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MOVE-TO-FRONT RULE FOR ACCESSING SEVERAL RECORDS

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The move-to-front rule is applied on a model where several records are accessed each time. The records will then be placed in the front positions randomly or with the former relative order between themselves preserved. Equilibrium distributions are explored. Comparison of expected stationary search costs is carried out.

1. INTRODUCTION

Assume that there are n records in a linear list denoted by $N = \{1,2,\ldots,n\}$, of which only one can be accessed each time, and that accesses are mutually independent. Searching for records wanted always starts at the front position of the list. The time spent until the record wanted has been found is the search cost, which is to be minimized. Since p_i, the probability that the record i is accessed, is unknown (for all i), self-organizing rules have been considered. The most famous one is the move-to-front rule which moves the record accessed to the first position with the relative order of the other records remaining unchanged. The stationary distribution of the list of records, the expectation, and the probability generating function of the stationary search cost under this rule have already been obtained by McCabe [8], Hendricks [6], and Fill and Holst [5]. The probability that a record is in a particular position under equilibrium has been derived by Burville and Kingman [1]. t-step transition probabilities have been derived by Nelson [9], Phatarfod [11], and Fill [3]. See Fill [3] for more on the spectral structure. Phatarfod [10] has calculated the nonzero eigenvalues of the transition probability matrix and their multiplicities. Fill [4] has considered the rate of convergence of the distribution of search cost. With respect to expected stationary search cost, the move-to-front rule has been proved to

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be worse than the transposition rule (Rivest [12]) and even any POS(i) rule, $i = 2,3,\dots,n-2$, which, after a record in the jth position is accessed, moves it up one position if $j \le i$ and moves it to the ith position if $j > i$ (Chong and Lam [2]). Nevertheless, the distribution of the list of records converges more rapidly under the move-to-front rule.

Valiveti, Oommen, and Zgierski [13] first considered a model for accessing several records each time and applied a kind of move-to-front rule to it. Access probabilities are, however, of special structure there, namely, whether a record is to be accessed is independent of the other records accessed.

In this paper, we consider instead a model with quite general access probabilities. Besides the move-to-front rule placing back the accessed records in the front positions with their relative order being random [13], we also consider another move-to-front rule which preserves their relative order before. For both cases, we find out the number of positive recurrent communication classes (Thms. 2 and 4) and the necessary and sufficient conditions for having only one such class (corollaries to Thms. 2 and 4). We also work out the recursive formulae for calculating the stationary distributions under both rules.

Using the definition of cost in [13], it is proved that the move-to-front rule with order preserved is better than that with random order.

2. MOVE-TO-FRONT RANDOMLY

Suppose that there are n records in a linear list. At each time one can access several records which will be moved randomly to the front of the list thereafter. Suppose that p_A is the probability that *just* the set of records A is to be accessed, and that different accesses are independent.

For example, suppose $n = 6$ and the records now have the order 1,5,6,3,2,4. With probability $p_{\{2,4,5\}}$ the three records 2, 4, and 5 are accessed. These three records are then placed randomly at the first, second, and third positions with record 1, 6, and 3 at the fourth, fifth, and sixth positions, respectively. The transition probability from state (1,5,6,3,2,4) to state (2,5,4,1,6,3) is

$$\frac{p_{\{2,4,5\}}}{3!} + \frac{p_{\{1,2,4,5\}}}{4!} + \frac{p_{\{1,2,4,5,6\}}}{5!} + \frac{p_{\{1,2,3,4,5,6\}}}{6!}.$$

2.1. All p_A Greater than Zero when $A \neq \varnothing$

We now assume that p_A's are greater than zero for all nonempty A.

To calculate the stationary distribution, define, for every permutation $\pi = (\pi(1), \pi(2), \dots, \pi(n))$,

$$Q(\{\pi(1), \pi(2), \dots, \pi(t-1)\}) := \sum_{B} p_B$$

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and

$$S_{(\pi(t),\pi(t+1),\ldots,\pi(n))} := \frac{1}{1 - Q(\{\pi(1),\pi(2),\ldots,\pi(t-1)\})} \times \sum_{j=1}^{n-t+1} \frac{S_{(\pi(t+j),\pi(t+j+1),\ldots,\pi(n))}}{j!} \sum_{B} p_{\{\pi(t),\ldots,\pi(t+j-1)\}\cup B}$$

where B runs over all subsets of $\{\pi(1),\pi(2),\ldots,\pi(t-1)\}$, with the conventions that

$$\{\pi(1),\pi(2),\ldots,\pi(t-1)\} := \varnothing \qquad \text{for } t = 1$$

and

$$S_{(\pi(t+j),\pi(t+j+1),\ldots,\pi(n))} := 1 \qquad \text{for } j = n-t+1, \quad t = 1,2,\ldots,n.$$

