertices, and we have e point from v_1 to v_2. Pick $\mathbf{a} = (a_1,\dots,a_k)\in\mathbb{Z}_{\geq 0}^k$ and $n\geq 0$, and consider the isotypical component of $\mathbb{C}[\mathrm{SL}_2^s \times \mathrm{SL}_2 \times \mathrm{SL}_2^{k-s}]^{\mathrm{SL}_2\times\mathrm{SL}_2}$:

$$\begin{aligned}&\big[V(a_1)\otimes\cdots\otimes V(a_s)\otimes V(n)\big]^{\mathrm{SL}_2} \otimes \big[V(n)\otimes V(a_{s+1})\otimes\cdots\otimes V(a_k)\big]^{\mathrm{SL}_2}\\ &\qquad\otimes \big[V(a_1)\otimes\cdots\otimes V(a_k)\big].\end{aligned} \tag{4.8}$$

The map $\mathrm{SL}_2^k \to \mathrm{SL}_2^s \times \mathrm{SL}_2 \times \mathrm{SL}_2^{k-s}$ which sends $(g_1,\dots,g_k)$ to $(g_1,\dots,g_s, Id, g_{s+1},\dots,g_k)$ induces the isomorphism of SL_2^k-algebras $\mathbb{C}[\mathrm{SL}_2^k]^{\mathrm{SL}_2} \cong \mathbb{C}[\mathrm{SL}_2^s \times \mathrm{SL}_2 \times \mathrm{SL}_2^{k-s}]^{\mathrm{SL}_2\times\mathrm{SL}_2}$ from Proposition 4.1. This algebra map is computed on the above component by plugging the $V(n)$ component into its "dual" $V(n)$. Since this map preserves the SL_2^k action, it must likewise map the $\mathbf{a}$ component of $\mathbb{C}[\mathrm{SL}_2^s \times \mathrm{SL}_2 \times \mathrm{SL}_2^{k-s}]^{\mathrm{SL}_2\times\mathrm{SL}_2}$ isomorphically onto the $\mathbf{a}$-component of $\mathbb{C}[\mathrm{SL}_2^k]^{\mathrm{SL}_2}$, so we obtain

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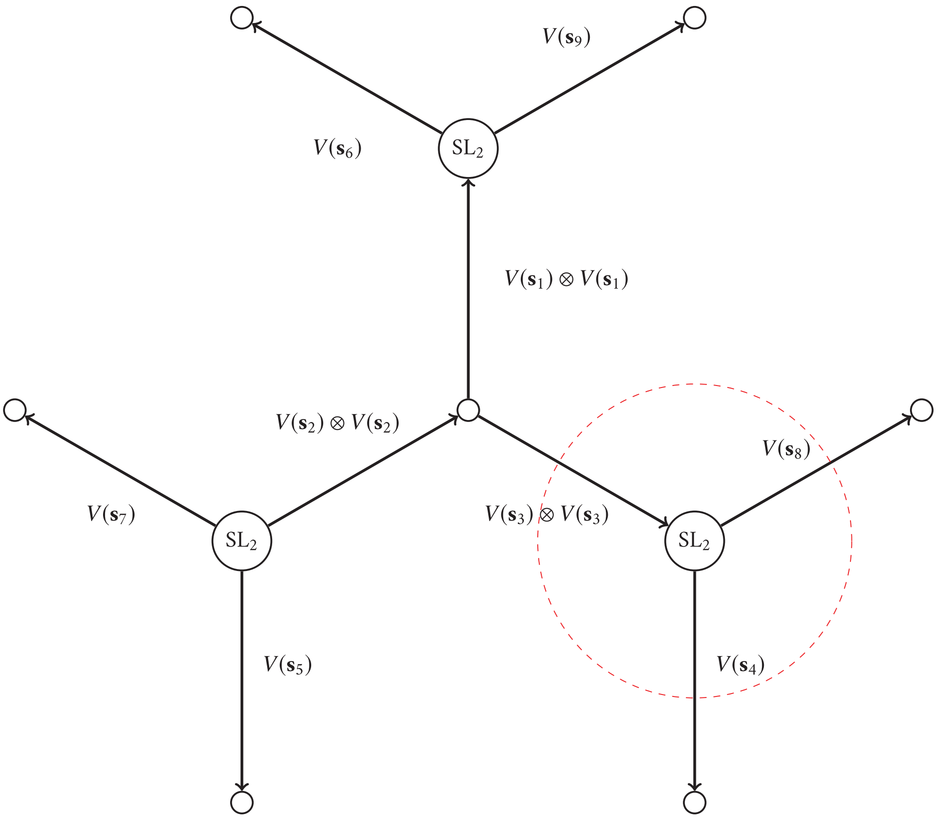

Figure 3: Isotypical components $W_\sigma(\mathbf{s}) \subset \mathbb{C}[M(\sigma)]^{G(\sigma)}$. The dotted circle contains those SL_2 representations that are acted on by the copy of SL_2 associated to the lower right trinode.

$[V(a_1) \otimes \cdots \otimes V(a_k)]^{SL_2} \otimes V(a_1) \otimes \cdots \otimes V(a_k)$ as a direct sum over n of the components above. ∎

We make use of the decompositions of the Rees algebras R and S from Section 3 to give a description of $\mathbb{C}[X_\sigma]$ in terms of the spaces $W_\sigma(\mathbf{s})$. By definition we have:

$$H^0(\overline{M}(\sigma), \mathcal{L}^{\otimes n}) = \bigotimes_{e \in E^\circ(\sigma)} H^0(\overline{SL}_2, \mathcal{O}(n)) \otimes \bigotimes_{\ell \in L(\sigma)} H^0(\mathbb{P}^2, \mathcal{O}(n)). \tag{4.9}$$

In particular, the same power n is used in the computations of the global sections for each line bundle. Recall that $H^0(\overline{SL}_2, \mathcal{O}(n)) = \bigoplus_{0 \le m \le n} V(m) \otimes V(m)t^n$ and $H^0(\mathbb{P}^2, \mathcal{O}(n)) = \bigoplus_{0 \le m \le n} V(m)t^n$. Since t^n is a placeholder that agrees across all components of the tensor product, we obtain:

$$H^0(\overline{M}(\sigma), \mathcal{L}^{\otimes n}) = \bigoplus_{\{\mathbf{s} \in \mathbb{Z}_{\ge 0}^{E(\sigma)} \mid \forall e \in E(\sigma),\ \mathbf{s}(e) \le n\}} \Big[\bigotimes_{e \in E^\circ(\sigma)} V(\mathbf{s}(e)) \otimes V(\mathbf{s}(e)) \otimes \bigotimes_{\ell \in L(\sigma)} V(\mathbf{s}(\ell)) \Big]. \tag{4.10}$$

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As a consequence, we obtain the following decomposition of the projective coordinate ring of X_σ:

$$\mathbb{C}[X_\sigma] = \mathbb{C}[\overline{M}(\sigma)]^{G(\sigma)} = \bigoplus_{n\geq 0} \bigoplus_{\{\mathbf{s}\in\mathbb{Z}_{\geq 0}^{E(\sigma)} \mid \forall e\in E(\sigma),\ \mathbf{s}(e)\leq n\}} W_\sigma(\mathbf{s})t^n. \tag{4.11}$$

Proposition 4.3 *The projective variety X_σ is a compactification of X.*

Proof We show that $X_\sigma = Proj(\mathbb{C}[X_\sigma])$ contains X as a dense, open subscheme. We consider the element $1t \in W_\sigma(\mathbf{0})t^1 \subset \mathbb{C}[X_\sigma]$, where $\mathbf{0} : E(\sigma) \to \mathbb{Z}_{\geq 0}$ is the weight, which assigns 0 to every edge of σ. As constructed, each graded component $\bigoplus_{\{\mathbf{s}\in\mathbb{Z}_{\geq 0}^{E(\sigma)} \mid \mathbf{s}(e)\leq n,\ \forall e\in E(\sigma)\}} W_\sigma(\mathbf{s})t^n \subset \mathbb{C}[X_\sigma]$ is a subspace of $\mathbb{C}[X]$, and the multiplication operation on these graded components is computed by the multiplication rule in $\mathbb{C}[X]$; this is a consequence of the Proposition 4.1 and the definition of $\mathbb{C}[X_\sigma]$. By inverting $1t$ we obtain $\frac{1}{1t^n}W_\sigma(\mathbf{s})t^n = \frac{1}{1t^m}W_\sigma(\mathbf{s})t^m$ for all $\mathbf{s}$ with $\mathbf{s}(e) \leq n, m, \forall e \in E(\sigma)$ in the 0-degree part of $\frac{1}{1t}\mathbb{C}[X_\sigma]$. It follows that $\left[\frac{1}{1t}\mathbb{C}[X_\sigma]\right]_0 \cong \mathbb{C}[X]$, and that the complement of the hypersurface $1t = 0$ in X_σ is $Spec(\mathbb{C}[X]) = X$. ∎

We let $D_\sigma \subset X_\sigma$ be the hypersurface defined by $1t \in \mathbb{C}[X_\sigma]$.

5 The cone C_σ of valuations on $\mathbb{C}[X]$

We describe the geometry of the hypersurface $D_\sigma \subset X_\sigma$. In order to construct its irreducible components and describe their intersections, we construct a cone C_σ of discrete valuations on $\mathbb{C}[X]$. We show that C_σ is simplicial and generated by distinguished valuations v_e, $e \in E(\sigma)$ (see Definition 5.1). In Section 6, we show that v_e is obtained by taking order of vanishing along a component of D_σ.

