Proceedings of the Royal Society of Edinburgh, **152**, 1589–1612, *2022*
DOI:10.1017/prm.2021.70

On a backward problem for nonlinear time fractional wave equations

Jia Wei He
Faculty of Mathematics and Computational Science, Xiangtan University, Hunan 411105, China (jwhe@gxu.edu.cn)

Yong Zhou
Faculty of Mathematics and Computational Science, Xiangtan University, Hunan 411105, China
Faculty of Information Technology, Macau University of Science and Technology, Macau 999078, China (yzhou@xtu.edu.cn)

(Received 9 February 2020; accepted 26 January 2021)

In this paper, we concern with a backward problem for a nonlinear time fractional wave equation in a bounded domain. By applying the properties of Mittag-Leffler functions and the method of eigenvalue expansion, we establish some results about the existence and uniqueness of the mild solutions of the proposed problem based on the compact technique. Due to the ill-posedness of backward problem in the sense of Hadamard, a general filter regularization method is utilized to approximate the solution and further we prove the convergence rate for the regularized solutions.

Keywords: fractional wave equation; backward problem; existence; regularization

2020 *Mathematics subject classification:* Primary: 35R11; 26A33

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain smooth boundary $\partial\Omega$ (being of $\mathcal{C}^2$ class for $N \leqslant 3$). We shall consider the following backward problem for time fractional wave equations

$$\begin{cases} \partial_t^\alpha u - \Delta u = f(t,x,u), & x \in \Omega,\ t \in (0,T], \\ u(t,x) = 0, & x \in \partial\Omega,\ t \in (0,T], \\ \partial_t u(0,x) = 0, & x \in \Omega, \\ u(T,x) = g(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\partial_t = \partial/\partial_t$ and ∂_t^α is the Caputo fractional derivative of order $\alpha \in (1,2)$ defined by (see [**15**, **23**])

$$\partial_t^\alpha u(t,x) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \partial_s^2 u(s,x)\, \mathrm{d}s, \quad t > 0,$$

provided that the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ stands for Gamma function. f is a nonlinear function which will be satisfied some suitable assumptions.

Due to the nonlocality of fractional derivative, which reveals a powerful tool for describing anomalous diffusion process, because it is fitted into the power-law behaviour of anomalous diffusion phenomena (including subdiffusion and superdiffusion, there are not applicable the Fick's law any more). As for the fractional wave equation while sometimes it is called the superdiffusion equation, it is also substituted for modelling the propagation of diffusive waves in viscoelastics solids frequently, see e.g. [**11**, **12**]. This is one of the reasons that many researchers pay attention to study these problems, see e.g. [**1**, **3**, **8**, **16**] and the related references therein.

If the finial value condition $u(T, x) = g(x)$ in problem (1.1) shall replace by an initial value condition $u(0, x) = y(x)$, then problem (1.1) is called the forward problem of time fractional wave equations. As we know, there are many papers coping with forward problems of time fractional wave equations, for example, Li and Wang [**9**] studied some regularity properties of time fractional stochastic wave equation which is forced by an additive space–time white noise. The regularity of weak solutions for time fractional wave equations has been studied by Otárola and Salgado [**13**]. As for backward problem, which is one of the main topics of inverse problem, we find that there are still few papers about backward problem for time fractional wave equation, Wei and Zhang [**22**] studied the existence, uniqueness and conditional stability for the backward problem, the Tikhonov regularization method has been used to solve regularized solution. Following this paper, Tuan *et al.* [**18**] considered some existence and regularity results for finial value problems (also called backward problems) with respect to linear function as well as a regularizing scheme by using a modified regularization method in [**17**], compared with these methods and conclusions, we improve the existence results on some weaker nonlinear functions in this paper, additionally, we find that it is hard to check a positive constant L_0 such that an estimate of Mittag-Leffler function is valid for some observed point $t = T$ in view of its the approximation form (see the discussion after (2.2)). In order to overcome this difficult, we propose a suitable concept of mild solutions. Huynh *et al.* [**5**] studied the regularized solution for an inhomogeneous problem in a general bounded domain by applying the fractional Landweber regularization method. Inspired by the above research studies, we will consider several existence results under some different conditions of nonlinear functions.

On the contrary, another issue worthy of consideration for backward problem about time fractional wave equation is seriously ill-posed in the sense of Hadamard, that is, even if a solution will exist and it is uniqueness, but it is not stable, in a word, it does not depend continuously on the given data. In order to achieve it at practical applications, many numerical methods are proposed to study the ill-posedness behaviour, the regularization solution and error analysis are also given. Additionally, one finds that the backward problems have emerged in optimal control, mathematical finance and so on. Some theoretical analyses are established to study these problems contained with the properties of solution of existence, uniqueness, regularity and convergence. In fact, our problem is seriously ill-posed, it urges us to prove the convergence rate for the regularized solutions. For more details about backward problems, for example, we refer to [**2**, **4**, **6**, **14**, **19**, **20**, **24**] and the related references therein.

The rest of this paper is as follows. In § 2, we introduce some concepts, preliminaries and the properties of Mittag-Leffler functions. In § 3, we derive the solution representation of problem (1.1), some useful properties of solution operators also will be discussed. Furthermore, several existence results are obtained in § 4, which do not necessarily satisfy Lipschitz condition or smoothness of nonlinear functions. In § 5, a regularization method is proposed to approximate the solutions.

2. Preliminaries

In this section, some preliminaries will be presented in order to derive the solution representation as well as our main results.

2.1. Fractional power spaces

We adopt the eigenvalues of the Laplacian operator $L = -\Delta$. Since the operator L is nonnegative and self-adjoint in Sobolev space $H_0^1(\Omega)$, there exists an orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions $\phi_k \in H_0^1(\Omega)$, $k = 1, 2, \ldots$, that correspond to eigenvalues

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots \nearrow \infty,$$

which satisfy

$$L\phi_k = \lambda_k \phi_k, \quad \text{in } \Omega, \qquad \phi_k = 0, \quad \text{on } \partial\Omega.$$

We first take the domain $\mathcal{H}^s(\Omega) = D(L^s)$ of the fractional power operator L^s, for $s \geqslant 0$, the space is introduced by

$$\mathcal{H}^s(\Omega) = \left\{ u \in L^2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^{2s} |u_k|^2 < \infty \right\},$$

as the Hilbert space of functions

$$u(t,x) := \sum_{k=1}^{\infty} u_k(t)\phi_k(x) = \sum_{k=1}^{\infty} (u, \phi_k)\phi_k(x) \in L^2(\Omega),$$

equipped with norm

$$\|u\|_{\mathcal{H}^s(\Omega)}^2 = \sum_{k=1}^{\infty} \lambda_k^{2s} |u_k|^2.$$

Let X, Y be two Banach spaces, $\mathcal{B}(X, Y)$ stands for the space of all linear bounded operators from X into Y. Now, we consider a Banach space X with the norm $\|\cdot\|_X$, specially, let the norm of space $L^2(\Omega)$ be given by $\|\cdot\|$ and inner product is defined as $(\cdot,\cdot)$. We denote by $C([0,T]; X)$ a Banach space of all continuous maps from $[0,T]$ into X with $\sup_{t\in[0,T]} \|u(t)\|_X < \infty$, $C^\eta((0,T]; X)$ stands for a Banach space of all weighted continuous functions mapping $(0,T]$ into X with exponent $\eta\in(0,\alpha]$ as follows

$$C^{\eta}((0,T];X)=\Big\{u\in C((0,T];X):\ \lim_{t\to 0+}t^{\eta}\|u(t)\|_X \text{ exists and finite}\Big\},$$

equipped with the norm

$$\|u\|_{C^{\eta}((0,T];X)}=\sup_{0\leqslant t\leqslant T}t^{\eta}\|u(t)\|_X.$$

Let $1\leqslant p<\infty$ and let $L^p(0,T;X)$ denote the space of the compositions of all the p-integrable Lebesgue measure functions equipped with the norm

$$\|u\|_{L^p(0,T;X)}=\left(\int_0^T\|u(t)\|_X^p\mathrm{d}t\right)^{1/p}<\infty,$$

and $L^{\infty}(0,T;X)$ stands for the space of essentially bounded functions.

2.2. Mittag-Leffler functions

Let us recall the Mittag-Leffler function $E_{\alpha,\beta}(\cdot)$, for more details, we refer to [7, 10].

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\quad \alpha>0,\beta\in\mathbb{R},\ z\in\mathbb{C}.$$

The function $E_{\alpha,\beta}(z)$ is an entire function, and so it is real analytic when restricted to the real line. Moreover, the approximation form of Mittag-Leffler function is given by

$$E_{\alpha,\beta}(z)=-\sum_{k=1}^{N}\frac{1}{\Gamma(\beta-\alpha k)}\frac{1}{z^k}+O\left(\frac{1}{z^{N+1}}\right),$$

with $|z|\to\infty$, $\mu\leqslant|\arg(z)|\leqslant\pi$ for $\mu>0$, and $N\in\mathbb{N}$. In particular,

$$E_{\alpha,1}(z)=-\frac{1}{\Gamma(1-\alpha)}\frac{1}{z}+O\left(\frac{1}{z^2}\right),\tag{2.1}$$

with $|z|\to\infty$, $\mu\leqslant|\arg(z)|\leqslant\pi$ for $\mu>0$.

Lemma 2.1 [15]. *Let $0<\alpha<2$, and $\beta\in\mathbb{R}$ be arbitrary. Suppose that μ is such that $\pi\alpha/2<\mu<\min\{\pi,\pi\alpha\}$. Then there exists a constant $M=M(\alpha,\beta,\mu)>0$ such that*

$$|E_{\alpha,\beta}(z)|\leqslant\frac{M}{1+|z|},\quad \mu\leqslant|\arg(z)|\leqslant\pi.$$

Lemma 2.2 [15]. *Let $0<\alpha$ and $\lambda,a>0$. Then*

(i) $\frac{d}{dt}(E_{\alpha}(-\lambda t^{\alpha}))=-\lambda t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha})$, *for $t>0$;*

(ii) $\frac{d}{dt}(t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha}))=t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda t^{\alpha})$, *for $t>0$.*

https://doi.org/10.1017/prm.2021.70 Published online by Cambridge University Press

LEMMA 2.3 [**1**]. *Let* $1 < \beta < 2$, $\beta' \in \mathbb{R}$ *and* $\lambda > 0$. *Assume that* $0 \leqslant \mu \leqslant 1$, $0 < \nu < \beta$. *Then there exists a positive constant* C_1 *such that*

$$\left|\lambda^{\mu} t^{\nu} E_{\beta,\beta'}(-\lambda t^{\beta})\right| \leqslant C_1\, t^{\nu-\beta\mu}, \quad t > 0.$$

LEMMA 2.4 [**22**, lemma 3.2]. *For* $1 < \alpha < 2$ *and any fixed* $T > 0$, *there is at most a finite index set* $\Theta = \{k_1, k_2, ..., k_n\}$ *such that* $E_{\alpha,1}(-\lambda_k T^{\alpha}) = 0$ *for* $k \in \Theta$ *and* $E_{\alpha,1}(-\lambda_k T^{\alpha}) \neq 0$ *for* $k \in \mathbb{N}\backslash\Theta$.

