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Liouville billiard tables and an inverse spectral result

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Abstract. We consider a class of billiard tables (X, g), where X is a smooth compact manifold of dimension two with smooth boundary ∂X and g is a smooth Riemannian metric on X, the billiard flow of which is completely integrable. The billiard table (X, g) is defined by means of a special double cover with two branched points and it admits a group of isometries $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Its boundary can be characterized by the string property; namely, the sum of distances from any point of ∂X to the branched points is constant. We provide examples of such billiard tables in the plane (elliptical regions), on the sphere $\mathbf{S}^2$, on the hyperbolic space $\mathbf{H}^2$, and on quadrics. The main result is that the spectrum of the corresponding Laplace–Beltrami operator with Robin boundary conditions involving a smooth function K on ∂X uniquely determines the function K, provided that K is invariant under the action of G.

1. *Introduction*

This paper is concerned with a class of smooth compact Riemannian manifolds of dimension two with smooth boundaries, the billiard flows of which are completely integrable. We call them *Liouville billiard tables of classical type*. For such billiard tables we prove that the following inverse spectral result is true.

Let (X, g) be a closed Riemannian manifold, $\dim X = 2$, with a C^∞ boundary $\Gamma := \partial X \neq \emptyset$. Let Δ be the 'positive' Laplace–Beltrami operator on (X, g). Given a real-valued function $K \in C^\infty(\Gamma)$, we consider the operator Δ with a domain of definition

$$\left\{ u \in H^2(X) : \frac{\partial u}{\partial n}\bigg|_\Gamma = Ku|_\Gamma \right\},$$

where $n(x)$, $x \in \Gamma$, is the inward unit normal to Γ with respect to the metric g. We denote this operator by Δ_K. This is a selfadjoint operator in $L^2(X)$ with discrete spectrum

$$\text{Spec}\, \Delta_K := \{\lambda_1 \leq \lambda_2 \leq \cdots\},$$

where each eigenvalue $\lambda = \lambda_j$ is repeated according to its multiplicity, and it solves the spectral problem

$$\begin{aligned} \Delta u &= \lambda u \quad \text{in } X, \\ \frac{\partial u}{\partial n}\bigg|_{\Gamma} &= Ku|_{\Gamma}. \end{aligned} \tag{1.1}$$

The manifolds we define admit a group of isometries G isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. We denote by $\mathrm{Symm}_G(\Gamma)$ the set of all real-valued functions $K \in C^\infty(\Gamma)$ which are invariant under the induced action of G on Γ. Consider the map

$$\mathrm{Symm}_G(\Gamma) \longrightarrow \mathbb{R}^{\mathbb{N}}, \tag{1.2}$$

assigning to each $K \in \mathrm{Symm}_G(\Gamma)$ the spectrum $\mathrm{Spec}(\Delta_K)$ of the boundary value problem (1.1). Guillemin and Melrose [**6**] have proved that for elliptical regions in $\mathbb{R}^2$ (Γ is an ellipse), the map (1.2) is one-to-one (injective). We generalize their result for Liouville billiard tables of classical type (see Definition 2.7).

Our main result is the following.

THEOREM 1. *Let (X, g) be a Liouville billiard table of classical type. Then the map (1.2) is one-to-one (injective).*

By a billiard table we mean a compact connected Riemannian manifold (X, g) of dimension two with smooth boundary. The corresponding dynamical system is the billiard flow. Denote by $H : T^*X \to \mathbb{R}$ the Hamiltonian corresponding to the metric g via the Legendre transformation. It will be shown that Hamiltonian systems corresponding to Liouville billiard tables are integrable. Hereafter, integrable means that there exists a real-valued function $I \in C^\infty(T^*X \setminus 0)$ which is constant on each (broken) bicharacteristic of H, and such that the differentials dH and dI are linearly independent at almost any $\varrho \in T^*X \setminus 0$. A (broken) bicharacteristic γ of H is a map

$$\gamma : [0, T) \setminus P \longrightarrow T^*\mathring{X} \setminus 0, \quad \mathring{X} = X \setminus \Gamma,$$

where P is either a finite subset $0 < t_1 < t_2 < \cdots < t_N < T$ of $[0, T)$ or the empty set, such that on each connected component of $[0, T) \setminus P$ the curve $\gamma(t) = (x(t), \xi(t))$ is an integral curve of the Hamiltonian vector field X_H of H in $T^*\mathring{X} \setminus 0$, and for $0 < t \in P$ we have $x(t) = x(t - 0) = x(t + 0) \in \Gamma$ and

$$\xi(t - 0)_{|T_{x(t)}\Gamma} = \xi(t + 0)_{|T_{x(t)}\Gamma}, \quad \langle \xi(t + 0), n(x(t)) \rangle > 0.$$

Recall from [**8**, §24.3], that for each $\varrho \in T^*\mathring{X} \setminus 0$ there exists at least one generalized bicharacteristic γ starting from ϱ such that any compact arc of γ can be approximated uniformly by (broken) bicharacteristics. Hence, I is constant on any such generalized bicharacteristic. In other words, I is invariant under the 'generalized billiard flow'. Note that in the analytic case, for each $\varrho \in T^*\mathring{X} \setminus 0$ there is only one generalized bicharacteristic issuing from ϱ.

Elliptical regions in $\mathbb{R}^2$ give a particular case of integrable billiard tables. We shall give several other examples on the sphere $\mathbf{S}^2$, the hyperbolic space $\mathbf{H}^2$, and on quadrics. There is a common property for all of them; namely, the existence of a special double cover with

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two branched points (see Proposition 2.1). It is more or less known that the corresponding billiard flows are integrable. The novelty here is that we propose a general construction for such billiard tables and we give explicit formulae for the covering maps. We also show that the Liouville billiard tables in the examples are of classical type (Definition 2.7). In particular, we obtain that Theorem 1 holds for all of them.

We prove that the boundary Γ of any Liouville billiard table has the string property; namely, the sum of distances from any point of Γ to the branched points is constant. In particular, elliptical regions are the only Liouville billiard tables in $\mathbb{R}^2$. In addition, a class of the billiard tables we define satisfies the so-called *strong evolution property* (see Proposition 4.9).

To prove Theorem 1 we use a result of Guillemin and Melrose in [**6, 7**] concerning the singularities of the distribution

$$Z_K(t) = \sum_{\lambda \in \mathrm{Spec}(\Delta_K)} \cos(t\sqrt{\lambda}) = \mathrm{tr}(\cos(t\sqrt{\Delta_K})), \quad t > 0. \tag{1.3}$$

It is well known that the singular support of $Z_K(t), t > 0$, is contained in the length spectrum $\mathcal{L}(X, g)$ of the corresponding billiard table. Recall that $\mathcal{L}(X, g)$ consists of the lengths of all closed generalized geodesics of (X, g). Suppose now that there is an 'invariant circle' S of the billiard ball map B in the co-ball bundle $B^*\Gamma$, such that the rotation number of the map $B : S \to S$ is rational. Then the generalized geodesics issuing from S are all closed and we denote by ℓ the corresponding common minimal length. Assume that S is a 'clean' submanifold of $B^*\Gamma$ and that there are no other closed geodesics with the same length ℓ. Then the integral

$$M = \int_S \frac{K}{\cos\phi}\, d\mu \tag{1.4}$$

can be recovered from the leading term of the asymptotic expansion of $\sigma(t) = Z_K(t) - Z_0(t)$ at $t = \ell$, where Z_0 is the trace (1.3) corresponding to the Neumann problem (see [**6**, Theorem 4.2]). Here ϕ is the angle between the initial vector of the corresponding geodesic issuing form S and the inward normal to Γ at the initial point of the geodesic. The measure μ on S coincides (up to multiplication with a constant) with the Leray form.

To prove Theorem 1 we look for an infinite sequence of such 'invariant circles' S_j approaching the glancing manifold as $j \to \infty$ (the boundary Γ is strictly geodesically convex). The main difficulty is in finding S_j so that the corresponding lengths ℓ_j are all 'simple' in the length spectrum of (X, g), which means that if γ is a closed generalized geodesic of length ℓ_j then γ is a broken geodesic issuing from S_j. To do this, we essentially use the properties of the corresponding billiard ball map (see §4). In this way, we recover the integral invariants M_j on S_j in (1.4) from the spectrum of Δ_K. Moreover, we obtain a simple formula for M_j in terms of the functions f and q defining the Liouville billiard table (see (5.1)). This allows us to recover K from the sequence M_j.

An interesting and difficult question is if similar results are valid for dimensions greater than or equal to 3. A construction for integrable billiard tables close to the one in this paper could be done for dimensions greater than or equal to 3. The corresponding results will be published elsewhere.

The paper is organized as follows. In §2 we give a general construction of Liouville billiard tables. In §3 we consider several examples on $\mathbf{S}^2$, $\mathbf{H}^2$, and on quadrics. In §4 we investigate the corresponding billiard ball map. The proof of the main theorem is given in §5.

2. *Liouville billiard tables*

In the present section we define a class of completely integrable billiard tables of dimension two. The construction we propose is influenced from the classification theorems of *Liouville surfaces* given in [**3, 9, 10, 12, 14**] and the classical examples of integrable billiards described in §3 (see [**1, 2, 4, 5, 16–18**]). By definition, a Liouville surface is a complete two-dimensional Riemannian manifold without boundary, the geodesic flow of which admits a quadratic in the velocity integral, which is functionally independent of the energy integral. The idea of using special covers first appears in [**12, 19**] (see §3, and [**3**] for a complete list of references). Anyway, to the best of our knowledge, a similar construction of integrable billiard tables has not been documented in the literature.

We consider two functions $f \in C^\infty(\mathbb{R})$, $f(x+1) = f(x)$, and $q \in C^\infty([-N, N])$, $N > 0$, such that:

(H_1) f is even, $f > 0$ if $x \notin \frac{1}{2}\mathbb{Z}$, and $f(0) = f(1/2) = 0$;

(H_2) q is even, $q < 0$ if $y \neq 0$, $q(0) = 0$ and $q''(0) < 0$;

(H_3) $f^{(2k)}(l/2) = (-1)^k q^{(2k)}(0)$, $l = 0, 1$, for every natural $k \in \mathbb{N}$.

In particular, if $f \sim \sum_{k=1}^\infty f_k x^{2k}$ is the Taylor expansion of f at 0, then, by (H_3), the Taylor expansion of q at 0 is $q \sim \sum_{k=1}^\infty (-1)^k f_k x^{2k}$.

Consider the quadratic forms

$$dg^2 = (f(x) - q(y))(dx^2 + dy^2) \tag{2.1}$$

$$dI^2 = (f(x) - q(y))(q(y)\,dx^2 + f(x)\,dy^2) \tag{2.2}$$

defined on the cylinder $C = \mathbf{T}^1 \times [-N, N]$, $\mathbf{T}^1 \stackrel{\Delta}{=} \mathbb{R}/\mathbb{Z}$.

