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Stability of static current sheets connecting plane magnetic null points

MANUEL NÚÑEZ

Departamento de Análisis Matemático. Universidad de Valladolid 47005 Valladolid, Spain

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The configuration created in the plane by the separation of a magnetic hyperbolic null point into two critical points connected by a current sheet is considered. The main parameters are the orders of the zeros of these new null points, which determine the local topology of the magnetic field. It is shown that when the magnetic field is static, the fluid tends to flow orthogonally to the field in the vicinity of the sheet endpoints. Moreover, the Lorentz force pushes one of them towards the other, so the configuration tends to collapse again into a single null point except when the order of both is precisely $\frac{1}{2}$.

1. Introduction

Magnetic critical points play a major role in our present understanding of the fast conversion of magnetic to kinetic energy. The reason is that in a magnetofluid of very small resistivity (large Reynolds number), such as most astrophysical plasmas, the magnetic field is practically frozen into the fluid. However, at two-dimensional X-type critical points, large gradients of the field and electric currents may occur, which allow the magnetic lines to break from the flow and reconnect. This sudden relaxation of the magnetic topology may rapidly release large quantities of magnetic energy to produce motion or heat. This process was invoked by Petschek (1964) and Syrovatskii (1971) in their models of fast reconnection, and since then a sizable literature has been produced on this subject (see e.g. Sulem et al. 1985; Priest and Forbes 1986; Priest and Lee 1990; Priest et al. 1994; Titov and Priest 1997; and references therein). It was recognized early that under certain circumstances, such as the head-on collision of two masses of plasma, the saddle point tends to break into two Y points connected by a singular line where the magnetic field is discontinuous and hence a current sheet appears. In fact, one does not need to invoke specific motions of the fluid for this to occur: even in the absence of flow, most magnetic static configurations contain discontinuities (Parker 1994, pp. 16–25); also, a network of magnetic flux cells develops this structure spontaneously (Parker 1994 pp. 167–170). From a strictly mathematical viewpoint, it has been shown that relaxation of magnetic fields containing saddle points tends to break them into Y points (Bajer 1989; Linardatos 1993). Recent numerical simulations (Friedel et al. 1997) confirm this fact. It is true that in principle (Klapper 1998) this process seems to require an infinite amount of time, but in fact the current density grows exponentially in time (Friedel et al. 1997), so we may safely assume that, within the limits set by a small but positive resistivity, a current sheet occurs. This resistivity will require that the current sheet have some thickness and not be an actual mathematical singularity, but study of the easier ideal case will provide a limit that, although it cannot be physically reached in either time or space, is similar enough to the real situation to allow deduction of some meaningful consequences. All the theoretical and computational studies mentioned above yield magnetic fields that are parallel to the current sheet and run in opposite senses at both sides of it. However, there is in principle no mathematical reason why a saddle point should evolve in this manner, because the only invariant is the index of the point, which may be conserved in other ways. One purpose of this paper is to prove that all other configurations are unstable, and therefore the usual picture seems to be justified. Finally, we deal here with quasistatic configurations, meaning essentially that the magnetic field is not supposed to change in time. Although this cannot be correct in the long run, it is justified by the fact that we are studying the stability of a certain topology, and in fact it is an accepted simplifying hypothesis for many models of fast reconnection.

2. The magnetic vector field

The main topological invariant is the *index* (see e.g. Coddington and Levinson 1984, pp. 398–403), which for an X-type critical point is -1. This means that if Λ is a closed curve enclosing it and no other neutral point,

$$\frac{1}{2\pi}\int_{\Lambda}\frac{B_1\,B_2'-B_2\,B_1'}{|\mathbf{B}|^2}\,ds \;=\; -1, \tag{1}$$

