Euro. Jnl of Applied Mathematics (2013), *vol.* 24, *pp.* 789–801. © Cambridge University Press 2013 789
doi:10.1017/S095679251300017X

Conservation laws for plane steady potential barotropic flow

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(*Received 24 March 2011; revised 3 May 2013; accepted 8 May 2013; first published online 3 June 2013*)

It is shown that the set of conservation laws for the nonlinear system of equations describing plane steady potential barotropic flow of gas is given by the set of conservation laws for the linear Chaplygin system. All the conservation laws of zero order for the Chaplygin system are found. These include both known and new nonlinear conservation laws. It is found that the number of conservation laws of the first order is not more than three, assuming that the laws do not depend on the velocity potential and are not non-obvious ones. The components of these conservation laws are quadratic with respect to the stream function and its derivatives. All the Chaplygin functions are found, for which the Chaplygin system has three non-obvious conservation laws of the first order that are independent of velocity potential. All such non-obvious first-order conservation laws are found.

Key words: Conservation laws; Steady barotropic flow; Chaplygin equation; Chaplygin system

1 Introduction

This paper is devoted to the calculation of conservation laws for the system of differential equations describing the plane steady potential barotropic (isentropic) flow of gas. The conservation laws are powerful tools for the study of nonlinear systems of differential equations and can be applied for the construction of weak solutions. Some additional conservation laws for steady potential flow were found early in the work of Loewner [7], where he identified some useful conservation laws for more general system (also see Morawetz [9]). The importance of entropy functions (i.e. where additional conservation laws hold) in constructing weak solutions for transonic flow problems has been recognised in some recent papers, such as Chen *et al.* [1].

We use the following basic concepts of the theory of conservation laws [2,10,14].

Let (S) be an arbitrary system of differential equations for $m \geqslant 1$ unknown functions $\mathbf{u} = (u_1, u_2, \ldots, u_m)$ of $n \geqslant 2$ independent variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Denote by $[S]$ the submanifold of the prolongation manifold [10,14], defined by the equations of system (S) and all its differential prolongations.

A conservation law for system (S) is a vector $\mathbf{A} = \mathbf{A}(\mathbf{x}, \mathbf{u}, \underset{1}{\mathbf{u}}, \underset{2}{\mathbf{u}}, \ldots) = (A_1, A_2, \ldots, A_n)$ such that

$$(\mathbf{D} \cdot \mathbf{A})_{[S]} = 0,$$

where $\mathbf{D} = (D_1, D_2, \ldots, D_n)$ and $D_j = D_{x_j}$ $(j = 1, 2, \ldots, n)$ – an operator of total differentiation with respect to the variable x_j. The highest order for the derivatives of the dependent functions $u_1, u_2, \ldots, u_m$ with respect to the independent variables $x_1, x_2, \ldots, x_n$, appearing in the expression $\mathbf{A}$, is called the order of the conservation law.

Any conservation law is determined within a constant factor. Any conservation law $\mathbf{A}$ is called trivial [10] if

$$(\mathbf{D} \cdot \mathbf{A})_{[S]} = \mathbf{D} \cdot \left(\mathbf{A}|_{[S]}\right) = 0.$$

Two conservation laws are equivalent if their linear combination is a trivial conservation law.

The set of conservation laws for system (S) is divided into non-intersecting classes of equivalent conservation laws. The conservation law can be trivial for two reasons [10]. The triviality of the first type consists of $\mathbf{A}|_{[S]} = 0$. This triviality is easy to eliminate if we consider only a projection of vector $\mathbf{A}$ on manifold $[S]$. The second possible type of triviality arises when the condition on divergence

$$\mathbf{D} \cdot \mathbf{A} \equiv 0$$

is valid for all functions $u_1, u_2, \ldots, u_m$ of variables $x_1, x_2, \ldots, x_n$ irrespective of whether or not these functions are solutions of the given system of differential equations. A description of all conservation laws of the second type is presented, for example, in [10, Theorem 4.24].

Generally speaking, by definition the trivial conservation law is a linear combination of trivial conservation laws of two specified types.

The set of conservation laws for system (S) is divided into non-intersecting classes of equivalent conservation laws.

A classification of conservation laws is of interest only within equivalence so that the conservation law will be understood as a class of equivalent conservation laws.

If system (S) is the linear system of differential equations, i.e. has a form

$$\mathbf{L}[\mathbf{u}] = 0, \tag{1.1}$$

where $\mathbf{L}$ is a linear operator then using the operator Green's formula we obtain

$$\mathbf{v} \cdot \mathbf{L}[\mathbf{u}] - \mathbf{u} \cdot \mathbf{L}^*[\mathbf{v}] = \mathbf{D} \cdot \mathbf{A}, \tag{1.2}$$

where $\mathbf{L}^*$ is a conjugate operator. It follows that any linear system (1.1) has the conservation law $\mathbf{A} = \mathbf{A}(\mathbf{u}, \mathbf{v}, \ldots)$, determined by the operator Green's formula (1.2), where $\mathbf{v} = \mathbf{v}(\mathbf{x})$ is any solution of the conjugate system of equations

$$\mathbf{L}^*[\mathbf{v}] = 0.$$

This functional arbitrariness of the conservation law, given by the function **v**, is a consequence of the linearity of the system of equations (1.1).