The involution $\sigma : (x, y) \mapsto (-x, -y)$ induces an involution of the cylinder C, that will be denoted by σ as well. We identify the points m and $\sigma(m)$ on the cylinder and denote by $\widetilde{C} \stackrel{\Delta}{=} C/\sigma$ the topological quotient space. Let $\pi : C \to \widetilde{C}$ be the corresponding projection. A point $x \in C$ is called *singular* if $\pi^{-1}(\pi(x)) = x$, otherwise it is a *regular* point of π. Obviously, the singular points are $F_1 = \pi(0, 0)$ and $F_1 = \pi(1/2, 0)$. Denote by $\mathbf{D}^2$ the unit disk $\{x_1^2 + x_2^2 \leq 1\}$ in $\mathbb{R}^2$.

PROPOSITION 2.1. *Suppose that f and g satisfy (H_1)–(H_3). Then the quotient space $\widetilde{C}$ is homeomorphic to the unit disk $\mathbf{D}^2$. There exists a unique differential structure on $\widetilde{C}$ such that the projection $\pi : C \to \widetilde{C}$ is a smooth map, π is a local diffeomorphism in the regular points, and the push-forward $\pi_* g$ gives a smooth Riemannian metric. The push-forward of the form I is also smooth.*

Remark 2.2. The uniqueness of the differential structure on $\widetilde{C}$ means that if $(\widetilde{C}, \mathcal{D}_1)$ and $(\widetilde{C}, \mathcal{D}_2)$ are two differential structures on the quotient space $\widetilde{C}$ satisfying the conditions of Proposition 2.1, then the identity map id : $\widetilde{C} \to \widetilde{C}$ is a diffeomorphism of the manifolds

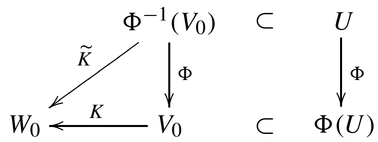
$(\widetilde{C}, \mathcal{D}_1)$ and $(\widetilde{C}, \mathcal{D}_2)$. In this way we obtain a unique Riemannian manifold, that we denote by $(X, \pi_* g)$.

We denote by Γ the boundary ∂X of X. We give the following definition.

Definition 2.3. The billiard table $(X, \pi_* g)$ is said to be Liouville.

To prove Proposition 2.1 we need the next simple lemma which will also be useful in the next section. Denote by $V \subset \mathbb{R}^2$ a neighborhood of the origin and by U the square $(-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon)$, $\varepsilon > 0$ in $\mathbb{R}^2$. We denote by (x, y) and (r, s) the coordinates in U and V, respectively.

LEMMA 2.4. *Let $\Phi : U \to V$ be a smooth mapping such that $\sigma^*\Phi = \Phi$ and $\Phi^{-1}(0, 0) = (0, 0)$. Suppose that for each $(r, s) \neq (0, 0)$, the set $\Phi^{-1}(r, s)$ consists of exactly two points and the differential $d\Phi|_{(x,y)}$ is non-degenerate at each of them. Consider the quadratic form (2.1), where f and q are smooth functions on the interval $(-\varepsilon, \varepsilon)$, $f \geq 0$ and $q \leq 0$, and suppose that the push-forward $d\tilde{g}^2 = \Phi_*(dg^2)$ is a well-defined smooth Riemannian metric on $\Phi(U)$. Then the following conditions are satisfied:*
(A_1) *f is even, $f > 0$ if $x \neq 0$, $f(0) = 0$, $f''(0) > 0$;*
(A_2) *q is even, $q < 0$ if $y \neq 0$, $q(0) = 0$;*
(A_3) *$f^{(2k)}(0) = (-1)^k q^{(2k)}(0)$, $k \in \mathbb{N}$.*
Moreover, there exist neighborhoods $V_0 \subset \Phi(U)$ and $W_0 \subset \mathbb{R}^2 = \{(u, v)\}$ of the origin in $\mathbb{R}^2$ and a diffeomorphism $K : V_0 \to W_0$, such that the map $\widetilde{K}$ given by



satisfies $\widetilde{K}(x, y) \stackrel{\Delta}{=} (K \circ \Phi)(x, y) = (x^2 - y^2, 2xy)$. In other words, setting $z = x + iy$ and $w = u + iv$ we obtain $\widetilde{K}(z) = z^2$. Conversely, if the mapping Φ is given by $\Phi : z \mapsto w = z^2$ and (A_1)–(A_3) are satisfied, then $\Phi_* g$ is well-defined and smooth, and the push-forward of the quadratic form (2.2) is also well-defined and smooth.

Proof. Suppose first that the push-forward $d\tilde{g}^2$ is well-defined and smooth. Then $\sigma^*(\Phi^*\tilde{g}) = \Phi^*\tilde{g}$, and $\sigma^* g = g$. The last equality shows that $f(-x) - q(-y) = f(x) - q(y)$, hence f and q are even functions. Moreover, $f(0) = q(0) = 0$ and $f > 0$ if $x \neq 0$, and $q < 0$ if $y \neq 0$, since $d\tilde{g}^2$ is a Riemannian metric. On the other hand, it is clear that the push-forward $\tilde{g}$ is well-defined and smooth on $\Phi(U) \setminus (0, 0)$, provided f and q are even.

Suppose that $\tilde{g}$ is smooth in $\Phi(U)$. Consider a conformal coordinate system $\{(u', v')\}$ in a neighborhood V_0 of the point $(r, s) = (0, 0)$. In other words, $d\tilde{g}^2 = \mu(u', v')(du'^2 + dv'^2)$ in these coordinates. Let $K_1 : V_0 \to W_0'$ be the corresponding transition function, $K_1(0, 0) = (0, 0)$, and $\widetilde{K}_1 \stackrel{\Delta}{=} K_1 \circ \Phi$. Since both (x, y) and (u', v') are conformal local coordinates of the metric dg^2 in $U \setminus (0, 0)$, the mapping $\widetilde{K}_1$ is conformal in $U \setminus (0, 0)$. Taking the right orientation and introducing the complex variables $z = x + iy$ and $w = u' + iv'$, we identify $\widetilde{K}_1$ with a holomorphic function $p(z)$.

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Then $p(-z) = p(z)$, and we have $p(z) = p_1(z^2)$, where $p_1(w)$ is holomorphic and $p_1(0) = 0$. If $(dp_1/dw)(0) = 0$, then for each $w_0' \neq 0$ close to 0 the preimage $p^{-1}(w_0')$ would consist of more than two points, which is not allowed by assumption. Hence, $(dp_1/dw)(0) \neq 0$ and shrinking W_0' if necessary we obtain a biholomorphic map $p_1 : W_0 \to W_0'$, where W_0 is a neighborhood of the origin in $\mathbf{C}$. It is clear that in the coordinates $w = u + iv$ (with transition function $w = p_1^{-1}(w')$) the map Φ is given by $z \mapsto w = z^2$. Let $d\tilde{g}^2 = \lambda(u, v)(du^2 + dv^2) = \lambda(w, \bar{w})\, dw\, d\bar{w}$ be the Riemannian metric $\tilde{g}$ in the new chart. We have $g = \Phi^*\tilde{g} = 4\lambda(w, \bar{w})|z|^2\, dz\, d\bar{z}$. Hence,

$$f(x) - q(y) = 4\lambda(u, v)(x^2 + y^2), \tag{2.3}$$

where $\lambda(u, v)$ is a smooth positive function, $u = x^2 - y^2$, $v = 2xy$. Let

$$\lambda \sim \lambda^{(1)} + (\lambda_1^{(2)} u + \lambda_2^{(2)} v) + \cdots, \qquad f \sim f_0 + f_1 x^2 + \cdots, \qquad q \sim q_0 + q_1 y^2 + \cdots$$

be the Taylor expansion of the corresponding functions in the points $u = v = 0$, $x = 0$, and $y = 0$, respectively. Substituting these series in (2.3) and comparing the coefficients of the homogeneous terms of x and y, we obtain that $f_0 = q_0$, $f_1 = -q_1 = 4\lambda^{(1)} > 0$ and $f_k = (-1)^k q_k$. Therefore, conditions (A_1)–(A_3) are satisfied.

Conversely, suppose that (A_1)–(A_3) are satisfied and let Φ be given by $z \mapsto w = z^2$. It is clear that the metric $d\tilde{g}^2 = \lambda(u, v)(du^2 + dv^2)$ is well-defined and smooth on $\Phi(U)\setminus 0$ and $4\lambda(u, v) = (f(x) - q(y))/(x^2 + y^2)$. We have

$$4\lambda(u, v) = f_1 + f_2(x^2 - y^2) + o(|z|^2) = f_1 + f_2 u + o(|w|),$$

where $\lambda(0, 0) = f_1/4$. Hence, λ is differentiable at $(u, v) = (0, 0)$. The proof that λ is smooth is straightforward. The same arguments show that the push-forward $\Phi_* I$ gives a smooth form in a neighborhood of the zero, and we complete the proof of Lemma 2.4. □

A variant of the last lemma is proved in [**11**].

Proof of Proposition 2.1. The first statement of the proposition is obvious. The points $A = (0, 0)$ and $B = (1/2, 0)$ on the cylinder C are fixed points of the involution σ. Denote by D_r^2 the disk $\{x^2 + y^2 < r^2\}$, where r is a fixed number, $0 < r < \min\{1/2, N\}$, and consider the map $\Phi : z \mapsto w = z^2$, where $z = x + iy$ and $w = u + iv$. Φ maps D_r^2 onto a neighborhood of $w = 0$. The coordinates $\{(u, v)\}$ give a differential structure in a neighborhood of the branched point $\pi(A)$. It follows from (H_1)–(H_3) and Lemma 2.4 that the push-forwards $\Phi_* g$ and $\Phi_* I$ are smooth and $\Phi_* g$ is positive definite. The same construction gives a differential structure in a neighborhood of the other branched point. The other points of the cylinder C are regular and the differential structure is induced from the differential structure on the cylinder C. It follows from Lemma 2.4 that the differential structure described above is unique in the sense of Remark 2.2. Proposition 2.1 is proved. □

PROPOSITION 2.5. *Liouville billiard tables are integrable.*

Proof. Obviously the metric $\pi_* g$ has Liouville form (2.1) in a neighborhood of any regular point of the cover π. It is easy to see that the form $\pi_* I$ gives a first integral of the geodesic flow of the metric $\pi_* g$ in a sufficiently small neighborhood of any regular point. Indeed, on the cover, $\pi_* g$ and $\pi_* I$ are given, respectively, by (2.1) and (2.2) in the coordinates (x, y). As functions on the cotangent bundle T^*C they have the form

$$H(x, y, p_1, p_2) = \frac{p_1^2 + p_2^2}{f(x) - q(y)} \tag{2.4}$$

$$I(x, y, p_1, p_2) = \frac{q(y)p_1^2 + f(x)p_2^2}{f(x) - q(y)}. \tag{2.5}$$

The functions H and I are in involution and their differentials are linearly independent almost everywhere. Hence, the geodesic flow of the Riemannian metric $\pi_* g$ is completely integrable. Let $\xi \in T_x X$, $x \in \Gamma$, and let $n(x)$ be the inward unit normal to Γ at the point x. Define an involution $\tau : TX|_\Gamma \to TX|_\Gamma$ by $(x, \xi) \mapsto (x, \xi - 2g(\xi, n(x))n(x))$. It follows from (2.1) and (2.2) that τ preserves the values of g and I. This completes the proof of Proposition 2.5. □

Remark 2.6. The corresponding analytic variants of Proposition 2.1, 2.5 and Lemma 2.4 are also true.