Note that $Q(\{\pi(1),\pi(2),\ldots,\pi(t-1)\})$ is the probability for a given request that only the records in $\{\pi(1),\pi(2),\ldots,\pi(t-1)\}$ are accessed, and $1 - Q(\{\pi(1),\pi(2),\ldots,\pi(t-1)\}) > 0$ is assured by the assumption that $A \neq \varnothing$ implies $p_A > 0$.

Example 1: Suppose $n = 4$. Then,

$$\begin{aligned} S_{(2,3,1,4)} &= \frac{1}{1-p_\varnothing}\left\{S_{(3,1,4)}p_{\{2\}} + S_{(1,4)}\frac{p_{\{2,3\}}}{2} + S_{(4)}\frac{p_{\{1,2,3\}}}{6} + \frac{p_{\{1,2,3,4\}}}{24}\right\} \\ &= \frac{1}{1-p_\varnothing}\left\{\frac{p_{\{2\}}}{1-p_\varnothing-p_{\{2\}}}\left[S_{(1,4)}(p_{\{3\}}+p_{\{2,3\}}) + S_{(4)}\frac{p_{\{1,3\}}+p_{\{1,2,3\}}}{2} + \frac{p_{\{1,3,4\}}+p_{\{1,2,3,4\}}}{6}\right] + S_{(1,4)}\frac{p_{\{2,3\}}}{2} + \frac{p_{\{1,2,3\}}}{6} + \frac{p_{\{1,2,3,4\}}}{24}\right\}, \end{aligned}$$

where

$$S_{(4)} = 1,$$

$$S_{(1,4)} = \frac{1}{1-p_\varnothing-p_{\{2\}}-p_{\{3\}}-p_{\{2,3\}}} \times \left[S_{(4)}(p_{\{1\}}+p_{\{1,2\}}+p_{\{1,3\}}+p_{\{1,2,3\}}) + \frac{p_{\{1,4\}}+p_{\{1,2,4\}}+p_{\{1,3,4\}}+p_{\{1,2,3,4\}}}{2}\right].$$

Note that the denominator $1 - p_\varnothing - p_{\{2\}} - p_{\{3\}} - p_{\{2,3\}}$ equals $p_{\{1\}} + p_{\{1,2\}} + p_{\{1,3\}} + p_{\{1,2,3\}} + p_{\{4\}} + p_{\{2,4\}} + p_{\{3,4\}} + p_{\{2,3,4\}} + p_{\{1,4\}} + p_{\{1,2,4\}} + p_{\{1,3,4\}} + p_{\{1,2,3,4\}}$.

THEOREM 1: *$S_{(\pi(1),\pi(2),\ldots,\pi(n))}$ is the stationary probability of the state π.*

The proof is given in the Appendix.

2.2. General Cases

Now we consider general p_A's. If some of the p_A's are zero, $Q(\{\pi(1),\pi(2),\ldots,\pi(t-1)\})$ may equal 1 for some π and t. Then the denominator in the definition of $S_{(\pi(t),\pi(t+1),\ldots,\pi(n))}$ may equal zero. We will now find out the necessary and sufficient condition for the stationary distribution to be unique and make amendment to the definition of $S_{(\pi(t),\pi(t+1),\ldots,\pi(n))}$ so that Theorem 1 still holds.

Let U be the union of all subsets A of N such that $p_A > 0$.

LEMMA 1: $|U| \leq m \Leftrightarrow$ *there exists* $\{a_1,a_2,\ldots,a_m\}$ *such that* $Q(\{a_1,a_2,\ldots,a_m\}) = 1$.

PROOF: If $|U| \leq m$, then there exist at least $n-m$ different numbers $b_1,b_2,\ldots,b_{n-m}$ in N which do not belong to U. So

$$\{i_1,i_2,\ldots,i_k\} \text{ contains any of } b_1,b_2,\ldots,b_{n-m} \Rightarrow p_{\{i_1,i_2,\ldots,i_k\}} = 0.$$

Thus,

$$Q(\{a_1,a_2,\ldots,a_m\}) = 1,$$

if $\{a_1,a_2,\ldots,a_m\} = \{1,2,\ldots,n\}\backslash\{b_1,b_2,\ldots,b_{n-m}\}$.