A conservation law for the linear system of equations (1.1) that is a linear combination of trivial conservation laws and conservation laws generated by the operator Green's formula (1.2) is called an obvious conservation law [2, 3], otherwise it is non-obvious conservation law.

2 Basic equations and relations

The plane steady potential barotropic flow of gas is described by the following equations [8, 13]:

$$\begin{aligned} &uu_x + vu_y + \tfrac{c^2}{\rho}\rho_x = 0, \quad uv_x + vv_y + \tfrac{c^2}{\rho}\rho_y = 0, \\ &u\rho_x + v\rho_y + \rho\left(u_x + v_y\right) = 0, \quad v_x - u_y = 0, \end{aligned} \tag{2.1}$$

where $\mathbf{x} = (x, y) \in R^2$, $\mathbf{u} = \mathbf{u}(\mathbf{x}) = (u, v)$ is the velocity, $\rho = \rho(\mathbf{x})$ is the density, $c^2 = \frac{dp}{d\rho}$, $c = c(\rho) > 0$ is the speed of sound, and $p = p(\rho)$ is the pressure.

Introduction of a velocity potential $\phi = \phi(\mathbf{x})$ and a stream function $\psi = \psi(\mathbf{x})$ by using formulas

$$\phi_x = u, \quad \phi_y = v, \quad \psi_x = -\rho v, \quad \psi_x = \rho u \tag{2.2}$$

allows us to write system (2.1) in terms of new variables ζ and σ

$$\zeta = \int \frac{\rho}{q} dq, \quad \sigma = \operatorname{arctg}\frac{v}{u}, \quad q = |u|. \tag{2.3}$$

This hodograph transformation recasts the initial system (2.1) into the form of the Chaplygin system [4, 12, 13]

$$\phi_\sigma = \psi_\zeta, \quad \phi_\zeta = -K(\zeta)\psi_\sigma \tag{2.4}$$

with the Chaplygin function

$$K(\zeta) = \frac{1 - M^2}{\rho^2}, \quad \left(K_\zeta(\zeta) > 0\right),$$

where $M = q/c$ is Mach number.

Elimination of velocity potential from equations (2.4) leads to the Chaplygin equation [4, 11, 13] for the stream function ψ

$$\psi_{\zeta\zeta} + K(\zeta)\psi_{\sigma\sigma} = 0. \tag{2.5}$$

We now find conservation laws for systems of differential equations (2.1), (2.4), and (2.5).

3 Transformation of conservation laws

Let us consider a correspondence for the conservation laws of the initial system (2.1) and the conservation laws of the Chaplygin system (2.4).

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The conservation laws for the system (2.1) are given by a relation

$$(\mathbf{D}\cdot\mathbf{B})_{[(2.1)]}=0, \tag{3.1}$$

here $\mathbf{B}=\mathbf{B}(\mathbf{x},\mathbf{u},\rho,\underset{1}{\mathbf{u}},\underset{1}{\rho},\underset{2}{\mathbf{u}},\underset{2}{\rho},\dots)=(B_1,B_2)$, $\mathbf{D}=\left(D_x,D_y\right)$.

The conservation laws for the Chaplygin system (2.4) are given by

$$(\mathbf{D}\cdot\mathbf{A})_{[(2.4)]}=0, \tag{3.2}$$

here $\mathbf{A}=\mathbf{A}(\zeta,\sigma,\psi,\phi,\underset{1}{\psi},\underset{1}{\phi},\underset{2}{\psi},\underset{2}{\phi},\dots)=(A_1,A_2)$, $\mathbf{D}=\left(D_\zeta,D_\sigma\right)$.

Theorem 3.1 *Let the Jacobian* $J=\frac{\partial(u,v)}{\partial(x,y)}\neq 0$. *Then any conservation law for the nonlinear system of equations (2.1) is one of the form* $\mathbf{B}=(B_1,B_2)$, *defined by*

$$\begin{aligned} B_1&=-\tfrac{1}{|\mathbf{u}|^2}\left(\left(u_yv-uv_y\right)A_1+\rho\left(uu_y+vv_y\right)A_2\right),\\ B_2&=\tfrac{1}{|\mathbf{u}|^2}\left(\left(u_xv-uv_x\right)A_1+\rho\left(uu_x+vv_x\right)A_2\right), \end{aligned} \tag{3.3}$$

where $\mathbf{A}=(A_1,A_2)$ *is the conservation law for the linear Chaplygin system (2.4). Conversely, if* $\mathbf{B}=(B_1,B_2)$ *is the conservation law for the nonlinear system (2.1), then* $\mathbf{A}=(A_1,A_2)$ *is the conservation law for the linear Chaplygin system (2.4), where*

$$A_1=\frac{q}{J}\left(q_xB_1+q_yB_2\right),\quad A_2=\frac{q^2}{\rho J}\left(\sigma_xB_1+\sigma_yB_2\right). \tag{3.4}$$

Proof. By direct calculations it is established that for any two functions B_1, B_2 the following identity holds:

$$\begin{aligned} D_xB_1+D_yB_2=&\frac{\rho J}{|\mathbf{u}|^2}D_\zeta\left(\frac{1}{J}\left(\left(uu_x+vv_x\right)B_1+\left(uu_y+vv_y\right)B_2\right)\right)\\ &+\frac{\rho J}{|\mathbf{u}|^2}D_\sigma\left(\frac{1}{\rho J}\left(-\left(u_xv-uv_x\right)B_1+\left(u_yv-uv_y\right)B_2\right)\right). \end{aligned}$$

The theorem follows.