From now on we consider Liouville billiard tables which 'resemble' that of the ellipse (see §3.1). We impose the following additional conditions:

(H_4) the boundary Γ of X is locally geodesically convex which amounts to $(dq/dy)(N) < 0$;

(H_5) $f(x) = f(1/2 - x)$ and f is strictly monotone on the interval $[0, 1/4]$;

(H_6) f and q are analytic and their critical points are non-degenerate.

Definition 2.7. Liouville billiard tables satisfying conditions (H_4)–(H_6) are said to be *of classical type*.

Remark 2.8. One of the consequences of (H_5) is that there is a group $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ acting on (X, g) by isometries.

Indeed, the involutions $\sigma_1 : (x, y) \mapsto (x, -y)$ and $\sigma_2 : (x, y) \mapsto (1/2 - x, y)$ induce isometries of the Liouville billiard table $(X, \pi_* g)$, which will be denoted by the same letters. Consider the group of isometries G generated by σ_1 and σ_2. The set of fixed points of σ_1 (σ_2) can be parameterized by a geodesic arc γ_1 (γ_2), which links two different points of Γ and it is orthogonal to Γ at these points. In the case of the ellipse, γ_1 and γ_2 are the corresponding axes.

Liouville billiard tables possess the so called string property. Namely, we have the following.

PROPOSITION 2.9. *Any broken geodesic in a Liouville billiard table starting from the singular point F_1 (F_2) passes through F_2 (F_1) after one reflection at Γ. Moreover, the sum of distances from any point of Γ to F_1 and F_2 is constant.*

The proposition will be proved at the end of §4.2.

We give examples of Liouville billiard tables of classical type in the next section.

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3. *Integrable billiard tables on surfaces of constant curvature and quadrics*

The aim of this section is to provide several examples of Liouville billiard tables of classical type. It is more or less known that the corresponding billiard flows are integrable. The main point here is that we give explicit formulae for the covering maps, the metric and the integral.

3.1. *Elliptical billiard tables in the Euclidean space.*

Consider the mapping $\Phi : \mathbf{T}^1 \times \mathbb{R} \to \mathbb{R}^2$, given by:

$$u = \epsilon \cosh 2\pi y \cos 2\pi x,$$
$$v = \epsilon \sinh 2\pi y \sin 2\pi x,$$

where $(x, y) \in \mathbf{T}^1 \times \mathbb{R}$ and $(u, v) \in \mathbb{R}^2$, $\epsilon > 0$ (see [**19**]). The images of the coordinate lines $y = y_0 =$ constant $\neq 0$ are confocal ellipses on the plane $\{(u, v)\}$, with foci $F_1 = (-\epsilon, 0)$ and $F_2 = (\epsilon, 0)$, while the images of the coordinate lines $x = x_0 =$ constant $\neq k/2$, $k \in \mathbb{Z}$, are confocal hyperbolas with the same foci F_1 and F_2. The mapping Φ is a double cover of the plane $\mathbb{R}^2$ with branched points F_1 and F_2. Let $ds^2 = du^2 + dv^2$ be the Euclidean metric on $\mathbb{R}^2$. The pull-back $g := \Phi^* s$ of the metric s has a Liouville form

$$dg^2 = (f(x) - q(y))(dx^2 + dy^2), \tag{3.1}$$

where $f(x) = 4\epsilon^2\pi^2 \sin^2 2\pi x$ and $q(y) = -4\epsilon^2\pi^2 \sinh^2 2\pi y$. The quadratic form

$$dI^2 := (f(x) - q(y))(q(y)\, dx^2 + f(x)\, dy^2) \tag{3.2}$$

is a first integral of the geodesic flow of the 'metric' g. Note that the quadratic forms g and I vanish at the points $(k/2, 0)$, $k \in \mathbb{Z}$. It follows from Proposition 2.4, 2.5 and Lemma 2.1 that the push-forward $\widetilde{I}$ of the form I gives a smooth integral of the elliptical billiard tables in the interior of the ellipses $\{y = \text{constant} \neq 0\}$. Moreover, $\widetilde{I}$ is an integral of any billiard table, the boundary of which consists of curves from the confocal family described above.

3.2. *Integrable billiard tables on the sphere.*

Integrable billiard tables on $\mathbf{S}^2$ and $\mathbf{H}^2$ have been considered earlier in [**2, 4**]. Here we give explicit formulae for the corresponding branched covers.

Denote by $\mathbf{S}^2 = \{x^2 + y^2 + z^2 = 1\}$ the unit sphere embedded in the Euclidean space $\mathbb{R}^3$. The metric g_1 is the restriction of the Euclidean one $dg_0^2 = dx^2 + dy^2 + dz^2$ on $\mathbf{S}^2$. The coordinates $\{(x, y)\}$ of the unit disk $\mathbf{D}_1^2 = \{x^2 + y^2 < 1\}$ give a parameterization of the positive half $(z > 0)$ of the sphere $\mathbf{S}^2$, and we can rewrite the metric g_1 in these coordinates. Let us consider the branched cover of the unit disk $\Phi_1 : \mathbb{R}/2\pi\mathbb{Z} \times (-1/k, 1/k) \to \mathbf{D}_1^2$ given by

$$x = \frac{k}{\sqrt{1 + k^2}} \sqrt{1 + v^2} \cos u,$$
$$y = kv \sin u,$$

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where k is a positive constant. The pull-back $\tilde{g}_1 \triangleq \Phi_1^* g_1$ has a Liouville form

$$d\tilde{g}_1^2 = k^2(f(u) - q(v))\left(\frac{du^2}{1 + k^2 \sin^2 u} + \frac{dv^2}{(1 + v^2)(1 - k^2 v^2)}\right),$$

where $f(u) = \sin^2 u$ and $q(v) = -v^2$. After an obvious change of the variables the metric $\tilde{g}_1$ takes form (2.1). Propositions 2.1, 2.5 and Lemma 2.4 show that the interior of any curve $\{v = \text{constant} > 0\}$ is a Liouville billiard table. It is easy to see that all these billiard tables are of classical type.

3.3. *Integrable billiard tables on the hyperbolic space.* Consider the hyperboloid of two sheets $\mathbf{H}^2 = \{-x^2 - y^2 + z^2 = 1\}$ embedded in the Minkowski space $\mathbb{R}^{2,1}$. The metric g_{-1} of the Hyperbolic space is the restriction of the Minkowski metric $dg_0^2 = dx^2 + dy^2 - dz^2$ on $\mathbf{H}^2$. The coordinates $\{(x, y)\}$ give a parameterization of the positive sheet $(z > 0)$ of the hyperboloid $\mathbf{H}^2$. Consider the branched cover of the plane $\Phi_{-1} : \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R} \to \mathbb{R}^2$ given by

$$x = \frac{k}{\sqrt{1 - k^2}}\sqrt{1 + v^2}\cos u,$$
$$y = kv \sin u,$$

where k is a constant in the interval $0 < k < 1$. The pull-back $\tilde{g}_{-1} = \Phi_{-1}^* g_{-1}$ has a Liouville form

$$d\tilde{g}_{-1}^2 = k^2(f(u) - q(v))\left(\frac{du^2}{1 - k^2 \sin^2 u} + \frac{dv^2}{(1 + v^2)(1 + k^2 v^2)}\right),$$

where $f(u) = \sin^2 u$ and $q(v) = -v^2$. As in §3.2 we obtain that the billiard tables inside the curves $\{v = \text{constant} \neq 0\}$ are Liouville billiard tables of classical type.

3.4. *Integrable billiard tables on quadratic surfaces.* Another class of integrable billiard tables appears on quadrics. Consider, for example, the ellipsoid

$$E = \left\{\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1\right\},$$

where $0 < c < b < a$. Denote by E_λ the corresponding one-parameter family of confocal quadrics, i.e.

$$E_\lambda = \left\{\frac{x^2}{a - \lambda} + \frac{y^2}{b - \lambda} + \frac{z^2}{c - \lambda} = 1\right\}.$$

Fix a real number $\lambda_2^0 \in (c, b)$, and consider the hyperboloid of one sheet $E_{\lambda_2^0}$. It can be easily seen that $E_{\lambda_2^0}$ intersects the ellipsoid E in two curves S_1 and S_1'. These curves are boundaries of two isometric regions D_1 and D_1' on the ellipsoid, diffeomorphic to the unit disk. It turns out that the billiard flows in D_1 and D_1' (equipped with the induced Euclidean metric from $\mathbb{R}^3$) are integrable. Fix a real number $\lambda_3^0 \in (b, a)$. The quadric $E_{\lambda_2^0}$

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is a hyperboloid of two sheets that intersects the ellipsoid E in two curves S_2 and S_2'. As in the previous case these curves bound two isometric regions D_2 and D_2' on the ellipsoid. The corresponding billiard tables are integrable (see also [**4**]). It is easy to see that the same construction gives integrable billiard tables on every fixed hyperboloid of two sheets $E_{\lambda_2^0}$. The boundaries of the billiard tables are given by the intersection curves of the hyperboloid $E_{\lambda_2^0}$ with the ellipsoids from the confocal family E_λ.