If we stretch Λ continuously so as to retain in its interior the current sheet generated by the decomposition of the null point, and if the magnetic field evolves continuously within Λ, then the integral in (1) remains continuous as a function of time. Since it takes only integer values, it is constant. Assuming that the only zeros of $\mathbf{B}$ are located at the endpoints of the current line Γ, those zeros must have orders α and β, with $\alpha+\beta=1$. As usual in this type of problem, we take advantage of the irrotational character of $\mathbf{B}$ outside Γ ($\partial B_x/\partial y - \partial B_y/\partial x = 0$) together with the general fact that $\mathbf{B}$ is solenoidal, to ensure that the function $f = B_y + iB_x$ is analytic outside Γ. Often (Biskamp 1993) a scalar potential vector ψ ($\partial\psi/\partial x = B_x$, $\partial\psi/\partial y = B_y$) is used instead of f. However, there is in general no guarantee that ψ exists at all, because the domain of definition minus Γ is not simply connected, so we shall avoid this approach. We intend to study the behaviour of $\mathbf{B}$ near Γ, and specifically near the endpoints, where the topology is non-trivial. Let us fix one of them at 0 and assume that in a small neighbourhood of it, after an eventual rotation, Γ is the negative real axis. (In fact, we replace Γ by its tangent vector at the endpoint). Then $f(z)$ behaves locally as $re^{i\alpha\phi}z^{\alpha}$, $r>0$, $0<\alpha<1$, where in the expression of z^{α} we take the main argument (belonging to $(-\pi,\pi)$), so that Γ is the branch line of f. $re^{i\alpha\phi}$ is the general expression for the complex number that multiplies the lower-order term in the expression of $f(z)$. This means that

$$\mathbf{B} = r|z|^{\alpha}(\cos[\alpha(\arg\; z+\phi)], \sin[\alpha(\arg\; z+\phi)]). \tag{2}$$

If we impose the condition that the normal component of $\mathbf{B}$ should be continuous at Γ then

