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PRESERVATION OF PROPERTIES UNDER MIXTURE

Henry W. Block, Yulin Li, and Thomas H. Savits
Department of Statistics
University of Pittsburgh
Pittsburgh, PA 15260
E-mail: {hwb, yulst12, tsavits}@stat.pitt.edu

In general, finite mixtures of distributions with increasing failure rates are not increasing. However, conditions have been given by Lynch [8] so that a mixture of distributions with increasing failure rates has increasing failure rate. We establish similar results for other standard classes and also give examples which show that although the assumptions are stringent, continuous mixtures of standard families of lifetime distributions do have increasing failure rates. We also show that the result of Lynch follows from Savits [12] and the techniques of the last-cited paper can be applied to other classes as well.

1. INTRODUCTION

It is well known that mixtures of distributions with decreasing failure rates have decreasing failure rates. However, if the distributions which are mixed have failure rates which do not decrease, many different types of behavior are possible. In Block, Mi, and Savits [4], a general result was given concerning the asymptotic behavior of a mixture. Under certain conditions, it was shown that the limit of the failure rate of a mixture was the same as the limit of the failure rate of the strongest mixed population (we often refer to these as subpopulations or components of the mixture). Block and Joe [2] extended this type of result to the asymptotic monotonicity of a mixture. In particular, if the strongest component has a failure rate which is eventually increasing, then the mixture failure rate has a similar property. See Block, Li, and Savits [3] for refinements of the previously cited articles as well as many examples. The overall behavior of the mixture has recently been studied by Block, Savits, and Wondmagegnehu [7]. These authors have determined that even for the

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mixture of distributions with increasing linear failure rates, many behaviors are possible. In fact, the mixture of two distributions with increasing linear failure rates can have four changes of monotonicity. Consequently, it is surprising that Lynch [8] obtained a closure theorem for mixtures of distributions with increasing failure rates.

In this article, we show several things. First, we discuss the conditions assumed in Lynch [8] and show that his result is a special case of Savits [12] with the correct interpretation. Next, we explain why the Lynch result does not apply to finite mixtures. We also show that similar closure theorems are possible for other nonparametric classes, including distributions that are increasing failure rate average (IFRA), new better than used (NBU), and decreasing mean residual life (DMRL). Bassan and Spizzichino [1] also are working on the DMRL case and may have a result similar to ours. We give examples to show that these results apply to several well-known distributions under continuous mixing.

We generally use the terms increasing and decreasing instead of the nondecreasing and nonincreasing, respectively. For ease of exposition, we also assume that our lifetime distributions have no mass at zero.

2. LYNCH'S RESULT

In Lynch [8], a result is given for mixtures of distributions with increasing failure rates (IFR). The result can be restated as follows. Let $\{\bar{F}(t|\theta) : \theta \geq 0\}$ be a family of survival functions of lifetime distributions with univariate parameter $\theta \geq 0$. Let M be the mixing distribution function on $[0,\infty)$. The mixture survival function has the form

$$\bar{F}_M(t) = \int \bar{F}(t|\theta)\, dM(\theta).$$

The main result of Lynch [8] is that if M has an IFR distribution and if $\bar{F}(t|\theta)$ is log concave in the variables (t,θ) (i.e., $\ln \bar{F}(t|\theta)$ is concave in (t,θ)) and is increasing in θ for each $t \geq 0$, then $F_M(t)$ is IFR. A converse result is also given. The bivariate condition on $\bar{F}(t|\theta)$ is a type of multivariate IFR condition, since, marginally, $\bar{F}(t|\theta)$ is log concave in t for each θ (which is univariate IFR) and, similarly, log concave in θ for each $t \geq 0$. This condition was studied by Savits [12] in a multivariate context. The main result of [12] is Theorem 3.4, which we state.

THEOREM 2.1: *The lifetime Θ is IFR if and only if $E[h(t,\Theta)]$ is log concave in t for all functions $h(t,\theta)$ which are log concave in (t,θ) and are increasing in θ for each $t \geq 0$.*

In the context of the present article, let Θ be the mixing variable and assume that it is IFR and also that $\bar{F}(t|\theta)$ is log concave in (t,θ) and increasing in θ. As a corollary to Theorem 2.1, it follows that $\bar{F}_M(t)$ is IFR.