LEMMA 2.5 [**22**, lemma 3.6]. *Let* $1 < \alpha < 2$. *Then there exist positive constants* M_-, M_+ *depending on* α, T *and finite eigenvalues* λ_k *with* $k \in \{k_1, k_2, \ldots, k_{n+m}\}\backslash\Theta$, $m \in \mathbb{N} \cup \{0\}$ *such that*

$$\frac{M_-}{\lambda_k} \leqslant |E_{\alpha,1}(-\lambda_k T^{\alpha})| \leqslant \frac{M_+}{\lambda_k}, \quad k \in \mathbb{N} \setminus \Theta.$$

Noting that, in view of the approximation form of Mittag-Leffler function (2.1) and lemma 2.4, there exists $L_0 > 0$ such that

$$E_{\alpha,1}(-\lambda_k T^{\alpha}) \leqslant \frac{1}{2\Gamma(1-\alpha)\lambda_k T^{\alpha}} < 0, \quad \lambda_k T^{\alpha} > L_0, \tag{2.2}$$

for $1 < \alpha < 2$, thus $E_{\alpha,1}(-\lambda_k T^{\alpha}) = 0$ only if $\lambda_k T^{\alpha} \leqslant L_0$. Since $\lim_{k\to\infty} \lambda_k = +\infty$, there are only finite λ_k satisfying $\lambda_k T^{\alpha} \leqslant L_0$ with $k \in \Theta$. According to the abovementioned discussions and related lemmas, we know that there exist some finite λ_k and T such that $E_{\alpha,1}(-\lambda_k T^{\alpha}) = 0$, for every $k \in \Theta$. Thus, throughout this paper, we shall get rid of the part of $k \in \Theta$ in λ_k and set $\mathcal{E}_{\alpha}(-\lambda_k T^{\alpha}) := E_{\alpha,1}(-\lambda_k T^{\alpha}) \neq 0$ for $k \in \mathbb{N}\backslash\Theta$, with using these notations $\mathcal{E}_{\alpha}(-\lambda_k t^{\alpha}) := E_{\alpha,1}(-\lambda_k t^{\alpha})$ and $\mathcal{E}_{\alpha,\beta}(-\lambda_k t^{\alpha}) := E_{\alpha,\beta}(-\lambda_k t^{\alpha})$, for $k \in \mathbb{N}\backslash\Theta$. Consequently, similarly to lemma 2.4, lemma 2.5 and (2.2), together above arguments and lemma 2.1, one can check the following inequalities obviously, for $k \in \mathbb{N}\backslash\Theta$, $t \geqslant 0$,

$$\frac{c_-}{1+\lambda_k T^{\alpha}} \leqslant |\mathcal{E}_{\alpha}(-\lambda_k T^{\alpha})| \leqslant \frac{c_+}{1+\lambda_k T^{\alpha}}, \quad |\mathcal{E}_{\alpha,\zeta}(-\lambda_k t^{\alpha})| \leqslant \frac{c_+}{1+\lambda_k t^{\alpha}}, \tag{2.3}$$

where $\zeta \in \mathbb{R}$,

$$c_- := \min\left\{(2|\Gamma(1-\alpha)|)^{-1}, M_-, M_- T^{\alpha}\right\},$$

and

$$c_+ := \max\{M, C_1, M_+, M_+(T^{\alpha} + \lambda_1^{-1})\}.$$

Clearly, we have $c_- < c_+$.

Noting that from (2.1) we see that it is hard to find a suitable constant L_0 such that (2.2) is satisfied in the actual application, that is, if T enough large such that $E_{\alpha,1}(-\lambda_k T^{\alpha})$ does not identically equal to zero, we do not need to get rid of the part of $k \in \Theta$ in λ_k and remain all $k \in \mathbb{N}$, however, from a view point of an actual observation, this situation may not be achieved such $T > (\lambda_1^{-1} L_0)^{1/\alpha}$ with an enough large constant L_0 from the approximation form (2.1). Consequently, we don't consider this case in the current paper and in order to overcome this difficult, we shall establish a suitable solution representation. In the sequel, set $\Pi = \mathbb{N}\backslash\Theta$, we will assume $k \in \Pi$ all the time.

https://doi.org/10.1017/prm.2021.70 Published online by Cambridge University Press

3. Solution representation

In this section, we first give a suitable mild solution definition of problem (1.1), and we will further study the properties of solution operators which derived from solution representation.

3.1. Definition of mild solution

Let u be the solution of initial-boundary value problems with respect to forward time fractional wave equations, passing the eigenvalues of the fractional Laplacian operator, it yields

$$\partial_t^\alpha u_k(t) + \lambda_k u_k(t) = f_k(t, u),$$

associated with the initial conditions $u_k(0) = (u_0, \phi_k) = u_{0k}, u_k'(0) = (u_1, \phi_k) = u_{1k}$ and $f_k(t, u) = (f(t, u), \phi_k)$. Then, from theorem 5.15 in [7], one obtains

$$u_k(t) = E_\alpha(-\lambda_k t^\alpha) u_{k0} + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (t-\tau)^\alpha) f_k(\tau, u)\, \mathrm{d}\tau. \tag{3.1}$$

By substituting $t = T$ into (3.1), it yields

$$u_k(T) = E_\alpha(-\lambda_k T^\alpha) u_{k0} + \int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (T-\tau)^\alpha) f_k(\tau, u)\, \mathrm{d}\tau.$$

Let $g_k = (g, \phi_k)$, since $E_\alpha(-\lambda_k T^\alpha)$ may be equal to zero for some $k \in \Theta$, and then, for $k \in \mathbb{N}\backslash\Theta$, we have

$$\begin{aligned} u_k(t) = {}& \frac{\mathcal{E}_\alpha(-\lambda_k t^\alpha)}{\mathcal{E}_\alpha(-\lambda_k T^\alpha)} \left(g_k - \int_0^T (T-\tau)^{\alpha-1} \mathcal{E}_{\alpha,\alpha}(-\lambda_k (T-\tau)^\alpha) f_k(\tau, u)\, \mathrm{d}\tau \right) \\ & + \int_0^t (t-\tau)^{\alpha-1} \mathcal{E}_{\alpha,\alpha}(-\lambda_k (t-\tau)^\alpha) f_k(\tau, u)\, \mathrm{d}\tau. \end{aligned}$$

Let us simple $u(t)(\cdot)$ instead of $u(t, x)$, for any $v \in L^2(\Omega)$, denote two operators by

$$\mathcal{S}_\alpha(t)v = \sum_{k=1, k\in\Pi}^{\infty} \frac{\mathcal{E}_\alpha(-\lambda_k t^\alpha)}{\mathcal{E}_\alpha(-\lambda_k T^\alpha)} (v, \phi_k)\phi_k,$$

and

$$\mathcal{P}_\alpha(t)v = t^{\alpha-1} \sum_{k=1, k\in\Pi}^{\infty} \mathcal{E}_{\alpha,\alpha}(-\lambda_k t^\alpha)(v, \phi_k)\phi_k.$$

From above arguments, let symbol $\circ$ be a composition operator as follows

$$\mathcal{S}_\alpha(t) \circ \mathcal{P}_\alpha(\varsigma)v = \sum_{k=1, k\in\Pi}^{\infty} \frac{\mathcal{E}_\alpha(-\lambda_k t^\alpha)}{\mathcal{E}_\alpha(-\lambda_k T^\alpha)} \mathcal{E}_{\alpha,\alpha}(-\lambda_k \varsigma^\alpha)(v, \phi_k)\phi_k, \quad v \in L^2(\Omega),$$

for $t, \varsigma \in [0, T]$, hence, we can find a mild solution of problem (1.1) in which its definition is given below.

https://doi.org/10.1017/prm.2021.70 Published online by Cambridge University Press

DEFINITION 3.1. For every $\eta \in (0, \alpha]$, a function u is called a mild solution of problem (1.1) if $u \in C^{\eta}((0,T]; L^2(\Omega))$ and it satisfies the integral equation

$$u(t) = \mathcal{S}_\alpha(t)g - \int_0^T \mathcal{S}_\alpha(t) \circ \mathcal{P}_\alpha(T-\tau) f(\tau, u)\, \mathrm{d}\mathrm{d}\tau + \int_0^t \mathcal{P}_\alpha(t-\tau) f(\tau, u)\, \mathrm{d}\tau. \quad (3.2)$$

3.2. Some properties

PROPERTY 1. $\mathcal{S}_\alpha(t)$ is a unbounded operator for the time $t = 0$ in $L^2(\Omega)$ while it belongs to $\mathcal{B}(\mathcal{H}^1(\Omega), L^2(\Omega))$.

Proof. Obviously, $\mathcal{S}_\alpha(t)$ is linear operator. If taking $v_n = \phi_n(x)$, $n \in \Pi$, in view of (2.3) and $\lim_{t\to 0} \mathcal{E}_\alpha(-\lambda_k t^\alpha) = 1$, we deduce that $\|v_n\| = 1$ and

$$\begin{aligned}
\|\mathcal{S}_\alpha(0)v_n\|^2 &= \sum_{k=1, k\in\Pi}^{\infty} \frac{1}{|\mathcal{E}_\alpha(-\lambda_k T^\alpha)|^2} |(v_n, \phi_k)|^2 \\
&\geqslant \frac{1}{c_+^2} \sum_{k=1, k\in\Pi}^{\infty} (1 + \lambda_k T^\alpha)^2 |(v_n, \phi_k)|^2 \\
&\geqslant \frac{1}{c_+^2} \|v_n\|^2 + \frac{T^{2\alpha}}{c_+^2} \lambda_n^2 \|v_n\|^2.
\end{aligned}$$

Therefore, it follows that $\|\mathcal{S}_\alpha(0)v_n\| > T^\alpha \lambda_n / c_+$. From $\lambda_n \to \infty$ as $n \to \infty$, it shows that $\mathcal{S}_\alpha(t)$ is not unbounded in $L^2(\Omega)$ at time $t = 0$. On the contrary, for any $v \in L^2(\Omega)$, the Sobolev embedding $\mathcal{H}^2(\Omega) \hookrightarrow L^2(\Omega)$ implies

$$\begin{aligned}
\|\mathcal{S}_\alpha(0)v\|^2 &\leqslant \frac{1}{c_-^2} \sum_{k=1, k\in\Pi}^{\infty} (1 + \lambda_k T^\alpha)^2 |(v, \phi_k)|^2 \\
&\leqslant \frac{2}{c_-^2} \|v\|^2 + \frac{2T^{2\alpha}}{c_-^2} \sum_{k=1, k\in\Pi}^{\infty} \lambda_k^2 |(v, \phi_k)|^2 \\
&\leqslant \frac{2C_2^2}{c_-^2} \|v\|^2_{\mathcal{H}^1(\Omega)} + \frac{2T^{2\alpha}}{c_-^2} \|v\|^2_{\mathcal{H}^1(\Omega)},
\end{aligned}$$

where C_2 is a positive constant, in addition, we use the inequality $(1+a)^2 \leqslant 2(1 + a^2)$, $a \in \mathbb{R}$. Thus, we show the desired results. □

PROPERTY 2. Let $v \in L^2(\Omega)$. Then $\mathcal{S}_\alpha(t)v$ is continuous on $L^2(\Omega)$ for all $t \in (0, T]$, that is $\mathcal{S}_\alpha(t)v \in C((0,T]; L^2(\Omega))$.