5.1 Valuations on $\mathbb{C}[M(\sigma)]$

We introduce a valuation $v_e : \mathbb{C}[X]\backslash\{0\} \to \mathbb{Z}$ for each edge $e \in E(\sigma)$. First, we recall the valuations $v : \mathbb{C}[\mathrm{SL}_2]\backslash\{0\} \to \mathbb{Z}$ and $deg : \mathbb{C}[\mathbb{A}^2]\backslash\{0\} \to \mathbb{Z}$ from Section 3. The space $M(\sigma)$ is the product $\prod_{e\in E^\circ(\sigma)} \mathrm{SL}_2 \times \prod_{\ell\in L(\sigma)} \mathbb{A}^2$. Accordingly, its coordinate ring carries a valuation $\bar{v}_e : \mathbb{C}[M(\sigma)]\backslash\{0\} \to \mathbb{Z}$ for each edge $e \in E(\sigma)$; this is computed by using v when $e \in E^\circ(\sigma)$ and deg when $e \in L(\sigma)$. The associated algebraic filtration by the spaces $\bar{F}^e_m = \{f \in \mathbb{C}[M(\sigma)] \mid \bar{v}_e(f) \geq -m\}$ are given by the following spaces:

$$\begin{aligned}\bar{v}_e, e \in E^\circ(\sigma):\ \bar{F}^e_m = &\Big[\bigotimes_{e'\in E^\circ(\sigma),\ e'\neq e} \mathbb{C}[\mathrm{SL}_2]\Big] \\ &\otimes \Big[\bigoplus_{0\leq n\leq m} V(n)\otimes V(n)\Big] \otimes \Big[\bigotimes_{\ell\in L(\sigma)} \mathbb{C}[\mathbb{A}^2]\Big],\end{aligned} \tag{5.1}$$

$$\bar{v}_\ell, \ell \in L(\sigma):\ \bar{F}^\ell_m = \Big[\bigotimes_{e\in E^\circ(\sigma)} \mathbb{C}[\mathrm{SL}_2]\Big] \otimes \Big[\bigoplus_{0\leq n\leq m} V(n)\Big] \otimes \Big[\bigotimes_{\ell'\in L(\sigma),\ \ell'\neq\ell} \mathbb{C}[\mathbb{A}^2]\Big]. \tag{5.2}$$

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We also have the following strict filtration spaces:

$$\bar{v}_e, e \in E^\circ(\sigma): \ \bar{F}^e_{<m} = \Big[\bigotimes_{e' \in E^\circ(\sigma),\ e' \neq e} \mathbb{C}[\mathrm{SL}_2]\Big] \otimes \Big[\bigoplus_{0 \le n < m} V(n) \otimes V(n)\Big] \otimes \Big[\bigotimes_{\ell \in L(\sigma)} \mathbb{C}[\mathbb{A}^2]\Big], \tag{5.3}$$

$$\bar{v}_\ell, \ell \in L(\sigma): \ \bar{F}^\ell_{<m} = \Big[\bigotimes_{e \in E^\circ(\sigma)} \mathbb{C}[\mathrm{SL}_2]\Big] \otimes \Big[\bigoplus_{0 \le n < m} V(n)\Big] \otimes \Big[\bigotimes_{\ell' \in L(\sigma),\ \ell' \neq \ell} \mathbb{C}[\mathbb{A}^2]\Big]. \tag{5.4}$$

Clearly, $\bar{F}^e_{<m} \subset \bar{F}^e_m$ for any $e \in E(\sigma)$. The reader can verify that the associated graded algebra of $\bar{v}_e, e \in E^\circ(\sigma)$ and $\bar{v}_\ell, \ell \in L(\sigma)$ are the coordinate rings of $\prod_{e' \in E^\circ(\sigma),\ e' \neq e} \mathrm{SL}_2 \times \mathrm{SL}_2^c \times \prod_{\ell \in L(\sigma)} \mathbb{A}^2$ and $\prod_{e \in E^\circ(\sigma)} \mathrm{SL}_2 \times \mathbb{A}^2 \times \prod_{\ell' \in L(\sigma),\ \ell' \neq \ell} \mathbb{A}^2$, respectively. We observe that $r deg : \mathbb{C}[\mathbb{A}^2]\backslash\{0\} \to \mathbb{R}$ is a valuation for any $r \in \mathbb{R}_{\ge 0}$, see Remark 2.4.

Definition 5.1 Let $\mathbf{r} \in \mathbb{R}^{E^\circ(\sigma)}_{\ge 0} \times \mathbb{R}^{L(\sigma)}$, and let $\bar{v}_\mathbf{r} : \mathbb{C}[M(\sigma)]\backslash\{0\} \to \mathbb{R}$ be the valuation $[\sum \mathbf{r}(e)\bar{v}_e]$ obtained using the sum operation described in Definition 2.3.

The valuations $\bar{v}_\mathbf{r}$ are built from the valuations v and deg, which can be computed entirely in terms of the representation theory of SL_2. The following lemma shows that this is also the case for $\bar{v}_\mathbf{r}$.

Lemma 5.1 *Let $\mathbf{s} \in \mathbb{Z}^{E(\sigma)}_{\ge 0}$ and $f \in \big[\bigotimes_{e \in E^\circ(\sigma)} V(\mathbf{s}(e)) \otimes V(\mathbf{s}(e)) \otimes \bigotimes_{\ell \in L(\sigma)} V(\mathbf{s}(\ell))\big] \subset \mathbb{C}[M(\sigma)]$, then $\bar{v}_\mathbf{r}(f)$ is computed by taking the "dot product" of $\mathbf{r}$ and $\mathbf{s}$ over the edges of σ:*

$$\bar{v}_\mathbf{r}(f) = \sum_{e \in E(\sigma)} -\mathbf{r}(e)\mathbf{s}(e) = -\langle \mathbf{r}, \mathbf{s}\rangle. \tag{5.5}$$

Furthermore, the filtration space $\bar{F}^\mathbf{r}_m = \{f \in \mathbb{C}[M(\sigma)] \mid \bar{v}_\mathbf{r}(f) \ge -m\}$ is the following sum:

$$\bar{F}^\mathbf{r}_m = \bigoplus_{\{\mathbf{s} \mid \langle \mathbf{r}, \mathbf{s}\rangle \le m\}} \Big[\bigotimes_{e \in E^\circ(\sigma)} V(\mathbf{s}(e)) \otimes V(\mathbf{s}(e)) \otimes \bigotimes_{\ell \in L(\sigma)} V(\mathbf{s}(\ell))\Big]. \tag{5.6}$$

Proof This is a direct consequence of the formula for computing the valuations $v : \mathbb{C}[\mathrm{SL}_2]\backslash\{0\} \to \mathbb{Z}$ and $deg : \mathbb{C}[\mathbb{A}^2]\backslash\{0\} \to \mathbb{Z}$, and Definition 2.3. In particular, $\langle \mathbf{r}, \mathbf{s}\rangle \le -m$ if and only if $-\langle \mathbf{r}, \mathbf{s}\rangle \ge -m$. ■

5.2 Valuations on $\mathbb{C}[X]$

In what follows, we place a partial ordering $\le$ on $\mathbb{Z}^{E(\sigma)}_{\ge 0}$, where $\mathbf{s} \le \mathbf{s}'$ if $\mathbf{s}(e) \le \mathbf{s}'(e)$, $\forall e \in E(\sigma)$. We let $v_\mathbf{r} : \mathbb{C}[X]\backslash\{0\} \to \mathbb{R}$ be the restriction of $\bar{v}_\mathbf{r}$ from $\mathbb{C}[M(\sigma)]$ to $\mathbb{C}[X]$.

Proposition 5.2 *The following hold for a tree σ, the associated decomposition $\mathbb{C}[X] = \bigoplus_{\mathbf{s} \in \mathbb{Z}^{E(\sigma)}_{\ge 0}} W_\sigma(\mathbf{s})$, and the valuation $v_\mathbf{r}$ for any $\mathbf{r} \in \mathbb{R}^{E^\circ(\sigma)}_{\ge 0} \times \mathbb{R}^{L(\sigma)}$:*

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(1) *for* $f \in W_\sigma(\mathbf{s})$, *we have* $v_{\mathbf{r}}(f) = -\langle \mathbf{r}, \mathbf{s}\rangle$,
(2) *for* $m \in \mathbb{R}$, *the filtration space* $F_m^{\mathbf{r}} = \{f \in \mathbb{C}[X] \mid v_{\mathbf{r}}(f) \geq -m\} = \bigoplus_{\{\mathbf{s} | \langle \mathbf{r}, \mathbf{s}\rangle \leq m\}} W_\sigma(\mathbf{s})$,
(3) *for* $f = \sum f_{\mathbf{s}}$ *with* $f_{\mathbf{s}} \in W_\sigma(\mathbf{s})$, *we have* $v_{\mathbf{r}}(f) = MIN\{-\langle \mathbf{r}, \mathbf{s}\rangle \mid f_{\mathbf{s}} \neq 0\}$,
(4) *for any* $\mathbf{s}, \mathbf{s}' \in \mathbb{Z}_{\geq 0}^{E(\sigma)}$, *we have* $W_\sigma(\mathbf{s})W_\sigma(\mathbf{s}') \subset \bigoplus_{\mathbf{s}'' \preceq \mathbf{s}+\mathbf{s}'} W_\sigma(\mathbf{s}'')$. *Furthermore, the* $\mathbf{s} + \mathbf{s}'$ *component of this product is always nonzero.*

Proof The valuations $\bar{v}_{\mathbf{r}}$ are all $G(\sigma)$-invariant so their filtration spaces $\bar{F}_m^{\mathbf{r}}$ are $G(\sigma)$-representations. We have $W_\sigma(\mathbf{s}) = \left[\bigotimes_{e \in E^\circ(\sigma)} V(\mathbf{s}(e)) \otimes V(\mathbf{s}(e)) \otimes \bigotimes_{\ell \in L(\sigma)} V(\mathbf{s}(\ell))\right]^{G(\sigma)}$, so (1) is a consequence of Lemma 5.1. Furthermore, to prove (2) we can compute $F_m^{\mathbf{r}} = \bar{F}_m^{\mathbf{r}} \cap \mathbb{C}[X] = \left[\bar{F}_m^{\mathbf{r}}\right]^{G(\sigma)} = \bigoplus_{\{\mathbf{s}|\langle \mathbf{r},\mathbf{s}\rangle \leq m\}} W_\sigma(\mathbf{s})$ by Lemma 5.1. Part (2) shows that $v_{\mathbf{r}}$ is adapted to the direct sum decomposition $\mathbb{C}[X] = \bigoplus_{\mathbf{s} \in \mathbb{Z}_{\geq 0}^{E(\sigma)}} W_\sigma(\mathbf{s})$ (recall this notion from Section 2), so part (3) follows as a consequence.

We know that $W_\sigma(\mathbf{s})W_\sigma(\mathbf{s}') \subset \bigoplus_{\mathbf{s}'' \preceq \mathbf{s}+\mathbf{s}'} W_\sigma(\mathbf{s}'')$ from properties of multiplication in $\mathbb{C}[\mathrm{SL}_2]$ and $\mathbb{C}[\mathbb{A}^2]$. For the second part of (4), we first observe that $F_{<m}^{\mathbf{r}} = \bigoplus_{\{\mathbf{s}|\langle \mathbf{r},\mathbf{s}\rangle < m\}} W_\sigma(\mathbf{s})$. Let v_σ be the valuation obtained from $\bar{v}_\sigma = [\sum_{e \in E(\sigma)} \bar{v}_e]$. Then, for any $f \in W_\sigma(\mathbf{s}'')$ with $\mathbf{s}'' \preceq \mathbf{s} + \mathbf{s}'$ we must have $v_\sigma(f) \geq -\sum_{e \in E(\sigma)} \mathbf{s}(e) + \mathbf{s}'(e)$, with equality if and only if $\mathbf{s}'' = \mathbf{s} + \mathbf{s}'$. Now (2) implies that the product of the components $W_\sigma(\mathbf{s})$ and $W_\sigma(\mathbf{s}')$ in the associated graded algebra $gr_\sigma(\mathbb{C}[X])$ of v_σ is projection onto the $W_\sigma(\mathbf{s} + \mathbf{s}')$ component. Since v_σ is a valuation, $gr_\sigma(\mathbb{C}[X])$ is a domain, so this product must be nonzero. ■

Recall the Berkovich analytification X^{an} of the affine variety X. Proposition 5.2 allows us to construct a distinguished subset of X^{an} associated to the tree σ.