This theorem allows us to reduce the problem of finding conservation laws for the nonlinear system of equations (2.1), describing the plane steady potential barotropic flow of gas, to the problem of finding conservation laws for the linear Chaplygin system (2.4). It should also be noted that the set of conservation laws for the Chaplygin system (2.4), which do not depend on the velocity potential ϕ, coincides with the set of the conservation laws for the Chaplygin equation (2.5).

4 Conservation laws of zero order

The conservation laws of zero order for the Chaplygin system (2.4) have the form of (3.2), in which

$$\mathbf{A}=\mathbf{A}(\zeta,\sigma,\psi,\phi)=(A_1,A_2). \tag{4.1}$$

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4.1 Obvious conservation laws

The Chaplygin system (2.4) has the form of (1.1) with the linear operator

$$\mathbf{L} = \begin{pmatrix} \partial_\zeta & -\partial_\sigma \\ K(\zeta)\,\partial_\sigma & \partial_\zeta \end{pmatrix}$$

and a dependent vector function $\mathbf{u} = (\psi, \phi)^T$.

In this case an adjoint operator is determined by the formula

$$\mathbf{L}^* = -\mathbf{L}^T,$$

and the operator Green's formula (1.2) takes the form

$$\mathbf{v} \cdot \mathbf{L}[\mathbf{u}] - \mathbf{u} \cdot \mathbf{L}^*[\mathbf{v}] = D_\zeta(v_1\psi + v_2\phi) + D_\sigma(-v_1\phi + K(\zeta)\,v_2\psi), \tag{4.2}$$

where $\mathbf{v} = (v_1, v_2)^T$ is any solution of the adjoint system of equations

$$\mathbf{L}^*[\mathbf{v}] = 0,$$

or, because of the similar structure of operators $\mathbf{L}$ and $\mathbf{L}^*$, the vector-function $\mathbf{w} = (v_2, v_1)^T$ is a corresponding solution of the Chaplygin system

$$\mathbf{L}[\mathbf{w}] = 0. \tag{4.3}$$

The Green's formula (4.2) generates an infinite number of obvious conservation laws of zero order for the Chaplygin system (2.4), determined by the vector $\mathbf{A} = (A_1, A_2)$ with components

$$A_1 = v_1\psi + v_2\phi, \quad A_2 = -v_1\phi + K(\zeta)\,v_2\psi. \tag{4.4}$$

It should be noted that by virtue of (3.3) and (3.4) the conservation laws for system (2.1) up to a trivial conservation law, obtained in [15], are generated by the first-order conservation laws (4.4). In fact, the conservation laws for system (2.1) of [15] are

$$B_1 = f\rho u + gv, \quad B_2 = f\rho v - gu, \tag{4.5}$$

where $\mathbf{w} = (g, f)^T$ is an arbitrary solution of the Chaplygin system (4.3).

For

$$v_1 = -f_\sigma, \quad v_2 = -g_\sigma,$$

the conservation laws for the Chaplygin system (2.4), generated by the conservation laws (4.5) and with the help of (3.4), have the form

$$\begin{aligned} A_1 &= D_\sigma(f\psi + g\phi) + (v_1\psi + v_2\phi), \\ A_2 &= -D_\zeta(f\psi + g\phi) + (-v_1\phi + K(\zeta)\,v_2\psi) \end{aligned}$$

and are the sum of a trivial conservation law and the conservation law (4.4).

4.2 Non-obvious conservation laws

Finding non-obvious conservation laws of zero order for the Chaplygin system (2.4) corresponds to solving (3.2) with (4.1).

Splitting (3.2) for parametric derivatives leads to a system of determining equations

$$\partial_\psi A_1 + \partial_\phi A_2 = 0, \quad \partial_\psi A_2 - K(\zeta)\,\partial_\phi A_1 = 0, \quad \partial_\zeta A_1 + \partial_\sigma A_2 = 0. \tag{4.6}$$

To investigate the consistency of this over-determined system it is necessary to prolong or differentiate it [10].