Proof. In fact, we just need to show that the series

$$\sum_{k=1, k\in\Pi}^{\infty} \frac{\mathcal{E}_\alpha(-\lambda_k t^\alpha)}{\mathcal{E}_\alpha(-\lambda_k T^\alpha)} (v, \phi_k)\phi_k(x)$$

is uniformly convergent on $L^2(\Omega)$ for any $v \in L^2(\Omega)$ and any $t \in [\delta, T]$ with $\delta > 0$.

https://doi.org/10.1017/prm.2021.70 Published online by Cambridge University Press

By virtue of (2.3), we get

$$\left|\frac{\mathcal{E}_\alpha(-\lambda_k t^\alpha)}{\mathcal{E}_\alpha(-\lambda_k T^\alpha)}\right| \leqslant \frac{c_+}{c_-}\frac{1+\lambda_k T^\alpha}{1+\lambda_k t^\alpha} \leqslant \frac{c_+ T^\alpha}{c_- t^\alpha}, \quad \text{for all } t \in (0,T]. \tag{3.3}$$

In the following, we know that $\mathcal{E}_\alpha(-\lambda_k t^\alpha)$ is uniform continuous since the identity

$$\mathcal{E}_{\alpha,1}(-z^2) = \int_0^\infty M_{\alpha/2}(\theta)\cos(z\theta)\,\mathrm{d}\theta, \quad z \in \mathbb{C},\ \alpha \in (1,2). \tag{3.4}$$

This above identity can be found in [**11**], where $M_\varrho(\cdot)$ is the Mainardi's Wright-type function defined by

$$M_\varrho(z) = \sum_{n=0}^\infty \frac{(-z)^n}{n!\Gamma(1-\varrho(n+1))}, \quad \varrho \in (0,1),\ z \in \mathbb{C}.$$

In fact, from the uniform continuity of $\cos(\sqrt{z})$ for $z \in \mathbb{R}_+$, we know that for any $\varepsilon > 0$ and each $k \in \mathbb{N}$, there exists a $\delta > 0$ such that, for $t_1, t_2 \in \mathbb{R}_+$ with $|t_2 - t_1| < \delta$,

$$\left|\cos\left(\sqrt{\lambda_k t_2^\alpha}\theta\right) - \cos\left(\sqrt{\lambda_k t_1^\alpha}\theta\right)\right| < \varepsilon.$$

Therefore, by virtue of (3.4), we have

$$|\mathcal{E}_\alpha(-\lambda_k t_2^\alpha) - \mathcal{E}_\alpha(-\lambda_k t_1^\alpha)| = \left|\int_0^\infty M_{\alpha/2}(\theta)(\cos(\sqrt{\lambda_k t_2^\alpha}\theta) - \cos(\sqrt{\lambda_k t_1^\alpha}\theta))\,\mathrm{d}\theta\right| < \varepsilon,$$

where we use the property

$$M_\varrho(\theta) \geqslant 0, \quad \int_0^\infty M_\varrho(\theta)\,\mathrm{d}\theta = 1.$$

It allows us to obtain the desired series which is uniformly convergent on $[\delta, T]$ by using the Cauchy convergence criterion. Now, for any $\varepsilon > 0$, there exists $M > 0$ such that for all positive integers p whereas $m \in \Pi$ and $m > M$

$$\sum_{k=m+1}^{m+p} |(v,\phi_k)|^2 < \left(\frac{c_- t^\alpha}{c_+ T^\alpha}\right)^2 \varepsilon, \quad \text{for all } t \in [\delta, T].$$

Let

$$S_m(t)v = \sum_{k=1}^m \frac{\mathcal{E}_\alpha(-\lambda_k t^\alpha)}{\mathcal{E}_\alpha(-\lambda_k T^\alpha)}(v,\phi_k)\phi_k(x).$$

Therefore, it yields that

$$\begin{aligned}\|S_{m+p}(t)v - S_m(t)v\|^2 &= \sum_{k=m+1}^{m+p}\left|\frac{\mathcal{E}_\alpha(-\lambda_k t^\alpha)}{\mathcal{E}_\alpha(-\lambda_k T^\alpha)}(v,\phi_k)\right|^2 \\ &\leqslant \left(\frac{c_+ T^\alpha}{c_- t^\alpha}\right)^2 \sum_{k=m+1}^{m+p} |(v,\phi_k)|^2 < \varepsilon.\end{aligned}$$

By the arbitrariness of ε, we deduce that the conclusion holds. □

https://doi.org/10.1017/prm.2021.70 Published online by Cambridge University Press

In the sequel, for convenience, we set $X := C^{\eta}((0,T];L^2(\Omega))$, $\eta = \alpha\gamma$ for $\gamma \in (0,1]$, and let operator $\mathcal{T}_\alpha$ be defined by

$$(\mathcal{T}_\alpha v)(t) = \int_0^T \mathcal{S}_\alpha(t) \circ \mathcal{P}_\alpha(T-\tau)v(\tau)\,\mathrm{d}\tau, \quad v \in X.$$

LEMMA 3.2. *Let $\gamma \in (0, \frac{1}{\alpha})$ such that $\eta \in (0,1)$. Then operator $\mathcal{T}_\alpha$ is a completely continuous operator from X into X.*

Proof. For every $n \in \Pi$, let $\Phi_n = \text{span}\{\phi_1(x), ..., \phi_n(x)\}$, since $\{\phi_k\}_{k=1}^{\infty}$ is an orthonormal basis in $L^2(\Omega)$, one finds that $L^2(\Omega)$ can be expressed by $\text{span}\{\phi_1(x), ..., \phi_n(x), ...\}$. Obviously, Φ_n is a finite-dimensional subspace of $L^2(\Omega)$. For every $n \in \Pi$, denote operators $\mathcal{S}_{\alpha,n}(t,\varsigma) : L^2(\Omega) \to \Phi_n$ by

$$\mathcal{S}_{\alpha,n}(t,\varsigma)v = \varsigma^{\alpha-1} \sum_{k=1, k\in\Pi}^{n} \frac{\mathcal{E}_\alpha(-\lambda_k t^\alpha)}{\mathcal{E}_\alpha(-\lambda_k T^\alpha)} \mathcal{E}_{\alpha,\alpha}(-\lambda_k \varsigma^\alpha)(v,\phi_k)\phi_k(x),$$

for $t \in (0,T]$, $\varsigma := T - \tau$ with $\tau \in [0,T]$. Observe that, $\mathcal{S}_{\alpha,n}(t,\varsigma)$ are linear finite-dimensional operators. Next, for every $n \in \mathbb{N}$, we define linear operators $\mathcal{T}_{\alpha,n}$ in the same way by

$$(\mathcal{T}_{\alpha,n} v)(t) = \int_0^T \mathcal{S}_{\alpha,n}(t, T-\tau)v(\tau)\,\mathrm{d}\tau, \quad v \in X.$$

Obviously, $\mathcal{T}_{\alpha,n}v$ are well-defined on X. Denote a bounded set on X by $U_r = \{v \in X : \ \|v\|_X \leqslant r\}$ for each positive constant r. We shall prove that for any positive constant r, the set $\{t^\eta(\mathcal{T}_{\alpha,n}v)(t), \ v \in U_r\}$ is relatively compact in X.

For any $v \in X$, it follows from the fact $|\mathcal{E}_{\alpha,\zeta}(-\lambda_k z^\alpha)| \leqslant c_+$, $\zeta \in \mathbb{R}$, $z > 0$ and (2.3) that

$$\begin{aligned}
&\|\mathcal{S}_{\alpha,n}(t,\varsigma)v\|^2 \\
&= \varsigma^{2(\alpha-1)} \sum_{k=1}^{n} \left| \frac{\mathcal{E}_\alpha^\gamma(-\lambda_k t^\alpha)}{\mathcal{E}_\alpha^\gamma(-\lambda_k T^\alpha)} \frac{\mathcal{E}_\alpha^{1-\gamma}(-\lambda_k t^\alpha)}{\mathcal{E}_\alpha^{1-\gamma}(-\lambda_k T^\alpha)} \mathcal{E}_{\alpha,\alpha}^\gamma(-\lambda_k\varsigma^\alpha)\mathcal{E}_{\alpha,\alpha}^{1-\gamma}(-\lambda_k\varsigma^\alpha) \right|^2 |(v,\phi_k)|^2 \\
&\leqslant \frac{c_+^4}{c_-^2} \varsigma^{2(\alpha-1)} \sum_{k=1}^{n} \left(\frac{1+\lambda_k T^\alpha}{1+\lambda_k t^\alpha} \right)^{2\gamma} \left(\frac{1+\lambda_k T^\alpha}{1+\lambda_k \varsigma^\alpha} \right)^{2-2\gamma} |(v,\phi_k)|^2 \\
&\leqslant \frac{c_+^4}{c_-^2} T^{2\alpha} t^{-2\alpha\gamma} \varsigma^{2(\alpha\gamma-1)} \|v\|^2. \qquad (3.5)
\end{aligned}$$

Therefore, by the assumption of $\eta \in (0,1)$, we have

$$\begin{aligned}
\|(\mathcal{T}_{\alpha,n}v)(t)\| &\leqslant \int_0^T \|\mathcal{S}_{\alpha,n}(t, T-\tau)v(\tau)\|\mathrm{d}\tau \\
&\leqslant \frac{c_+^2}{c_-} T^\alpha t^{-\eta} \int_0^T (T-\tau)^{\eta-1} \|v(\tau)\|\mathrm{d}\tau \qquad (3.6) \\
&\leqslant \frac{c_+^2 \pi}{c_- \sin(\pi\eta)} T^\alpha t^{-\eta} \|v\|_X,
\end{aligned}$$

https://doi.org/10.1017/prm.2021.70 Published online by Cambridge University Press

where we use $B(\eta, 1-\eta) = \pi/\sin(\pi\eta)$, and $B(\cdot,\cdot)$ is the Beta function.

To begin with, we deduce that $\lim_{t\to 0+} t^{\eta}(\mathcal{T}_{\alpha,n}v)(t)$ exists and is finite. In fact, from (3.5), one can see that the representation

$$t^{\eta}\varsigma^{\alpha-1}\sum_{k=1,k\in\Pi}^{n}\frac{\mathcal{E}_{\alpha}(-\lambda_k t^{\alpha})}{\mathcal{E}_{\alpha}(-\lambda_k T^{\alpha})}\mathcal{E}_{\alpha,\alpha}(-\lambda_k\varsigma^{\alpha})(v,\phi_k)\phi_k(x)$$

is integrable bounded for $\varsigma = T-\tau$ with respect to a.e. $\tau\in[0,T]$ on $L^2(\Omega)$ which implies that $t^{\eta}\mathcal{T}_{\alpha,n}(t)v$ is uniformly bounded on $L^2(\Omega)$. Let

$$\mathcal{S}_{\alpha,n1}(t,\varsigma)v = \varsigma^{\alpha-1}\frac{\mathcal{E}_{\alpha}(-\lambda_k t^{\alpha})}{\mathcal{E}_{\alpha}(-\lambda_k T^{\alpha})}\mathcal{E}_{\alpha,\alpha}(-\lambda_k\varsigma^{\alpha})(v,\phi_k).$$

It follows from the uniform continuity of $\mathcal{E}_{\alpha}(-\lambda_k t^{\alpha})$ that $\mathcal{S}_{\alpha,n1}(t,\varsigma)v$ is also uniformly continuous for every $k=1,2,\ldots,n$, $k\in\Pi$ and all $t\in(0,T]$. On the contrary, we know

$$\int_0^T |\mathcal{S}_{\alpha,n1}(t,T-\tau)v(\tau)|\,\mathrm{d}\tau \leqslant \frac{c_+^2}{c_-}t^{-\eta}T^{\alpha}\int_0^T (T-\tau)^{\alpha-1}\|v(\tau)\|\mathrm{d}\tau \leqslant \frac{c_+^2}{c_-}t^{-\eta}T^{\alpha}\|v\|_X.$$

Therefore, $t^{\eta}\mathcal{S}_{\alpha,n1}(t,\cdot)v$ is uniformly bounded for $t\in[0,T]$, which deduce that $t^{\eta}\mathcal{S}_{\alpha,n}(t,\cdot)v$ is uniformly continuous for $t\in(0,T]$ and thus $t^{\eta}\mathcal{T}_{\alpha,n}(t)v$ is uniformly continuous for $t\in(0,T]$ which implies from (3.6) that $\lim_{t\to 0+} t^{\eta}\mathcal{T}_{\alpha,n}(t)v$ exists and is finite. Let $z(0) := \lim_{t\to 0+} t^{\eta}(\mathcal{T}_{\alpha,n}v)(t)$, hence we deduce $z(0)$ is well-defined.