Corollary 5.3 *There is a continuous map* $\phi_\sigma : \mathbb{R}_{\geq 0}^{E^\circ(\sigma)} \times \mathbb{R}^{L(\sigma)} \to X^{an}$ *which takes* $\mathbf{r}$ *to* $v_{\mathbf{r}}$.

Proof In Proposition 5.2, we have shown that there is such a map ϕ_σ. Thus, it remains to establish that this map is continuous. Using the definition of the topology on X^{an}, it suffices to show that any evaluation function ev_f, $f \in \mathbb{C}[X]$, pulls back to a continuous function on $\mathbb{R}_{\geq 0}^{E^\circ(\sigma)} \times \mathbb{R}^{L(\sigma)}$. By part (3) of Proposition 5.2, we have $ev_f(v_{\mathbf{r}}) = MIN\{-\langle \mathbf{r}, \mathbf{s}\rangle \mid f_{\mathbf{s}} \neq 0\}$, where $f_{\mathbf{s}}$ denotes the $W_\sigma(\mathbf{s})$ component of f; this function is piecewise-linear in $\mathbf{r}$ and therefore continuous. ■

Suppose that a tree σ' is obtained from σ by contracting a nonleaf edge $e \in E^\circ(\sigma)$. There is a natural inclusion, $i_e : \mathbb{R}_{\geq 0}^{E^\circ(\sigma')} \times \mathbb{R}^{L(\sigma')} \to \mathbb{R}_{\geq 0}^{E^\circ(\sigma)} \times \mathbb{R}^{L(\sigma)}$, by regarding $\mathbb{R}_{\geq 0}^{E^\circ(\sigma')} \times \mathbb{R}^{L(\sigma')}$ as the weightings of σ, which are 0 on e.

Lemma 5.4 *For* $\mathbf{r} \in \mathbb{R}_{\geq 0}^{E^\circ(\sigma')} \times \mathbb{R}^{L(\sigma')}$, $v_{\mathbf{r}} = v_{i_e(\mathbf{r})}$. *As a consequence,* $\mathbb{R}_{\geq 0}^{E^\circ(\sigma')} \times \mathbb{R}^{L(\sigma')}$ *can be regarded as a face of* $\mathbb{R}_{\geq 0}^{E^\circ(\sigma)} \times \mathbb{R}^{L(\sigma)}$.

Proof This follows by direct computation using Lemma 4.2 and Proposition 5.2. ■

Definition 5.2 Let $\mathcal{T}(n)$ denote the complex $\bigcup_\sigma \mathbb{R}_{\geq 0}^{E^\circ(\sigma)} \times \mathbb{R}^{L(\sigma)}$ obtained as the push-out of the diagram of inclusions defined by the maps i_e.

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The maps ϕ_σ glue together to define a continuous map $\Phi : \mathcal{T}(n) \to X^{an}$. Let C_σ denote the image, $\phi_\sigma(\mathbb{R}_{\geq 0}^{E^\circ(\sigma)} \times \mathbb{R}^{L(\sigma)})$. In Section 7, we show that the evaluation functions $ev_{p_{ij}} \circ \Phi : \mathcal{T}(n) \to \mathbb{R}$ map a tree to its *dissimilarity vector* in $\mathbb{R}^{\binom{n}{2}}$. The dissimilarity vector is a complete invariant of a metric tree, so Φ must be an injective map.

6 The geometry of $D_\sigma \subset X_\sigma$

With the valuations $v_{\mathbf{r}}$ and the decomposition $\mathbb{C}[X] = \bigoplus_{\mathbf{s}\in\mathbb{Z}_{\geq 0}^{E(\sigma)}} W_\sigma(\mathbf{s})$, we have two useful tools for understanding the geometry of the compactification X_σ. In this section, we show that D_σ is reduced, and we give a recipe to decompose D_σ into irreducible components.

6.1 The ideal $I_S \subset \mathbb{C}[X_\sigma]$

Much of our understanding of the divisor D_σ is derived from the decomposition of the projective coordinate ring of X_σ into the spaces $W_\sigma(\mathbf{s})$:

$$\mathbb{C}[X_\sigma] = \bigoplus_{n\geq 0} \bigoplus_{\{\mathbf{s}\in\mathbb{Z}_{\geq 0}^{E(\sigma)} \mid \forall e\in E(\sigma),\ \mathbf{s}(e)\leq n\}} W_\sigma(\mathbf{s})t^n. \tag{6.1}$$

This decomposition enables us to define a set of distinguished ideals in $\mathbb{C}[X_\sigma]$.

Definition 6.1 For $S \subset E(\sigma)$, a subset of the edges of σ, let $I_S \subset \mathbb{C}[X_\sigma]$ be the following vector space:

$$I_S = \bigoplus_{n\geq 0} \bigoplus_{\{\mathbf{s}\in\mathbb{Z}_{\geq 0}^{E(\sigma)} \mid \forall e\in E(\sigma),\ \mathbf{s}(e)\leq n,\ \exists e'\in S,\ \mathbf{s}(e')<n\}} W_\sigma(\mathbf{s})t^n. \tag{6.2}$$

Proposition 6.1 *For any $S \subset E(\sigma)$, $I_S \subset \mathbb{C}[X_\sigma]$ is a prime ideal.*

Proof By definition, I_S is a homogeneous ideal, and $W_\sigma(\mathbf{s})W_\sigma(\mathbf{s}') \subset \bigoplus_{\mathbf{s}''\preceq\, \mathbf{s}+\mathbf{s}'} W_\sigma(\mathbf{s}'')$ by Proposition 5.2. Now, suppose $fg \in I_S$ for $f, g \in \mathbb{C}[X_\sigma]$, homogeneous elements of degrees n and m, respectively. We view f and g as regular functions on X with the property that $v_e(f) \geq -n$ and $v_e(g) \geq -m$, $\forall e \in E(\sigma)$. If $fg \in I_S$, it must be the case that $v_e(fg) = v_e(f) + v_e(g) > -(n+m)$ for some $e \in E(\sigma)$. But by Proposition 5.2 this can only happen if $v_e(f) > -n$ or $v_e(g) > -m$, so we conclude that $f \in I_S$ or $g \in I_S$. ■

We let $D_S \subset X_\sigma$ be the zero locus of I_S. Clearly, we have that $S \subset S'$ implies that $I_S \subset I_{S'}$ and $D_S \supset D_{S'}$. The following proposition shows that D_σ is built from the irreducible, reduced subvarieties D_S.

Proposition 6.2 *For any tree σ the following hold:*

(1) $I_{S\cup S'} = I_S + I_{S'}$, $D_S \cap D_{S'} = D_{S\cup S'}$,
(2) *if $S \neq S'$ then $I_S \neq I_{S'}$.*

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Proof All of the ideals I_S are sums of the spaces $W_\sigma(\mathbf{s})t^n$, so for both (1) and (2) it suffices to check membership on these spaces. We have $W_\sigma(\mathbf{s})t^n \subset I_{S\cup S'}$ if and only if there is some $e \in S \cup S'$ with $\mathbf{s}(e) < n$; this happens if and only if $W_\sigma(\mathbf{s})t^n$ is either in I_S or $I_{S'}$. To show that $I_S, I_{S'}$ are distinct prime ideals, we only need to show that there is some element which is in I_S, but not in $I_{S'}$. We define a weighting ω_S of $E(\sigma)$ by non-negative integers so that $0 \neq W_\sigma(\omega_S)t^4 \subset \mathbb{C}[X_\sigma]$, $\omega_S(e) = 4, \forall e \in S$ and $\omega_S(e) = 2, \forall e \notin S$. Clearly, $W_\sigma(\omega_S)t^4 \subset I_{S'}$ but $W_\sigma(\omega_S)t^4 \not\subset I_S$. This reduces the question to showing that $W_\sigma(\omega_S) \neq 0$, which is handled by the following lemma. ■

For the proof of the following lemma, see 4.6.

Lemma 6.3 *For any subset $S \subset E(\sigma)$, the invariant space $W_\sigma(\omega_S) \subset \mathbb{C}[X]$ is nonzero.*

Proof It suffices to show that any tensor product $V(n_1) \otimes \cdots \otimes V(n_k)$ where $n_i \in \{2, 4\}$ and $k \geq 3$ contains an invariant. If $k = 3$, the Pieri rule (equation (3.3)) shows that this is the case. Suppose that this holds up to $k - 1$. The tensor product decomposition $V(n_1) \otimes V(n_2) = \bigoplus V(m)$ induces a decomposition of the k-fold tensor product:

$$V(n_1) \otimes \cdots \otimes V(n_k) = \bigoplus V(m) \otimes V(n_3) \otimes \cdots \otimes V(n_k). \tag{6.3}$$

Here, the sum is over all m so that n_1, n_2, m satisfy the Pieri rule. For whatever combination of 2 and 4 are given by n_1, n_2, we know we can have m be 2 or 4 as necessary from the case $k = 3$. But then $V(m) \otimes \cdots \otimes V(n_k)$ contains an invariant by the induction hypothesis, so $V(n_1) \otimes \cdots \otimes V(n_k)$ does as well. ■

Corollary 6.4 *If σ is a trivalent tree, then the D_S are the intersections of the irreducible components of the reduced divisor D_σ. In particular, D_σ is of combinatorial normal crossings type.*

Proof If σ is trivalent, then $|E(\sigma)| = dim(X_\sigma) = 2n - 3$. Picking any ordering on the elements $e_i \in E(\sigma)$ we can form an increasing chain of distinct subsets $S_i = \{e_1, \ldots, e_i\}$. From Proposition 6.2, we know that the corresponding ideals I_{S_i} form an increasing chain of distinct prime ideals. It follows that $height(I_S) = codim(D_S) = |S|$. In particular, $codim(D_e) = 1$ for any $e \in E(\sigma)$, and $D_S = \cap_{e\in S} D_e$. Finally, we observe that $\langle 1t \rangle \subset \mathbb{C}[X_\sigma]$ is $\cap_{e\in E(\sigma)} I_e$; it follows that $D_\sigma = \cup_{e\in E(\sigma)} D_e$. ■

Remark 6.5 We briefly sketch how to show that $D_\sigma \subset X_\sigma$ is a combinatorial normal crossings divisor when σ is not trivalent. Choose a trivalent tree σ' that surjects onto σ by collapsing some subset of edges. Using methods from Section 7, we use σ' to define a flat degeneration of X_σ to a toric variety $Y_{\sigma',\sigma}$. Here, $Y_{\sigma',\sigma}$ is the projective toric variety associated to a polytope $P_{\sigma',\sigma}$ defined by intersecting the cone $P_{\sigma'}$ from Definition 7.1 with the half spaces defined by requiring that the weights on the edges coming from σ are less than or equal to 1. This polytope is then considered with respect to the lattice L_σ from 7.1. The degeneration restricts to a degeneration of X to the expected toric variety for the trivalent tree σ' and also degenerates each component $D_S \subset D_\sigma$ to a toric orbit closure $Y_S \subset Y_{\sigma',\sigma}$ corresponding to a face $P_{\sigma',\sigma}(S) \subset P_{\sigma',\sigma}$. Now one can combinatorially determine that the faces $P_{\sigma',\sigma}(S)$, and therefore the components $D_S \subset D_\sigma$, have the correct dimension.