After the first prolongation, the following equations are added

$$\partial_\psi^2 A_1 + K(\zeta)\,\partial_\phi^2 A_1 = 0, \quad \partial_\psi\partial_\sigma A_1 - \partial_\phi\partial_\zeta A_1 = 0, \quad \partial_\psi\partial_\zeta A_1 + K(\zeta)\,\partial_\phi\partial_\sigma A_1 = 0. \tag{4.7}$$

After the second prolongation, the following equations are added

$$\begin{aligned} &\partial_\phi\partial_\psi\partial_\zeta A_1 + K(\zeta)\,\partial_\phi^2\partial_\sigma A_1 = 0, \quad \partial_\phi\left(\partial_\zeta^2 A_1 + K(\zeta)\,\partial_\sigma^2 A_1\right) = 0, \\ &\left(\partial_\zeta K(\zeta)\right)\partial_\phi^2 A_1 + K(\zeta)\left(\partial_\phi^2\partial_\zeta A_1 - \partial_\phi\partial_\psi\partial_\sigma A_1\right) = 0. \end{aligned} \tag{4.8}$$

As a result, the system is an involution and it is possible to find its general solution. From the last equation of system (4.8), the second equation of (4.7) and conditions $\partial_\zeta K(\zeta) > 0$ for the Chaplygin function, it follows that

$$\partial_\phi^2 A_1 = 0. \tag{4.9}$$

From this relation, the first equations of systems (4.7) and (4.8) respectively are reduced to

$$\partial_\psi^2 A_1 = 0, \quad \partial_\phi\partial_\psi\partial_\zeta A_1 = 0, \tag{4.10}$$

and the second equation of (4.6) gives

$$\partial_\phi\partial_\psi\partial_\sigma A_1 = 0. \tag{4.11}$$

The general solution of systems (4.6)–(4.11) is easily found and has the form

$$\begin{aligned} A_1 &= 2c\psi\phi + (v_1\psi + v_2\phi) + D_\sigma f, \\ A_2 &= c\left(K(\zeta)\,\psi^2 - \phi^2\right) + (-v_1\phi + K(\zeta)\,v_2\psi) - D_\zeta f, \end{aligned}$$

where c is an arbitrary constant, $f = f(\zeta, \sigma, \psi, \phi)$ is an arbitrary function and the vector function $\mathbf{w} = (v_2, v_1)^T$ is any solution of system (4.3).

Thus, the Chaplygin system (4.3) has a unique solution up to the obvious conservation laws

$$A_1 = 2\psi\phi, \quad A_2 = K(\zeta)\,\psi^2 - \phi^2. \tag{4.12}$$

The corresponding conservation law for system (2.1) is obtained from (4.12) by (3.3) and has the form

$$B_1 = -\frac{1}{|\mathbf{u}|^2}\left[2\psi\phi\left(u_y v - u v_y\right) + \rho\left(u u_y + v v_y\right)\left(K(\zeta)\,\psi^2 - \phi^2\right)\right],$$

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$$B_2 = \frac{1}{|\mathbf{u}|^2}\left[2\psi\phi\left(u_x v - u v_x\right) + \rho\left(u u_x + v v_x\right)\left(K\left(\zeta\right)\psi^2 - \phi^2\right)\right].$$

For system (2.1) this is the non-local conservation law of the first order with two non-local variables ψ and ϕ.

5 Conservation laws of the first order

The conservation laws of the first order for the Chaplygin system (2.4) is given by (3.2), in which

$$\mathbf{A} = \mathbf{A}\left(\zeta, \sigma, \psi, \phi, \psi_\zeta, \psi_\sigma\right) = \left(A_1, A_2\right). \tag{5.1}$$

Substitution of (5.1) in (3.2) and splitting of parametric derivatives lead to a system of determining equations

$$\begin{aligned} &K\left(\zeta\right)\partial_{\psi_\zeta}A_1 - \partial_{\psi_\sigma}A_2 = 0, \quad \partial_{\psi_\sigma}A_1 + \partial_{\psi_\zeta}A_2 = 0, \\ &\partial_\zeta A_1 + \psi_\zeta\partial_\psi A_1 - K\left(\zeta\right)\psi_\sigma\partial_\phi A_1 + \partial_\sigma A_2 + \psi_\sigma\partial_\psi A_2 + \psi_\zeta\partial_\phi A_2 = 0. \end{aligned} \tag{5.2}$$

After the first continuation, the following equations are added

$$\begin{aligned} &\partial^2_{(\psi_\sigma)^2}A_1 + K\left(\zeta\right)\partial^2_{(\psi_\zeta)^2}A_1 = 0, \quad \partial_{\psi_\zeta}\Phi_1 - \partial_{\psi_\sigma}\Phi_2 + \partial_\psi A_1 + \partial_\phi A_2 = 0, \\ &\partial_{\psi_\sigma}\Phi_1 + K\left(\zeta\right)\partial_{\psi_\zeta}\Phi_2 - K\left(\zeta\right)\partial_\phi A_1 + \partial_\psi A_2 = 0, \\ &\Phi_1 = \partial_\zeta A_1 + \psi_\zeta\partial_\psi A_1 - K\left(\zeta\right)\psi_\sigma\partial_\phi A_1, \quad \Phi_2 = \partial_\sigma A_1 + \psi_\sigma\partial_\psi A_1 + \psi_\zeta\partial_\phi A_1. \end{aligned} \tag{5.3}$$

Next, we consider two special cases: (1) the conservation laws are linear in the derivatives ψ_ζ, ψ_σ; (2) the conservation laws do not depend on the potential ϕ.