$$\cos[\alpha(\pi+\phi)] = \cos[\alpha(-\pi+\phi)], \tag{3}$$

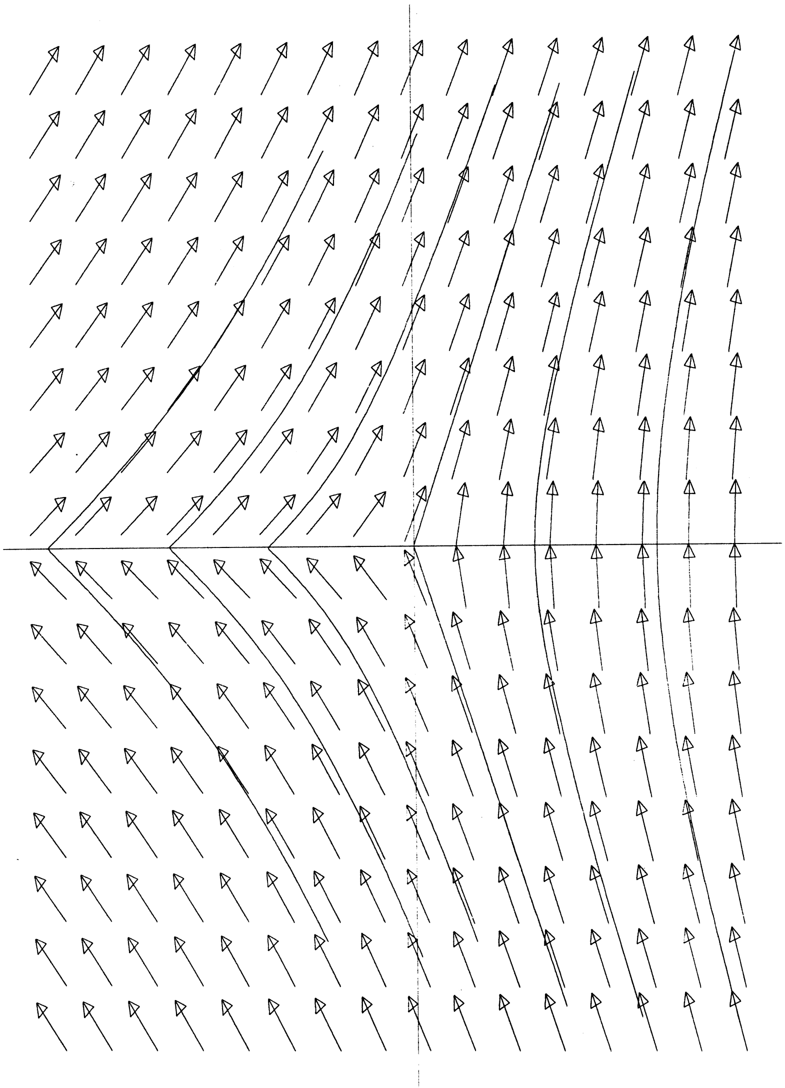

Figure 1. Magnetic field directions for $\alpha = \frac{1}{4}$, i.e. $\alpha < \frac{1}{2}$.

which means that $\sin\,\alpha\phi\,\sin\alpha\pi = 0$. Since $0 < \alpha < 1$, either $\phi = 0$ or $\phi = \pi/\alpha$, i.e. $e^{i\alpha\phi} = \pm 1$. Thus **B** is in fact unique except for a real factor, which does not modify the topology of the vector field. We therefore assume that

$$\mathbf{B} = |z|^{\alpha}(\cos\,\alpha \arg\,z, \sin\,\alpha \arg\,z). \tag{4}$$

This vector field possesses some straight integral lines, which emanate from 0 (separatrices). If one of them is given by $\arg\,z = \lambda$ then, the vectors $(\sin\,\alpha\lambda, \cos\,\alpha\lambda)$ and $(\cos\,\lambda, \sin\,\lambda)$ must be parallel. This amounts to

$$\cot\,\alpha\lambda = \tan\,\lambda. \tag{5}$$

There are two solutions within $(-\pi, \pi)$ to this equation if $\alpha \leqslant \frac{1}{2}$, and four if $\frac{1}{2} < \alpha < 1$. In the limiting case $\alpha = \frac{1}{2}$, Γ itself is a separatrix, and the normal component

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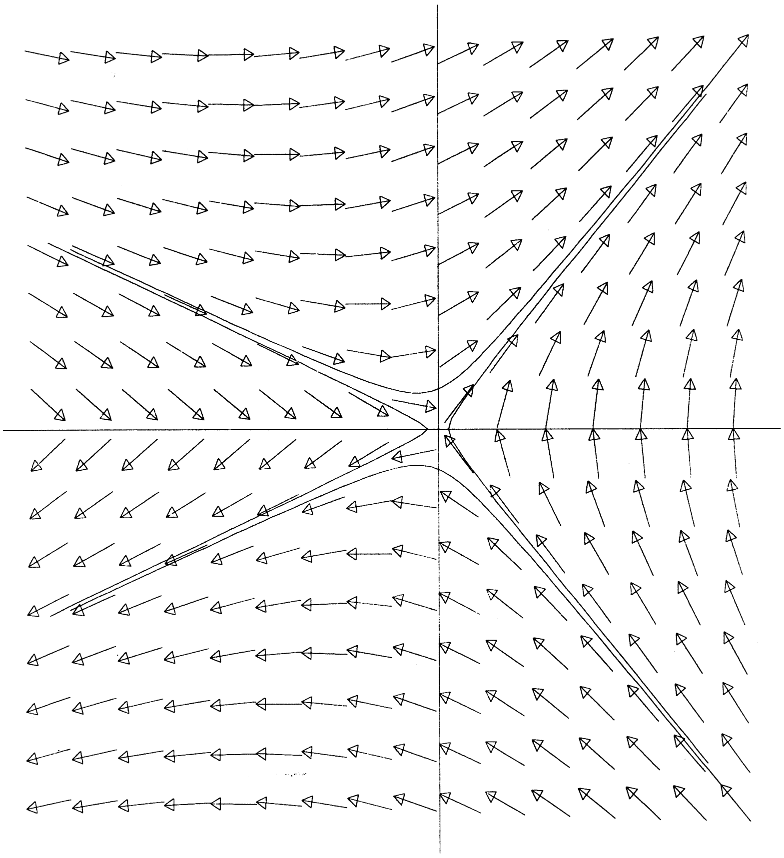