Conversely, if there exists $\{a_1,a_2,\ldots,a_m\}$ such that $Q(\{a_1,a_2,\ldots,a_m\}) = 1$, let

$$\{b_1,b_2,\ldots,b_{n-m}\} = \{1,2,\ldots,n\}\backslash\{a_1,a_2,\ldots,a_m\}.$$

Then,

$$\{i_1,i_2,\ldots,i_k\} \text{ contains any of } b_1,b_2,\ldots,b_{n-m} \Rightarrow p_{\{i_1,i_2,\ldots,i_k\}} = 0.$$

Hence, each $b_j \notin U$ and $|U| \leq m$. ■

THEOREM 2: *The number of irreducible subchains is* $(n-|U|)!$, *that is, there are* $(n-|U|)!$ *positive recurrent communication classes with the complement of their union consisting of transient states.*

PROOF: Suppose $|U| = n$. Choose $i_1,i_2,\ldots,i_{k(1)}$ such that $p_{\{i_1,i_2,\ldots,i_{k(1)}\}} > 0$. Of course, $\{i_1,i_2,\ldots,i_{k(1)}\} \subseteq U$. If $k(1) < n$, then $U\backslash\{i_1,i_2,\ldots,i_{k(1)}\}$ is nonempty. So we can always choose $i_{k(1)+1},\ldots,i_{k(2)}$ from this nonempty set with $k(2) > k(1)$ and choose (if necessary) $j_1^1,\ldots,j_{m(1)}^1$ from $\{i_1,i_2,\ldots,i_{k(1)}\}$ such that

$$p_{\{j_1^1,\ldots,j_{m(1)}^1,i_{k(1)+1},\ldots,i_{k(2)}\}} > 0.$$

If $k(2) < n$, then we can always choose $i_{k(2)+1},\ldots,i_{k(3)}$ from $U\backslash\{i_1,\ldots,i_{k(2)}\}$ with $k(3) > k(2)$ and choose (if necessary) $j_1^2,\ldots,j_{m(2)}^2$ from $\{i_1,\ldots,i_{k(2)}\}$ such that

$$p_{\{j_1^2,\ldots,j_{m(2)}^2,i_{k(2)+1},\ldots,i_{k(3)}\}} > 0.$$

So after finite steps we obtain $i_1,\ldots,i_{k(l)}$ such that $k(l) = n$ and

$$p_{\{i_1,\ldots,i_{k(1)}\}} > 0, p_{\{j_1^1,\ldots,j_{m(1)}^1,i_{k(1)+1},\ldots,i_{k(2)}\}} > 0,\ldots, p_{\{j_1^{l-1},\ldots,j_{m(l-1)}^{l-1},i_{k(l-1)+1},\ldots,i_{k(l)}\}} > 0.$$

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Starting at any state, if we access the records $j_1^{l-1}, \ldots, j_{m(l-1)}^{l-1}, i_{k(l-1)+1}, \ldots, i_{k(l)}$, and then access $j_1^{l-2}, \ldots, j_{m(l-2)}^{l-2}, i_{k(l-2)+1}, \ldots, i_{k(l-1)}$, and then $\ldots$, and at last access $i_1, \ldots, i_{k(1)}$, the order of the n records will then become

$$i_1, \ldots, i_{k(1)}, i_{k(1)+1}, \ldots, i_{k(2)}, \ldots, i_{k(l-1)+1}, \ldots, i_{k(l)} (= i_n)$$

with positive probability

$$\frac{p_{\{i_1,\ldots,i_{k(1)}\}}}{k(1)!} \; \frac{p_{\{j_1^1,\ldots,j_{m(1)}^1, i_{k(1)+1},\ldots,i_{k(2)}\}}}{(k(2)-k(1)+m(1))!} \cdots \frac{p_{\{j_1^{l-1},\ldots,j_{m(l-1)}^{l-1}, i_{k(l-1)+1},\ldots,i_{k(l)}\}}}{(k(l)-k(l-1)+m(l-1))!}.$$

Of course, every irreducible subchain must contain this state. So there is only one irreducible subchain.