For any $w\in\mathfrak{U}_r := \{y\in C([0,T];L^2(\Omega)) :\ \|y\|_{C([0,T];L^2(\Omega))}\leqslant r\}$, $r>0$, set

$$v(t) = t^{-\eta}w(t), \quad \text{for } t\in(0,T].$$

Thus, $v\in U_r$. Define

$$(\mathfrak{T}_{\alpha,n}w)(t) = \begin{cases} t^{\eta}(\mathcal{T}_{\alpha,n}v)(t), & \text{for } t\in(0,T],\\ z(0), & \text{for } t=0.\end{cases}$$

Thenceforth, it remains to show that $\{\mathfrak{T}_{\alpha,n}w : w\in\mathfrak{U}_r\}$ is relatively compact. Observe that, from (3.6), $\|\mathfrak{T}_{\alpha,n}w\|_{C([0,T];L^2(\Omega))}\leqslant c_+^2T^{\alpha}r/c_-$. Thus, we conclude that the set $\mathfrak{U}_r$ is uniformly bounded. In order to prove that the set $\{\mathfrak{T}_{\alpha,n}w,\ w\in\mathfrak{U}_r\}$ is equicontinuous, we need to proof that $\mathcal{S}_{\alpha,n}(t,\cdot)$ is continuous in the uniform operator topology on $L^2(\Omega)$ for all $t>0$. For this purpose, we need to show the compactness and strong continuity of this operator. By applying (2.3), for any $v\in L^2(\Omega)$ and any $\delta>0$ such that $t\in[\delta,T]$, it follows that

$$\|\mathcal{S}_{\alpha,n}(t,\cdot)v\|^2 \leqslant T^{2(\alpha-1)}\sum_{k=1,k\in\Pi}^{n}\frac{c_+^4}{c_-^2}\left(\frac{1+\lambda_k T^{\alpha}}{1+\lambda_k t^{\alpha}}\right)^2|(v,\phi_k)|^2 \leqslant \left(\frac{c_+^2T^{2\alpha-1}}{c_-\delta^{\alpha}}\right)^2\|v\|^2. \tag{3.7}$$

By virtue of the range $R(\mathcal{S}_{\alpha,n}(t,\cdot)v)$ finite, we thus conclude that the operator $\mathcal{S}_{\alpha,n}(t,\cdot)$ are compact operators on $L^2(\Omega)$ for every $n\in\Pi$. In addition, in view of

(3.4) and (3.7), for any $v \in L^2(\Omega)$ and any $\delta > 0$, for $t_1, t_2 \in [\delta, T]$ with $t_1 < t_2$, we get

$$\begin{aligned}
&\|\mathcal{S}_{\alpha,n}(t_2,\cdot)v - \mathcal{S}_{\alpha,n}(t_1,\cdot)v\|^2 \\
&\quad \leqslant c_+^2 T^{2(\alpha-1)} \sum_{k=1,k\in\Pi}^{n} \left| \frac{\mathcal{E}_\alpha(-\lambda_k t_2^\alpha) - \mathcal{E}_\alpha(-\lambda_k t_1^\alpha)}{\mathcal{E}_\alpha(-\lambda_k T^\alpha)} \right|^2 |(v,\phi_k)|^2 \\
&\quad \leqslant 2\left(\frac{c_+^2 T^{2\alpha-1}}{c_-\delta^\alpha}\right)^2 \|v\|^2.
\end{aligned}$$

Therefore, in view of uniform continuity of $\mathcal{E}_\alpha(-\lambda_k t^\alpha)$, by applying the property of series, we thus obtain

$$\|\mathcal{S}_{\alpha,n}(t_2,\cdot)v - \mathcal{S}_{\alpha,n}(t_1,\cdot)v\| \to 0 \quad \text{as } t_2 \to t_1,$$

which shows that $\mathcal{S}_{\alpha,n}(t,\cdot)v$ are strong continuous for all $t \in [\delta, T]$. Consequently, we conclude the desired proof.

Now, for $t_1 = 0$, $0 < t_2 \leqslant T$, it is easy to see that

$$\|(\mathfrak{T}_{\alpha,n}w)(t_2) - (\mathfrak{T}_{\alpha,n}w)(0)\| \to 0, \quad \text{as } t_2 \to 0.$$

For any $0 < t_1 < t_2 \leqslant T$, we have

$$\begin{aligned}
&\|(\mathfrak{T}_{\alpha,n}w)(t_2) - (\mathfrak{T}_{\alpha,n}w)(t_1)\| \\
&\quad = \|t_2^\eta(\mathcal{T}_{\alpha,n}v)(t_2) - t_1^\eta(\mathcal{T}_{\alpha,n}v)(t_1)\| \\
&\quad \leqslant |t_2^\eta - t_1^\eta| \|(\mathcal{T}_{\alpha,n}v)(t_2)\| + t_1^\eta \|(\mathcal{T}_{\alpha,n}v)(t_2) - (\mathcal{T}_{\alpha,n}v)(t_1)\| \\
&\quad := I_1 + I_2.
\end{aligned}$$

By the inequality $a^\rho - b^\rho \leqslant (a-b)^\rho$ for $0 < b < a$ and $\rho \in [0,1]$, obviously from (3.6) and $|t_2^\eta - t_1^\eta| \leqslant (t_2 - t_1)^\eta$, we get $I_1 \to 0$ as $t_2 \to t_1$. As for I_2, by virtue of the continuity in the uniform operator topology of $\mathcal{S}_{\alpha,n}(t,\cdot)$ for all $t > 0$, we obtain

$$\begin{aligned}
\|(\mathfrak{T}_{\alpha,n}w)(t_2) - (\mathfrak{T}_{\alpha,n}w)(t_1)\| &\leqslant \int_0^T \|(\mathcal{S}_{\alpha,n}(t_2, T-\tau) - \mathcal{S}_{\alpha,n}(t_1, T-\tau))w(\tau)\| \mathrm{d}\tau \\
&\leqslant \frac{T^{1-\eta} r}{1-\eta} \sup_{\varsigma\in[0,T]} \|\mathcal{S}_{\alpha,n}(t_2,\varsigma) - \mathcal{S}_{\alpha,n}(t_1,\varsigma)\|_{\mathcal{B}(L^2(\Omega))} \\
&\to 0, \quad \text{as } t_2 \to t_1.
\end{aligned}$$

That means $I_2 \to 0$ as $t_2 \to t_1$ which the right-hand side of the aforementioned inequality tends to zero independently of $w \in \mathfrak{U}_r$. From above arguments, one can easily deduce that the set $\{\mathfrak{T}_{\alpha,n}w,\ w \in \mathfrak{U}_r\}$ is equicontinuous. Thus, according to Ascoli–Arzelà theorem, we conclude that operators $\mathfrak{T}_{\alpha,n}$ are compact on $C([0,T]; L^2(\Omega))$ as well as operators $\mathcal{T}_{\alpha,n}$ are compact on X.

Now, we prove that $\mathcal{T}_{\alpha,n}$ converge uniformly to $\mathcal{T}_\alpha$ whenever n tends to infinite. Indeed, for any $v \in L^2(\Omega)$ and any $\gamma \in (0,1)$, by applying lemma 2.3 with respect to $\mu \in (0,1)$ and $\varsigma = T-\tau \in (0,T]$, we first have

$$
\begin{aligned}
&\|\mathcal{S}_{\alpha,n}(t,\varsigma)v - \mathcal{S}_\alpha(t)\circ\mathcal{P}_\alpha(\varsigma)v\|^2 \\
&\quad\leqslant \varsigma^{2(\alpha-1)}\sum_{k=n+1}^{\infty}\left|\frac{\mathcal{E}_\alpha(-\lambda_k t^\alpha)}{\mathcal{E}_\alpha(-\lambda_k T^\alpha)}\mathcal{E}_{\alpha,\alpha}(-\lambda_k\varsigma^\alpha)\right|^2 |(v,\phi_k)|^2 \\
&\quad\leqslant \frac{c_+^{4-2\gamma}}{c_-^2}T^{2\alpha}t^{-2\eta}\varsigma^{2(\eta-1)}\sum_{n=N+1}^{\infty}\left|\lambda_k^{-\mu\gamma}(\lambda_k^{\mu}\mathcal{E}_{\alpha,\alpha}(-\lambda_k\varsigma^\alpha))^\gamma\right|^2|(v,\phi_k)|^2 \\
&\quad\leqslant \frac{c_+^{6-2\gamma}}{c_-^2}T^{2\alpha}t^{-2\eta}\varsigma^{2(\eta-1)-2\eta\mu}\lambda_{N+1}^{-2\mu\gamma}\|v\|^2.
\end{aligned}
$$

Therefore, for any $v \in X$, by Hölder inequality, we get

$$
\begin{aligned}
\|(\mathcal{T}_\alpha v)(t) - (\mathcal{T}_{\alpha,n}v)(t)\| &\leqslant \int_0^T \|(\mathcal{S}_{\alpha,n}(t,T-\tau) - \mathcal{S}_\alpha(t)\circ\mathcal{P}_\alpha(T-\tau))v(\tau)\|\mathrm{d}\tau \\
&\leqslant \frac{c_+^{3-\gamma}}{c_-}T^\alpha t^{-\eta}\lambda_{n+1}^{-\mu\gamma}\int_0^T (T-\tau)^{\eta(1-\mu)-1}\|v(\tau)\|\mathrm{d}\tau \\
&\leqslant \frac{c_+^{3-\gamma}}{c_-}T^{\alpha-\eta\mu}t^{-\eta}\lambda_{n+1}^{-\mu\gamma}B\big(\eta(1-\mu),1-\eta\big)\|v\|_X,
\end{aligned}
\tag{3.8}
$$

which implies that

$$
\|\mathcal{T}_\alpha v - \mathcal{T}_{\alpha,n}v\|_X \leqslant \frac{c_+^{3-\gamma}}{c_-}T^{\alpha-\eta\mu}\lambda_{nN+1}^{-\mu\gamma}B\big(\eta(1-\mu),1-\eta\big)\|v\|_X
$$

Noting that, from the asymptotic property of the eigenvalue with $\lambda_{n+1} \to \infty$ as $n \to \infty$, we thus get $\mathcal{T}_{\alpha,n}v \to \mathcal{T}_\alpha v$ in X. It means that the operator $\mathcal{T}_\alpha$ is a compact operator from X into X.