Figure 2. Magnetic field directions for $\alpha = \frac{3}{4}$, i.e. $\alpha > \frac{1}{2}$.

of the field vanishes there. In general, the tangential components $\pm|z|^{\alpha}\sin\alpha\pi$ differ in the current density $2|z|^{\alpha}\sin\alpha\pi$. The local topology of the field is as follows: for $\alpha < \frac{1}{2}$, we get essentially a parallel field of directions (Fig. 1). The field lines, however, are non-differentiable at Γ. For $\alpha > \frac{1}{2}$, the topology is similar to that of the null X points (Fig. 2). Only the field lines fail again to be smooth at Γ. The connection between both types of points shows how the original X point breaks into two new ones: one of the separatrices transforms into a section of parallel field lines, transverse to the current sheet Γ, and the rest is left pretty much as it was. The limiting case $\alpha = \frac{1}{2}$ is somewhat different (Fig. 3): the field lines are antiparallel at Γ, so that the normal component is 0. Now it is the null point that explodes into Γ, while the topology of the field lines does not change.

3. Local behaviour of the fluid velocity

If, as stated in Sec. 1, we assume that the magnetic field is static and the fluid incompressible then the induction equation for the velocity **u** becomes

$$\nabla \times (\mathbf{u} \times \mathbf{B}) = \mathbf{u} \cdot \nabla \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{u} = \mathbf{0}, \tag{6}$$

which is an ordinary differential equation along every magnetic field line. There are two complementary approaches to this equation. One is to write $\mathbf{u} \times \mathbf{B}$ as $u_{\perp} B \mathbf{e}_z$,

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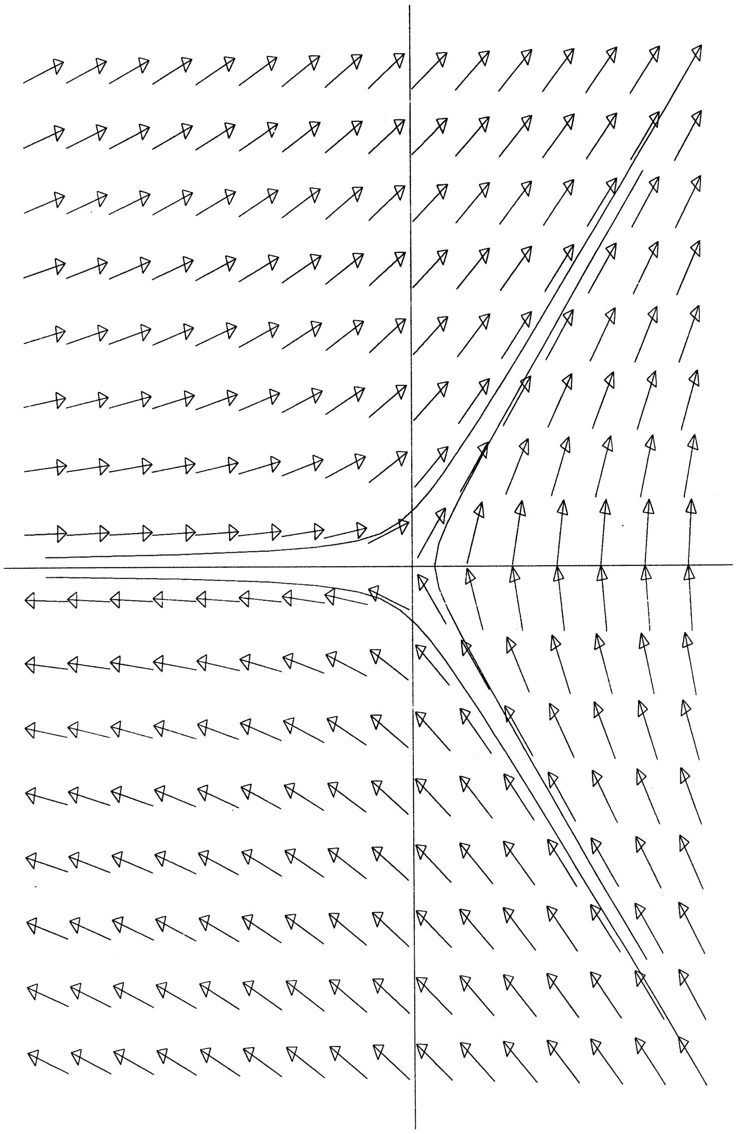