5.1 Conservation laws linear in derivatives

We consider the conservation laws for the Chaplygin system that are linear with respect to the derivatives and have the following form:

$$\begin{aligned} &A_1 = f_1\left(\zeta, \sigma, \psi, \phi\right)\psi_\zeta + f_2\left(\zeta, \sigma, \psi, \phi\right)\psi_\sigma + f_3\left(\zeta, \sigma, \psi, \phi\right), \\ &A_2 = g_1\left(\zeta, \sigma, \psi, \phi\right)\psi_\zeta + g_2\left(\zeta, \sigma, \psi, \phi\right)\psi_\sigma + g_3\left(\zeta, \sigma, \psi, \phi\right). \end{aligned} \tag{5.4}$$

Substituting (5.4) in (5.2) and (5.3) and the splitting of the parametric derivative ψ_ζ, ψ_σ yield an over-determined system

$$\begin{aligned} &g_2 = K\left(\zeta\right)f_1, \quad g_1 = -f_2, \\ &\partial_\phi f_2 = \partial_\psi f_1, \quad \partial_\psi f_3 + \partial_\phi g_3 = \partial_\sigma f_2 - \partial_\zeta f_1, \\ &K\left(\zeta\right)\partial_\phi f_3 - \partial_\psi g_3 = \partial_\zeta f_2 + K\left(\zeta\right)\partial_\sigma f_1, \\ &\partial_\zeta f_3 + \partial_\sigma g_3 = 0. \end{aligned} \tag{5.5}$$

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After the first prolongation, the following equations are added

$$\begin{aligned}
&\partial_\zeta\partial_\phi g_3+\partial_\zeta^2 f_1+K(\zeta)\left(\partial_\sigma^2 f_1-\partial_\sigma\partial_\phi f_3\right),\\
&\partial_\phi^2 g_3+\partial_\zeta\partial_\phi f_1-\partial_\sigma\partial_\psi f_1+\partial_\phi\partial_\psi f_3,\\
&\partial_\phi\partial_\psi g_3+\partial_\zeta\partial_\psi f_1+K(\zeta)\left(\partial_\sigma\partial_\phi f_1-\partial_\phi^2 f_3\right)=0.
\end{aligned} \tag{5.6}$$

After the second prolongation, the following equations are added

$$\begin{aligned}
&\left(\partial_\zeta K(\zeta)\right)\partial_\phi P=0, \quad \partial_\psi^2 P=\partial_\zeta\partial_\psi P=\partial_\sigma\partial_\psi P=0,\\
&\partial_\zeta^2 P+K(\zeta)\,\partial_\sigma^2 P=0, \quad P=\partial_\phi f_3-\partial_\sigma f_1.
\end{aligned} \tag{5.7}$$

The system (5.5)–(5.7) is in involution and its general solution is easily found. Substituting this solution into (5.4) gives all the conservation laws for the Chaplygin system, linear with respect to the derivatives

$$\begin{aligned}
A_1&=-D_\sigma\left[Q+a\psi^2\int K(\zeta)\,d\zeta-\int v_1 d\sigma\right]+2a\psi\phi+(v_1\psi+v_2\phi),\\
A_2&=D_\zeta\left[Q+a\psi^2\int K(\zeta)\,d\zeta-\int v_1 d\sigma\right]+a\left(K(\zeta)\,\psi^2-\phi^2\right)+\\
&\quad+(-v_1\phi+K(\zeta)\,v_2\psi).
\end{aligned}$$

Here $Q=Q(\zeta,\sigma,\psi,\phi)$ is an arbitrary function, a is an arbitrary constant, and $\mathbf{w}=(v_2,v_1)^T$ is any solution of the Chaplygin system (4.3).

It follows that all the conservation laws of the form (5.4) for the Chaplygin system (2.4) are given by zero-order conservation laws, defined by (4.4) and (4.12), up to trivial conservation laws of the first order.

5.2 Conservation laws independent on potential

We now find the conservation laws of the first order for the Chaplygin system which do not depend on the velocity potential ϕ. We consider the conservation laws of the form

$$\mathbf{A}=\mathbf{A}\left(\zeta,\sigma,\psi,\psi_\zeta,\psi_\sigma\right)=(A_1,A_2). \tag{5.8}$$

The set of conservation laws of type (5.8) for the Chaplygin system (2.4) coincides with the set of conservation laws of the first order for the Chaplygin equation (2.5).

The obvious conservation laws for the Chaplygin equation (2.5), generated by the operator Green's formula are

$$A_1=v\psi_\zeta-v_\zeta\psi, \quad A_2=K(\zeta)\,(v\psi_\sigma-v_\sigma\psi),$$

where $v=v(\zeta,\sigma)$ is any solution of equation (2.5). Substitution of these conservation laws in (3.3) gives the conservation laws for system (2.1).

Next we solve the classification problem of the Chaplygin equation (2.5) for the non-obvious conservation laws of the first order.