Suppose $|U| = n - q$ with $q \geq 1$. Then there exist q different numbers $b_1, \ldots, b_q$ such that each $b_j \notin U$. That means these q records will never be accessed and their relative order remains unchanged. So if the Markov chain starts at a state with b_i preceding b_j, it can never reach any states with b_j preceding b_i. Eventually, these q records will be in the last q positions of the list, and, if we take them away, we encounter the case $|U'| = n'$ where $n' = n - q$. Hence, there are just $q!$ irreducible subchains. ■

Corollary: *The stationary distribution is unique*

$\Leftrightarrow$ *there is only one irreducible subchain,*
$\Leftrightarrow |U| \geq n - 1,$
$\Leftrightarrow Q(\{\pi(1), \pi(2), \ldots, \pi(n-2)\}) < 1$ for any permutation π.

If $|U| = n$, $Q(\{\pi(1), \pi(2), \ldots, \pi(n-1)\}) < 1$ *for any permutation* π. *So Theorem* 1 *is valid. Moreover, in case of* $|U| = n - 1$, *if we make a little amendment to the definition of* $S_{(\pi(t),\pi(t+1),\ldots,\pi(n))}$ *by letting* $S_{(\pi(n))} = 1$, *Theorem* 1 *remains valid.*

Example 2: Suppose $n = 4$, $p_{\{1,2\}} > 0$, $p_{\{3,4\}} > 0$, $p_{\{1,2\}} + p_{\{3,4\}} = 1$. Then $|U| = n$. The unique irreducible subchain is $H =$

$$\{(1,2,3,4),(1,2,4,3),(2,1,3,4),(2,1,4,3),$$
$$(3,4,1,2),(3,4,2,1),(4,3,1,2),(4,3,2,1)\}.$$

One can easily check that $S_{(\pi(1),\pi(2),\pi(3),\pi(4))}$ is zero if $(\pi(1), \pi(2), \pi(3), \pi(4)) \notin H$. It can also be seen that $|U| = n$ does not imply that the chain itself is irreducible.

Example 3: Suppose $n = 4$, $p_{\{1,2\}} > 0$, $p_{\{2,3\}} > 0$, $p_{\{1,2\}} + p_{\{2,3\}} = 1$. Then $|U| = n - 1$. The unique irreducible subchain is $J =$

$$\{(1,2,3,4),(2,1,3,4),(2,3,1,4),(3,2,1,4)\}.$$

Again, $S_{(\pi(1),\pi(2),\pi(3),\pi(4))}$ is zero if $(\pi(1), \pi(2), \pi(3), \pi(4)) \notin J$.

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Example 4: Suppose $n = 4$, $p_{\{1\}} > 0$, $p_{\{1,2\}} > 0$, $p_{\{1\}} + p_{\{1,2\}} = 1$. Then $|U| = n - 2$. There are two irreducible subchains:

$$\{(1,2,3,4),(2,1,3,4)\} \quad \text{and} \quad \{(1,2,4,3),(2,1,4,3)\}.$$

The denominator in the definition of $S_{(3,4)}$ is zero. Now let $S_{(3,4)} = 1$. Then,

$$S_{(1,2,3,4)} = p_{\{1\}} + \frac{p_{\{1,2\}}}{2}, \qquad S_{(2,1,3,4)} = \frac{p_{\{1,2\}}}{2},$$

and

$$S_{(1,2,3,4)} + S_{(2,1,3,4)} = 1.$$

$S_{(1,2,3,4)}$ and $S_{(2,1,3,4)}$ form a stationary distribution and will also be the limiting distribution if the chain starts from a state with 3 preceding 4.

The same argument can be made for the states (1,2,4,3) and (2,1,4,3). Or, if we take away from the list the records 3 and 4, the remaining record set satisfies $|U'| = n' = 2$, thus the chain is now irreducible with stationary probabilities

$$S_{(1,2)} = p_{\{1\}} + \frac{p_{\{1,2\}}}{2} \quad \text{and} \quad S_{2,1} = \frac{p_{\{1,2\}}}{2}.$$

3. MOVE-TO-FRONT WITH ORDER PRESERVED

In this section, we assume that the accessed records will be moved to front preserving the relative order in which they were arranged before the access. The other assumptions remain the same as in the first paragraph of Section 2.

3.1. All p_A Greater than Zero when $A \neq \emptyset$

Again, we first assume that all the p_A's are greater than zero for all $A \neq \emptyset$.