Figure 3. Magnetic field directions for $\alpha = \frac{1}{2}$.

where $u_\perp$ is the component of $\mathbf{u}$ normal to $\mathbf{B}$ (with a consistent sign). Since

$$\nabla \times (u_\perp B \mathbf{e}_z) = \nabla(u_\perp B) \times \mathbf{e}_z = \mathbf{0}, \tag{7}$$

$u_\perp B$ must be constant; and since $B(z) = |z|^\alpha$, $u_\perp$ = const $|z|^{-\alpha}$. In particular, at Γ, where the magnetic field lines make an angle (say) ϕ, the normal components of $\mathbf{u}$ above and below Γ make an angle $\pi - \phi$, having the same magnitude. For $\alpha = \frac{1}{2}$, the normal components are opposite, which means that $\mathbf{u}$ itself must be discontinuous there. Either there is a net influx of plasma into Γ, or Γ expels fluid (remember that we are dealing with the ideal limit). Obviously, if $\mathbf{u}$ is a solution of the induction equation, then so is $-\mathbf{u}$; therefore both possibilities exist. However, from the physical viewpoint, plasma flow into the current sheet is far more likely,

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because, as stated in Sec. 1 the sheet itself may have been created by the collision of two masses of plasma at the original null point, analogously to the Sweet–Parker model (Sweet 1958; Parker 1963). Conservation of mass then requires that the plasma escape rapidly through the endpoints (or is absorbed for $-\mathbf{u}$). If $\alpha \neq \frac{1}{2}$ then $\mathbf{u}$ may be continuous at Γ for a unique choice of the parallel component $u_\parallel$, in which case $\mathbf{u}$ must have the direction of Γ. Moreover, unless $\mathbf{u}$ is everywhere parallel to $\mathbf{B}$, $u_\perp$ tends to infinity near the null point; this is an ideal effect that in a more realistic (resistive) setting would mean only a larger velocity there.

The equivalent equation

$$\mathbf{u} \cdot \nabla \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{u} = \mathbf{0}, \tag{8}$$

provides a more refined information of the behaviour of $\mathbf{u}$ at the separatrices. After some manipulations, we find

$$\nabla \mathbf{B} = \alpha |z|^{\alpha-1} \begin{pmatrix} -\sin[(1-\alpha)\arg z] & \cos[(1-\alpha)\arg z] \\ \cos[(1-\alpha)\arg z] & \sin[(1-\alpha)\arg z] \end{pmatrix}, \tag{9}$$

so that if we take s as the arclength parameter, $\mathbf{B} \cdot \nabla = B\, d/ds$, we have

$$\frac{d\mathbf{u}}{ds} = \frac{\alpha}{|z|} \begin{pmatrix} -\sin[(1-\alpha)\arg z] & \cos[(1-\alpha)\arg z] \\ \cos[(1-\alpha)\arg z] & \sin[(1-\alpha)\arg z]) \end{pmatrix} \mathbf{u}. \tag{10}$$

$\mathbf{u}$ is in principle determined by an arbitrary initial condition at every field line. However, initial conditions that would not yield $u_\perp B = \text{const}$ correspond to non-solenoidal velocities, and therefore in those cases (8) is not equivalent to $\nabla \times (\mathbf{u} \times \mathbf{B}) = \mathbf{0}$. Thus the apparent freedom of choice is not really so, but anyway general properties of (8) provide additional information on how $\mathbf{u}$ behaves near the critical point. At the only lines including it, i.e. the separatrices, we have $\arg z = \lambda =$ constant: when $\mathbf{B}$ points away from $\mathbf{0}$, the equation has the form

$$\frac{d\mathbf{u}}{ds} = \frac{\alpha}{s} \mathsf{A}\mathbf{u}, \tag{11}$$

where A is a constant matrix with eigenvalues $1, -1$, and respective eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} \cos(1-\alpha)\lambda \\ \sin(1-\alpha)\lambda + 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} \cos(1-\alpha)\lambda \\ \sin(1-\alpha)\lambda - 1 \end{pmatrix}. \tag{12}$$