It should be observed that in addition to the bivariate IFR assumption, the assumption that the mixing distribution is IFR precludes the application of this result to finite or even discrete mixtures since distributions with flat parts are not IFR.

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3. CLOSURE RESULTS

We now establish some results similar to the Lynch result for other reliability classes.

3.1 IFRA

We give a direct proof of this result, but as in the IFR case, the result also follows from a comment in Savits [12]. The details of this are given in a note following the theorem.

In Block and Savits [5], the following lemma is used to show that the convolution of IFRA distributions is IFRA.

LEMMA 3.1 (Block and Savits [5]): *F is IFRA iff*

$$\int h(t)\, dF(t) \le \left\{ \int h^{\alpha}(t/\alpha)\, dF(t) \right\}^{1/\alpha}$$

for all $0 < \alpha < 1$ *and all nonnegative increasing functions h.*

We now state the IFRA result.

THEOREM 3.1: *Assume* $\bar{F}(t|\theta)$ *is increasing in* θ *for each* $t \ge 0$ *and satisfies*

$$\bar{F}(\alpha t|\alpha\theta) \ge \bar{F}^{\alpha}(t|\theta) \tag{3.1}$$

for all $0 < \alpha < 1$ *and for all* $t \ge 0$ *and* $\theta \ge 0$. *Also assume that the mixing distribution* $M(\theta)$ *is an IFRA distribution. Then,* F_M *is IFRA. Conversely, if* F_M *is IFRA whenever* $\bar{F}(t|\theta)$ *satisfies the above two conditions, then M is IFRA.*

PROOF: The converse part follows by taking $\bar{F}(t|\theta) = I(t < \theta)$. The direct part of the proof follows because for $0 < \alpha < 1$,

$$\begin{aligned} \bar{F}_M(\alpha t) &= \int \bar{F}(\alpha t|\theta)\, dM(\theta) \\ &\ge \int \bar{F}^{\alpha}(t|\theta/\alpha)\, dM(\theta) \\ &\ge \left[\int \bar{F}(t|\theta)\, dM(\theta) \right]^{\alpha} \\ &= \bar{F}_M^{\alpha}(t), \end{aligned}$$

where in the first inequality we made use of the inequality in the statement of the theorem, and in the second inequality we used Lemma 3.1 applied to *M*. ■

Note 3.1:

1. From inequality (3.1) and the monotonicity of $\bar{F}$ in θ, we obtain

$$\bar{F}(\alpha t|\theta) \ge \bar{F}(\alpha t|\alpha\theta) \ge \bar{F}^{\alpha}(t|\theta);$$

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that is, $\bar{F}(t|\theta)$ is IFRA for each $\theta \geq 0$.

2. Inequality (3.1) can be written as

$$\log \bar{F}(\alpha t|\alpha\theta) \geq \alpha \log \bar{F}(t|\theta), \qquad 0 < \alpha < 1,$$

a condition which is called log subhomogeneous in Savits [12]. Theorem 3.1 can be shown to follow from the last paragraph of Savits [12] and the results of Block and Savits [6]. Also see Savits [11].

3. It is easy to see that if $\bar{F}(t|\theta)$ is log concave in (t,θ), then it is log subhomogeneous in (t,θ).

3.2 NBU

An NBU closure result can also be given. The proof requires the following lemma.

Lemma 3.2: *F is NBU iff*

$$\int g(\alpha t)h[(1-\alpha)t]\,dF(t) \leq \int g(t)\,dF(t) \int h(t)\,dF(t)$$

for all nonnegative, increasing functions g and h and all $0 < \alpha < 1$.