Let $\{A_1, A_2, \ldots, A_m\}$ be a partition of N, where some A_i may be null, and α_i be a permutation of A_i, $i = 1,2,\ldots,m$. As a convention, α_i is empty if $A_i = \emptyset$. Then define, for every $i = 1,2,\ldots,m-1$,

$$\begin{aligned} R(\alpha_1,\alpha_2,\ldots,\alpha_m) &\equiv R(A_1,A_2,\ldots,A_m) \\ &\equiv R(\alpha_1,\alpha_2,\ldots,\alpha_i,A_{i+1},A_{i+2},\ldots,A_m) := \sum_{A:\forall i, A\cap A_i = A_i \text{ or } \emptyset} p_A. \end{aligned}$$

Note that $R(\alpha_1,\alpha_2,\ldots,\alpha_m)$ is the probability for a given request that only unions of $\emptyset, A_1, A_2, \ldots, A_m$ are accessed.

Define

$$T_{\alpha_1,\alpha_2,\ldots,\alpha_m} := 1$$

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when each α_i contains not more than one element, and

$$T_{\alpha_1,\alpha_2,\ldots,\alpha_m} := \frac{1}{1-R(\alpha_1,\alpha_2,\ldots,\alpha_m)} \sum p_{\cup_{i=1}^m B_i} T_{\beta_1,\beta_1',\ldots,\beta_m,\beta_m'},$$

where the summation runs over all β_i's, $i = 1,\ldots,m$, such that, assuming $\alpha_i \equiv (a_1^i, a_2^i, \ldots, a_{n(i)}^i)$,

$$\left.\begin{aligned} \beta_i &:= (a_1^i, a_2^i, \ldots, a_{l(i)}^i), \\ B_i &:= \{a_1^i, a_2^i, \ldots, a_{l(i)}^i\}, \\ \beta_i' &:= (a_{l(i)+1}^i, a_{l(i)+2}^i, \ldots, a_{n(i)}^i), \end{aligned}\right\} \qquad \textbf{(1)}$$

where $0 \le l_i \le n_i$ for each i and $0 < l(i) < n(i)$ for at least one i (i.e., cutting at least one of the permutations $\alpha_1, \alpha_2, \ldots, \alpha_m$). Note that $T_{\alpha_1,\alpha_2,\ldots,\alpha_m}$ does not depend on the order of α_i's in its subscript, and equals $T_{\alpha_2,\alpha_3,\ldots,\alpha_m}$ if α_1 is empty.

Example 5 (to be compared with Example 1): Suppose $n = 4$. Then,

$$\begin{aligned} &T_{(2,3,1,4)} \\ &\quad= \frac{1}{1-p_\varnothing-p_{\{1,2,3,4\}}}\{p_{\{2\}}T_{(2),(3,1,4)} + p_{\{2,3\}}T_{(2,3),(1,4)} + p_{\{1,2,3\}}T_{(2,3,1),(4)}\} \\ &\quad= \frac{1}{1-p_\varnothing-p_{\{1,2,3,4\}}} \\ &\qquad\times\Bigg\{\frac{p_{\{2\}}}{1-p_\varnothing-p_{\{2\}}-p_{\{1,3,4\}}-p_{\{1,2,3,4\}}} \\ &\qquad\quad\times[(p_{\{3\}}+p_{\{2,3\}})T_{(2),(3),(1,4)} + (p_{\{1,3\}}+p_{\{1,2,3\}})T_{(2),(3,1),(4)}] \\ &\qquad\quad+\frac{p_{\{2,3\}}}{1-p_\varnothing-p_{\{2,3\}}-p_{\{1,4\}}-p_{\{1,2,3,4\}}} \\ &\qquad\quad\times[(p_{\{2\}}+p_{\{1,2,4\}})T_{(2),(3),(1,4)} + (p_{\{1\}}+p_{\{1,2,3\}})T_{(2,3),(1),(4)} \\ &\qquad\qquad+p_{\{1,2\}}T_{(2),(3),(1),(4)}] \\ &\qquad\quad+\frac{p_{\{1,2,3\}}}{1-p_\varnothing-p_{\{1,2,3\}}-p_{\{4\}}-p_{\{1,2,3,4\}}} \\ &\qquad\quad\times[(p_{\{2\}}+p_{\{2,4\}})T_{(2),(3,1),(4)} + (p_{\{2,3\}}+p_{\{2,3,4\}})T_{(2,3),(1),(4)}]\Bigg\}, \end{aligned}$$