Denote by $(\lambda_1, \lambda_2, \lambda_3)$ the standard elliptical coordinates $\lambda_1 < c < \lambda_2 < b < \lambda_3 < a$. The relation between the elliptical coordinates λ_i and the Cartesian coordinates x, y and z is given by the formulae

$$x^2 = \frac{(a-\lambda_1)(a-\lambda_2)(a-\lambda_3)}{(a-b)(a-c)},$$
$$y^2 = \frac{(b-\lambda_1)(b-\lambda_2)(b-\lambda_3)}{(b-a)(b-c)},$$
$$z^2 = \frac{(c-\lambda_1)(c-\lambda_2)(c-\lambda_3)}{(c-a)(c-b)}.$$

Taking $\lambda_1 = c - \theta_1^2$, $\lambda_2 = c\cos^2\theta_2 + b\sin^2\theta_2$, and $\lambda_3 = b\cos^2\theta_3 + a\sin^2\theta_3$ we obtain a single-valued mapping $(\theta_1, \theta_2, \theta_3) \mapsto (x, y, z)$ given by the formulae

$$x = \left(\sqrt{((a-c)+\theta_1^2)((a-c)\cos^2\theta_2 + (a-b)\sin\theta_2)}\cos\theta_3\right)/\sqrt{a-c},$$
$$y = \sqrt{(a-c)+\theta_1^2}\cos\theta_2\sin\theta_3,$$
$$z = \left(\theta_1\sin\theta_2\sqrt{(b-c)\cos^2\theta_3 + (a-c)\sin^2\theta_3}\right)/\sqrt{a-c}.$$

This mapping is a branched cover of the plane $\mathbb{R}^3$ (see [**11**]). It is not hard to see that this cover gives a double branched cover of the billiard tables described above. We give the formulae for the corresponding metrics on the cover, while the integral is given by the push-down of (2.2).

3.4.1. *Integrable billiard tables on the ellipsoid.* We have two types of billiard tables on the ellipsoid.

1. Setting $\theta_1 =$ constant, $u = \theta_3$, and $v = \sqrt{b-c}\sin\theta_2$ we obtain a branched cover of the first family of integrable billiard tables on the ellipsoid. In coordinates (u, v) we obtain

$$dg^2 = (f(u)-q(u))\left(\frac{(b\cos^2 u + a\sin^2 u)\,du^2}{(b\cos^2 u + a\sin^2 u) - c} + \frac{(b-v^2)\,dv^2}{(a-(b-v^2))((b-v^2)-c)}\right),$$

where $f(u) = (a-b)\sin^2 u$, $q(v) = -v^2$ and $-\sqrt{b-c} < v < \sqrt{b-c}$.

2. Taking $\theta_1 =$ constant, $u = \theta_2$, and $v = \sqrt{a-b}\sin\theta_3$ we obtain a cover of the second family of billiard tables on the ellipsoid. On the cover the metric is

$$dg^2 = (f(u)-q(u))\left(\frac{(c\sin^2 u + b\cos^2 u)\,du^2}{a-(c\sin^2 u + b\cos^2 u)} + \frac{(b+v^2)\,dv^2}{(a-(b+v^2))((b+v^2)-c)}\right),$$

where $f(u) = (b-c)\sin^2 u$, $q(v) = -v^2$ and $-\sqrt{a-b} < v < \sqrt{a-b}$.

3.4.2. *Integrable billiard tables on the hyperboloid of two sheets.* Fix $\alpha \in (b, a)$ and consider the hyperboloid of two sheets given by the equation $\lambda_3 = \text{constant}$. Taking $u = \theta_2$ and $v = \theta_1$, we obtain a branched cover of the family of billiard tables on the hyperboloid of two sheets. On the cover we have

$$dg^2 = (f(u) - q(u))\left(\frac{(\alpha - (c\sin^2 u + b\cos^2 u))\,du^2}{a - (c\sin^2 u + b\cos^2 u)} + \frac{(\alpha - (c - v^2))\,dv^2}{(1 + v^2)(b - (c - v^2))}\right),$$

where $f(u) = (b - c)\sin^2 u$, $q(v) = -v^2$ and $-1 < v < 1$.

It is easy to see the following.

PROPOSITION 3.1. *The Liouville billiard tables considered in §3 are of classical type.*

Applying Theorem 1 to the billiard tables considered in this section we obtain the following.

COROLLARY 3.2. *The map (1.2) is one-to-one for each of the billiard tables in §3.1–§3.4.*

4. *Properties of the billiard ball map*

Let $(X, \tilde{g})$ be a billiard table with boundary Γ. Consider the co-ball bundle $B^*\Gamma = \{(x, \xi) \in T^*\Gamma : \|\xi\|_x \le 1\}$, where the Hamiltonian $(x, \xi) \to \|\xi\|_x^2$ is quadratic with respect to ξ and is given by the Legendre transformation of the induced Riemannian metric on Γ. Denote by $\overset{\circ}{B}{}^*\Gamma$ the interior of $B^*\Gamma$. Let $S^*X = \{(x, \xi) \in T^*X : \tilde{H}(x, \xi) = 1\}$ be the co-sphere bundle of X, where $\widetilde{H}$ stands for the Hamiltonian function on T^*X corresponding to the Riemannian metric $\tilde{g}$ via the Legendre transformation. Set $\Sigma^+ = \{(x, \xi) \in S^*X|_\Gamma : \langle \xi, n(x)\rangle > 0\}$, $n(x)$ being the unit inward normal to Γ at x.

The corresponding billiard ball map B is defined as follows [**1**]. Take (x, ξ) in the interior of $B^*\Gamma$ and denote by $\xi_+ \in \Sigma_x^+$ the co-vector uniquely determined by $\xi_+|_{T_x\Gamma} = \xi$. Consider the integral curve, $\exp(tX_{\widetilde{H}})(x, \xi_+)$, of the Hamiltonian vector field $X_{\widetilde{H}}$ starting at (x, ξ_+). If it transversally intersects $S^*X|_\Gamma$ at a time $t_1 > 0$ and lies entirely in the interior $S^*\overset{\circ}{X}$ of S^*X for $t \in (0, t_1)$, we set $(y, \eta_-) = J(x, \xi_+) = \exp(t_1 X_{\widetilde{H}})(x, \xi_+)$, and define $B(x, \xi) = (y, \eta)$, where $\eta = \eta_-|_{T_y\Gamma}$. We denote by $\widetilde{B}^*\Gamma$ the set of all such points (x, ξ). As in [**13**] we can write B in an invariant form as follows. Consider the pull-back ω_1 in $T^*X|_\Gamma$ of the symplectic form ω in T^*X via the inclusion map. The projection along the characteristics of ω_1 induces a smooth map $\pi_1 : S^*X|_\Gamma \to B^*\Gamma$ and we denote by $\pi_1^+ : \overset{\circ}{B}{}^*\Gamma \to \Sigma^+$ an inverse map to π_1. Then we can write $B = \pi_1 \circ J \circ \pi_1^+$. In this way we obtain a symplectic map $B : \widetilde{B}^*\Gamma \to B^*\Gamma$, which is analytic if the billiard table is analytic. We extend B as the identity mapping on the boundary $S^*\Gamma$ of $B^*\Gamma$.

4.1. *The phase portrait of the integral.*

Suppose now that $(X, \tilde{g})$ is a Liouville billiard table where $\tilde{g} = \pi_*(g)$. We identify the boundary Γ of a Liouville billiard table with the circle $\{(x, N) : x \in \mathbf{T}^1\}$ on the cylinder C with coordinates $\{(x, y)\}$. Consider the natural coordinates $\{(x, y, p_1, p_2)\}$ of the cotangent bundle T^*C and set $\Gamma_N = \{y = N\} \subset T^*C$. Then Γ_N is diffeomorphic to $T^*X|_\Gamma$ and using (2.4) we identify

$$S^*X|_\Gamma \cong \{(x, N, p_1, p_2) \in \Gamma_N : p_1^2 + p_2^2 = f(x) - q(N)\}.$$

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Moreover, we identify $T^*\Gamma$ with $T^*\mathbf{T}^1$ via the inclusion map $(x, p_1) \longmapsto (x, N; p_1, 0)$. The billiard ball map $B : \widetilde{B}^*\Gamma \to B^*\Gamma$ preserves the symplectic form $\omega_0 = dp_1 \wedge dx$ of $T^*\Gamma$. It is easy to see that the characteristics of the pull-back $\omega|_{\Gamma_N}$ to Γ_N of the symplectic form $\omega = dp_1 \wedge dx + dp_2 \wedge dy$ are spanned by the vectors $\partial/\partial p_2$. Then $\pi_1(x, N, p_1, p_2) = (x, p_1)$ on $S^*X|_\Gamma$,

$$B^*\Gamma \cong \{(x, p_1) \in T^*\mathbf{T}^1 : p_1^2 - f(x) + q(N) \le 0\},$$

and the map $\pi_1^+ : B^*\Gamma \to \Sigma^+$ is given by

$$(x, p_1) \longmapsto (x, N; p_1, -\sqrt{f(x) - q(N) - p_1^2}).$$

Consider the function $\mathcal{I} = I \circ \pi_1^+$ on $B^*\Gamma$, where I is the integral (2.5). In the coordinates $\{(x, p_1)\}$ it is given by

$$\mathcal{I}(x, p_1) = f(x) - p_1^2. \tag{4.1}$$

By construction, $\mathcal{I}$ is a smooth function in $\mathring{B}^*\Gamma$ and it is analytic if the Liouville billiard is of classical type.

LEMMA 4.1. *The function $\mathcal{I}(x, p_1) = f(x) - p_1^2$ is constant on the trajectories of the billiard ball map B.*

The lemma follows immediately from Proposition 2.5.

It is easy to see that $q(N) \le \mathcal{I}(x, p_1) \le f(x)$. Fix a value h of the integral $\mathcal{I}(x, p_1)$, $q(N) \le h \le \max f$ and denote by $S_h = \{\mathcal{I}(x, p_1) = h\}$ the corresponding level set in $B^*\Gamma$. The set of glancing points of the billiard ball map coincides with the set of constant level $\{\mathcal{I}(x, p_1) = q(N)\}$.

Consider now the critical points of the integral $\mathcal{I}(x, p_1)$ in $B^*\Gamma$. Using (H_1) (and (H_5) if the Liouville billiard table is of classical type), we easily obtain the following.