With the help of characteristic variables

$$\lambda = \int \sqrt{-K(\zeta)}d\zeta + \sigma, \quad \mu = \int \sqrt{-K(\zeta)}d\zeta - \sigma \tag{5.9}$$

equation (2.5) is written as

$$\psi_{\lambda\mu} - \frac{1}{4}\theta'(\zeta)\left(\psi_\lambda + \psi_\mu\right) = 0, \tag{5.10}$$

where

$$\theta = \theta(\zeta) = \frac{1}{\sqrt{-K(\zeta)}}, \tag{5.11}$$

and ζ is implicitly defined by the equation

$$\lambda + \mu = 2\int \sqrt{-K(\zeta)}d\zeta.$$

A set of conservation laws of the first order, $\mathbf{A} = \left(A_1\left(\zeta, \sigma, \psi, \psi_\zeta, \psi_\sigma\right), A_2\left(\zeta, \sigma, \psi, \psi_\zeta, \psi_\sigma\right)\right)$ for equation (2.5) such that

$$\left(D_\zeta A_1 + D_\sigma A_2\right)_{[(2.5)]} = 0$$

lies in the set of conservation laws with components

$$\begin{aligned} A_1 &= F_1(\lambda, \mu, \psi, \xi, \eta) + F_2(\lambda, \mu, \psi, \xi, \eta), \\ A_2 &= \sqrt{-K(\zeta)}\left(F_1(\lambda, \mu, \psi, \xi, \eta) - F_2(\lambda, \mu, \psi, \xi, \eta)\right), \end{aligned} \tag{5.12}$$

where λ, μ are defined by (5.9),

$$\xi = \frac{1}{2}\left(\frac{\psi_\zeta}{\sqrt{-K(\zeta)}} + \psi_\sigma\right), \quad \eta = \frac{1}{2}\left(\frac{\psi_\zeta}{\sqrt{-K(\zeta)}} - \psi_\sigma\right)$$

and the vector $\mathbf{F} = \left(F_1\left(\lambda, \mu, \psi, \psi_\lambda, \psi_\mu\right), F_2\left(\lambda, \mu, \psi, \psi_\lambda, \psi_\mu\right)\right)$ is the conservation law of the first order for equation (5.10) such that

$$\left(D_\lambda F_1 + D_\mu F_2\right)_{[(5.10)]} = 0.$$

The Laplace invariants for equation (5.10) are [5, 6, 11]

$$k = h = \frac{1}{16}\left(\left(\theta'\right)^2 - 2\theta\theta''\right). \tag{5.13}$$

The Chaplygin equation (2.5) with the Chaplygin function $K(\zeta) \neq 0$ is equivalent to the Laplace equation for $k = h = 0$. From (5.13) this is possible only if $K(\zeta) = -\frac{1}{(a\zeta+b)^4}$, where $a \neq 0$, b are arbitrary constants. In this case, the Chaplygin equation has infinitely many non-obvious conservation laws of the first order.

If $K(\zeta) = -\frac{1}{(a\zeta+b)^4}$ then the transformation $\psi = \frac{\omega(\lambda,\mu)}{\lambda+\mu}$ reduces equation (5.10) to the form $\omega_{\lambda\mu} = 0$. It follows that in this case the non-obvious conservation laws of the first

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order for equation (2.5) are determined by formulas

$$A_1 = f_1\left(a\sigma + \frac{1}{a\zeta+b}, (a\zeta+b)\,\psi_\zeta - \frac{1}{a\zeta+b}\psi_\sigma + a\psi\right) + f_2\left(a\sigma - \frac{1}{a\zeta+b}, (a\zeta+b)\,\psi_\zeta + \frac{1}{a\zeta+b}\psi_\sigma + a\psi\right),$$

$$A_2 = \frac{1}{(a\zeta+b)^2} f_1\left(a\sigma + \frac{1}{a\zeta+b}, (a\zeta+b)\,\psi_\zeta - \frac{1}{a\zeta+b}\psi_\sigma + a\psi\right) - \frac{1}{(a\zeta+b)^2} f_2\left(a\sigma - \frac{1}{a\zeta+b}, (a\zeta+b)\,\psi_\zeta + \frac{1}{a\zeta+b}\psi_\sigma + a\psi\right),$$

where f_1 and f_2 are arbitrary analytic functions.

If $h \neq 0$, then the first Ovsyannikov invariant $I_1 = \frac{k}{h} = 1$ [2, 11, 14]. Using classification of conservation laws of the first order for linear differential equations of the second order with two independent variables [2, 3], we get two partial results for equation (5.10). The first partial result for equation (5.10) has no more than three non-obvious conservation laws of the first order and their components are quadratic functions on stream function ψ and its derivatives ψ_λ, ψ_μ. The second partial result for equation (5.10) has three non-obvious conservation laws of the first order if and only if the second Ovsyannikov invariant I_2 is identically constant, i.e.

$$I_2 = \frac{1}{h}(\ln h)_{\lambda\mu} = \gamma = \text{ const.} \tag{5.14}$$

The classifying equation (5.14), which determines all the Chaplygin functions $K(\zeta)$, is written by using (5.11), in the form

$$4\theta\left(\theta\left(\ln\left((\theta')^2 - 2\theta\theta''\right)\right)'\right)' = \gamma\left((\theta')^2 - 2\theta\theta''\right). \tag{5.15}$$