Hence a basis of the space of solutions is formed by

$$\{s^{\alpha}\mathbf{v}_1, s^{-\alpha}\mathbf{v}_2\}. \tag{13}$$

It is easy to check that $\mathbf{v}_1$ is parallel to $\mathbf{B}$ and that $\mathbf{v}_2$ is orthogonal to it. Since $B = |z|^{\alpha} = s^{\alpha}$, the condition $u_\perp B = \text{const}$ is automatically satisfied along the separatrix.

If $\mathbf{B}$ points away from $\mathbf{0}$, the equation becomes

$$\frac{d\mathbf{u}}{ds} = -\frac{\alpha}{s} \mathsf{A}\mathbf{u}, \tag{14}$$

so that the basis is now $\{s^{-\alpha}\mathbf{v}_1, s^{\alpha}\mathbf{v}_2\}$. In this case, $\mathbf{v}_2$ is parallel to $\mathbf{B}$ and $\mathbf{v}_1$ is orthogonal to it: again $u_\perp B = \text{const}$. Hence, in addition to the known fact that $|u_\perp| \sim |z|^{-\alpha}$, we know that the parallel component $u_\parallel$ behaves as $|z|^{\alpha}$: the approach of the plasma to the null point slows precisely as the magnitude of $\mathbf{B}$. The velocity field is mostly orthogonal to the magnetic field, which makes the possibility mentioned before of $\mathbf{u}$ being continuous at Γ rather unlikely, because in that case

$u_\parallel$ should be of the order of $u_\perp$. It is far more plausible that there is a net influx (or outflux) of plasma into Γ.

4. The Lorentz force

We shall now investigate whether if this magnetic structure is stable when the fluid motion is taken into account. Thus we must consider the possible evolution of the topology described before by studying the Lorentz force inherent to it: this, together with the gradient of a possible hydrostatic pressure within the fluid, will control the plasma motion, and (since, in the ideal case, magnetic field lines are frozen into the fluid) the initial evolution of the magnetic field as well. This study is made mathematically more difficult by the fact that, since $\mathbf{B}$ is irrotational outside Γ, the current $\mathbf{J} = \nabla \times \mathbf{B}$ is not an ordinary function but rather a generalized one or distribution. To find it, recall that if g is a one-variable function of class $\mathscr{C}^\infty$ except for a finite jump at $x = 0$ then its differential in the sense of distributions is $g' + [g_+(0) - g_-(0)]\delta_0$, where δ_0 is the Dirac measure centred at 0. We may apply this theorem to the component B_x as a function of y, which has a jump at $y = 0$. It is found that

$$\frac{\partial B_x}{\partial y}(z) = \alpha|z|^{\alpha-1}\cos[(1-\alpha)\arg z] + 2|z|^\alpha \sin(\alpha\pi)\,\delta_0(y), \tag{15}$$

where $\delta_0(y)$ is the Dirac measure centred at Γ: the action of this measure upon a smooth function ψ is given by $\langle\delta_0(y), \psi\rangle = \int_\Gamma \psi(x, 0)\,dx$.

Since B_y is continuous and smooth as a function of x,

$$\frac{\partial B_y}{\partial x}(z) = \alpha|z|^{\alpha-1}\cos[(1-\alpha)\arg z], \tag{16}$$

which means

$$\mathbf{J} = \nabla \times \mathbf{B} = (0, 0, -2|z|^\alpha \sin(\alpha\pi)\,\delta_0(y)). \tag{17}$$