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6.2 D_S as a GIT quotient

Now, we describe each reduced, irreducible subvariety $D_S \subset X_\sigma$ as a GIT quotient in the style of Section 4. We select a direction on each $e \in E(\sigma)$ and define the following product space associated to $S \subset E(\sigma)$:

$$\overline{M}(\sigma, S) = \prod_{e \in E^\circ(\sigma) \setminus S} \overline{\mathrm{SL}}_2 \times \prod_{e \in E^\circ(\sigma) \cap S} (\mathbb{P}^1 \times \mathbb{P}^1) \times \prod_{\ell \in L(\sigma) \setminus S} \mathbb{P}^2 \times \prod_{\ell \in L(\sigma) \cap S} \mathbb{P}^1. \tag{6.4}$$

The space $\overline{M}(\sigma, S)$ is naturally a closed, reduced, irreducible subspace of $\overline{M}(\sigma)$. We let $\mathcal{L}_S$ be the restriction of the line bundle $\mathcal{L}$ to this subspace. Following Section 3, we have:

$$\mathcal{L}_S = \tag{6.5}$$

$$\left[\boxtimes_{e \in E^\circ(\sigma) \setminus S} \mathcal{O}(1)\right] \boxtimes \left[\boxtimes_{e \in E^\circ(\sigma) \cap S} \mathcal{O}(1) \boxtimes \mathcal{O}(1)\right] \boxtimes \left[\boxtimes_{\ell \in L(\sigma) \setminus S} \mathcal{O}(1)\right] \boxtimes \left[\boxtimes_{\ell \in L(\sigma) \cap S} \mathcal{O}(1)\right].$$

where:

$$H^0(\overline{\mathrm{SL}}_2, \mathcal{O}(m)) = F_m = \bigoplus_{0 \le n \le m} V(n) \otimes V(n), \tag{6.6}$$

$$H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(m) \boxtimes \mathcal{O}(m)) = V(m) \otimes V(m), \tag{6.7}$$

$$H^0(\mathbb{P}^2, \mathcal{O}(m)) = \bigoplus_{0 \le n \le m} V(n), \tag{6.8}$$

$$H^0(\mathbb{P}^1, \mathcal{O}(m)) = V(m), \tag{6.9}$$

as SL_2-representations.

Proposition 6.6 *The space D_S is isomorphic to the GIT quotient $\overline{M}(\sigma, S) /\!/_{\mathcal{L}_S} G(\sigma)$. In particular, there is an isomorphism of graded algebras:*

$$\bigoplus_{m \ge 0} H^0(\overline{M}(\sigma, S), \mathcal{L}_S^{\otimes m})^{G(\sigma)} \cong \mathbb{C}[X_\sigma]/I_S. \tag{6.10}$$

Proof Using the descriptions of the components of the graded coordinate rings of $\overline{M}(\sigma, S)$ and $\overline{M}(\sigma)$ above and in Section 3, we can form the following exact sequence.

$$0 \to \bigoplus_{m \ge 0} J_m \to \bigoplus_{m \ge 0} H^0(\overline{M}(\sigma), \mathcal{L}^{\otimes m}) \to \bigoplus_{m \ge 0} H^0(\overline{M}(\sigma, S), \mathcal{L}_S^{\otimes m}) \to 0, \tag{6.11}$$

where J_m is the direct sum of components of the form

$$\begin{aligned} &\Big[\bigotimes_{e \in E^\circ(\sigma) \setminus S} V(\mathbf{s}(e)) \otimes V(\mathbf{s}(e))\Big] \otimes \Big[\bigotimes_{e \in E^\circ(\sigma) \cap S} V(\mathbf{s}(e)) \otimes V(\mathbf{s}(e))\Big] \\ &\otimes \Big[\bigotimes_{\ell \in L(\sigma) \setminus S} V(\mathbf{s}(\ell))\Big] \otimes \Big[\bigotimes_{\ell \in L(\sigma) \cap S} V(\mathbf{s}(\ell))\Big], \end{aligned} \tag{6.12}$$

for $\mathbf{s} \in \mathbb{Z}_{\ge 0}^{E(\sigma)}$ with $\mathbf{s}(e) \le m$ for all $e \in E(\sigma)$ and $\mathbf{s}(e) < m$ (or $\mathbf{s}(\ell) < m$) for some $e \in S$. The group $G(\sigma)$ is reductive. This implies that the subsequence of invariants of 6.11 is

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a direct summand. As a consequence, taking the $G(\sigma)$-invariants of (6.11) produces the exact sequence:

$$0 \to I_S \to \mathbb{C}[X_\sigma] \to \bigoplus_{m\geq 0} H^0(\overline{M}(\sigma, S), \mathcal{L}_S^{\otimes m})^{G(\sigma)} \to 0, \tag{6.13}$$

which proves the proposition. ∎

Remark 6.7 Proposition 6.6 gives an interesting interpretation of the compactification $D_\sigma \subset X_\sigma \supset X$, where "going to infinity" in the direction of an edge $e \in E(\sigma)$ has one passing from $SL_2 \subset \overline{SL_2}$ to $\mathbb{P}^1 \times \mathbb{P}^1$. This latter space is a $\mathbb{G}_m$-quotient of the singular 2×2 matrices SL_2^c. So in the boundary D_σ, we see a variant of a quiver variety made from singular matrices, as opposed to $X \subset X_\sigma$, which is made with elements of SL_2.

6.3 The valuations ord_{D_e}

Now, we relate the divisorial valuations associated to the irreducible components of D_σ to the cone of valuations C_σ constructed in Section 5. In Proposition 3.3, we see that taking order along the boundary divisor $D = \mathbb{P}^1 \times \mathbb{P}^1 \subset \overline{SL_2}$ produces the valuation $v : \mathbb{C}[SL_2]\setminus\{0\} \to \mathbb{Z}$. An identical statement holds for the boundary copy of $\mathbb{P}^1$ in $\mathbb{P}^2 \supset \mathbb{A}^2$ and the valuation $deg : \mathbb{C}[\mathbb{A}^2]\setminus\{0\} \to \mathbb{Z}$.

We have a distinguished irreducible divisor $\overline{M}(\sigma, e) \subset \overline{M}(\sigma)$ for each edge $e \in E(\sigma)$ coming from the construction in 6.2. As a product space, $\overline{M}(\sigma, e)$ is obtained by replacing the copy of $\overline{SL_2}$ or $\mathbb{P}^2$ at the edge e with its boundary divisor. Since $M(\sigma)$ is a dense, open subspace of $\overline{M}(\sigma)$, both the coordinate ring $\mathbb{C}[M(\sigma)]$ and its ring of $G(\sigma)$-invariants $\mathbb{C}[X]$ inherit the valuation $ord_{\overline{M}(\sigma,e)}$. The divisor D_e is then obtained from $\overline{M}(\sigma, e)$ as the GIT quotient. The following proposition relates the valuations obtained from these divisors.

Proposition 6.8 *For any $e \in E(\sigma)$, the valuations v_e, $ord_{\overline{M}(\sigma,e)}$, and ord_{D_e} coincide on $\mathbb{C}[X]$.*

Proof By definition we have $\bar{v}_e = ord_{\overline{M}(\sigma,e)}$, where $\bar{v}_e : \mathbb{C}[M(\sigma)]\setminus\{0\} \to \mathbb{Z}$ is from Section 5; so it follows that $v_e = ord_{\overline{M}(\sigma,e)}$ on $\mathbb{C}[X]$. Let $\bar{\eta}_e$ be the generic point of $\overline{M}(\sigma, e) \subset \overline{M}(\sigma)$, with local ring $\mathcal{O}_{\bar{\eta}_e}$ and maximal ideal $\langle \bar{t}_e \rangle \subset \mathcal{O}_{\bar{\eta}_e}$. Then, the local ring $\mathcal{O}_{\eta_e}$ at the generic point η_e of D_e is the ring of $G(\sigma)$-invariants in $\mathcal{O}_{\bar{\eta}_e}$, and furthermore $\bar{t}_e \in \mathcal{O}_{\eta_e}$. Computing v_e must then coincide with computing ord_{D_e}, as both valuations amount to measuring $\bar{t}_e$ degree. ∎

6.4 X_σ is Fano

We finish this section with the observation that the compactifications X_σ are all of Fano type. For simplicity, we focus on the case when σ is a trivalent tree.

Proposition 6.9 *For σ, a trivalent tree, and $D_\sigma \subset X_\sigma$, the associated boundary divisor of the compactification, we have*

$$-K_{X_\sigma} = 3D_\sigma. \tag{6.14}$$

In particular, since D_σ is ample, X_σ is Fano.

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We prove Proposition 6.9 in Section 7.2. The proof uses some material from Section 7.2 to produce a toric degeneration of $\mathbb{C}[X_\sigma]$. Moreover, we use a result of Watanabe [Wat81] which allows a computation for the anticanonical class of the *Proj* of a positively graded Cohen–Macaulay algebra.

7 The tropical geometry of *X*

We recall the tropical variety $\mathrm{Trop}(I_{2,n})$ obtained from the homogeneous Plücker ideal $I_{2,n}$. The ideal $I_{2,n}$ vanishes on the Plücker generators $p_{ij} \in \mathbb{C}[X]$, $1 \leq i < j \leq n$, and defines the Plücker embedding $\mathrm{Gr}_2(\mathbb{C}^n) \subset \mathbb{P}(\wedge^2(\mathbb{C}^n))$ of the Grassmannian of 2-planes. We show that the map $ev_n = (\ldots, \text{evpi j}, \ldots) : \text{Xan} \to \mathbb{R}^{\binom{n}{2}}$ defined by the Plücker generators maps $\mathcal{T}(n)$ (Definition 5.2) isomorphically onto $\mathrm{Trop}(I_{2,n})$.