To end this proof, we show that $\mathcal{T}_\alpha$ is a continuous operator. In fact, let $\{v_i\}_{i=1}^\infty \subset U_r$ and $v \in U_r$ with $\lim_{i\to\infty} v_i = v$ in U_r. Similarly to (3.5), it yields

$$
\|\mathcal{S}_\alpha(t)\circ\mathcal{P}_\alpha(\varsigma)v\| \leqslant \frac{c_+^2}{c_-}T^\alpha t^{-\eta}\varsigma^{\eta-1}\|v\|. \tag{3.9}
$$

Hence, we get

$$
\|\mathcal{S}_\alpha(t)\circ\mathcal{P}_\alpha(T-\tau)(v_i(\tau)-v(\tau))\|^2 \leqslant \frac{2c_+^2}{c_-^2}T^\alpha t^{-\eta}(T-\tau)^{\eta-1}(\|v_i(\tau)\|^2 + \|v(\tau)\|^2),
$$

which together Lebesgue's dominated convergence theorem and the same way as in arguments above, we get

$$
\|(\mathcal{T}_\alpha v_i)(t) - (\mathcal{T}_\alpha v)(t)\| \leqslant \int_0^T \|\mathcal{S}_\alpha(t)\circ\mathcal{P}_\alpha(T-\tau)(v_i(\tau)-v(\tau))\|\mathrm{d}\tau \to 0,
$$

as $i \to \infty$.

Hence, $\mathcal{T}_\alpha v_i \to \mathcal{T}_\alpha v$ as $i \to \infty$. Then, $\mathcal{T}_\alpha$ is continuous and we conclude that $\mathcal{T}_\alpha$ is completely continuous. The proof is completed. □

https://doi.org/10.1017/prm.2021.70 Published online by Cambridge University Press

Similarly to property 2 and lemma 3.2, it is not difficult to check the following lemma.

LEMMA 3.3. *The operator $\mathcal{S}_\alpha(t)$ is compact for every $t \in (0,T]$ and continuous in the uniform operator topology on $\mathcal{B}(L^2(\Omega))$ for all $t \in (0,T]$, and $\mathcal{P}_\alpha(t)$ is compact for every $t \geqslant 0$ and continuous in the uniform operator topology on $\mathcal{B}(L^2(\Omega))$ for all $t \geqslant 0$, respectively.*

4. Existence and uniqueness results

In this section, the existence and uniqueness results of mild solutions for the present problem are considered. To achieve this goal, we need the following assumption.

(Hf1) There exists a positive condition L_f such that $f : (0,T] \times L^2(\Omega) \to L^2(\Omega)$ is continuous with respect to u and it is measurable with respect to t and satisfies

$$\|f(t,u)\| \leqslant L_f \|u\|, \quad \forall u \in L^2(\Omega).$$

THEOREM 4.1. *Let $\gamma \in (0, \frac{1}{\alpha})$ and $g \in \mathcal{H}^{1-\gamma}(\Omega)$. Assume that (Hf1) holds. Then problem (1.1) has at least one mild solution provided with*

$$\kappa := c_+ L_f T^{\alpha-\eta} B(\alpha, 1-\eta) + \frac{c_+^2 \pi}{c_- \sin(\pi\eta)} L_f T^\alpha \leqslant \frac{1}{2}.$$

Proof. For each $r > 0$, denote a set

$$B_r = \{u \in X : \ \|u\|_X \leqslant r\}.$$

Clearly, B_r is a bounded closed and convex subset of X. To achieve the aim of this theorem, we need to show that the operator equation $u = \mathcal{F}u$ has a solution in B_r, where $\mathcal{F}$ is defined as

$$(\mathcal{F}u)(t) = \mathcal{S}_\alpha(t)g - (\mathcal{T}_\alpha f)(t) + (\mathcal{Q}_\alpha f)(t),$$

and $\mathcal{Q}_\alpha$ is defined by

$$(\mathcal{Q}_\alpha f)(t) = \int_0^t \mathcal{P}_\alpha(t-\tau) f(\tau, u(\tau))\, \mathrm{d}\tau.$$

Claim I. The operator $\mathcal{F} : C((0,T]; L^2(\Omega)) \to C((0,T]; L^2(\Omega))$ is well-defined. Indeed, from property 2, we know that $\mathcal{S}_\alpha(t)g \in C((0,T]; L^2(\Omega))$ for $g \in L^2(\Omega)$ so it is for $g \in \mathcal{H}^{1-\gamma}(\Omega)$. In view of lemma 3.2, we see that $(\mathcal{T}_\alpha f)(t) \in C((0,T]; L^2(\Omega))$. From the assumption of f, for any $t_1, t_2 \in [0,T]$ with $t_1 < t_2$, it yields

$$\begin{aligned}\|(\mathcal{Q}_\alpha f)(t_2) - (\mathcal{Q}_\alpha f)(t_1)\| &\leqslant \left\| \int_0^{t_1} (\mathcal{P}_\alpha(t_2-\tau) - \mathcal{P}_\alpha(t_1-\tau)) f(\tau, u(\tau))\, \mathrm{d}\tau \right\| \\ &\quad + \left\| \int_{t_1}^{t_2} \mathcal{P}_\alpha(t_2-\tau) f(\tau, u(\tau))\, \mathrm{d}\tau \right\| \\ &:= J_1 + J_2.\end{aligned}$$

https://doi.org/10.1017/prm.2021.70 Published online by Cambridge University Press

For J_1, from lemma 3.3, we have

$$J_1 \leqslant \int_0^{t_1} \|(\mathcal{P}_\alpha(t_2-\tau)-\mathcal{P}_\alpha(t_1-\tau))u(\tau)\|\mathrm{d}\tau$$
$$\leqslant \frac{L_f t_1^{1-\eta}}{1-\eta}\|u\|_X \sup_{\tau\in[0,t_1]} \|\mathcal{P}_\alpha(t_2-\tau)-\mathcal{P}_\alpha(t_1-\tau)\|_{\mathcal{B}(L^2(\Omega))}.$$

which implies that $J_1 \to 0$ as $t_2 \to t_1$. As for J_2, we have

$$J_2 \leqslant \int_{t_1}^{t_2} (t_2-\tau)^{\alpha-1}\sqrt{\sum_{k=1,k\in\Pi}^{\infty} |\mathcal{E}_{\alpha,\alpha}(-\lambda_k(t_2-\tau)^\alpha)|^2\,|f_k(\tau,u(\tau))|^2\mathrm{d}\tau}$$
$$\leqslant L_f c_+ \int_{t_1}^{t_2} (t_2-\tau)^{\alpha-1}\|u(\tau)\|\mathrm{d}\tau$$
$$\leqslant \frac{L_f c_+}{1-\eta}\|u\|_X (t_2-t_1)^{\alpha-\eta} \to 0, \quad \text{as } t_2 \to t_1.$$

Thus, $\mathcal{Q}_\alpha f \in C([0,T];L^2(\Omega))$. Combining with above arguments, for any $u \in C((0,T];L^2(\Omega))$, we obtain $\mathcal{F}u \in C((0,T];L^2(\Omega))$.

Claim II. The operator $\mathcal{F}u \in B_r$ for any $u \in B_r$.

From the inequality $(1+a)^b \leqslant 1+a^b$ and $(1+c)^2 \leqslant 1+c^2$ for $a,c \geqslant 0$ and $b \in [0,1]$, it is clear from lemma 2.3 and (2.3) that

$$\|\mathcal{S}_\alpha(t)g\|^2 \leqslant \frac{c_+^2}{c_-^2}\sum_{k=1,k\in\Pi}^{\infty}\left(\frac{1+\lambda_k T^\alpha}{1+\lambda_k t^\alpha}\right)^2 |(g,\phi_k)|^2$$
$$\leqslant \frac{c_+^2}{c_-^2}T^{2\eta}t^{-2\eta}\sum_{k=1,k\in\Pi}^{\infty}\left(\frac{1+\lambda_k T^\alpha}{1+\lambda_k t^\alpha}\right)^{2(1-\gamma)} \lambda_k^{-2(1-\gamma)}\lambda_k^{2(1-\gamma)}|(g,\phi_k)|^2$$
$$\leqslant \frac{2c_+^2}{c_-^2}T^{2\eta}t^{-2\eta}\sum_{k=1,k\in\Pi}^{\infty}\left(\lambda_k^{2(1-\gamma)}+T^{2\alpha(1-\gamma)}\right)|(g,\phi_k)|^2.$$

The embedding $\mathcal{H}^{1-\gamma}(\Omega) \hookrightarrow L^2(\Omega)$ implies

$$\|\mathcal{S}_\alpha(t)g\| \leqslant C_T t^{-\eta}\|g\|_{\mathcal{H}^{1-\gamma}(\Omega)}, \tag{4.1}$$

where $C_T = c_+\sqrt{2C_3}(T^\eta + T^\alpha)/c_-$ and C_3 is a positive constant. Therefore, we deduce that $\|\mathcal{S}_\alpha(\cdot)g\|_X \leqslant C_T\|g\|_{\mathcal{H}^{1-\gamma}(\Omega)}$.

It is similar to (3.6), for any $u \in B_r$, that

$$\|(\mathcal{T}_\alpha f)(t)\| \leqslant \frac{c_+^2}{c_-}T^\alpha L_f t^{-\eta}\int_0^T (T-\tau)^{\eta-1}\|u(\tau)\|\mathrm{d}\tau \leqslant \frac{c_+^2\pi}{c_-\sin(\pi\eta)}T^\alpha L_f t^{-\eta}\|u\|_X,$$

which deduces $\|\mathcal{T}_\alpha f\|_X \leqslant c_+^2 L_f T^\alpha r/c_-$. Moreover, from (2.3), we have

$$\|(\mathcal{Q}_\alpha f)(t)\| \leqslant c_+ L_f \int_0^t (t-\tau)^{\alpha-1}\|u(\tau)\|\mathrm{d}\tau \leqslant c_+ L_f t^{\alpha-\eta}B(\alpha,1-\eta)\,r.$$

https://doi.org/10.1017/prm.2021.70 Published online by Cambridge University Press

Therefore, one can selected r large enough such that

$$C_T\|g\|_{\mathcal{H}^{1-\gamma}(\Omega)}+\frac{c_+^2\pi}{c_-\sin(\pi\eta)}L_fT^\alpha r+c_+L_fT^\alpha B(\alpha,1-\eta)r\leqslant r,$$

and then we get

$$\|\mathcal{F}u\|_X\leqslant\|\mathcal{S}_\alpha(\cdot)g\|_X+\|\mathcal{T}_\alpha f\|_X+\|\mathcal{Q}_\alpha f\|_X\leqslant r.$$

This implies that $\mathcal{F}(B_r)\subseteq B_r$.

Claim III. Operator $\mathcal{F}$ is completely continuous.

Obviously, from lemmas 3.2 and 3.3, we just need to show that $\mathcal{F}$ is a completely continuous operator. Indeed, by lemma 3.2, for every $t\in[0,T]$, it is sufficient to prove that $\mathcal{Q}_\alpha$ is completely continuous in X. Since $\mathcal{P}_\alpha(t)$ is compact for every $t\in[0,T]$ in view of lemma 3.3, we can structure a family of finite dimension compact operators as the same way in lemma 3.2 by

$$(\mathcal{Q}_{\alpha,n}f)(t)=\int_0^t\mathcal{P}_{\alpha,n}(t-\tau)f(\tau,u(\tau))\,\mathrm{d}\tau,$$

for every $n\in\Pi$, in which

$$\mathcal{P}_{\alpha,n}(t)v=t^{\alpha-1}\sum_{k=1,k\in\Pi}^{n}\mathcal{E}_{\alpha,\alpha}(-\lambda_kt^\alpha)(v,\phi_k)\phi_k,\quad v\in L^2(\Omega).$$

It is clear that the $H^n(t)=\{t^\eta(\mathcal{Q}_{\alpha,n}f)(t):u\in B_r\}$ are relatively compact for every $t\in[0,T]$.