PROPOSITION 4.2. *The critical points of the integral $\mathcal{I}(x, p_1)$ are given by $P_i = (x_i, 0)$, where x_i are the critical points of f. The point P_i is non-degenerate in the Morse sense if and only if $f''(x_i) \ne 0$. If $f''(x_i) > 0$, then P_i is non-degenerate of index 1 (‘hyperbolic’ point). If $f''(x_i) < 0$, then P_i is non-degenerate of index 2 (‘elliptic’ point). The points $A_1 = (0, 0)$ and $A_2 = (1/2, 0)$ are non-degenerate of index 1. Each $h \in (q(N), 0)$ is a regular value of the integral $\mathcal{I}(x, p_1)$ and the corresponding level set S_h consist of two circles. The level set S_0 consists of the critical points A_1 and A_2, and four different arcs connecting them and containing only regular points of the integral $\mathcal{I}(x, p_1)$. Moreover, if the Liouville billiard table is of classical type, then each $h \in (0, f(1/4))$ is a regular value of $\mathcal{I}(x, p_1)$, the corresponding level set S_h consists of two circles, and $S_{f(1/4)}$ consists of two non-degenerate critical points of index 2.*

If $f(x) - h > 0$ for $x \in [a, b]$, then $p_1 = \pm\sqrt{f(x) - h}$ is a parameterization of S_h, and we set $S_h^\pm[a, b] = \{(x, \pm\sqrt{f(x) - h}) : x \in [a, b]\}$. In a neighborhood of these curves we can introduce local coordinates $\{(x, \mathcal{I})\}$. The pull-back of ω_0 to $S_h^\pm[a, b]$ has the form

$$\omega_0 = dp_1 \wedge dx = \pm\left(\frac{dx}{2\sqrt{f(x) - \mathcal{I}}}\right) \wedge d\mathcal{I}.$$

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We consider the following 1-form on S_h:

$$\lambda_h \triangleq \begin{cases} \dfrac{dx}{\sqrt{f(x)-h}} & p_1 > 0, \\ -\dfrac{dx}{\sqrt{f(x)-h}} & p_1 < 0. \end{cases}$$

If h is a regular value of the integral $\mathcal{I}(x, p_1)$, the form λ_h gives a smooth 1-form on S_h which is invariant under the Hamiltonian flow of $\mathcal{I}$. λ_h is called the *Leray form*. It is easy to see that the Leray form λ_h is invariant with respect to any symplectic transformation defined in a tubular neighborhood of S_h and preserving the function $\mathcal{I}(x, p_1)$. We supply S_h with orientation by means of the Leray form.

4.2. *Rotation function for Liouville billiard tables of classical type.* Fix a regular value h of the integral $\mathcal{I}(x, p_1)$ and consider a connected component $S_h^0 \cong \mathbf{T}^1$ of the level set S_h. Suppose that the billiard ball map B is defined on S_h^0 and preserves it inducing a diffeomorphism $B|_{S_h^0} : S_h^0 \to S_h^0$. The Leray form λ_h provides S_h^0 with a smooth positive measure which is invariant with respect to the map $B|_{S_h^0}$. By means of λ_h we introduce a periodic coordinate $\{s \bmod \mu_0\}$ ($\mu_0 = \int_{S_h^0} \lambda_h$) on S_h^0 such that $ds = \lambda_h$. It is clear that $B|_{S_h^0}$ takes s to $s + \mu$, where μ is a constant on S_h^0. The number $\rho|_{S_h^0} = \mu/\mu_0$ is the *rotation number* of the map $B|_{S_h^0}$. The rotation number $\rho|_{S_h^0}$ depends on the choice of the orientation on S_h^0 and it is defined modulo 1. It is clear that

$$\rho|_{S_h^0} = \frac{\int_{\widehat{PQ}} \lambda_h}{\int_{S_h^0} \lambda_h},$$

were P is an arbitrary point on S_h^0, $Q = B(P)$, and $\widehat{PQ}$ is an arc in S_h^0 connecting P and Q.

We will prove (see Proposition 4.3) that there is at most a finite set $\Xi(q)$ of values $h_1, \dots, h_l \in (q(N), 0)$ such that the billiard ball map B is well-defined on S_h for each $h \in (q(N), 0) \setminus \Xi(q)$, but it is not defined on the level sets S_{h_i}. Geometrically this means that the geodesics issuing from the sets S_{h_i} do not reach the boundary Γ again and they stay forever in the interior of the billiard table. If $h \in (q(N), 0) \setminus \Xi(q)$ then the billiard ball map B preserves the connected components of the level sets S_h. Moreover, the involution $(x, p_1) \xrightarrow{\iota} (x, -p_1)$ interchanges the connected components of S_h and we have $\iota B \iota = B^{-1}$. In particular, the rotation number of the restriction of B to each of the components of S_h is the same and we denote it by $\rho^-(h)$. If $0 < h < f(1/4)$ then the billiard ball map interchanges the components of the level sets S_h. In this case we consider the square B^2 of the billiard ball map and denote the corresponding rotation function by $\rho^+(h)$, $h \in (0, f(1/4))$. In what follows, we consider the cases $h \in (q(N), 0)$, $h \in (0, f(1/4))$, and $h = 0$ separately.

Given a value h of the restriction of integral I on S^*C, we consider the invariant set of bicharacteristic flow $T_h = \{H = 1, I = h\} \subset S^*C$. Passing to TC by the Legendre transformation, we determine $T_h = \{g = 1, I = h\} \subset TC$ (in the coordinates (x, y) on C)

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by the system

$$g(v, v) = (f(x) - q(y))(\dot{x}^2 + \dot{y}^2) = 1,$$
$$I(v, v) = (f(x) - q(y))(q(y)\dot{x}^2 + f(x)\dot{y}^2) = h,$$

where $v = \dot{x}\partial_x + \dot{y}\partial_y$ denotes the velocity of a geodesic curve in T_h. This yields

$$\dot{x} = \pm\frac{\sqrt{f(x) - h}}{f(x) - q(y)},$$
$$\dot{y} = \pm\frac{\sqrt{h - q(y)}}{f(x) - q(y)}.$$

It is easy to see that this dynamical system corresponds to the projections on the base C of the bicharacteristics lying in the invariant set T_h. Changing the parameterization we obtain

$$\frac{dx}{d\tau} = \pm\sqrt{f(x) - h},$$
$$\frac{dy}{d\tau} = \pm\sqrt{h - q(y)}. \tag{4.2}$$

If $h < q(N)$, then the corresponding geodesics never touch the boundary of the billiard table.

Case A. Suppose that $q(N) < h < 0$. Consider the caustics of the bicharacteristics lying in T_h, i.e. the envelope of the geodesics which are projections of bicharacteristics in T_h. Note that the geodesics tangent to a caustic of T_h may never reach the boundary Γ. It follows from (4.2) that the caustics corresponding to T_h, $q(N) < h < 0$, can be identified with the curves $\{y = y(h)\}$ on C, where $y(h)$ are the positive solutions of the equation $q(y) = h$. Denote by $y_m(h)$ the maximal of them, and suppose that y_m is not a critical point of the function $q(y)$. Then $\{y = y_m(h)\}$ is a caustic corresponding to the broken geodesics issuing from the invariant set $S_h \subset \mathring{B}^*\Gamma$, of the billiard ball map. Indeed, for each $\nu \in S_h$, the broken bicharacteristic γ issuing from ν ($\gamma(0) = \pi^+(\nu)$) lies in T_h. Consider the corresponding broken geodesic $(x(\tau), y(\tau))$, $y(0) = N$, parameterized by $\tau \geq 0$. In view of (4.2) it lies in the annulus $y_m \leq y(\tau) \leq N$. Moreover, $dx/d\tau(\tau) > 0$ for each $\tau \geq 0$, and there exist $\tau_1 > \tau(h) > 0$ such that $dy/d\tau(\tau(h)) = 0$, $dy/d\tau(\tau) < 0$ for $\tau \in [0, \tau(h))$, $dy/d\tau(\tau) > 0$ in $\tau \in (\tau(h), \tau_1]$, and $y(\tau_1) = N$. By the same argument to any other solution y_0 of $q(y) = h$ correspond caustics for T_h such that the geodesics tangent to $\{y = y_0\}$ never reach the boundary ($y(\tau)$ oscillates between two different zeros of $q(y) = h$).

Consider the rotation function $\rho^-(h)$, $q(N) < h < 0$, of the billiard ball map corresponding to the geodesics issuing from S_h, the caustic of which is given by $\{y = y_m(h)\}$.

PROPOSITION 4.3. *Suppose that $y_m(h)$, $q(N) < h < 0$, is not a critical point of q. Then*

$$\rho^-(h) = \left(2\int_{y_m(h)}^{N} \frac{dy}{\sqrt{h - q(y)}}\right)\left(\int_0^1 \frac{dx}{\sqrt{f(x) - h}}\right)^{-1}.$$

Proof. Denote by $P_1 \in \Gamma$ the point with coordinates $(0, N)$ on the cylinder C. Consider the solution $\gamma(\tau) = (x(\tau), y(\tau))$ of the system

$$\frac{dx}{d\tau} = \sqrt{f(x) - h},$$
$$\frac{dy}{d\tau} = -\sqrt{h - q(y)},$$

with initial data $\gamma(0) = P_1$. The curve γ touches the caustic $\{y = y_m(h)\}$ at

$$\tau(h) = \int_{y_m(h)}^{N} \frac{dy}{\sqrt{h - q(y)}}.$$

Denote by Q_1 the point $\gamma(\tau(h))$ and consider the solution $\bar{\gamma}(\tau)$ of the system

$$\frac{dx}{d\tau} = \sqrt{f(x) - h},$$
$$\frac{dy}{d\tau} = \sqrt{h - q(y)},$$

with initial data $\bar{\gamma}(0) = Q_1$. This curve intersects the boundary Γ at a point $P_2 = (x(h), N)$. In the coordinate chart $\{(x, I)\}$ (defined in a neighborhood of the connected component of S_h), the billiard ball map takes the point $(0, h)$ to $(x(h), h)$. Therefore, the rotation function is

$$\rho^-(h) = \frac{\int_0^{x(h)} \lambda_h}{\int_0^1 \lambda_h}, \tag{4.3}$$

where λ_h is the Leray form. It follows from (4.2) that

$$2\tau(h) = \int_0^{x(h)} \frac{dx}{\sqrt{f(x) - h}}$$

which completes the proof of Proposition 4.3. □

We consider a subset $\Xi(q) = \{h_1, \ldots, h_l \in (q(N), 0)\}$ of the set of critical points of q, defined recurrently as follows. We take $h_1 = q(N_1)$, where $N_1 \in (0, N)$ is the closest critical point of q to N (if such a point does not exist we set $\Xi(q) = \emptyset$). Then given h_k, $k \geq 1$, we define by recurrence $h_{k+1} = q(N_{k+1})$, where $N_{k+1} < N_k$ is the closest critical point of the function q to N_k on the interval $(0, N)$, such that $q(N_{k+1}) > h_k$. If such a point N_{k+1} does not exist we set $\Xi(q) = \{h_1, \ldots, h_k \in (q(N), 0)\}$.