7.1 The tropical Grassmannian

The tropical Grassmannian variety $\mathrm{Trop}(I_{2,n})$ was introduced by Speyer and Sturmfels in [SS04]. It is one of the best understood tropical varieties in part because the Plücker relations are known to be a tropical basis for $I_{2,n}$. For any $1 \leq i, j, k, \ell \leq n$ in cyclic order we have:

$$p_{ij}p_{k\ell} - p_{i\ell}p_{jk} + p_{ik}p_{j\ell} = 0. \tag{7.1}$$

The tropical variety $\mathrm{Trop}(I_{2,n})$ is then the set of tropical solutions $\mathbf{d} = (\ldots, d_{ij}, \ldots) \in \mathbb{R}^{\binom{n}{2}}$ of the following tropical polynomials:

$$MIN\{d_{ij} + d_{k\ell}, d_{i\ell} + d_{jk}, d_{ik} + d_{j\ell}\}. \tag{7.2}$$

Using a variant of [SS04, Theorem 4.2], it is then possible to use a solution $\mathbf{d} \in \mathrm{Trop}(I_{2,n})$ to reconstruct a unique tree σ with n labeled leaves along with a corresponding real weight vector $\mathbf{r} \in \mathbb{R}_{\geq 0}^{E^\circ(\sigma)} \times \mathbb{R}^{L(\sigma)}$ such that d_{ij} is the sum of the negatives $-\mathbf{r}(e)$ of the edges e in the unique path in σ between the leaves i and j. We let $\mathbf{d} : \mathbb{R}_{\geq 0}^{E^\circ(\sigma)} \times \mathbb{R}^{L(\sigma)} \to \mathbb{R}^{\binom{n}{2}}$ be the function that takes a metric tree to the vector of negatives of pairwise distances between its leaves. The tuple $\mathbf{d}(\mathbf{r})$ is called the *dissimilarity vector* of $\mathbf{r}$ (see [Man11, Man12, PS04]).

Now, we show that $\mathrm{Trop}(I_{2,n})$ can be realized as the image of $\mathcal{T}(n)$ under the evaluation map defined by the Plücker generators. We let $ev_n : X^{an} \to \mathbb{R}^{\binom{n}{2}}$ be the map that sends $v \in X^{an}$ to $(\ldots, v(p_{ij}), \ldots)$. Recall the continuous map $\Phi : \mathcal{T}(n) \to X^{an}$ defined in Section 5.2.

Proposition 7.1 *The composition $ev_n \circ \Phi : \mathcal{T}(n) \to \mathbb{R}^{\binom{n}{2}}$ is an isomorphism of the complex of polyhedral cones onto* $\mathrm{Trop}(I_{2,n})$*. In particular, $ev_n(v_{\mathbf{r}})$ is equal to the dissimilarity vector $\mathbf{d}(\mathbf{r}) \in \mathbb{R}^{\binom{n}{2}}$.*

Proof A variant of this proposition appears in [Man11]. First, consider the decomposition of $\mathbb{C}[X]$ given by its characterization as the ring of SL_2-invariants in $\mathbb{C}[\mathbb{A}^2 \times \cdots \times \mathbb{A}^2]$:

$$\mathbb{C}[X] = \bigoplus_{\mathbf{a} \in \mathbb{Z}_{\geq 0}^n} [V(a_1) \otimes \cdots \otimes V(a_n)]^{\mathrm{SL}_2}. \tag{7.3}$$

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The invariant space with $a_k = 0$ except for $a_i = a_j = 1$ is 1-dimensional (see Section 3), and it is spanned by the Plücker generator p_{ij}. Moreover, $[V(a_1) \otimes \cdots \otimes V(a_n)]^{\mathrm{SL}_2}$ is the space $W_{\sigma_n}(\mathbf{s}')$, where σ_n is the unique tree with n leaves and one internal vertex, and $\mathbf{s}'$ is the unique weighting of the edges of σ_n where $\mathbf{s}'(\ell_i) = a_i$. Now we fix a tree σ, and note that σ_n can be obtained from σ by repeatedly collapsing internal edges. The following decomposition is a consequence of this observation and Lemma 4.2:

$$(7.4) \qquad [V(a_1) \otimes \cdots \otimes V(a_n)]^{\mathrm{SL}_2} = \bigoplus_{\{\mathbf{s} \in \mathbb{Z}_{\geq 0}^{E(\sigma)} \mid \mathbf{s}(\ell_i) = a_i\}} W_\sigma(\mathbf{s}).$$

For the invariant space containing p_{ij}, exactly one $\mathbf{s}$ in this decomposition can have $W_\sigma(\mathbf{s}) \neq 0$. Using (4.6), we observe that the $\mathbf{s}$ with this property satisfies $\mathbf{s}(e) = 1$ if e is in the unique path from i to j and $\mathbf{s}(e) = 0$ otherwise. Now, Proposition 5.2 implies that $v_{\mathbf{r}}(p_{ij}) = -\langle \mathbf{r}, \mathbf{s} \rangle$ for this $\mathbf{s}$, which is precisely the sum of the $-\mathbf{r}(e)$ for e in the unique path from i to j. The characterization of $\mathrm{Trop}(I_{2,n})$ given in [SS04] now implies that the image of C_σ under the map ev_n is precisely the vectors $\mathbf{d} \in \mathbb{R}^{\binom{n}{2}}$ coming from trees with topology and labeling given by σ. Each map $\mathbf{r} \to v_{\mathbf{r}}(p_{ij})$ is linear, so ev_n maps C_σ linearly onto its image. ∎

7.2 Associated graded algebras from $\mathcal{T}(n)$

Now, we will compute the associated graded algebras of the valuations $v_{\mathbf{r}} \in \Phi(\mathcal{T}(n))$. Lemma 5.4 allows us to regard any valuation $v_{\mathbf{r}} \in C_{\sigma'}$ as $v_{i_e(\mathbf{r})} \in C_\sigma$, where σ' is obtained from σ by contracting the edge e. Repeatedly using Lemma 5.4, therefore, allows us to consider only trivalent trees when we compute with the valuations $v_{\mathbf{r}}$. For now, we assume that σ is trivalent. We can see from (4.6) that each space $W_\sigma(\mathbf{s}) \subset \mathbb{C}[X]$ in this case is a tensor product of invariant spaces of the form $[V(i) \otimes V(j) \otimes V(k)]^{\mathrm{SL}_2}$. The Pieri rule (3.3) then implies the following lemma.

Lemma 7.2 *For σ a trivalent tree, $W_\sigma(\mathbf{s})$ is multiplicity-free. In particular, $W_\sigma(\mathbf{s}) = \mathbb{C}$ if for every vertex $v \in V(\sigma)$ with edges e_1, e_2, e_3, the triple $\mathbf{s}(e_1), \mathbf{s}(e_2), \mathbf{s}(e_3)$ satisfies the conditions of the Pieri rule, and $W_\sigma(\mathbf{s}) = 0$ otherwise.*

Definition 7.1 For this definition, see (3.3). Let $L_\sigma \subset \mathbb{Z}^{E(\sigma)}$ be the sublattice of those points ω with the property that $\omega(e_1), \omega(e_2), \omega(e_3)$ satisfy the *parity condition* whenever e_1, e_2, e_3 share a common vertex. Let $P_\sigma \subset \mathbb{R}_{\geq 0}^{E(\sigma)}$ be the polyhedral cone of those points ω with the property that $\omega(e_1), \omega(e_2), \omega(e_3)$ satisfy the *triangle inequalities* whenever e_1, e_2, e_3 share a common vertex. Finally, let S_σ be the saturated affine semigroup $P_\sigma \cap L_\sigma$.

The coordinate algebra $\mathbb{C}[X]$ can now be expressed as a direct sum of one-dimensional spaces $W_\sigma(\mathbf{s})$:

$$(7.5) \qquad \mathbb{C}[X] = \bigoplus_{\mathbf{s} \in S_\sigma} W_\sigma(\mathbf{s}).$$

Choose one nonzero vector $b_{\mathbf{s}} \in W_\sigma(\mathbf{s})$ for each $\mathbf{s} \in S_\sigma$ so that $\mathbb{C}[X] = \bigoplus_{\mathbf{s} \in S_\sigma} \mathbb{C} b_{\mathbf{s}}$. Proposition 5.2 implies that multiplication of basis members has a lower-triangular expansion: $b_{\mathbf{s}} b_{\mathbf{s}'} = \sum_{\mathbf{s}'' \preceq \mathbf{s}+\mathbf{s}'} C_{\mathbf{s},\mathbf{s}'}^{\mathbf{s}''} b_{\mathbf{s}''}$, where $\preceq$ indicates that $\mathbf{s}''(e) \leq \mathbf{s}(e) + \mathbf{s}'(e)$ for

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every $e \in E(\sigma)$. We call $\mathbb{B}_\sigma = \{b_{\mathbf{s}} \mid \mathbf{s} \in S_\sigma\}$ a *branching basis* of $\mathbb{C}[X]$ corresponding to σ. The following is immediate from Definition 2.2 and part (2) of Proposition 5.2.

Proposition 7.3 *A branching basis $\mathbb{B}_\sigma \subset \mathbb{C}[X]$ is adapted to every valuation in C_σ.*

Let the tree σ be equipped with an orientation on its edges, and let $S \subset E^\circ(\sigma)$ be some subset of nonleaf edges. We define two affine schemes attached to these data:

$$M(\sigma, S) = \prod_{e \in E^\circ(\sigma)\setminus S} \mathrm{SL}_2 \times \prod_{e \in S} \mathrm{SL}_2^c \times \prod_{\ell \in L(\sigma)} \mathbb{A}^2, \tag{7.6}$$

$$X(S) = M(\sigma, S) /\!/ G(\sigma). \tag{7.7}$$

Here $G(\sigma)$ acts on $M(\sigma, S)$ by the same recipe used on $M(\sigma)$.

Proposition 7.4 *For $v_{\mathbf{r}} \in C_\sigma$, let $S \subset E^\circ(\sigma)$ be the set of edges for which $\mathbf{r}(e) \neq 0$; we have the following:*

(1) *the product $b_{\mathbf{s}} b_{\mathbf{s}'}$ in $gr_{\mathbf{r}}(\mathbb{C}[X])$ is the subsum of $\sum_{\mathbf{s}'' \leq \mathbf{s}+\mathbf{s}'} C_{\mathbf{s},\mathbf{s}'}^{\mathbf{s}''} b_{\mathbf{s}''}$ consisting of those terms $\mathbf{s}''$ where $\mathbf{s}''(e) = \mathbf{s}(e) + \mathbf{s}'(e)$ when $e \in S$ and $\mathbf{s}''(e) \leq \mathbf{s}(e) + \mathbf{s}'(e)$ when $e \notin S$.*
(2) *$gr_{\mathbf{r}}(\mathbb{C}[X]) \cong \mathbb{C}[X(S)]$,*
(3) *the Plücker generators p_{ij} are a Khovanskii basis for any $v_{\mathbf{r}}$.*

Proof First, we observe that Proposition 5.2 implies that the equivalence classes of the basis members $b_{\mathbf{s}} \in \mathbb{B}_\sigma$ are still a basis of $gr_{\mathbf{r}}(\mathbb{C}[X])$. Indeed, any component $F_m^{\mathbf{r}}/F_{>m}^{\mathbf{r}} \subset gr_{\mathbf{r}}(\mathbb{C}[X])$ is a quotient of the span of $\mathbb{B}_\sigma \cap F_m^{\mathbf{r}}$ by the span of $\mathbb{B}_\sigma \cap F_{>m}^{\mathbf{r}}$. In particular, for any $f = \sum C_{\mathbf{s}} b_{\mathbf{s}} \in \mathbb{C}[X]$, the equivalence class $\bar{f} \in gr_{\mathbf{r}}(\mathbb{C}[X])$ is computed by taking the subsum of only those terms $C_{\mathbf{s}} b_{\mathbf{s}}$ for which $-\langle \mathbf{r}, \mathbf{s}\rangle$ is minimal. Part (1) follows from this observation.