On the contrary, applying lemma 2.3 with respect to $\mu\in(0,1)$, and (2.3), we get

$$\begin{aligned}\|(\mathcal{Q}_\alpha f)(t)-(\mathcal{Q}_{\alpha,n}f)(t)\|&=\left\|\int_0^t(\mathcal{P}_\alpha(t-\tau)-\mathcal{P}_{\alpha,n}(t-\tau))f(\tau,u(\tau))\,\mathrm{d}\tau\right\|\\&\leqslant c_+L_f\lambda_{n+1}^{-\mu}\int_0^t(t-\tau)^{\alpha-1-\alpha\mu}\,\|u(\tau)\|\,\mathrm{d}\tau\\&\leqslant c_+L_fB(\alpha(1-\mu),1-\eta)t^{\alpha(1-\mu)-\eta}\lambda_{n+1}^{-2\mu}\,r,\end{aligned}$$

which implies that $\|\mathcal{Q}_\alpha f-\mathcal{Q}_{\alpha,n}f\|_X\to0$, as $n\to\infty$. Consequently, we derive that set $H^n(t)$ are arbitrarily close to the set $H(t)=\{t^\eta(\mathcal{Q}_\alpha f)(t):u\in B_r\}$. Thus, $H(t)$ is relatively compact in X for every $t\in[0,T]$. Therefore, $\mathcal{Q}_\alpha$ is a compact operator.

Furthermore, by using the same ways as in claim I and lemma 3.2, one can check that the set $H(t)$ is equicontinuous. Next, we will show that $\mathcal{Q}_\alpha$ is continuous.

Let $\{u_m\}_{m=1}^\infty\subset B_r$ be a sequence and $u\in B_r$ such that $\lim_{m\to\infty}u_m=u$, hence from the continuity assumption of f, it yields

$$\lim_{m\to\infty}f(t,u_m(t))=f(t,u(t)),\quad t\in(0,T],$$

and

$$\|f(\tau,u_m(\tau))-f(\tau,u(\tau))\|\leqslant L_f\|u_m(\tau)\|+L_f\|u(\tau)\|\leqslant2L_ft^{-\eta}\,r,$$

https://doi.org/10.1017/prm.2021.70 Published online by Cambridge University Press

which makes that $(t-\tau)^{\alpha-1}\tau^{-\eta} \in L^1(0,t)$ for a.e., $\tau \in (0,t)$. Therefore, Lebesgue's dominated convergence theorem implies

$$\begin{aligned}\|(\mathcal{Q}_\alpha f_m)(t) - (\mathcal{Q}_\alpha f)(t)\| &= \left\| \int_0^t \mathcal{P}_\alpha(t-\tau)(f(\tau,u_m(\tau)) - f(\tau,u(\tau)))\,\mathrm{d}\tau \right\| \\ &\leqslant c_+ \int_0^t (t-\tau)^{\alpha-1} \|f(\tau,u_m(\tau)) - f(\tau,u(\tau))\|\,\mathrm{d}\tau \\ &\to 0 \quad \text{as } m \to \infty.\end{aligned}$$

Hence, $\|\mathcal{Q}_\alpha f_m - \mathcal{Q}_\alpha f\|_X \to 0$ as $m \to \infty$ which shows that $\mathcal{Q}_\alpha$ is continuous, and then operator $\mathcal{F}$ is also continuous. By Ascoli–Arzelà theorem, we know that $\mathcal{F}$ is completely continuous. Consequently, Schauder's fixed point theorem shows that $\mathcal{F}$ has at least one fixed point on B_r, and then problem (1.1) has a mild solution. The proof is completed. □

Remark 4.2. Noting that the above existence result does not need to assume the Lipschitz type condition or smoothness of nonlinear functions, that is, the assumption condition of existence result is weaker than which in paper [**5**]. Specially, there is a common technique to study a PDE by transforming it into an abstract differential equation, and the concept introduced of mild solutions will be more convenient and useful to deal with such abstract problem. From this point of view, for an existence of nonlinear problem it is not necessary to assume that the function f has a smoother requirement likely [**1**] where f is continuously differentiable.

Remark 4.3. If the following condition

$$\|f(t,u)\| \leqslant L'_f \|u\|_X, \quad \forall u \in X, \tag{4.2}$$

substitutes for f in (Hf1) for some constant $L'_f > 0$, then the operator $\mathcal{T}_\alpha$ is also completely continuous. Obviously, the above condition is stronger than the condition (Hf1). However, we can pick a different range of η by $\eta \in [1,\alpha]$, that is $\gamma \in [\frac{1}{\alpha},1]$. By repeating the above proof process in theorem 4.1, we also establish an existence result of mild solutions (see below). In addition, we also remark that there exists a solution on $C^\alpha((0,T];L^2(\Omega))$ for $\eta = \alpha$ $(\gamma = 1)$.

Theorem 4.4. *Let $\gamma \in [\frac{1}{\alpha},1]$ and $g \in \mathcal{H}^{1-\gamma}(\Omega)$. Assume that (Hf1) holds with respect to f satisfying (4.2). Then problem (1.1) has at least one mild solution provided with*

$$\frac{c_+^2}{c_-\eta} L'_f T^{\eta+\alpha} + c_+ L'_f T^{\alpha+\eta} \leqslant \frac{1}{2}.$$

(Hf2) There exists a positive condition L''_f such that $f : (0,T] \times L^2(\Omega) \to L^2(\Omega)$ satisfies the following condition

$$\|f(t,u) - f(t,v)\| \leqslant L''_f \|u - v\|, \quad \forall u, v \in X.$$

Theorem 4.5. *Assume that the hypotheses of theorem 4.1 and (Hf2) hold. Then problem (1.1) has a unique mild solution.*

https://doi.org/10.1017/prm.2021.70 Published online by Cambridge University Press

Proof. The existence can be found in theorem 4.1, in the following, we will check the uniqueness of solution. For any $u, v \in X$, similarly to (3.6), we have

$$\begin{aligned}
\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\| &= \left\| \int_0^t \mathcal{P}_\alpha(t-\tau)(f(\tau, u(\tau)) - f(\tau, v(\tau)))\,\mathrm{d}\tau \right\| \\
&\quad + \left\| \int_0^T \mathcal{S}_\alpha(t) \circ \mathcal{P}_\alpha(T-\tau)(f(\tau, u(\tau)) - f(\tau, v(\tau)))\,\mathrm{d}\tau \right\| \\
&\leqslant c_+ L_f'' \int_0^t (t-\tau)^{\alpha-1} \|u(\tau) - v(\tau)\|\,\mathrm{d}\tau \\
&\quad + \frac{c_+^2}{c_-} T^\alpha L_f'' t^{-\eta} \int_0^T (T-\tau)^{\eta-1} \|u(\tau) - v(\tau)\|\,\mathrm{d}\tau,
\end{aligned}$$

which implies that $\|\mathcal{F}u - \mathcal{F}v\|_X \leqslant C_T' \|u - v\|_X$, where $C_T' = c_+ L_f'' T^\alpha B(\alpha, 1-\eta) + c_+^2 \pi L_f''/(c_- \sin(\pi\eta))$. Therefore, choosing L_f'' such that $C_T' \leqslant \kappa$ which given in theorem 4.1, we deduce that $\mathcal{F}$ is a contraction operator. Thus, the uniqueness of mild solution follows. □

(Hf3) There exist a positive constant p with $p > \max\{1/\eta, 1\}$ and a positive function $\vartheta(\cdot) \in L^p(0,T)$ such that $f : (0,T] \times L^2(\Omega) \to L^2(\Omega)$ is continuous with respect to u and it is measurable with respect to t and satisfies

$$\|f(t,u)\| \leqslant \vartheta(t), \quad \forall u \in X,\ t \in (0,T]. \tag{4.3}$$

THEOREM 4.6. *Let $g \in \mathcal{H}^{1-\gamma}(\Omega)$ for $\gamma \in (0,1]$. Assume that (Hf3) holds. Then problem (1.1) has at least one mild solution $u \in C^\eta((0,T]; L^2(\Omega)) \cap L^q(0,T; L^2(\Omega))$ for $1 < q < 1/\eta$.*

Proof. Let us return the proof of the compactness of $\mathcal{T}_\alpha$. By repeating the proving process of lemma 3.2, there is a similar method to show that $\mathcal{T}_{\alpha,n}$ converge uniformly to $\mathcal{T}_\alpha$ as $n \to \infty$. Indeed, for any $u \in L^2(\Omega)$, by applying lemma 2.3 with respect to $\mu \in (0, 1 - (1/p\eta))$, one can use the same way as in (3.8) that

$$\begin{aligned}
&\|(\mathcal{T}_\alpha f)(t) - (\mathcal{T}_{\alpha,n} f)(t)\| \\
&\quad \leqslant \frac{c_+^2}{c_-} T^\alpha t^{-\eta} \lambda_{n+1}^{-\gamma\mu} \int_0^T (T-\tau)^{\eta(1-\mu)-1} \|f(\tau, u(\tau))\| \mathrm{d}\tau \\
&\quad \leqslant \frac{c_+^2}{c_-} T^\alpha t^{-\eta} \lambda_{n+1}^{-\gamma\mu} \int_0^T (T-\tau)^{\eta(1-\mu)-1} \vartheta(\tau)\,\mathrm{d}\tau \\
&\quad \leqslant \frac{c_+^2}{c_-} \left(\frac{p-1}{\eta p(1-\mu) - 1} \right)^{1-1/p} T^{\alpha + \eta(1-\mu) - 1/p} t^{-\eta} \lambda_{n+1}^{-\gamma\mu} \|\vartheta\|_{L^p(0,T)},
\end{aligned} \tag{4.4}$$

which implies that $\|\mathcal{T}_\alpha f - \mathcal{T}_{\alpha,n} f\|_X \to 0$ as $n \to \infty$.

Next, we just show that the operator $\mathcal{F}$ maps B_r into itself, the remains of the proof of existence result similarly follows to theorem 4.1. In fact, we have for any

https://doi.org/10.1017/prm.2021.70 Published online by Cambridge University Press

$u \in B_r$, in view of (3.9) it yields

$$\begin{aligned} \|(\mathcal{T}_\alpha f)(t)\| &\leqslant \frac{c_+^2}{c_-} T^\alpha t^{-\eta} \int_0^T (T-\tau)^{\eta-1} \vartheta(\tau)\,\mathrm{d}\tau \\ &\leqslant \frac{c_+^2}{c_-} \left(\frac{p-1}{\eta p-1}\right)^{1-1/p} T^{\alpha+\eta-1/p} t^{-\eta} \|\vartheta\|_{L^p(0,T)}. \end{aligned} \tag{4.5}$$

Moreover, from (2.3), we get

$$\|(\mathcal{Q}_\alpha f)(t)\| \leqslant c_+ \int_0^t (t-\tau)^{\alpha-1} \vartheta(\tau)\,\mathrm{d}\tau \leqslant c_+ t^{\alpha-1/p} \|\vartheta\|_{L^p(0,T)}. \tag{4.6}$$

Therefore, one can select r large enough such that

$$\begin{aligned} &C_T \|g\|_{\mathcal{H}^{s(1-\gamma)}(\Omega)} + \frac{c_+^2}{c_-} \left(\frac{p-1}{\eta p-1}\right)^{1-1/p} T^{\alpha+\eta-1/p} \|\vartheta\|_{L^p(0,T)} \\ &\quad + c_+ T^{\alpha+\eta-1/p} \|\vartheta\|_{L^p(0,T)} \leqslant r, \end{aligned}$$

and then we get that $\mathcal{F}(B_r) \subseteq B_r$.