The general solution of the ordinary differential equation (5.15) is given in [11] and has the form

$$K(\zeta) = \frac{\alpha_0}{(\alpha_1\zeta + \alpha_2)^4} K_0\left(\frac{\alpha_3\zeta + \alpha_4}{\alpha_1\zeta + \alpha_2}\right), \tag{5.16}$$

where α_j ($j = 0, 1, 2, 3, 4$) are arbitrary complex constants satisfying $\alpha_0(\alpha_1\alpha_4 - \alpha_2\alpha_3) \neq 0$ and the generating function $K_0(\zeta)$ is such that

1. for $\gamma = 0$ there will be two generating functions

or

$$K_0(\zeta) = -\frac{1}{\zeta^2} \tag{5.17}$$

or

$$K_0(\zeta) = (J_0(t))^4, \tag{5.18}$$

where t is implicitly defined by equation $\zeta J_0(t) - Y_0(t) = 0$ and $J_0(t)$ is the Bessel function of the first kind and order zero, $Y_0(t)$ is the Bessel function of the second kind and order zero.

https://doi.org/10.1017/S095679251300017X Published online by Cambridge University Press

2. For $\gamma \neq 0$ it is convenient to introduce a new parameter α instead of γ. We put

$$\frac{2}{\gamma} = \alpha(\alpha+1), \quad \alpha \neq -1, 0. \tag{5.19}$$

In this case, there will be three generating functions:

$$K_0(\zeta) = -\zeta^{-\frac{4\alpha}{2\alpha+1}} \left(\alpha \neq -1, \quad -\frac{1}{2}, \quad 0\right) \tag{5.20}$$

or

$$K_0(\zeta) = e^{\zeta}, \quad \left(\alpha = -\frac{1}{2}\right) \tag{5.21}$$

or

$$K_0(\zeta) = (P_\alpha(t))^4, \tag{5.22}$$

where t is implicitly defined by equation $\zeta P_\alpha(t) - Q_\alpha(t) = 0$, $P_\alpha(t)$ is the first kind Legendre function of degree α, and $Q_\alpha(t)$ is the second kind Legendre function of degree α.

The results for the classification of linear differential equations of the second order with two independent variables with respect to the conservation laws of the first order, obtained in [2], allow us to find all non-obvious conservation laws of the first order for equation (5.10) with canonical Chaplygin functions $K(\zeta) = K_0(\zeta)$ given by formula (5.17)–(5.22), and, consequently, by (5.12), and for Chaplygin equation (2.5).

If $K(\zeta) = -\frac{1}{\zeta^2}$ then

$$\begin{aligned} F_1 &= e^{-\frac{\lambda+\mu}{2}} \left((c_1 + c_2\mu) \left(\psi_\mu - \left(\lambda + \frac{1}{4}\right)\psi\right)^2 \right. \\ &\quad \left. + (c_1\lambda + c_3\mu)\left(\psi_\mu - \left(\lambda + \frac{1}{4}\right)\psi\right)\psi + \left(c_3\left(\lambda\mu - \frac{1}{2}\right) - c_2\lambda^2\mu\right)\psi^2\right) \\ F_2 &= e^{-\frac{\lambda+\mu}{2}} \left((c_3 - c_2\lambda) \left(\psi_\lambda - \left(\mu + \frac{1}{4}\right)\psi\right)^2 \right. \\ &\quad \left. + (c_1\lambda + c_3\mu)\left(\psi_\lambda - \left(\mu + \frac{1}{4}\right)\psi\right)\psi + \left(c_1\left(\lambda\mu - \frac{1}{2}\right) - c_2\lambda\mu^2\right)\psi^2\right), \end{aligned} \tag{5.23}$$

where c_1, c_2 and c_3 are arbitrary constants.

If $K(\zeta) = (J_0(t))^4$ then

$$\begin{aligned} F_1 &= J_0(i\tau)\left(\left(c_1 e^{2\mu} + c_2\right)\left(4\psi_\mu + (2\tau - 1)\psi\right)^2 \right. \\ &\quad - 2\left(c_1 e^{-2\lambda} + c_3 e^{-2\mu}\right)\left(4\psi_\mu + (2\tau-1)\psi\right)\psi \\ &\quad \left. + 4e^{-2\mu}\left(c_3(\tau - 2) - c_2\tau e^{-2\lambda}\right)\psi^2\right) \\ F_2 &= J_0(i\tau)\left(\left(c_3 e^{2\lambda} - c_2\right)(4\psi_\lambda + (2\tau-1)\psi)^2 \right. \\ &\quad - 2\left(c_1 e^{-2\lambda} + c_3 e^{-2\mu}\right)(4\psi_\lambda + (2\tau - 1)\psi)\psi \\ &\quad \left. + 4e^{-2\lambda}\left(c_1(\tau-2) + c_2\tau e^{-2\mu}\right)\psi^2\right), \end{aligned} \tag{5.24}$$

where $\tau = e^{-2(\lambda+\mu)}$, $i^2 = -1$, and c_1, c_2 and c_3 are arbitrary constants.