For part (2), note that $\mathbb{C}[X(S)]$ also has a decomposition into the spaces $W_\sigma(\mathbf{s})$. Indeed, the coordinate rings of SL_2^c and SL_2 have exactly the same isotypical decomposition, however their multiplication rules are different. In particular, the dominant weight decomposition defines a grading on $\mathbb{C}[\mathrm{SL}_2^c]$. This implies that for the components corresponding to $e \in S$, the only $W_\sigma(\mathbf{s}'')$ that contribute to the expansion of $W_\sigma(\mathbf{s}) W_\sigma(\mathbf{s}')$ are those with $\mathbf{s}''(e) = \mathbf{s}'(e) + \mathbf{s}(e)$; this proves part (2).

For part (3), we select $\mathbf{r}'$ with $\mathbf{r}'(e) > 0$ for every $e \in E^\circ(\sigma)$. In this case, the expansion of $b_{\mathbf{s}} b_{\mathbf{s}'} \in gr_{\mathbf{r}'}(\mathbb{C}[X])$ only has a $\mathbf{s} + \mathbf{s}'$ component. Since $\mathbb{C}$ is algebraically closed, it follows that $gr_{\mathbf{r}'}(\mathbb{C}[X]) \cong \mathbb{C}[S_\sigma]$ (see [ES96]). But this top component is always there when this multiplication is carried out in $gr_{\mathbf{r}}(\mathbb{C}[X])$; so it follows (see Lemma 2.5) that $gr_{\mathbf{r}'}(gr_{\mathbf{r}}(\mathbb{C}[X])) \cong \mathbb{C}[S_\sigma]$. This means that a generating set of $\mathbb{C}[S_\sigma]$ can be lifted to a generating set of $gr_{\mathbf{r}}(\mathbb{C}[X])$. The following lemma then implies that (the equivalence classes of) the Plücker generators generate any $gr_{\mathbf{r}}(\mathbb{C}[X])$. ∎

Lemma 7.5 *The affine semigroup S_σ is generated by the weightings ω_{ij} $(1 \leq i < j \leq n)$ that assign 1 to every edge on the unique path between leaf i and leaf j and 0 elsewhere.*

Proof This is a standard result, see e.g., [HMM11, Proposition 4.6]. We give a short conceptual proof. Consider a trinode with edges e_1, e_2, e_3 weighted with n_1, n_2, n_3, which satisfy the Pieri rules (3.3). We can find a graph on three vertices corresponding

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to the three leaves of the trinode such that when we view the edges of the graph as passing through the trinode's edges, we recover n_i as the number of paths passing through e_i. The number of paths from i to j is $x_{ij} = \frac{1}{2}(n_i + n_j - n_k)$. Notice that $x_{ij} + x_{ik} = n_i$. This proves the lemma for the case $n = 3$. Now take $\mathbf{s} \in S_\sigma$, and for each trivalent vertex $v \in V(\sigma)$, extract the paths associated to the edges connected to v. For two vertices v, v' connected by an edge e, this process yields the same number of paths in e, so we may glue these paths together any way we like. The result is a graph on n vertices (the leaves of σ). Since this graph is a union of edges, $\mathbf{s}$ can be realized as a sum of the ω_{ij}. ∎

Finally, we prove Proposition 6.9:

Proof of Proposition 6.9 By Proposition 6.8, $\mathbb{C}[X_\sigma] = \bigoplus_{n\geq 0} H^0(X_\sigma, nD_\sigma)$, so it follows that D_σ is ample. The projective coordinate rings of $\overline{\mathrm{SL}_2}$ and $\mathbb{P}^2$ are both normal; as a consequence, algebra $\mathbb{C}[\overline{M}(\sigma)]$ is normal. Since X_σ is a GIT quotient of $\overline{M}(\sigma)$, we must have that $\mathbb{C}[X_\sigma]$ is normal as well.

The basis $\mathbb{B}_\sigma \subset \mathbb{C}[X]$ induces a basis in $\bar{\mathbb{B}}_\sigma \subset \mathbb{C}[X_\sigma]$. The members of $\bar{\mathbb{B}}_\sigma$ are labeled by elements $(\mathbf{s}, m)$ in the semigroup $\bar{S}_\sigma \subset S_\sigma \times \mathbb{Z}$. Here $\mathbf{s} \in S_\sigma$ and for each edge $e \in E(\sigma)$ we have $\mathbf{s}(e) \leq m$. It is straightforward to show that this semigroup is generated by the elements $(\omega_{ij}, 1)$ where ω_{ij} is as in Lemma 7.5. The proof of Proposition 7.4 can be applied to $\mathbb{C}[X_\sigma]$ to show that $\mathbb{C}[X_\sigma]$ has associated graded algebra $\mathbb{C}[\bar{S}_\sigma]$.

Next, we show that $\mathbb{C}[X_\sigma]$ is a Gorenstein algebra. The algebra $\mathbb{C}[\bar{S}_\sigma]$ is a flat degeneration of $\mathbb{C}[X_\sigma]$, so by the argument in [LM16, Proposition 3.7], it suffices to show that $\mathbb{C}[\bar{S}_\sigma]$ is Gorenstein. The algebra $\mathbb{C}[\bar{S}_\sigma]$ is a normal affine semigroup algebra, so we use [BH93, Corollary 6.3.8] to show that it is Gorenstein. We consider $(\omega, 3) \in \bar{S}_\sigma$, where $\omega(e) = 2$ for all $e \in E(\sigma)$. If $(\tau, m) \in \bar{S}_\sigma$ is in the relative interior of the semigroup, we must have $\tau(e) < m$ and for any three edges e, f, g meeting at a vertex we need $\tau(e) < \tau(f) + \tau(g)$. Given these inequalities, it is straightforward to check that in this case $(\tau, m) - (\omega, 3)$ is still in $\bar{S}_\sigma$. This proves that $\mathbb{C}[\bar{S}_\sigma]$ and $\mathbb{C}[X_\sigma]$ are Gorenstein algebras.

Finally, we apply [Wat81, Corollary 2.9] to X_σ and its projective coordinate ring $\mathbb{C}[X_\sigma]$. Since $(\omega, 3)$ is degree 3 in $\mathbb{C}[\bar{S}_\sigma]$, as a consequence, the a-invariant of $\mathbb{C}[\bar{S}_\sigma]$ is -3. This information can be recovered from the Hilbert function of $\mathbb{C}[\bar{S}_\sigma]$, which agrees with the Hilbert function of $\mathbb{C}[X_\sigma]$. It follows that the a-invariant of $\mathbb{C}[X_\sigma]$ is -3 as well. Furthermore, D_σ is a multiplicity-free sum of irreducible divisors, so $K_{X_\sigma} + 3D_\sigma = 0$ in $CL(X_\sigma)$. ∎

7.3 Initial ideals from $\mathrm{Trop}(I_{2,n})$

Now, we relate the associated graded algebras of the valuations $v_{\mathbf{r}} \in \mathcal{T}(n)$ to initial ideals $in_{\mathbf{d}(\mathbf{r})}(I_{2,n})$ associated to points in the tropical variety $\mathrm{Trop}(I_{2,n})$. Let $\mathbf{x} = \{x_{ij} \mid 1 \leq i < j \leq n\}$.

Proposition 7.6 *For any $v_{\mathbf{r}} \in \mathcal{T}(n)$ the following hold:*

(1) *the valuation $v_{\mathbf{r}}$ coincides with the weight quasi-valuation $\mathfrak{v}_{\mathbf{d}(\mathbf{r})}$ (see 2.3),*
(2) *the associated graded algebra $\mathrm{gr}_{\mathbf{r}}(\mathbb{C}[X])$ is isomorphic to $\mathbb{C}[\mathbf{x}]/in_{\mathbf{d}(\mathbf{r})}(I_{2,n})$,*

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(3) *if* $\mathbf{r} \in C_\sigma \subset \mathcal{T}(n)$ *satisfies* $\mathbf{r}(e) \neq 0$, $\forall e \in E^\circ(\sigma)$, *then* $in_{\mathbf{d}(\mathbf{r})}(I_{2,n})$ *is the prime binomial ideal that vanishes on the generators* $[\omega_{ij}]$ *of the affine semigroup algebra* $\mathbb{C}[S_\sigma]$.

Proof Part (3) follows from (2). Both (1) and (2) are a consequence of Theorem2.7 and Proposition 7.4. ∎

8 Maximal rank valuations and Newton–Okounkov cones of X

In this section, we use the divisor $D_\sigma \subset X_\sigma$ to construct maximal rank valuations on $\mathbb{C}[X]$, establishing D_σ in the theory of *Newton–Okounkov* bodies for the Grassmannian variety. In particular, we show that S_σ can be realized as the value semigroup of a valuation on $\mathbb{C}[X]$, which can be extracted from D_σ.

8.1 Maximal rank valuations on $\mathbb{C}[X]$

There are many constructions of valuations on the Plücker algebra $\mathbb{C}[X]$ with representation-theoretic interpretations. Alexeev and Brion [AB04] give a construction in terms of Lusztig's dual canonical basis for any flag variety. Kaveh [Kav15] then shows that the dual canonical basis construction can be recovered from a *Parshin point* (see 8.2) construction on a Bott-Samuelson resolution of the flag variety. There are also many constructions of valuations coming from the theory of *birational sequences*, which utilize the Lie algebra action [FFL17b], [FFL17c], and [FFL17a]. Finally, cluster algebras [GHKK18, BFF+18, RW19] provide another organizing tool for valuations. The construction we give here is distinct from these approaches, and follows [Man16] and can be derived from [KM19].