Proof: The sufficiency of the condition follows easily by letting $g(t) = 1_{(a,\infty)}(t)$ and $h(t) = 1_{(b,\infty)}(t)$ for $a, b > 0$. The inequality becomes

$$\bar{F}[\max(a/\alpha, b/(1-\alpha))] \leq \bar{F}(a)\bar{F}(b)$$

and choosing $\alpha = a/(a+b)$ yields NBU. The necessity uses a standard technique (e.g., see Marshall and Shaked [9]). First, from the NBU inequality and the fact that $(a+b) \leq \max(a/\alpha, b/(1-\alpha))$ for all $0 < \alpha < 1$, the inequality holds for indicator functions of the intervals of the type (a,∞) and (b,∞) and this can also be extended to any pair of open or closed intervals of this type. It then follows that the theorem holds for finite weighted sums of such indicator functions. Since nonnegative increasing functions are increasing limits of these finite sums, the result follows by the monotone convergence theorem. ■

Theorem 3.2: *Assume* $\bar{F}(t|\theta)$ *is increasing in* θ *for each* $t \geq 0$ *and satisfies*

$$\bar{F}(t|\theta) \leq \bar{F}(\alpha t|\alpha\theta)\bar{F}[(1-\alpha)t|(1-\alpha)\theta] \tag{3.2}$$

for all $0 < \alpha < 1$ *and all* $t, \theta \geq 0$*, and also that the mixing distribution M is NBU. Then,* F_M *is NBU. Conversely, if* F_M *is NBU whenever* $\bar{F}(t|\theta)$ *satisfies the above two conditions, then M is NBU.*

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PROOF: The converse part follows by choosing $\bar{F}(t|\theta) = I(t < \theta)$. For the direct part of the theorem, let $0 < \alpha < 1$. Then,

$$\begin{aligned}\bar{F}_M(t) &= \int \bar{F}(t|\theta)\, dM(\theta)\\ &\leq \int \bar{F}(\alpha t|\alpha\theta)\bar{F}[(1-\alpha)t|(1-\alpha)\theta]\, dM(\theta)\\ &\leq \int \bar{F}(\alpha t|\theta)\, dM(\theta) \int \bar{F}[(1-\alpha)t|\theta]\, dM(\theta)\\ &= \bar{F}_M(\alpha t)\bar{F}_M[(1-\alpha)t].\end{aligned}$$

Thus, $\bar{F}_M$ is NBU. ■

Note 3.2:

1. It is easy to show that the conditions of the theorem imply that $\bar{F}(t|\theta)$ is NBU for each $\theta \geq 0$.
2. The inequality (3.2) in the theorem is also called log subadditivity.

3.3 DMRL

A similar result holds for DMRL. The usual definition that a lifetime with survival function $\bar{F}$ is DMRL is that the quantity $\int_t^\infty \bar{F}(u)\, du/\bar{F}(t)$ is decreasing in t. It is not hard to show that this is equivalent to $\int_t^\infty \bar{F}(u)\, du$ being log concave in t. This follows since the right-hand derivative of the log of the second quantity is equal to the negative of the reciprocal of the first quantity and the right-hand derivative of a concave function is a decreasing function.

THEOREM 3.3: *Assume $\bar{F}(t|\theta), \theta \geq 0$, is increasing in θ for each $t \geq 0$, $\int_t^\infty \bar{F}(u|\theta)\, du$ is log concave in (t,θ), and $M(\theta)$ is IFR. Then, F_M is DMRL. Conversely, if F_M is DMRL whenever $\bar{F}(t|\theta)$ is increasing in θ for each $t \geq 0$ and $\int_t^\infty \bar{F}(u|\theta)\, du$ is log concave in (t,θ), then M is DMRL.*

PROOF: The converse part follows by taking $\bar{F}(t|\theta) = I(t < \theta)$. The direct part of the proof follows from Theorem 3.4 of Savits [12]. In fact, from that theorem we know that

$$\begin{aligned}\int_t^\infty \bar{F}_M(u)\, du &= \int_t^\infty \left[\int_0^\infty \bar{F}(u|\theta)\, dM(\theta)\right] du\\ &= \int_0^\infty \left[\int_t^\infty \bar{F}(u|\theta)\, du\right] dM(\theta)\end{aligned}$$

is log concave in t. Hence, F_M is DMRL. ■

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Note 3.3:

1. The condition that $\int_t^\infty \bar{F}(u|\theta)\,du$ is log concave in (t,θ) implies that it is log concave in t for each $\theta \geq 0$. Thus, $\bar{F}(t|\theta)$ is DMRL for each $\theta \geq 0$.
2. It follows from Prekopa [10] that if $\bar{F}(t|\theta)$ is log concave in (t,θ), then $\int_t^\infty \bar{F}(u|\theta)\,du$ is log concave in (t,θ) provided the integral exists.