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where

$$\begin{aligned}
T_{(2),(3),(1),(4)} &= 1,\\
T_{(2),(3),(1,4)} &= \frac{(p_{\{1\}}+p_{\{1,2\}}+p_{\{1,3\}}+p_{\{1,2,3\}})T_{(2),(3),(1),(4)}}{1-p_{\varnothing}-p_{\{2\}}-p_{\{3\}}-p_{\{2,3\}}-p_{\{1,4\}}-p_{\{1,2,4\}}-p_{\{1,3,4\}}-p_{\{1,2,3,4\}}}\\
&= \frac{p_{\{1\}}+p_{\{1,2\}}+p_{\{1,3\}}+p_{\{1,2,3\}}}{p_{\{1\}}+p_{\{1,2\}}+p_{\{1,3\}}+p_{\{1,2,3\}}+p_{\{4\}}+p_{\{2,4\}}+p_{\{3,4\}}+p_{\{2,3,4\}}},\\
T_{(2),(3,1),(4)} &= \frac{p_{\{3\}}+p_{\{2,3\}}+p_{\{3,4\}}+p_{\{2,3,4\}}}{p_{\{3\}}+p_{\{2,3\}}+p_{\{3,4\}}+p_{\{2,3,4\}}+p_{\{1\}}+p_{\{1,2\}}+p_{\{1,4\}}+p_{\{1,2,4\}}},\\
T_{(2,3),(1),(4)} &= \frac{p_{\{2\}}+p_{\{1,2\}}+p_{\{2,4\}}+p_{\{1,2,4\}}}{p_{\{2\}}+p_{\{1,2\}}+p_{\{2,4\}}+p_{\{1,2,4\}}+p_{\{3\}}+p_{\{1,3\}}+p_{\{3,4\}}+p_{\{1,3,4\}}}.
\end{aligned}$$

Here we have repeatedly cut the permutations in the subscript of T until each of them contains not more than one element.

THEOREM 3: $T_{(\pi(1),\pi(2),\dots,\pi(n))}$ *is the stationary probability of the state* $\pi = (\pi(1),\pi(2),\dots,\pi(n))$.

Again, the proof is given in the Appendix.

3.2. General Cases

Suppose only $p_{A_j}, j=1,2,\dots,s(\geq 1)$, are greater than zero. Consider the combinations of $A_1,\dots,A_s$, that is, the pairwise disjoint sets $\tilde{A}_1\cap\cdots\cap\tilde{A}_s$ where $\tilde{A}_j = A_j$ or $A_j', j=1,2,\dots,s$. Here A_j' is the complement $N\backslash A_j$. Since each A_j can be expressed as a union of some $\tilde{A}_1\cap\cdots\cap\tilde{A}_s$, the algebra (i.e., field) generated by $\{A_1,\dots,A_s\}$ just consists of the unions of any sets having the form $\tilde{A}_1\cap\cdots\cap\tilde{A}_s$.

LEMMA 2:

$$\begin{aligned}
&\max\{|\tilde{A}_1\cap\cdots\cap\tilde{A}_s| : \tilde{A}_j = A_j \textit{ or } A_j' \textit{ for all } j=1,2,\dots,s\}\\
&\quad = \max\{|B_i| : \{B_1,B_2,\dots,B_k\} \textit{ is a partition of } N \textit{ such that } R(B_1,B_2,\dots,B_k)\\
&\qquad = 1 \textit{ and } 1\leq i\leq k\}. \qquad \textbf{(2)}
\end{aligned}$$

PROOF: Recall

$$R(B_1,B_2,\dots,B_k) = \sum_{G\subseteq\{1,2,\dots,k\}} p_{\cup_{g\in G}B_g}.$$

If this equals 1, then each A_j is a union $\cup_{g\in G_j} B_g$ with $G_j\subseteq\{1,2,\dots,k\}$. Suppose $i\in G_1$, and consider the combination $\tilde{A}_1\cap\cdots\cap\tilde{A}_s$, where $\tilde{A}_1 := A_1$ and, for $2\leq j\leq s$,

$$\tilde{A}_j := \begin{cases} A_j & \text{if } i\in G_j,\\ A_j' & \text{otherwise.}\end{cases}$$

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Then $\tilde{A}_1 \cap \cdots \cap \tilde{A}_s \supseteq B_i$. One can argue similarly if $i \notin G_1$ by setting $\tilde{A}_1 := A'_1$. Thus, the LHS of Eq. (2) is not less than the RHS of Eq. (2).