We pick a trivalent tree σ and recall that branching basis $\mathbb{B}_\sigma \subset \mathbb{C}[X]$ constructed in Section 7.2. Each member $b_\mathbf{s} \in \mathbb{B}_\sigma$ spans one of the spaces $W_\sigma(\mathbf{s})$, and there is a bijection between the members of $\mathbb{B}_\sigma$ and the elements of the semigroup S_σ. We select a total ordering $<$ on $E(\sigma)$; this induces a total ordering on S_σ and the basis $\mathbb{B}_\sigma$, which we also denote by $<$. In particular $\mathbf{s} < \mathbf{s}'$ if $-\mathbf{s}(e_i) < -\mathbf{s}'(e_i)$, where e_i is the first edge (according to $<$) where $\mathbf{s}$ and $\mathbf{s}'$ disagree. Now we define a function $\mathfrak{v}_{\sigma,<} : \mathbb{C}[X]\backslash\{0\} \to \mathbb{Z}^{E(\sigma)}$ as follows:

$$\mathfrak{v}_{\sigma,<}\Big(\sum C_\mathbf{s} b_\mathbf{s}\Big) = MIN\{\mathbf{s} \mid C_\mathbf{s} \neq 0\}, \tag{8.1}$$

where MIN is taken with respect to the ordering $<$ on S_σ. The following is essentially proved in [Man16].

Proposition 8.1 *The function* $\mathfrak{v}_{\sigma,<}$ *is a discrete valuation on* $\mathbb{C}[X]$ *of rank* $2n-3$ *adapted to* $\mathbb{B}_\sigma$ *with value semigroup* S_σ *and Newton–Okounkov cone* P_σ.

Proof Let $F_\mathbf{s}^{\sigma,<} = \bigoplus_{\mathbf{s}\leq\mathbf{s}'} W_\sigma(\mathbf{s}')$, then $F_\mathbf{s}^{\sigma,<} = \{f \mid \mathfrak{v}_{\sigma,<}(f) \geq \mathbf{s}\}$ by definition; this shows that $\mathfrak{v}_{\sigma,<}$ is adapted to $\mathbb{B}_\sigma$. Furthermore, for any $\mathbf{s}, \mathbf{s}'$ the product $F_\mathbf{s}^{\sigma,<} F_{\mathbf{s}'}^{\sigma,<}$ is a subspace of $F_{\mathbf{s}+\mathbf{s}'}^{\sigma,<}$ by Proposition 5.2. This implies that $\mathfrak{v}_{\sigma,<}$ is a quasi-valuation with value set equal to S_σ. To show that it is actually a valuation, we observe that part (4) of Proposition 5.2 implies that $\mathfrak{v}_{\sigma,<}(b_\mathbf{s} b_{\mathbf{s}'}) = \mathbf{s} + \mathbf{s}'$. This, in turn, implies that

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$\mathfrak{v}_{\sigma,<}(fg) = \mathfrak{v}_{\sigma,<}(f) + \mathfrak{v}_{\sigma,<}(g)$, as $\mathfrak{v}_{\sigma,<}$ will only see the values of the top components of f and g according to the ordering $<$. ■

It is also possible to show that there is a rank $|E(\sigma)|$ valuation $\mathfrak{v}_{\sigma,<} : \mathbb{C}[X]\backslash\{0\} \to \mathbb{Z}^{E(\sigma)}$ for any nontrivalent σ. In fact, one can apply [KM19, Theorem 4] to the integral generators of the extremal rays of $ev_n(C_\sigma) \subset \mathrm{Trop}(I_{2,n})$ to recover any such valuation. For this construction, one needs to compute the values $v_{e_i}(p_{ij})$ for each edge $e_i \in E(\sigma)$ and each member of the Khovanskii basis of Plücker generators $p_{ij} \in \mathbb{C}[X]$. This makes a $(2n-3) \times \binom{n}{2}$ matrix $M_{\sigma,<}$ which captures all the information of $\mathfrak{v}_{\sigma,<}$.

Proposition 8.1 shows that the branching basis $\mathbb{B}_\sigma$ is adapted to the valuation $\mathfrak{v}_{\sigma,<}$. Now, we describe a different basis which is also adapted to $\mathfrak{v}_{\sigma,<}$. Lemma 7.5 shows that the Plücker generators p_{ij} give a Khovanskii basis for $\mathfrak{v}_{\sigma,<}$, so a basis of standard monomials in the p_{ij} will be adapted to $\mathfrak{v}_{\sigma,<}$ as well. We say that σ is a planar tree if the cyclic ordering on the leaves of σ give an embedding of σ into the plane. In the proof of Lemma 7.5 we can choose to decompose a weighting of σ in a planar way, in particular, it is always possible to construct the paths in a noncrossing way. Furthermore, this reconstruction process shows that any two distinct planar arrangements give distinct weightings of σ. Let $\mathbb{B}_+$ be the set of monomials p^α in the Plücker generators such that for any i, j, k, ℓ in cyclic order $\alpha_{ik}\alpha_{j\ell} = 0$; such monomials correspond to planar graphs on $[n]$ [HMSV09]. Our remarks imply the following proposition.

Proposition 8.2 *The set $\mathbb{B}_+$ is an adapted basis of $\mathfrak{v}_{\sigma,<}$ for any planar σ. Furthermore, $\mathbb{B}_+$ and $\mathbb{B}_\sigma$ are related by upper-triangular transformations with respect to the ordering on S_σ induced by $<$.*

8.2 A Parshin point construction of $\mathfrak{v}_{\sigma,<}$

In order to make a connection with the theory of Newton–Okounkov bodies, we present $\mathfrak{v}_{\sigma,<}$ as a so-called Parshin point valuation (see [Kav15, KK12, LM09]). Roughly speaking, a Parshin point provides a higher-rank generalization for the construction of a discrete valuation from a prime divisor on a normal variety. Instead of taking degree along one height 1 prime, one takes successive degrees along a flag of subvarieties.

Definition 8.1 (Parshin point valuation) Let $p \in Y$ be a point in a variety of dimension $dim(Y) = n$, and $V_1 \supset \cdots \supset V_n = \{p\}$ be a flag of irreducible subvarieties. We further assume that V_i is locally cut out of V_{i-1} at p by t_i. This information defines a Parshin point, which we denote by $\mathcal{F}$. For $f \in \mathbb{C}(Y)$ we define a valuation $\mathfrak{v}_\mathcal{F}$ as follows. Let $s_1 = ord_{t_1}(f)$, $f_1 = t_1^{-s_1} f\,|_{V_1}$ and then continue this way to get $s_i = ord_{t_i}(f_{i-1})$, $f_i = t_i^{-s_i} f_{i-1}\,|_{V_i}$. We set $\mathfrak{v}_\mathcal{F}(f) = (s_1, \ldots, s_n)$.

Let σ be a trivalent tree, and let $<$ be a total ordering on $E(\sigma)$. We use $<$ to define a flag of subvarieties on X_σ. Using the total ordering $<$, we can label the edges $E(\sigma)$: $e_1, \ldots, e_{2n-3}$. This defines a flag $D_{e_1} \supset D_{e_1,e_2} \supset \cdots \supset D_{E(\sigma)}$, where D_S is the subvariety defined in Section 6. In particular, $D_{E(\sigma)}$ is the point in X_σ defined by the maximal ideal $I_{E(\sigma)}$.

Proposition 8.3 *The Parshin point $D_{e_1} \supset D_{e_1,e_2} \supset \cdots \supset D_{E(\sigma)}$ defines a valuation $\mathfrak{w}_{\sigma,<} : \mathbb{C}[X]\backslash\{0\} \to \mathbb{Z}^{E(\sigma)}$ which coincides with $\mathfrak{v}_{\sigma,<}$.*

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Proof We start by considering the ideal $J_{E(\sigma)} \subset \mathbb{C}[\overline{M}(\sigma)]$:

$$J_{E(\sigma)} = \bigoplus_{n\geq 0} \bigoplus_{\{\mathbf{s}\in\mathbb{Z}_{\geq 0}^{E(\sigma)} \mid \forall e\in E(\sigma), \mathbf{s}(e)\leq n,\ \exists e', \mathbf{s}(e')<n\}} \Big[\bigotimes_{e\in E^{\circ}(\sigma)} V(\mathbf{s}(e))\otimes V(\mathbf{s}(e))\Big]$$

$$\otimes\Big[\bigotimes_{\ell\in L(\sigma)} V(\mathbf{s}(\ell))\Big]t^n. \tag{8.2}$$

Notice that $J_{E(\sigma)} \cap \mathbb{C}[X_\sigma] = I_{E(\sigma)}$, $J_{E(\sigma)}$ is $G(\sigma)$-fixed, and $\mathcal{O}_{J_{E(\sigma)}}^{G(\sigma)} = \mathcal{O}_{I_{E(\sigma)}}$, where these are the local rings for the corresponding points on $\overline{M}(\sigma)$ and X_σ, respectively. We work with the space $\overline{M}(\sigma)$ because it has the advantage of being smooth. Let t_e be the local equation for the prime divisor $\overline{M}(\sigma, e)$ (this element can be taken to coincide with $\bar{t}_e$ from Proposition 6.8). Note that t_e is $G(\sigma)$-fixed, so $t_e \in \mathcal{O}_{I_{E(\sigma)}}$. Furthermore, $\overline{M}(\sigma)$ is a product over $e \in E(\sigma)$, so we must have $ord_{t_e}(t_{e'}) = 0$ when $e \neq e'$. The subvarieties $D_{e_1,\dots,e_k}$ with their local equations t_{e_k} define a Parshin point of X_σ; in particular t_{e_k} locally cuts out $D_{e_1,\dots,e_k}$ in $D_{e_1,\dots,e_{k-1}}$, because this is the case for the corresponding ideals in $\mathcal{O}_{J_{E(\sigma)}}$.