4. EXAMPLES

The following examples show that the conditions on the mixed distributions for the various closure theorems are reasonable and are satisfied by many standard distributions. It is enough to show that these satisfy the bivariate log concavity condition. The monotonicity condition is obvious.

Example 4.1 (Weibull Distribution): The Weibull distribution is given by

$$\bar{F}(t) = \exp(-\lambda t^\alpha),$$

where $\alpha, \lambda > 0$. When $\alpha > 1$, it is IFR. Reparameterize it as

$$\bar{F}(t|\theta) = \exp(-t^\alpha/\theta^{\alpha-1}),$$

where $\theta > 0$ and $\alpha > 1$. Let $H(t,\theta) = -\ln \bar{F}(t|\theta) = t^\alpha/\theta^{\alpha-1}$. It can then be shown that H is a convex function of (t,θ). Therefore, $\bar{F}$ is log concave in (t,θ).

Example 4.2 (Exponential Power Distribution): The exponential power distribution is given by

$$\bar{F}(t) = \exp\{1 - \exp[(\lambda t)^\alpha]\},$$

where $\alpha, \lambda > 0$. When $\alpha > 1$, this is IFR. Reparameterize it as

$$\bar{F}(t|\theta) = \exp\{1 - \exp(t^\alpha/\theta^{\alpha-1})\},$$

where $\alpha > 1$ and $\theta > 0$. Set $H(t,\theta) = -\ln \bar{F}(t|\theta) = \exp(t^\alpha/\theta^{\alpha-1}) - 1$. Since $t^\alpha/\theta^{\alpha-1}$ is convex in (t,θ) and the exponential function is increasing and convex, it follows that $H(t|\theta)$ must be convex. Therefore, $\bar{F}$ is log concave in (t,θ).

Example 4.3 (Gompertz Distribution): The Gompertz distribution is given by

$$\bar{F}(t) = \exp\left\{\frac{\beta}{\alpha}[1 - \exp(\alpha t)]\right\},$$

where $\alpha, \beta > 0$. This distribution is IFR. We reparameterize it as

$$\bar{F}(t|\theta) = \exp\{\theta\beta[1 - \exp(t/\theta)]\},$$

where $\theta, \beta > 0$. Set $H(t,\theta) = -\ln \bar{F}(t|\theta) = \theta\beta[\exp(t/\theta) - 1]$. Since the Hessian matrix

$$D^2H = \frac{\beta \exp(t/\theta)}{\theta^3}\begin{pmatrix} \theta^2 & -\theta t \\ -\theta t & t^2 \end{pmatrix}$$

is nonnegative definite, H is convex in (t,θ) and so $\bar{F}(t|\theta)$ is log concave in (t,θ).

Example 4.4 (Generalized Gamma Distribution): The density of the generalized gamma distribution is

$$f(t) = \alpha\lambda^{\beta}t^{\alpha\beta-1}\exp(-\lambda t^{\alpha})/\Gamma(\beta),$$

where $\alpha, \beta, \lambda > 0$. When $\alpha > 1$ and $\beta > 1$, this is IFR. We reparameterize it as

$$f(t|\theta) = \frac{\alpha t^{\alpha\beta-1}}{\theta^{(\alpha-1)\beta}}\exp\left(-\frac{t^{\alpha}}{\theta^{\alpha-1}}\right)\Big/\Gamma(\beta),$$

where $\alpha > 1$, $\beta > 1$, and $\theta > 0$. Denote its survival function by $\bar{F}(t|\theta)$ and let $\bar{G}(t)$ be the survival function of the IFR gamma distribution with density

$$g(t) = t^{\beta-1}\exp(-t)/\Gamma(\beta)$$

for $\beta > 1$. It follows that $\bar{F}(t|\theta) = \bar{G}(t^{\alpha}/\theta^{\alpha-1})$. Since $-\ln \bar{G}(t)$ is convex and increasing, and $t^{\alpha}/\theta^{\alpha-1}$ is convex in (t,θ), it follows that $\bar{F}(t|\theta)$ is log concave in (t,θ) when $\alpha > 1$ and $\beta > 1$.

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