PROPOSITION 4.4. *The rotation function $\rho^-(h)$ is strictly increasing in a neighborhood of the point $q(N)$ and $\lim_{h\to q(N)+0} \rho^-(h) = 0$. The rotation function is analytic on the set $(q(N), 0) \setminus \Xi(q)$, and $\lim_{h\to h_i \pm 0} \rho^-(h) = +\infty$ for every $h_i \in \Xi(q)$.*

Proof. It is clear that $y_m(h) \to N$ as $h \to q(N)+0$, hence $\lim_{h\to q(N)+0} \rho^- = 0$. We have

$$\tau(h) = \int_{y_m(h)}^{N} \frac{dy}{\sqrt{h - q(y)}} = -2\int_0^{\sqrt{h-q(N)}} \frac{dy}{dq}(h - w^2)\, dw,$$

where $w^2 = h - q(y)$. Hence,

$$\tau'(h) = -\frac{dy}{dq}(q(N))\frac{1}{\sqrt{h - q(N)}} - 2\int_0^{\sqrt{h-q(N)}} \frac{d^2y}{dq^2}(h - w^2)\, dw.$$

This shows that $\tau'(h) \to +\infty$ as $h \to q(N)+0$. We have

$$\frac{d\rho^-}{dh}(h) = \left(2\tau'(h)\int_0^1 \frac{dx}{\sqrt{f(x)-h}} - \tau(h)\int_0^1 \frac{dx}{(f(x)-h)^{3/2}}\right) \times \left[\left(\int_0^1 \frac{dx}{\sqrt{f(x)-h}}\right)^2\right]^{-1}.$$

Therefore, $(\rho^-)'(h) > 0$ if h is sufficiently close to $q(N)$.

Consider the closest critical point N_1 to N of the function q and the critical value $h_1 = q(N_1)$. It is clear that $q(y)$ is strictly monotone on the interval $[N_1, N]$. Moreover, in view of (H_6), there exists $\delta > 0$ such that

$$\frac{dy}{dq}(h) = \frac{R(h)}{\sqrt{h_1 - h}}, \quad h \in (h_1 - \delta, h_1],$$

where the function $R(h)$ is continuous and $R(h) < c_1 < 0$ on $(h_1 - \delta, h_1]$. We have

$$\begin{aligned}\tau(h) &= \int_{y_m(h)}^{N_1+\delta} \frac{dy}{\sqrt{h-q(y)}} + \int_{N_1+\delta}^{N} \frac{dy}{\sqrt{h-q(y)}} \\ &= -2\int_0^{\sqrt{h-q(N_1+\delta)}} \frac{dy}{dq}(h-w^2)\,dw + r_1(h) \\ &= -2\int_0^{\sqrt{h-q(N_1+\delta)}} \frac{R(h-w^2)}{\sqrt{(h_1-h)+w^2}}\,dw + r_1(h),\end{aligned}$$

where $r_1(h)$ is smooth in a neighborhood of h_1. Hence,

$$\tau(h) > -2c_1 \int_0^{\sqrt{h-q(N_1+\delta)}} \frac{1}{\sqrt{(h_1-h)+w^2}}\,dw = c_1 \ln(h_1-h) + O(1).$$

Therefore, $\lim_{h\to h_1-0} \rho^-(h) = +\infty$. Analogously we prove $\lim_{h\to h_1+0} \rho^-(h) = +\infty$. The same arguments can be applied to any of the points of $\Xi(q)$. This completes the proof of Proposition 4.4. □

Case B. Suppose that $0 < h < f(1/4)$. In this case the caustics of the set T_h can be identified with the curves $\{x = x(h)\}$, where $x(h)$ are the solutions of the equation $f(x) = h$ in the interval $(0, 1/2)$. Denote by $x_m(h)$ the minimal of them. The billiard ball map changes the connected components of the level set S_h. Denote by $\rho^+(h)$ the rotation function of the square of the billiard ball map.

PROPOSITION 4.5. *The rotation function $\rho^+(h)$ of the iterated billiard ball map on the interval* $0 < h < f(1/4)$ *is*

$$\rho^+(h) = \left(\int_{-N}^{N} \frac{dy}{\sqrt{h-q(y)}}\right)\left(\int_{x_m(h)}^{1/2-x_m(h)} \frac{dx}{\sqrt{f(x)-h}}\right)^{-1}. \tag{4.4}$$

Proof. Suppose that the iterated billiard ball map takes the point $(x_m(h), h)$ to $(x'(h), h)$. As in the proof of Proposition 4.3 we show that

$$2\int_{-N}^{N} \frac{dy}{\sqrt{h-q(y)}} = 2k\int_{x_m(h)}^{1/2-x_m(h)} \frac{dx}{\sqrt{f(x)-h}} \pm \int_{x_m(h)}^{x'(h)} \frac{dx}{\sqrt{f(x)-h}},$$

where k is an integer. Therefore, the rotation function is given by formula (4.4). Proposition 4.5 is thus proved. □

COROLLARY 4.6. *The rotation function* $\rho^+(h)$ *is analytic on the interval* $(0, f(1/4))$.

Remark 4.7. Let $f \sim f(1/4) - \alpha_1(x - 1/4)^2 - \alpha_2(x - 1/4)^4 + \cdots$ be the Taylor expansion of f at $x = 1/4$. As in the proof of Proposition 4.4 it can be seen that $\lim_{h\to 0-0} \rho^- = 1/2$, $\lim_{h\to 0+0} \rho^+ = 1$ and

$$\lim_{h\to f(1/4)-0} \rho^+ = \frac{\sqrt{\alpha_1}}{\pi} \int_{-N}^{N} \frac{dy}{\sqrt{f(1/4) - q(y)}}.$$

The existence and the exact values of these limits will not be used later and we omit the proof.

Case C. Let $h = 0$. Consider the coordinates $\{(x, p_1)\}$ on $T^*\Gamma$ described in §4.1. The level set $S_0 = \{\mathcal{I}(x, p_1) = 0\}$ consists of the points $A_1 = (0, 0)$ and $A_2 = (0, 1/2)$, and the arcs $S_0^{\pm}(0, 1/2) = \{(x, \pm\sqrt{f(x)}) : x \in (0, 1/2)\}$ and $S_0^{\pm}(1/2, 1) = \{(x, \pm\sqrt{f(x)}) : x \in (1/2, 1)\}$. The billiard ball map B is defined and analytic in a regular neighborhood of the singular level S_0. It is clear that B maps $S_0^+(0, 1/2)$ to $S_0^+(1/2, 1)$ and preserves the Leray form λ_0 defined on them. Using λ_0 we introduce the variable $s = \int_{1/2}^{x} \lambda_0$ and identify $S_0^+(0, 1/2)$ with $\mathbb{R}$. It is clear that $\lambda_0 = ds$ and, therefore, the square B^2 of the billiard ball map is simply a translation $s \mapsto s + \alpha$, where α is a constant. If $\alpha = 0$, we obtain that all geodesics issuing from $S_0^+(0, 1/2)$ are periodic with the same primary length. If $\alpha \neq 0$ the geodesics issuing from $S_0^+(0, 1/2)$ are not periodic. We have proved the following lemma.

LEMMA 4.8. *For each* $T > 0$ *there are at most finitely many* $L \in (0, T)$ *such that* L *is the length of a closed broken geodesic issuing from* S_0.

Proof of Proposition 2.9. Consider a broken geodesic γ in X starting from F_1, and denote by $P \in \Gamma$ its first point of contact with the boundary. Suppose that γ is different from γ_1. Then the intersection points of γ with $B^*\Gamma$ lie in S_0 and they are different from the critical points A_1 and A_2. In the coordinates (x, y) in C we have $F_1 = (0, 0)$, $F_2 = (1/2, 0)$, and we can suppose that $P = (x_1, N)$ with $0 < x_1 < 1/2$ (the case $P = (x_1, -N)$, $0 < x_1 < 1/2$, is treated in the same way). Then γ is given by the solution $\bar{\gamma}(\tau) = (x(\tau), y(\tau))$ of the system

$$\frac{dx}{d\tau} = \sqrt{f(x)},$$
$$\frac{dy}{d\tau} = \pm\sqrt{-q(y)},$$

such that $\lim_{\tau\to-\infty} \bar{\gamma}(\tau) = (0, 0)$, and there exists $\tau_0 \in \mathbb{R}$ such that $y(\tau_0) = N$, $y'(\tau) > 0$ for $\tau < \tau_0$ and $y'(\tau) < 0$ for $\tau > \tau_0$. We have to prove that $\lim_{\tau\to+\infty} \bar{\gamma}(\tau) = (1/2, 0)$. Using that $-q(y) = Cy^2(1 + O(y^2))$, $C > 0$, near $y = 0$ we obtain that $\lim_{\tau\to+\infty} y(\tau) = 0$. In the same way we prove that $\lim_{\tau\to+\infty} x(\tau) = 1/2$. Differentiating with respect to $P \in \Gamma$ we prove that the sum of distances from $P \in \Gamma$ to F_1 and F_2 is $|\widehat{F_1 P}| + |\widehat{F_2 P}| = C_0$, where $C_0 > 0$ is a positive constant. □

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Suppose that (H_4) holds. As we have seen in §4.2 (Case A), the caustics of the billiard trajectories can be identified with the curves $\{y = y(h)\}$ on C, where $y(h)$ is a positive solution of the equation $q(y) = h$, and h is a fixed real number in $(q(N), 0)$. If h is close to $q(N)$, the curve $\{y = y(h)\}$ (defined uniquely) is a boundary of another Liouville billiard table $(X_h, g|_{X_h})$, $X_h \subset X$, defined by functions f_h, and q_h such that $f_h \equiv f$, and q_h is given by the restriction of the function q on the interval $[-y(h), y(h)]$. It is clear that X_h and X share the same caustics $\Gamma_{h'} = \{y = y(h')\} \subset X_h$, $h < h'$, near the boundary Γ_h of X_h. Hence, we have proved the following.

PROPOSITION 4.9. *Near the boundary, Liouville billiard tables satisfying* (H_4) *inherit from the ellipses the so-called* strong evolution *property. It means that* a caustic of a caustic is again a caustic.

4.3. *Length spectrum.* Let P_1 be a point in Γ with coordinates (x_1, N) on the cylinder C. Fix a value $h \notin \Xi(q)$ of the integral $\mathcal{I}(x, p_1)$ of the billiard ball map, and consider the geodesic $\gamma_{(P_1,h)}(t)$ such that $\gamma_{(P_1,h)}(0) = P_1$, $I(\gamma(0), \dot\gamma(0)) = h$ and $g(\dot\gamma(0), \partial_x) \geq 0$. Denote by $P_2 = (x_2, N)$ the next point of intersection of γ with Γ. Let $l(P_1, h)$ be the length of the geodesic segment of γ connecting P_1 with P_2. We are interested in the behavior of $l(P_1, h)$ as h approaches the exceptional set $\Xi(q)$.

LEMMA 4.10. *We have:*
(a) $l(P_1, h) \to \infty$ *as* h *tends to* $h_i \pm 0$, $h_i \in \Xi(q)$;
(b) *for each* $0 < \epsilon \ll 1$, *there is* $C_\epsilon > 0$ *such that* $l(P_1, h) > C_\epsilon$ *for each* $h > q(N) + \epsilon$, $h \notin \Xi(q)$.