If we consider a partition consisting of all $\tilde{A}_1 \cap \cdots \cap \tilde{A}_s$'s, we immediately obtain the equality. ■

THEOREM 4: *The number of irreducible subchains is*

$$\prod_{\tilde{A}_j = A_j \text{ or } A'_j} (|\tilde{A}_1 \cap \cdots \cap \tilde{A}_s|!).$$

PROOF: First assume that the product above is 1. Then each $\{i\}$ can be expressed as $A_{1,i} \cap \cdots \cap A_{s,i}$ with

$$A_{j,i} = \begin{cases} A_j & \text{if } i \in A_j \\ A'_j & \text{if } i \notin A_j \end{cases}.$$

Let

$$m_{j,i} = \begin{cases} 1 & \text{if } i \in A_j \\ 0 & \text{if } i \notin A_j \end{cases}$$

and let m_i be the binary number with $m_{1,i}$ as the first digit, $m_{2,i}$ as the second digit, $\ldots$, and $m_{s,i}$ as the last and sth digit. Note that $m_i = m_k$ implies $i = k$. So, if $i \neq k$, we may assume without loss of generality that $m_i > m_k$, and there thus exists a t such that

$$A_{1,i} = A_{1,k}, \ldots, A_{t,i} = A_{t,k}$$

and

$$A_{t+1,i} = A_{t+1}, A_{t+1,k} = A'_{t+1}.$$

If we first access records A_s, and then access records A_{s-1}, and then $\ldots$, and at last access records A_1, the record i will then surely precede record k since A_{t+1} contains i and does not contain k while both i and k are or are not in $A_1, \ldots, A_t$. So, if we let $\pi = (\pi(1), \ldots, \pi(n))$ be the permutation of 1,2,$\ldots$,n such that

$$m_{\pi(1)} > m_{\pi(2)} > \cdots > m_{\pi(n)},$$

the state π can be reached by the chain starting anywhere. Therefore, every irreducible subchain must contain π. Thus, there is only one irreducible subchain.

Now suppose that among the sets having the form $\tilde{A}_1 \cap \cdots \cap \tilde{A}_s$, there are just r of them having more than one element. Denote these r sets as $H_1, \ldots, H_r$. Note that the H_i's are pairwise disjoint. Denote

$$H_i = \{h_1^i, \ldots, h_{k(i)}^i\}.$$

If $H_i = A'_1 \cap \cdots \cap A'_s$, then all records other than $h_1^i, \ldots, h_{k(i)}^i$ will eventually be accessed, and thus $h_1^i, \ldots, h_{k(i)}^i$ will eventually be placed together in the last $k(i)$ positions of the list. Otherwise, $h_1^i, \ldots, h_{k(i)}^i$ must be accessed together. (If, for ex-

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ample, we could access record h_1^i without accessing h_2^i, then there would exist an A_j such that $h_1^i \in A_j$ and $h_2^i \notin A_j$. This contradicts the fact that $H_i = \tilde{A}_1 \cap \cdots \cap \tilde{A}_s \subseteq \tilde{A}_j$, no matter whether $\tilde{A}_j = A_j$ or $\tilde{A}_j = A_j'$.) Again, they will eventually be placed together in the list. If we treat each H_i as a single record h^i, then the record set becomes

$$\{h^1, \ldots, h^r\} \cup \left(\{1, 2, \ldots, n\} \setminus \bigcup_{i=1}^{r} H_i\right),$$

which has only one irreducible subchain. Moreover, if we change record h^i back to H_i, noting that the relative order of records within H_i always remains unchanged, we know that the number of irreducible subchains is

$$\prod_{i=1}^{r} (|H_i|!) = \prod_{\tilde{A}_i = A_i \text{ or } A_i'} (|\tilde{A}_1 \cap \cdots \cap \tilde{A}_s|!). \qquad \blacksquare$$

COROLLARY: *The stationary distribution is unique*

$\Leftrightarrow$ *there is just one irreducible subchain,*
$\Leftrightarrow$ *the algebra generated by* $\{A_1, \ldots, A_s\}$ *is the finest one with respect to* N,
$\Leftrightarrow$ $R(B_1, B_2, \ldots, B_k) < 1$ *for every partition* $\{B_1, B_2, \ldots, B_k\}$, *unless each* $|B_i| \leq 1$,
$\Leftrightarrow$ *the definition of* $T_{\alpha_1, \ldots, \alpha_m}$ *and Theorem* 3 *are valid.*

Example 6 (to be compared with Example 3): Suppose $n = 4$, $p_{\{1,2\}} > 0$, $p_{\{2,3\}} > 0$, $p_{\{1,2\}} + p_{\{2,3\}} = 1$. The conditions in the corollary to Theorem 4 are satisfied. The irreducible subchain is

$$\{(2,1,3,4), (2,3,1,4)\}.$$

For states π other than (2,1,3,4) and (2,3,1,4), $T_\pi = 0$.