If $K(\zeta) = -\zeta^{-\frac{4\alpha}{2\alpha+1}}$, $\alpha \mp -1, -\frac{1}{2}, 0$ then

$$\begin{aligned}
F_1 &= (\lambda+\mu)^{-2\alpha}\left(\left(c_1 + c_2\mu + c_3\mu^2\right)\left(\psi_\mu + \left(\tau - \frac{1}{4}\right)\psi\right)^2\right.\\
&\quad - \alpha(\alpha+1)\psi\left(\left(c_2 + c_3(\mu-\lambda)\right)\left(\psi_\mu + \left(\tau - \frac{1}{4}\right)\psi\right)\right.\\
&\quad \left.\left. - \left(\frac{\tau}{\lambda+\mu}\left(-c_1 + c_2\lambda - c_3\lambda^2\right) + \frac{c_3}{2}\right)\psi\right)\right)\\
F_2 &= (\lambda+\mu)^{-2\alpha}\left(\left(-c_1 + c_2\lambda - c_3\lambda^2\right)\left(\psi_\lambda + \left(\tau - \frac{1}{4}\right)\psi\right)^2\right.\\
&\quad - \alpha(\alpha+1)\psi\left(\left(c_2 + c_3(\mu-\lambda)\right)\left(\psi_\lambda + \left(\tau - \frac{1}{4}\right)\psi\right)\right.\\
&\quad \left.\left. - \left(\frac{\tau}{\lambda+\mu}\left(c_1 + c_2\mu + c_3\mu^2\right) - \frac{c_3}{2}\right)\psi\right)\right),
\end{aligned} \tag{5.25}$$

where $\tau = \frac{\alpha(\alpha+1)}{\lambda+\mu}$ and c_1, c_2 and c_3 are arbitrary constants.

If $K(\zeta) = e^\zeta$ then the components F_1, F_2 of the conservation law of the first order for equation (5.10) are determined by formulas (5.25), in which one has to put $\alpha = -\frac{1}{2}$.

If $K(\zeta) = (P_\alpha(t))^4$ then

$$\begin{aligned}
F_1 &= P_\alpha(\xi)\left(\left(c_1 e^{2\gamma\mu} - c_2 e^{\gamma\mu} + c_3\right)\left(\phi_\mu + \left(\tau - \frac{1}{4}\right)\phi\right)^2\right.\\
&\quad - 2\left(c_2 e^{\gamma\mu} - c_3\left(1 + e^{\gamma(\lambda+\mu)}\right)\right)\left(\phi_\mu + \left(\tau - \frac{1}{4}\right)\phi\right)\phi\\
&\quad \left. + \left(\tau^2\left(-c_1 + c_2 e^{\gamma\lambda} - c_3 e^{2\gamma\lambda}\right)e^{2\gamma\mu} + c_3\gamma\right)\phi^2\right),\\
F_2 &= P_\alpha(\xi)\left(\left(-c_1 e^{-2\gamma\lambda} + c_2 e^{-\gamma\lambda} - c_3\right)\left(\phi_\lambda + \left(\tau e^{\gamma(\lambda+\mu)} - \frac{1}{4}\right)\phi\right)^2\right.\\
&\quad - 2e^{-\gamma(\lambda+\mu)}\left(c_2 e^{\gamma\mu} - c_3\left(1 + e^{\gamma(\lambda+\mu)}\right)\right)\left(\phi_\lambda + \left(\tau e^{\gamma(\lambda+\mu)} - \frac{1}{4}\right)\phi\right)\phi\\
&\quad \left. + \left(\tau^2\left(c_1 e^{2\gamma\mu} - c_2 e^{\gamma\mu} + c_3\right) - c_3\gamma\right)\phi^2\right),
\end{aligned} \tag{5.26}$$

where α and β are related by (5.19), $\tau = \frac{2}{e^{\beta(\lambda+\mu)}-1}$, $\xi = -\mathrm{cth}\frac{\lambda+\mu}{\alpha(\alpha+1)}$ and c_1, c_2 and c_3 are arbitrary constants.

6 Conclusion

Consequently, the Chaplygin equation (2.5) has at most three non-obvious conservation laws of the first order and their components are quadratic functions of the stream function ψ and its derivatives ψ_ζ, ψ_σ. The Chaplygin equation (2.5) has three non-obvious conservation laws of the first order only for the Chaplygin functions of the form (5.16) with the function $K_0(\zeta)$ given by any of the formulas (5.17), (5.18), (5.20)–(5.22). The substitution of (5.23)–(5.26) into (5.12) gives the non-obvious first-order conservation laws for the Chaplygin equation (2.5) with canonical Chaplygin functions; these are the first-order conservation laws, independent of the velocity potential, for the Chaplygin system (2.4).

By virtue of (3.3) these conservation laws generate nonlinear, non-local (with non-local variable – stream function ψ) conservation laws of the first order for the nonlinear system (2.1), describing the plane steady potential barotropic gas flow.

Acknowledgements

This work was supported by the Government grant of the Russian Federation for support of research projects implemented by leading scientists in the Federal State Budget Educational Institution of Higher Professional Education, Lomonosov Moscow State University under the agreement No. 11.G34.31.0054, and by the grant of Russian Foundation for Basic Research No. 11-01-12075-ofi-m-2011.

We thank an anonymous referee for helpful references.

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