Proof. Suppose first that $h < 0$. It follows from (4.2) that

$$\begin{aligned} l(P_1, h) &\triangleq \int_0^{\tau_0} \left\{(f(x) - q(y))\left(\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2\right)\right\}^{1/2} d\tau \\ &= \int_0^{\tau_0} \{(f(x) - q(y))(|f(x) - h| + |h - q(y)|)\}^{1/2} d\tau \\ &= \int_0^{\tau_0} (f(x) - q(y))\, d\tau = \int_0^{\tau_0} f(x)\, d\tau - \int_0^{\tau_0} q(y)\, d\tau \\ &= \int_{x_1}^{x_2} \frac{f(x)}{\sqrt{f(x) - h}}\, dx - 2\int_{y_m(h)}^{N} \frac{q(y)}{\sqrt{h - q(y)}}\, dx \\ &\geq -2\int_{y_m(h)}^{N} \frac{q(y)}{\sqrt{h - q(y)}}\, dy > 0. \end{aligned} \tag{4.5}$$

Similarly, if $h > 0$, then

$$l(P_1, h) \geq -\int_{-N}^{N} \frac{q(y)}{\sqrt{h - q(y)}}\, dy.$$

The integral

$$l_0^-(h) \triangleq -2\int_{y_m(h)}^{N} \frac{q(y)}{\sqrt{h - q(y)}}\, dy$$

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is a positive continuous function of h on $[q(N)+\epsilon, 0) \setminus \Xi(q)$. Moreover, it is easy to see that $\lim_{h\to h_i\pm 0} l_0^- = \infty$ and $\lim_{h\to 0-0} l_0^- = l_0 > 0$. The integral

$$l_0^+(h) \triangleq -\int_{-N}^{N} \frac{q(y)}{\sqrt{h-q(y)}}\,dy$$

is a positive continuous function of h on $[0, f(1/4)]$. This completes the proof of Lemma 4.10. □

Let $h \notin \Xi(q)$ be a regular value of the integral $\mathcal{I}(x, p_1)$, and let the rotation number $\rho^{\pm}(h) \in \mathbb{Q}$ be rational. Then for each $\nu \in S_h$ there is a closed broken bicharacteristic issuing from ν, and we denote by γ_ν the corresponding primitive broken closed geodesic. It follows from the definition of the rotation number (see the beginning of §4) that the number of vertices of γ_ν is independent of the choice of ν on S_h, and the length function $\nu \to L(\gamma_\nu)$ is continuous in S_h. Since the length spectrum $\mathcal{L}(X, g)$ (the set of lengths of all closed generalized geodesics) of the billiard table (X, g) has measure 0 in $\mathbb{R}$, we obtain that the continuous function $\nu \to L(\gamma_\nu) \in \mathcal{L}(X, g)$ is constant on each connected component of S_h. If $h < 0$ and the broken closed geodesic $[0, L(\gamma)] \ni t \to \gamma(t)$ issues from one of the components of S_h, then $[0, L(\gamma)] \ni t \to \gamma(L(\gamma) - t)$ issues from the other component. In particular, the primary length of the closed broken geodesics issuing from S_h is constant and we denote it by $l(h)$. The same is true for $h \in (0, f(1/4))$ since B interchanges the connected components of S_h in this case.

We consider now the closed broken geodesics issuing from S_h for h close to 0 and to $f(1/4)$. Denote by $\mathcal{L}_b(X, g)$ the set of lengths $L(\gamma)$ of all closed generalized geodesics γ having at least one common point with the boundary. By definition the set $L(\gamma)\mathbb{N}^*$ is also contained in $\mathcal{L}_b(X, g)$.

LEMMA 4.11. *There exist neighborhoods U_0 and U_1 in $\mathbb{R}$ of 0 and $f(1/4)$, respectively, such that for each $T > 0$ there are at most finitely many $L \in \mathcal{L}_b(X, g) \cap (0, T)$ which are lengths of closed broken geodesics issuing from S_h, $h \in U_0 \cup U_1$.*

Proof. We shall first prove the lemma for h in a neighborhood of 0. In view of Lemma 4.8 we can exclude the singular level S_0 from our consideration.

Suppose that there is an infinite sequence $\{x_j\}_{j=1}^{\infty}$ with $\lim_{j\to\infty} x_j = 0$ such that the geodesics issuing from the level sets $S_{x_j} = \{\mathcal{I}(x, p_1) = x_j\} \subset B^*\Gamma$ are all closed ($\rho^{\pm}(x_j) \in \mathbb{Q}$) and $\lim_{j\to\infty} l(x_j) = T_0$. Denote by n_j the number of vertices of the primitive closed geodesic issuing from $\nu \in S_{x_j}$, i.e. the smallest positive integer such that $B^{n_j}(\nu) = (\nu)$, which is independent of ν. Since the set of lengths $l(x_j)$, $j \in \mathbb{N}$, is bounded, using Lemma 4.10(b), we obtain that the set $\{n_j\}_{j\in\mathbb{N}}$, is bounded as well. Hence, we can suppose that $n_j = n$ does not depend on $j \geq 1$.

Choose sufficiently small neighborhoods $W_0 \subset W$ in $B^*\Gamma$ of one of the critical points, say A_1, such that $B^n(W_0) \subset W$. Then B^n is analytic in W_0 and $B^n(q, p) = (q, p)$ for each $(q, p) \in S_{x_j} \cap W_0$. On the other hand, by Proposition 4.2, A_1 is a non-degenerate critical point of $\mathcal{I}(q, p)$ of index 1 (hyperbolic), and we can provide W with analytic local coordinates $(q, p) \in \mathbb{R}^2$ such that $(q, p)(A_1) = (0, 0)$ and $\mathcal{I}(q, p) = q^2 - p^2$ in W. Then there is an open cone Γ_0 with a vertex at $(0, 0)$ such that any ray in Γ_0 starting from the origin intersects infinitely many level curves $S_{x_j} \cap W_0$. This implies

$B^n(q,p) = (q,p)$ in W_0. Since B is analytic in the connected component U of A_1 in the complement of $\bigcup_{j=1}^{l} S_{h_j}$, $h_j \in \Xi(q)$, in $B^*\Gamma$, we obtain $B^n(\nu) = \nu$ for each $\nu \in U$. Moreover, the length of the closed geodesic corresponding to the periodic orbit $\{\nu, B(\nu), \ldots, B^{n-1}(\nu)\}$ of B does not depend on $\nu \in U$, and we denote it by T. Taking into account Lemma 4.10(a), we obtain that $\Xi(q) = \emptyset$. Hence, B^n is the identity mapping in $B^*\Gamma$. This contradicts the strict geodesic convexity of Γ. The same is true in the case when h is close to $f(1/4)$, since the corresponding critical point is non-degenerate elliptic. □

We now investigate the lengths of the closed broken geodesics approximating Γ with winding number $m = 1$. Given a value h of the integral $\mathcal{I}(x, p_1)$, close to $q(N)$, and such that $\rho^-(h)$ is rational, we denote by $l(h)$ the length of the primitive closed broken geodesics issuing from S_h. Consider a sequence $\{p_k\}_{k=1}^{\infty}$ tending to $q(N)$ and such that $\rho^-(p_k) = 1/k$, $k \geq M_0$, where $M_0 \gg 1$ is a fixed natural number.

LEMMA 4.12. *There is $M_0 \gg 1$ such that for each $k \geq M_0$,*

$$l(p_k) < l(p_{k+1}) < L(\Gamma) \quad \text{and} \quad l(p_k) \to L(\Gamma) \quad \textit{as } k \to \infty.$$

Moreover, if $\rho^-(h) = m/n < 1/M_0$, $(m,n) = 1$, and $m \geq 2$, then $l(h) > L(\Gamma)$.

Proof. Set $t_0 = L(\Gamma)/2\pi$ and denote by $(r, \varphi) \in [t_0 - \delta_0, t_0] \times (\mathbb{R}/2\pi\mathbb{Z})$ the 'action-angle' coordinates corresponding to the smooth foliation of invariant 'circles' S_h for $h \geq q(N)$ and close to $q(N)$. The billiard ball map is given by $B(r,\varphi) = (r, \varphi + \mu(r))$ for $r \in [t_0 - \delta_0, t_0]$, where μ is smooth in the open interval and continuous in the closed one. Taking into account that Γ is strictly geodesically convex, we are going to show that μ is given by the derivative $\mu(r) = \tau'(r)$, where $\tau(r) = -Q(r)^{3/2}$ for $r \in [t_0 - \delta_0, t_0]$, Q is smooth in a neighborhood of $r = t_0$ and $Q(t_0) = 0$, $Q'(t_0) < 0$. To this end we make use of an approximate interpolating Hamiltonian ζ of B [**13**]. The function ζ is smooth in a neighborhood of t_0, $\zeta(t_0, \varphi) = 0$, $\partial\zeta/\partial r(t_0, \varphi) < 0$ and $R(r,\varphi) = B(r,\varphi) - \exp(H_{-\zeta^{3/2}})(r,\varphi)$ can be extended as a smooth function across t_0 and it vanishes to infinite order at $r = t_0$. Here

$$\exp(H_{-\zeta^{3/2}})(r,\varphi) = \left(r, \varphi - \frac{3}{2}\zeta(r,\varphi)^{1/2}\frac{\partial\zeta}{\partial r}(r,\varphi) + O((t_0 - r))\right)$$

stands for the time-one-flow of the Hamiltonian $-\zeta^{3/2}$. We are going to prove that $\zeta(r,\varphi) = \zeta(r,0)(1 + R_0(r,\varphi))$, where R_0 is smooth and vanishes to infinite order at $r = t_0$. Obviously, we have

$$\mu(r) = -\frac{3}{2}\zeta(r,\varphi)^{1/2}\frac{\partial\zeta}{\partial r}(r,\varphi)(1 + O((t_0 - r)^{1/2}).$$

On the other hand, $B^*\zeta = \zeta + O(\zeta^\infty)$, and we get

$$\zeta(r, \varphi + \mu(r)) = \zeta(r,\varphi) + O((t_0 - r)^\infty).$$

Expanding the smooth function ζ in Fourier series

$$\zeta(r,\varphi) = \sum_{k\in\mathbb{Z}} f_k(r)e^{ik\varphi},$$

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we obtain

$$f_k(r)(e^{ik\mu(r)} - 1) = O_k((t_0 - r)^\infty)$$

which implies $f_k(r) = O_k((t_0 - r)^\infty)$ for $k \neq 0$. Since f_k is smooth we get $(d^j f_k/dr^j)(t_0) = 0$ for each $j \geq 0$, $k \neq 0$. Hence $\zeta(r, \varphi) - \zeta(r, 0)$ vanishes to infinite order at $r = t_0$, and $Q(r) = \zeta(r, 0) + O((t_0 - r)^\infty)$ is smooth in $[t_0 - \delta_0, t_0]$.