We construct D_S as a GIT quotient of the space $\overline{M}(\sigma, S)$ in 6.2. We let $\overline{M}(\sigma, S)^o \subset \overline{M}(\sigma, S)$ be the subvariety obtained by the same product construction, only replacing $\overline{\mathrm{SL}}_2$ with SL_2 (respectively, $\mathbb{P}^2$ with $\mathbb{A}^2$ where appropriate) whenever e (respectively, ℓ) $\notin S$. Now we choose $b_{\mathbf{s}} \in \mathbb{C}[X]$. Proposition 6.8 shows that $ord_{t_{e_1}} = -\mathbf{s}(e_1)$. Regarding $t_{e_1}^{\mathbf{s}(e_1)} b_{\mathbf{s}}$ as a function on $\overline{M}(\sigma, e_1)^o$, we use the same argument in 6.8 to show that $ord_{t_{e_2}}(t_{e_1}^{\mathbf{s}(e_1)} b_{\mathbf{s}}) = 0 - \mathbf{s}(e_2)$, where t_{e_2} locally cuts out $\overline{M}(\sigma, e_1, e_2)$ along the boundary of $\overline{M}(\sigma, e_1)^o$. Continuing this way, we obtain the valuation $\mathfrak{w}_{\sigma,<}$, which has the property that $\mathfrak{w}_{\sigma,<}(b_{\mathbf{s}}) = \mathfrak{v}_{\sigma,<}(b_{\mathbf{s}})$ for any $b_s \in \mathbb{B}_\sigma$. Since both valuations are maximal rank, and since they take the same distinct values on the basis $\mathbb{B}_\sigma$, they must coincide (see Proposition 2.2). ∎

9 Example

Let us consider the simplest case: $n = 4$. There are three trivalent trees with four ordered leaves (Figures 4–6). The Tropical Grassmannian $\mathrm{Trop}(I_{2,4}) \subset \mathbb{R}^6$ is a fan with three five-dimensional cones $\overline{B}_{\sigma_1}, \overline{B}_{\sigma_2}, \overline{B}_{\sigma_3}$ glued along a four-dimensional lineality space, which can be constructed as the image of the map

$$\psi : \mathbb{R}^4 \to \mathbb{R}^6, (x_1, x_2, x_3, x_4) \mapsto (x_1 + x_2, x_1 + x_3, x_1 + x_4, x_2 + x_3, x_2 + x_4, x_3 + x_4) :$$

$$\overline{B}_{\sigma_1} : w_{13} + w_{24} = w_{14} + w_{23} \leq w_{12} + w_{34}, \quad \mathrm{image}(\psi) + \mathbb{R}_{\geq 0} e_{13} + e_{14} + e_{23} + e_{24}$$

$$\overline{B}_{\sigma_2} : w_{12} + w_{34} = w_{14} + w_{23} \leq w_{13} + w_{24}, \quad \mathrm{image}(\psi) + \mathbb{R}_{\geq 0} e_{12} + e_{14} + e_{23} + e_{34}$$

$$\overline{B}_{\sigma_3} : w_{12} + w_{34} = w_{13} + w_{24} \leq w_{14} + w_{23}, \quad \mathrm{image}(\psi) + \mathbb{R}_{\geq 0} e_{12} + e_{13} + e_{24} + e_{34},$$

where w_{ij} are the coordinates of $\mathbb{R}^6 = \mathbb{R}^{\binom{4}{2}}$ and e_{ij} are the standard basis of $\mathbb{R}^6 = \mathbb{R}^{\binom{4}{2}}$. Note that $\mathrm{image}(\psi)$ is spanned by $\psi(e_1) = (1,1,1,0,0,0), \psi(e_2) = (1,0,0,1,1,0), \psi(e_3) = (0,1,0,1,0,1), \psi(e_4) = (0,0,1,0,1,1)$.

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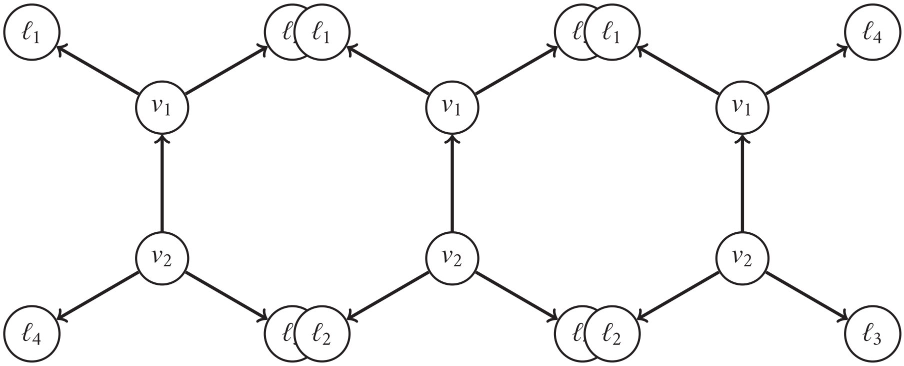

Figure 4: $\sigma_1 = (\{1,2\},\{3,4\})$. *Figure 5*: $\sigma_2 = (\{1,3\},\{2,4\})$. *Figure 6*: $\sigma_3 = (\{1,4\},\{2,3\})$.

Now, we see how to recover the tropical geometry and Newton–Okounkov information from the compactification $X \subset X_{\sigma_1} \supset D_{\sigma_1}$. From each component $D_e \subset D_{\sigma_1}$, $(e \in E(\sigma_1))$, we get a vector in $\text{Trop}(I_{2,4}) \subset \mathbb{R}^6$ using Proposition 6.8 and Proposition 7.1:

$$\begin{aligned}
D_{\ell_1} \mapsto\ & (ord_{D_{\ell_1}}(p_{12}), ord_{D_{\ell_1}}(p_{13}), ord_{D_{\ell_1}}(p_{14}), ord_{D_{\ell_1}}(p_{23}), ord_{D_{\ell_1}}(p_{24}),\\
& ord_{D_{\ell_1}}(p_{34})) = (v_{\ell_1}(p_{12}), v_{\ell_1}(p_{13}), v_{\ell_1}(p_{14}), v_{\ell_1}(p_{23}), v_{\ell_1}(p_{24}), v_{\ell_1}(p_{34}))\\
& = (1,1,1,0,0,0) = \psi(e_1)\\
D_{\ell_2} \mapsto\ & (ord_{D_{\ell_2}}(p_{12}), ord_{D_{\ell_2}}(p_{13}), ord_{D_{\ell_2}}(p_{14}), ord_{D_{\ell_2}}(p_{23}), ord_{D_{\ell_2}}(p_{24}),\\
& ord_{D_{\ell_2}}(p_{34})) = (v_{\ell_2}(p_{12}), v_{\ell_2}(p_{13}), v_{\ell_2}(p_{14}), v_{\ell_2}(p_{23}), v_{\ell_2}(p_{24}), v_{\ell_2}(p_{34}))\\
& = (1,0,0,1,1,0) = \psi(e_2)\\
D_{\ell_3} \mapsto\ & (ord_{D_{\ell_3}}(p_{12}), ord_{D_{\ell_3}}(p_{13}), ord_{D_{\ell_3}}(p_{14}), ord_{D_{\ell_3}}(p_{23}), ord_{D_{\ell_3}}(p_{24}),\\
& ord_{D_{\ell_3}}(p_{34})) = (v_{\ell_3}(p_{12}), v_{\ell_3}(p_{13}), v_{\ell_3}(p_{14}), v_{\ell_3}(p_{23}), v_{\ell_3}(p_{24}), v_{\ell_3}(p_{34}))\\
& = (0,1,0,1,0,1) = \psi(e_3)\\
D_{\ell_4} \mapsto\ & (ord_{D_{\ell_4}}(p_{12}), ord_{D_{\ell_4}}(p_{13}), ord_{D_{\ell_4}}(p_{14}), ord_{D_{\ell_4}}(p_{23}), ord_{D_{\ell_4}}(p_{24}),\\
& ord_{D_{\ell_4}}(p_{34})) = (v_{\ell_4}(p_{12}), v_{\ell_4}(p_{13}), v_{\ell_4}(p_{14}), v_{\ell_4}(p_{23}), v_{\ell_4}(p_{24}), v_{\ell_4}(p_{34}))\\
& = (0,0,1,0,1,1) = \psi(e_4)\\
D_{e^\circ} \mapsto\ & (ord_{D_{e^\circ}}(p_{12}), ord_{D_{e^\circ}}(p_{13}), ord_{D_{e^\circ}}(p_{14}), ord_{D_{e^\circ}}(p_{23}), ord_{D_{e^\circ}}(p_{24}),\\
& ord_{D_{e^\circ}}(p_{34})) = (v_{e^\circ}(p_{12}), v_{e^\circ}(p_{13}), v_{e^\circ}(p_{14}), v_{e^\circ}(p_{23}), v_{e^\circ}(p_{24}), v_{e^\circ}(p_{34}))\\
& = (0,1,1,1,1,0),
\end{aligned}$$

where e° is the nonleaf edge.

We describe them as row vectors of the following matrix, which span the maximal cone $\overline{B}_{\sigma_1}$ of $\text{Trop}(I_{2,4})$.

$$\begin{pmatrix} ord_{D_{\ell_1}} \\ ord_{D_{\ell_2}} \\ ord_{D_{\ell_3}} \\ ord_{D_{\ell_4}} \\ ord_{D_{e^\circ}} \end{pmatrix} = \begin{array}{c} \begin{matrix} p_{12} & p_{13} & p_{14} & p_{23} & p_{24} & p_{34} \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

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For Newton–Okounkov information, we fix a total order $<$ on $E(\sigma)$, for example, $\ell_1 > \ell_2 > \ell_3 > \ell_4 > e^\circ$, the order we used for the tropical geometry above. It corresponds to the flag of subvarieties (Parshin point), $D_{\ell_1} \supset D_{\ell_1,\ell_2} \supset \cdots \supset D_{E(\sigma)}$ and the valuation $\mathfrak{v}_{\sigma,<}$ (see Proposition 8.3). Now we compute the values of Plücker generators under this valuation:

$\mathfrak{v}_{\sigma,<}(p_{12}) = (1,1,0,0,0)$.
$\mathfrak{v}_{\sigma,<}(p_{13}) = (1,0,1,0,1)$.
$\mathfrak{v}_{\sigma,<}(p_{14}) = (1,0,0,1,1)$.
$\mathfrak{v}_{\sigma,<}(p_{23}) = (0,1,1,0,1)$.
$\mathfrak{v}_{\sigma,<}(p_{24}) = (0,1,0,1,1)$.
$\mathfrak{v}_{\sigma,<}(p_{34}) = (0,0,1,1,0)$.

These coincide with the column vectors of the matrix above, which generate the semigroup S_σ. Thus, from the perspective of the compactification X_σ, we have a unified understanding of the tropical geometry and Newton–Okounkov theory for the affine cone X of the Grassmannian $Gr_2(\mathbb{C}^4)$.

Moreover, these generators of S_σ correspond to the parametrization of the toric variety defined by $\mathbb{C}[S_\sigma]$:

$$(\mathbb{C}^*)^5 \to \mathbb{C}^6, (t_1,t_2,t_3,t_4,t_5) \mapsto (t_1t_2, t_1t_3t_5, t_1t_4t_5, t_2t_3t_5, t_2t_4t_5, t_3t_4).$$

The ideal presenting $\mathbb{C}[S_\sigma]$ is equal to the kernel of the homomorphism,

$$\mathbb{C}[p_{12},p_{13},p_{14},p_{23},p_{24},p_{34}] \to \mathbb{C}[y_{e_1},y_{e_2},y_{e_3},y_{e_4},y_{e_5}],$$

$$p_{12} \mapsto y_{e_1}y_{e_2}, p_{13} \mapsto y_{e_1}y_{e_3}y_{e_5}, \ldots, p_{34} \mapsto y_{e_3}y_{e_4},$$

which is the the principal ideal generated by $p_{13}p_{24} - p_{14}p_{23}$. Now, this ideal is equal to the initial ideal $in_\sigma(I_{2,4})$ associated to the cone B_σ in $\mathrm{Trop}(I_{2,4})$.

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