For the Lorentz force $(\nabla \times \mathbf{B}) \times \mathbf{B}$, we must take into account that $\mathbf{B}$ itself is discontinuous: we are not dealing with the product of a smooth function and a distribution, but rather the product of two distributions, which is a much harder notion. Although general theories exist (Colombeau 1984), in our case the very physical meaning of the magnitudes will indicate the path to take. For positive diffusivity η, the magnetic field $\mathbf{B}_\eta$ is smooth and in a certain sense (Klapper and Young 1995), it may be considered to be the smoothing on a scale of order $\eta^{1/2}$ of the ideal field $\mathbf{B}_0$: it is the convolution $\mathbf{B}_\eta = \rho_\eta * \mathbf{B}_0$, where ρ_η is a *regularizing function*, i.e. a bell-shaped smooth function around $(0, 0)$ of integral 1 and support tending to $(0, 0)$ for small η. ρ_η tends weakly to the Dirac measure $\delta_{(0,0)}$ as $\eta \to 0$. It is also known that $\nabla \times (\rho_\eta * \mathbf{B}_0) = \rho_\eta * \nabla \times \mathbf{B}_0$ tends weakly to $\nabla \times \mathbf{B}_0$. Now we may interpret $\nabla \times \mathbf{B}_0 \times \mathbf{B}_0$ as the limit of $[\nabla \times (\rho_\eta * \mathbf{B}_0)] \times (\rho_\eta * \mathbf{B}_0)$ as $\eta \to 0$, (i.e. the ideal Lorentz force is the limit of the resistive Lorentz forces as the diffusivity tends to zero).

For $\rho_\eta * \mathbf{B}_0$, since B_x is odd as a function of y, and B_y is continuous, at the points of Γ, $\rho_\eta * B_x$ vanishes and $\rho_\eta * B_y$ tends to $|z|^\alpha \cos\alpha\pi$. Hence $[\nabla \times (\rho_\eta * \mathbf{B}_0)] \times (\rho_\eta * \mathbf{B}_0)$ tends to

$$\mathbf{J} \times \mathbf{B} = |z|^{2\alpha}(\sin(2\alpha\pi)\,\delta_0(y), 0, 0), \tag{18}$$

which is the Lorentz force. It is directed along Γ, to the right if $\alpha < \frac{1}{2}$, to the left

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if $\alpha > \frac{1}{2}$, and vanishes for $\alpha = \frac{1}{2}$. As expected, it is always directed towards the acute angle of the magnetic field: if we write the Lorentz force as $\mathbf{B} \cdot \nabla \mathbf{B} - \frac{1}{2} \nabla B^2$, then the dominant term $\mathbf{B} \cdot \nabla \mathbf{B}$ is essentially the curvature vector of the magnetic field line, which is singular at the sharp turn in Γ, and always directed towards the centre of curvature, i.e. the acute angle. Since at one end of the current line $\alpha \leqslant \frac{1}{2}$ and at the other $\alpha \geqslant \frac{1}{2}$, both forces fit perfectly: they tend to displace the plasma from the side of higher order to that of lower order, except for $\alpha = \frac{1}{2}$, when there is no force. For the non-ideal case, one must expect a force localized along a band connecting the null points, and directed towards the point of lower exponent. Since, as stated before, the flow takes the magnetic field lines with it, it seems that one of the critical points is pushed towards the other and the configuration tends to collapse again into a single X point, except in the equilibrium case $\alpha = \frac{1}{2}$.

5. Conclusions

We have examined the possible configurations created by the break-up of a magnetic hyperbolic neutral point into two null points connected by a current sheet Γ. The topology in a neighbourhood of any of the endpoints depends solely on the order α of the zero at this point; there exist three different topologies, corresponding to $\alpha < \frac{1}{2}$, $\alpha = \frac{1}{2}$ and $\alpha > \frac{1}{2}$. For a quasistatic magnetic field, if the plasma velocity is not parallel to $\mathbf{B}$, there is a net influx of plasma towards the current sheet, and the fluid tends to escape rapidly near the endpoints. The Lorentz force is concentrated in Γ, and tends to push the null point of higher order into that of lower order; since in the ideal case the fluid takes the magnetic lines along with it, the configuration is likely to collapse again into a single null point, except in the case of equilibrium $\alpha = \frac{1}{2}$.

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