Given an invariant circle S_h of B with a rational rotation number $\rho^-(h) = \rho = m/n$, where $m, n \in \mathbb{N}$, $(m, n) = 1$, and ρ is close to 0, we consider the common primitive length $\ell(\rho) = l(h)$ of the family of broken geodesics issuing from S_h. Using a simple argument about the symplectic invariance of the length spectrum (see [**15**, §4, (4.2)–(4.6)], with $Q_0 = 0$), we obtain that

$$\frac{\ell(\rho)}{2\pi m} = \frac{\mathcal{S}(\rho)}{\rho},$$

where $\mathcal{S}(\rho)$ is the Legendre transformation of $\tau(r)/2\pi$. In other words,

$$\mathcal{S}(\rho) = r(\rho)\rho - \tau(r(\rho))/2\pi,$$

where $[0, \varepsilon_0] \ni \rho \to r(\rho) \in [t_0 - \delta_0, t_0]$, $r(0) = t_0$, is the inverse function to the 'frequency map' $r \to \tau'(r)/2\pi$. It is easy to see that

$$\frac{d}{d\rho}\left(\frac{\mathcal{S}(\rho)}{\rho}\right) = \frac{\tau(r(\rho))}{2\pi\rho^2} < 0,$$

which implies the first statement. To prove the second, we observe that

$$r(\rho) = t_0 - C\rho^2(1 + O(\rho)), \quad \tau(r(\rho))/\rho = -C_0\rho^2(1 + O(\rho)),$$
$$C, C_0 > 0, \quad \text{as } \rho \to +0.$$

Then $\ell(\rho) = 2\pi m\mathcal{S}(\rho)/\rho = m(L(\Gamma) - \rho^2 s(\rho))$, where s is smooth in a neighborhood of 0 and independent of m, which completes the proof of the lemma. □

5. *Proof of the main theorem*

First we recall some facts about the singularities of the distribution Z_K given by (1.3). It is known that the singular support of $Z_K(t)$, $t > 0$, is contained in the length spectrum $\mathcal{L}(X, g)$ of the corresponding billiard table. We also denote by $Z_0(t)$ the corresponding distribution for the Laplace–Beltrami operator with Neumann boundary conditions. As in [**6**] and [**7**], we consider the distribution $\sigma(t) = Z_K(t) - Z_0(t)$, $t > 0$. We observe that the singular support of σ on $\mathbb{R}_+$ is contained in the set $\mathcal{L}_b(X, g)$ which consists of the lengths of all closed generalized geodesics having at least one common point with the boundary. More precisely, we prove that the contribution in σ of any closed geodesic γ lying entirely in $\mathring{X}$ is C^∞. Indeed, let B be a pseudo-differential operator of order 0 the wavefront of which is contained in a small conic neighborhood of γ in $T^*\mathring{X} \setminus \{0\}$ and let $\zeta \in C_0^\infty(\mathbb{R})$ have support in a small neighborhood of $L(\gamma)$. Consider the distribution

$$\sigma_B(t) = \operatorname{tr}(\cos(t\sqrt{\Delta_K})B) - \operatorname{tr}(\cos(t\sqrt{\Delta_0})B),$$

where Δ_D is the Laplace–Beltrami operator in X with Neumann boundary conditions. Since the parametrics of $\cos(t\sqrt{\Delta_K})B$ and $\cos(t\sqrt{\Delta_D})B$ differ by a smooth function for

$t \in [-T, T]$, $T > L(\gamma)$, choosing the wavefront set of B sufficiently small, we obtain that σ_B is a smooth function in that interval.

Consider now a regular level S_h of the integral $\mathcal{I}$, where $h \notin \Xi(q)$ is a fixed real number in the interval $q(N) < h < f(1/4)$. It is clear that if the rotation number $\rho^-(h)$ is rational, say $\rho^-(h) = m/n$, $0 \leq m < n$, then all the geodesics issuing from S_h are closed. The corresponding primitive geodesics have n vertices, their winding number is m, and they have the same primary length $l(h)$. Moreover, if h is sufficiently close to $q(N)$, then $d\rho^-/dh > 0$ in a neighborhood of $h = q(N)$ (see Proposition 4.4), which implies that S_h is a *clean submanifold* for the iterated billiard ball map B^n. If there are no other broken geodesics of length $l(h)$, using Theorem 4.2 [**6**], we recover from the leading term of the asymptotic expansion of $\sigma(t)$ at $t = l(h)$ the integral

$$M(h) \triangleq \int_{S_h} \frac{K}{\cos\phi}\, d\mu_{S_h}.$$

In the last formula ϕ is the angle between the initial vector of the corresponding geodesic issuing form S_h and the inward normal to the boundary of the billiard at the initial point of the geodesic. The measure μ_{S_h} on S_h coincides (up to multiplication with a constant) with the Leray form λ_h defined in §4.1 . It is easy to see that

$$\cos\phi = \sqrt{\frac{h - q(N)}{f(x) - q(N)}}.$$

Therefore,

$$M(h) = \frac{1}{\sqrt{h - q(N)}} \int_0^1 \frac{K(x)}{\sqrt{f(x) - h}} \sqrt{f(x) - q(N)}\, dx \tag{5.1}$$

is a spectral invariant.

LEMMA 5.1. *There exists a strictly monotone sequence* $\{p_k\}_{k=1}^{\infty}$ *such that:*

(a) $p_k \to q(N) + 0$ *as* $k \to \infty$*;*

(b) *the geodesics issuing from* S_{p_k} *are closed and* $l(p_k) \neq L(\Gamma)$*;*

(c) *the primitive closed geodesics issuing from* S_{p_k} *are the only closed broken geodesics in* X *with length* $l(p_k)$.

Proof. The lemma follows from the properties of the Liouville billiard tables proved in the previous sections.

Given $h > q(N)$ close to $q(N)$ and such that $\rho^-(h)$ is rational, we consider the closed broken geodesics γ issuing from S_h and denote by $L(\gamma)$ the length of γ (then $h = I(\gamma(t), \dot\gamma(t))$). Using Lemma 4.12 we choose M_0 such that the geodesics γ issuing from S_{p_k}, $\rho^-(p_k) = 1/k$, $k \geq M_0$, are the only closed geodesics satisfying

$$L(\gamma) \in [l(p^0), L(\Gamma)), \quad h = I(\gamma(t), \dot\gamma(t)) \in (q(N), p^0], \quad p^0 = p_{M_0}.$$

Obviously the sequence $\{p_k\}_{k \geq M_0}$ satisfies items (a) and (b) of Lemma 5.1. In view of Lemma 4.10, there exist $\delta_i > 0$ $(i = 1, \ldots, l)$ such that for each $h \in (h_i - \delta_i, h_i + \delta_i)$ the length $l(h) > L(\Gamma)$. Denote by K the set $U_0 \cup U_1 \bigcup_j (h_j - \delta_j, h_j + \delta_j)$ where U_0 and U_1 are the neighborhoods from Lemma 4.11. To prove (c) it is sufficient to show that there are only finitely many $k_1, \ldots, k_r \in (p^0, f(1/4)] \setminus K$ such that $\rho^{\pm}(k_j) \in \mathbb{Q}$ and

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$l(k_j) \in [l(p^0), L(\Gamma))$. Taking ϵ such that $0 < \epsilon \le \rho^-(p^0) = 1/M_0 \ll 1$ and applying the second statement of Lemma 4.10, we find $J_0 > 0$ such that if the number of the vertices of a closed geodesic γ is greater than J_0 and $h = I(\gamma(t), \dot\gamma(t)) \in (p^0, f(1/4)]$, then $L(\gamma) > L(\Gamma)$. Therefore, if $h \in (p^0, f(1/4)] \setminus K$, $l(h) < L(\Gamma)$, and $\rho^\pm(h)$ is rational, then

$$\rho^\pm(h) = \frac{u}{v} \pmod 1 \quad \text{with } v < J_0. \tag{5.2}$$

Relation (5.2) gives a finite number of equations on h. The rotation function $\rho^\pm(h)$ is analytic on $(p^0, f(1/4)] \setminus K$ and we obtain that each equation in (5.2) has only finitely many solutions h in that set. Therefore, there is only a finite number of values $k_1, \dots, k_r \in (p^0, f(1/4)]$ such that $l(k_i) \in [l(p^0), L(\Gamma))$. Choosing M_0 sufficiently large we complete the proof of Lemma 5.1. □

Suppose that $\mathrm{Spec}(\Delta_{K_1}) = \mathrm{Spec}(\Delta_{K_2})$, where K_1 and K_2 are smooth functions on ∂X invariant under the action of the group G. It follows from (5.1) that

$$\int_0^1 \frac{K(x)}{\sqrt{f(x) - p_k}} \sqrt{f(x) - q(N)}\, dx \equiv 0,$$

where $K = K_1 - K_2$ and $q(N) < p_k < 0$, $\lim_{k\to\infty}(p_k) = q(N)$. Denote by Ω the complement of $[0, f(1/4)] \cup \{z \in \mathbb{C} : \arg z = 3\pi/2\}$ in $\mathbb{C}$ and consider the analytic function

$$V(h) \triangleq \int_0^1 \frac{K(x)}{\sqrt{f(x) - h}} \sqrt{f(x) - q(N)}\, dx, \quad h \in \Omega.$$

Then $V(h) \equiv 0$, and we obtain

$$V_1(h) \triangleq \int_0^{1/4} \frac{K(x)}{\sqrt{f(x) - h}} \sqrt{f(x) - q(N)}\, dx \equiv 0.$$

We are going to show that K has zeros at $x = 0, \frac{1}{4}, \frac{1}{2}$, and $\frac{3}{4}$. Let $K(x) = K_0 + O(x)$ as $x \to 0$. Since $f(x) = f_1 x^2 + O(x^3)$, $f_1 > 0$, using the identity $V_1(h) \equiv 0$, we get for $h < 0$

$$0 = K_0 \int_0^1 \frac{ds}{\sqrt{s^2 - h}} + O(1) = -\frac{K_0}{2} \log(-h) + O(1),$$

which implies $K_0 = 0$. In the same way we prove that $K(1/4) = 0$. Then we have

$$V_1(h) = \int_0^{f(1/4)} \frac{\widetilde{K}(t)}{\sqrt{t - h}}\, dt \equiv 0$$

where $t = f(x)$ and $\widetilde{K}(t) = K(x)\sqrt{f(x) - q(N)}/f'(x)$ is continuous on the interval $[0, f(1/4)]$. Finally, the arguments used in [**6**] show that $K \equiv 0$. Indeed, differentiating the function $V_1(h)$ with respect to h at the point $h = q(N)$ we obtain

$$\int_0^{f(1/4)} \frac{\widetilde{K}(t)}{\sqrt{t - q(N)}} (t - q(N))^{-k}\, dt = 0,$$

for $k = 0, 1, \dots$. The last equality and the Stone–Weierstrass Theorem show that $\widetilde{K} \equiv 0$. This proves Theorem 1.

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