Finally, we will check that $u \in L^q(0,T;L^2(\Omega))$. In fact, one see from (2.3), the assumptions of f, and (4.1), (4.5), (4.6) that

$$\begin{aligned} &\|\mathcal{S}_\alpha(t) g\|_{L^q(0,T;L^2(\Omega))} + \|(\mathcal{T}_\alpha f)(t)\|_{L^q(0,T;L^2(\Omega))} + \|(\mathcal{Q}_\alpha f)(t)\|_{L^q(0,T;L^2(\Omega))} \\ &\leqslant \left(\frac{1}{1-\eta q}\right)^{1/q} \Bigg(C_T T^{1/q-\eta} \|g\|_{\mathcal{H}^{1-\gamma}(\Omega)} + c_+ T^{\alpha+1/q-1/p} \|\vartheta\|_{L^p(0,T)} \\ &+ \frac{c_+^2}{c_-} \left(\frac{p-1}{\eta p-1}\right)^{1-1/p} T^{\alpha+1/q-1/p} \|\vartheta\|_{L^p(0,T)} \Bigg) < \infty, \end{aligned}$$

which implies $u \in L^q(0,T;L^2(\Omega))$. Hence, the proof is completed. □

(Hρ) There exists a positive function $\rho(t) \in L^1(0,T)$ such that

$$\Lambda_\rho := \int_0^T (T-\tau)^{-1} \rho(\tau)\,\mathrm{d}\tau < \infty.$$

Noting that this function of (Hρ) will exist, for example, $\rho(t) = T - t$ for $t \in (0,T]$.

THEOREM 4.7. *Let $g \in \mathcal{H}^1(\Omega)$. Suppose that there exists a positive function $\vartheta(\cdot)$ satisfying (Hρ) such that $f : (0,T] \times L^2(\Omega) \to L^2(\Omega)$ is continuous with respect to u and it is measurable with respect to t and satisfies* (4.3). *Then the mild solutions belongs to $C([0,T];L^2(\Omega))$ for some $\eta \in (0,\alpha)$.*

Proof. According to the assumptions of f, it is not difficult to check that there exists at least one mild solution $u \in C^\eta((0,T];L^2(\Omega))$. In the sequel, we shall show

https://doi.org/10.1017/prm.2021.70 Published online by Cambridge University Press

$u \in C([0,T]; L^2(\Omega))$. Now, for any $0 \leqslant t_1 < t_2 \leqslant T$, it follows that

$$\begin{aligned} \|u(t_2) - u(t_1)\| \leqslant \|\mathcal{S}_\alpha(t_2)g - \mathcal{S}_\alpha(t_1)g\| + \|(\mathcal{T}_\alpha f)(t_1) - (\mathcal{T}_\alpha f)(t_2)\| \\ + \|(\mathcal{Q}_\alpha f)(t_2) - (\mathcal{Q}_\alpha f)(t_1)\|. \end{aligned} \tag{4.7}$$

Noting that if $g \in \mathcal{H}^2(\Omega)$, then by property 1, $\mathcal{S}_\alpha(t)g$ is bounded in $L^2(\Omega)$ for all $t \in [0,T]$. Hence, we first obtain that $\mathcal{S}_\alpha(\cdot)g \in C([0,T]; L^2(\Omega))$.

On the contrary, by using (i) in lemma 2.2, we have

$$\begin{aligned} \|(\mathcal{T}_\alpha f)(t_1) - (\mathcal{T}_\alpha f)(t_2)\| &\leqslant \frac{c_+^2}{c_-} \left| \int_{t_1}^{t_2} z^{\alpha-1-\eta} \mathrm{d}z \right| \int_0^T (T-\tau)^{\eta-1} \|f(\tau, u(\tau))\| \mathrm{d}\tau \\ &\leqslant \frac{c_+^2}{c_-} \frac{1}{\alpha - \eta} (t_2^{\alpha-\eta} - t_1^{\alpha-\eta}) T^\eta \Lambda_\vartheta. \end{aligned}$$

Next, we shall estimate the last case in (4.7). To begin with, by using (ii) in lemma 2.2, it follows that

$$\begin{aligned} \left\|(\mathcal{Q}_\alpha f)(t_2) - (\mathcal{Q}_\alpha f)(t_1)\right\| &\leqslant c_+ \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \vartheta(\tau) \mathrm{d}\tau \\ &\quad + c_+ \int_0^{t_1} \left| \int_{t_1-\tau}^{t_2-\tau} z^{\alpha-2} \mathrm{d}z \right| \vartheta(\tau)\, \mathrm{d}\tau \\ &\leqslant \frac{c_+ \alpha}{\alpha - 1} (t_2 - t_1)^{\alpha-1} \|\vartheta\|_{L^1(0,T)}. \end{aligned}$$

Thus, together with arguments above, let $t_2 \to t_1$, it is clear that $u(t_2) \to u(t_1)$ in $L^2(\Omega)$.

Moreover, setting $\varsigma = T - \tau$, for any $v \in L^2(\Omega)$ it yields

$$\|\mathcal{S}_\alpha(t) \circ \mathcal{P}_\alpha(\varsigma) v\|^2 \leqslant \frac{c_+^4}{c_-^2} \varsigma^{2(\alpha-1)} \sum_{k=1, k\in\Pi}^{\infty} \left(\frac{1 + \lambda_k T^\alpha}{1 + \lambda_k \varsigma^\alpha} \right)^2 |(v, \phi_k)|^2 \leqslant \frac{c_+^4}{c_-^2} T^{2\alpha} \varsigma^{-2} \|v\|^2.$$

Therefore, we have

$$\|(\mathcal{T}_\alpha f)(t)\| \leqslant \frac{c_+^2}{c_-} T^\alpha \int_0^T (T-\tau)^{-1} \|f(\tau, u(\tau))\| \, \mathrm{d}\tau \leqslant \frac{c_+^2}{c_-} T^\alpha \Lambda_\vartheta.$$

In addition, one has

$$\|(\mathcal{Q}_\alpha f)(t)\| \leqslant c_+ \int_0^t (t-\tau)^{\alpha-1} \vartheta(\tau)\, \mathrm{d}\tau \leqslant c_+ T^\alpha \Lambda_\vartheta.$$

Consequently, we deduce that $u \in C([0,T]; L^2(\Omega))$. The proof is completed. □

https://doi.org/10.1017/prm.2021.70 Published online by Cambridge University Press

5. Regularization

Let $\mathbf{R}(\epsilon, \lambda_k)$ be identity to

$$\mathbf{R}(\epsilon, \lambda_k) = \frac{|\mathcal{E}_\alpha(-\lambda_k T^\alpha)|^2}{|\mathcal{E}_\alpha(-\lambda_k T^\alpha)|^2 + \epsilon\lambda_k^2}, \quad \epsilon > 0,\ k \in \Pi,$$

and let

$$C_\epsilon = \begin{cases} \mathbf{C}_1 \epsilon^{\sigma/4}, & 0 < \sigma < 4, \\ \mathbf{C}_2 \epsilon, & \sigma \geqslant 4. \end{cases}$$

where $\mathbf{C}_1 = C(\sigma, c_-)$, $\mathbf{C}_2 = C(\sigma, c_-, \lambda_1) > 0$ for $\sigma > 0$.

Since $\mathcal{S}_\alpha(t)$ is not bounded linear operator on $L^2(\Omega)$ at time $t = 0$, it means that problem (1.1) is not stable on $L^\infty(0, T; L^2(\Omega))$, and it can lead to the general ill-posed problem, in the sequel, we define a family of regularizing operators $\mathcal{S}_\alpha^\epsilon(t)$ with the main idea of a general filter regularization method by

$$\mathcal{S}_\alpha^\epsilon(t) v = \sum_{k=1, k\in\Pi}^{\infty} \mathbf{R}(\epsilon, \lambda_k) \frac{\mathcal{E}_\alpha(-\lambda_k t^\alpha)}{\mathcal{E}_\alpha(-\lambda_k T^\alpha)} (v, \phi_k)\phi_k.$$

Then we can obtain the following regularized solution by the same way as in above theorems

$$\mathbf{u}^\epsilon(t) = \mathcal{S}_\alpha^\epsilon(t) g^\epsilon - (\mathcal{T}_\alpha^\epsilon f)(t, \mathbf{u}^\epsilon) + (\mathcal{Q}_\alpha f)(t, \mathbf{u}^\epsilon),$$

where g^ϵ is a noisy final data and $\epsilon > 0$ is a noise level which is assumed to satisfy

$$\|g^\epsilon - g\| \leqslant \epsilon, \tag{5.1}$$

and hence we can rewrite it as follows

$$\begin{aligned} \mathbf{u}^\epsilon(t, x) = & \sum_{k=1, k\in\Pi}^{\infty} \mathbf{R}(\epsilon, \lambda_k) \frac{\mathcal{E}_\alpha(-\lambda_k t^\alpha)}{\mathcal{E}_\alpha(-\lambda_k T^\alpha)} g_k^\epsilon \phi_k(x) \\ & - \sum_{k=1, k\in\Pi}^{\infty} \mathbf{R}(\epsilon, \lambda_k) \frac{\mathcal{E}_\alpha(-\lambda_k t^\alpha)}{\mathcal{E}_\alpha(-\lambda_k T^\alpha)} \\ & \times \left[\int_0^T (T-\tau)^{\alpha-1} \mathcal{E}_{\alpha,\alpha}(-\lambda_k (T-\tau)^\alpha) f_k(\tau, \mathbf{u}^\epsilon(\tau))\, \mathrm{d}\tau \right] \phi_k(x) \\ & + \sum_{k=1, k\in\Pi}^{\infty} \left[\int_0^t (t-\tau)^{\alpha-1} \mathcal{E}_{\alpha,\alpha}(-\lambda_k (T-\tau)^\alpha) f_k(\tau, \mathbf{u}^\epsilon(\tau))\, \mathrm{d}\tau \right] \phi_k(x). \end{aligned}$$

https://doi.org/10.1017/prm.2021.70 Published online by Cambridge University Press

Let us introduce the function u_α by

$$u_\alpha(t,x) = \sum_{k=1,k\in\Pi}^{\infty} \mathbf{R}(\epsilon,\lambda_k)\frac{\mathcal{E}_\alpha(-\lambda_k t^\alpha)}{\mathcal{E}_\alpha(-\lambda_k T^\alpha)} g_k\phi_k(x) - \sum_{k=1,k\in\Pi}^{\infty} \mathbf{R}(\epsilon,\lambda_k)\frac{\mathcal{E}_\alpha(-\lambda_k t^\alpha)}{\mathcal{E}_\alpha(-\lambda_k T^\alpha)} \times \left[\int_0^T (T-\tau)^{\alpha-1}\mathcal{E}_{\alpha,\alpha}(-\lambda_k(T-\tau)^\alpha) f_k(\tau,u(\tau))\,\mathrm{d}\tau\right]\phi_k(x) + \sum_{k=1,k\in\Pi}^{\infty}\left[\int_0^t (t-\tau)^{\alpha-1}\mathcal{E}_{\alpha,\alpha}(-\lambda_k(T-\tau)^\alpha) f_k(\tau,u(\tau))\,\mathrm{d}\tau\right]\phi_k(x).$$

THEOREM 5.1. *Assume that* $u(0)\in\mathcal{H}^\sigma(\Omega)$ *for any* $\sigma>0$ *and there exists a positive constant* $\mathbf{M}$ *such that*

$$\|u(0)\|_{\mathcal{H}^\sigma(\Omega)} \leqslant \mathbf{M}.$$

Furthermore, there exists a positive function $\vartheta(\cdot)$ *satisfying (*$H\rho$*) such that* f *is continuous with respect to* u *and it is measurable with respect to* t*, satisfies* (4.3) *and the following conditions*

$$\|f(t,u)-f(t,v)\| \leqslant \vartheta(t)\|u-v\|, \quad \forall u,v\in L^2(\Omega), \quad t\in(0,T].$$

If $(\frac{c_+^2}{c_-}+c_+T^\alpha)\Lambda_\vartheta<1$*, then*

$$\|\mathbf{u}^\epsilon-u\|_{L^\infty(0,T;L^2(\Omega))} \leqslant \left[1-\left(\frac{c_+^2}{c_-}+c_+T^\alpha\right)\Lambda_\vartheta\right]^{-1}\left[\frac{c_+^3\sqrt{\epsilon}}{2c_-}+c_+\mathbf{M}C_\epsilon\right].$$

Proof. According to the assumption of f, it is easy to check that there is a unique solution $\mathbf{u}^\epsilon\in L^\infty(0,T;L^2(\Omega))$ and $\mathbf{u}^\epsilon-u\in L^\infty(0,T;L^2(\Omega))$ for each $\epsilon>0$. In fact, we only need to give an exact upper bound of $\|\mathbf{u}^\epsilon-u\|_{L^\infty(0,T;L^2(\Omega))}$.

By the triangle inequality, we have

$$\|\mathbf{u}^\epsilon(t)-u(t)\| \leqslant \|\mathbf{u}^\epsilon(t)-u_\alpha(t)\|+\|u_\alpha(t)-u(t)\|.$$

First, we estimate $\|\mathbf{u}^\epsilon(t)-u_\alpha(t)\|$.

Indeed, in view of the inequalities in lemma 2.5, we have

$$\frac{1}{|\mathcal{E}_\alpha(-\lambda_k T^\alpha)|^2+\epsilon\lambda_k^2} \leqslant \frac{1}{c_-^2/\lambda_k^2+\epsilon\lambda_k^2} \leqslant \frac{1}{2c_-\sqrt{\epsilon}}.$$

Thus, by virtue of inequality $z/(z+a)\leqslant 1$ for any $z,a\geqslant 0$, it yields that

$$|\mathbf{R}(\epsilon,\lambda_k)|\leqslant 1, \quad \left|\mathbf{R}(\epsilon,\lambda_k)\frac{\mathcal{E}_\alpha(-\lambda_k t^\alpha)}{\mathcal{E}_\alpha(-\lambda_k T^\alpha)}\right| \leqslant \frac{c_+^3}{2c_-\sqrt{\epsilon}},$$

https://doi.org/10.1017/prm.2021.70 Published online by Cambridge University Press

which implies from the assumption of f and (5.1) that

$$\begin{aligned}\|\mathbf{u}^\epsilon(t)-u_\alpha(t)\| &\leqslant \frac{c_+^3}{2c_-\sqrt{\epsilon}}\|g^\epsilon-g\|+\frac{c_+^2}{c_-}\int_0^T(T-\tau)^{-1}\|f(\tau,\mathbf{u}^\epsilon(\tau))-f(\tau,u(\tau))\|\,\mathrm{d}\tau\\ &\quad+c_+\int_0^t(t-\tau)^{\alpha-1}\|f(\tau,\mathbf{u}^\epsilon(\tau))-f(\tau,u(\tau))\|\,\mathrm{d}\tau\\ &\leqslant \frac{c_+^3}{2c_-\sqrt{\epsilon}}\|g^\epsilon-g\|+\frac{c_+^2}{c_-}\int_0^T(T-\tau)^{-1}\vartheta(\tau)\|\mathbf{u}^\epsilon(\tau)-u(\tau)\|\,\mathrm{d}\tau\\ &\quad+c_+\int_0^t(t-\tau)^{\alpha-1}\vartheta(\tau)\|\mathbf{u}^\epsilon(\tau)-u(\tau)\|\,\mathrm{d}\tau\\ &\leqslant \frac{c_+^3\sqrt{\epsilon}}{2c_-}+\left(\frac{c_+^2}{c_-}+c_+T^\alpha\right)\Lambda_\vartheta\|\mathbf{u}^\epsilon-u\|_{L^\infty(0,T;L^2(\Omega))}.\end{aligned}$$

Next, we estimate $\|u_\alpha(t)-u(t)\|$.

Obviously, let us return the initial value $u_k(0)$, it yields

$$\begin{aligned}\|u_\alpha(t)-u(t)\| &= \sqrt{\sum_{k=1,k\in\Pi}^{\infty}\left|(\mathbf{R}(\epsilon,\lambda_k)-1)\frac{\mathcal{E}_\alpha(-\lambda_k t^\alpha)}{\mathcal{E}_\alpha(-\lambda_k T^\alpha)}\right|}\\ &\qquad\overline{\times\left[h_k-\int_0^T(T-\tau)^{\alpha-1}\mathcal{E}_{\alpha,\alpha}(-\lambda_k(T-\tau)^\alpha)f_k(\tau,u(\tau))\,\mathrm{d}\tau\right]}\\ &\leqslant c_+\sqrt{\sum_{k=1,k\in\Pi}^{\infty}\left|\frac{\epsilon\lambda_k^2}{|\mathcal{E}_\alpha(-\lambda_k T^\alpha)|^2+\epsilon\lambda_k^2}\right|^2(u(0),\phi_k)^2}\\ &\leqslant c_+\sup_{k\in\mathbb{N}\cap\Pi}A(k)\|u(0)\|_{\mathcal{H}^\sigma(\Omega)},\end{aligned}$$

where

$$A(k)=\frac{\epsilon\lambda_k^{2-\sigma}}{|\mathcal{E}_\alpha(-\lambda_k T^\alpha)|^2+\epsilon\lambda_k^2}.$$

It follows from [**21**, lemma 2.5] and lemma 2.5 that

$$A(k)\leqslant\frac{\epsilon\lambda_k^{4-\sigma}}{c_-^2+\epsilon\lambda_k^4}\leqslant\begin{cases}\mathbf{C}_1\,\epsilon^{\sigma/4}, & 0<\sigma<4,\\ \mathbf{C}_2\,\epsilon, & \sigma\geqslant 4.\end{cases}$$

Therefore, we have

$$\begin{aligned}\|\mathbf{u}^\epsilon-u\|_{L^\infty(0,T;L^2(\Omega))} &\leqslant \frac{c_+^3\sqrt{\epsilon}}{2c_-}+\left(\frac{c_+^2}{c_-}+c_+T^\alpha\right)\Lambda_\vartheta\|\mathbf{u}^\epsilon-u\|_{L^\infty(0,T;L^2(\Omega))}\\ &\quad+c_+\mathbf{M}C_\epsilon.\end{aligned}$$

Consequently, since $(c_+^2/c_-+c_+T^\alpha)\Lambda_\vartheta<1$, we deduce the desired conclusion. □

Acknowledgement

The authors are very grateful to the editor and reviewers for their valuable comments. Project supported by the National Natural Science Foundation of China (12071396) and the Fundo para o Desenvolvimento das Ciências e da Tecnologia of Macau (Grant No. 0074/2019/A2).

References

1 E. Alvarez, C. Gal, V. Keyantuo and M. Warma. Well-posedness results for a class of semi-linear super-diffusive equations. *Nonlinear Anal.* **181** (2019), 24–61.

2 D. Chen, B. Hofmann and J. Zou. Regularization and convergence for ill-posed backward evolution equations in Banach spaces. *J. Differ. Equ.* **265** (2018), 3533–3566.

3 J. W. He and L. Peng. Approximate controllability for a class of fractional stochastic wave equations. *Comput. Math. Appl.* **78** (2019), 1463–1476.

4 J. Huang, G. Wang and J. Xiong. A maximum principle for partial information backward stochastic control problems with applications. *SIAM J. Control. Optim.* **48** (2009), 2106–2117.

5 L. N. Huynh, Y. Zhou, D. O'Regan and N. H. Tuan. Fractional Landweber method for an initial inverse problem for time-fractional wave equations. *Appl. Anal* **100** (2021), 860–878.

6 J. Jia, J. Peng, J. Gao and Y. Li. Backward problem for a time-space fractional diffusion equation. *Inverse Probl. Imaging* **12** (2018), 773–800.

7 A. A. Kilbas, H. M. Srivastava and J. J. Trujillo. *Theory and applications of fractional differential equations*, vol. 204, pp. 323 (Amsterdam: Elsevier Science B.V., 2006).

8 I. Kim, K. H. Kim and S. Lim. An $L_q(L_p)$-theory for the time fractional evolution equations with variable coefficients. *Adv. Math.* **306** (2017), 123–176.

9 Y. Li, Y. Wang and W. Deng. Galerkin finite element approximations for stochastic space–time fractional wave equations. *SIAM J. Numer. Anal.* **55** (2017), 3173–3202.

10 F. Mainardi. Fractional relaxation-oscillation and fractional diffusion-wave phenomena. *Chaos, Solitons Fractals* **7** (1996), 1461–1477.

11 F. Mainardi. *Fractional calculus and waves in linear viscoelasticity, an introduction to mathematical models* (London: Imperial College Press, 2010).

12 F. Mainardi and P. Paradisi. A model of diffusive waves in viscoelasticity based on fractional calculus. *Proc. 36th IEEE Conf. Decis. Control* **5** (1997), 4961–4966.

13 E. Otárola and A. J. Salgado. Regularity of solutions to space–time fractional wave equations: a PDE approach. *Fract. Calc. Appl. Anal.* **21** (2018), 1262–1293.

14 L. Peng and Y. Huang. On nonlocal backward problems for fractional stochastic diffusion equations. *Comput. Math. Appl.* **78** (2019), 1450–1462.

15 I. Podlubny. *Fractional differential equations* (San Diego: Academic Press, 1999).

16 K. Sakamoto and M. Yamamoto. Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. *J. Math. Anal. Appl.* **382** (2011), 426–447.

17 N. H. Tuan, V. Ao, L. N. Huynh and Y. Zhou. Regularization of a backward problem for the inhomogeneous time-fractional wave equation. *Math. Method. Appl. Sci.* **43** (2020), 5450–5463.

18 N. H. Tuan, A. Debbouche and T. B. Ngoc. Existence and regularity of final value problems for time fractional wave equations. *Comput. Math. Appl.* **78** (2019), 1396–1414.

19 N. H. Tuan, L. N. Huynh, T. B. Ngoc and Y. Zhou. On a backward problem for nonlinear fractional diffusion equations. *Appl. Math. Lett.* **92** (2019), 76–84.

20 J. Wang, T. Wei and Y. Zhou. Optimal error bound and simplified Tikhonov regularization method for a backward problem for the time-fractional diffusion equation. *J. Comput. Appl. Math.* **279** (2015), 277–292.

21 T. Wei and J. Wang. A modified quasi-boundary value method for the backward time-fractional diffusion problem. *ESAIM: Math. Model. Numer. Anal.* **48** (2014), 603–621.

22 T. Wei and Y. Zhang. The backward problem for a time-fractional diffusion-wave equation in a bounded domain. *Comput. Math. Appl.* **75** (2018), 3632–3648.

23 Y. Zhou. *Basic theory of fractional fifferential equations* (Singapore: World Scientific, 2014).

24 Y. Zhou, J. W. He, B. Ahmad and N. H. Tuan. Existence and regularity results of a backward problem for fractional diffusion equations. *Math. Meth. Appl. Sci.* **42** (2019), 6775–6790.