Example 7 (to be compared with Example 2): Suppose $n = 4$, $p_{\{1,2\}} > 0$, $p_{\{3,4\}} > 0$, $p_{\{1,2\}} + p_{\{3,4\}} = 1$. Now there are $2! \times 2!$ irreducible subchains:

$$\{(1,2,3,4), (3,4,1,2)\}, \quad \{(2,1,3,4), (3,4,2,1)\},$$

$$\{(1,2,4,3), (4,3,1,2)\}, \quad \{(2,1,4,3), (4,3,2,1)\}.$$

For all π's other than these states, $T_\pi = 0$. Note that the denominator in the definition of $T_{(1,2),(3,4)}$ is zero. If we let, however, $T_{(1,2),(3,4)}$ be 1, then

$$T_{(1,2,3,4)} = p_{\{1,2\}} T_{(1,2),(3,4)} = p_{\{1,2\}},$$

$$T_{(3,4,1,2)} = p_{\{3,4\}} T_{(1,2),(3,4)} = p_{\{3,4\}}.$$

These form a stationary distribution and will also be the limiting distribution if the chain starts from a state with 1 preceding 2 as well as with 3 preceding 4.

The same argument can be made for the other irreducible subchains.

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4. COST COMPARISON

In this section, we only consider the case that $p_A > 0$ for all $A \neq \emptyset$.

As in [13], when the permutation of the whole set of records is π, the access cost for the subset A of required records is

$$\sum_{a \in A} \pi^{-1}(a).$$

Thus, the expected cost with respect to π is

$$\mathrm{E}[\mathrm{cost}|\pi] = \sum_{A \subseteq N} p_A \sum_{a \in A} \pi^{-1}(a) = \sum_{i=1}^{n} \pi^{-1}(i) f_i,$$

where

$$f_i := \sum_{A \ni i} p_A.$$

Obviously, if f_i's are known, the optimal permutation is such that $\pi^{-1}(i) \leq \pi^{-1}(j) \Leftrightarrow f_i \geq f_j$.

The next theorem is not a surprise.

THEOREM 5: *Move-to-front rule with order preserved is better than that with random order with respect to expected stationary search cost.*

PROOF: Similar to [7], since the expected stationary search cost is now

$$\begin{aligned}
&\sum_{i,j:i \neq j} P\{i \text{ precedes } j \text{ in stationary}\} f_j \\
&\quad = \sum_{i,j:i<j} (P\{i \text{ precedes } j \text{ in stationary}\} f_j + P\{j \text{ precedes } i \text{ in stationary}\} f_i) \\
&\quad = \sum_{i,j:i<j} [\, f_i + P\{i \text{ precedes } j \text{ in stationary}\}(\, f_j - f_i)],
\end{aligned}$$

it suffices to prove that

$$f_i \geq f_j \Rightarrow P^{(2)}\{i \text{ precedes } j\} \geq P^{(1)}\{i \text{ precedes } j\},$$

or equivalently

$$\frac{\sum_{B \subseteq N \setminus \{i,j\}} p_{\{i\} \cup B}}{\sum_{B \subseteq N \setminus \{i,j\}} p_{\{j\} \cup B}} \geq 1 \Rightarrow \frac{P^{(2)}\{i \text{ precedes } j\}}{P^{(2)}\{j \text{ precedes } i\}} \geq \frac{P^{(1)}\{i \text{ precedes } j\}}{P^{(1)}\{j \text{ precedes } i\}},$$

where $P^{(1)}$ and $P^{(2)}$ represent the stationary distributions under move-to-front rule with random order and move-to-front rule with order preserved, respectively.

By Theorem 1 and Lemma 4 in the Appendix,

$$P^{(1)}\{i \text{ precedes } j\} = S_{(i,j)}^{N\backslash\{i,j\}} = S_{(i,j)}.$$

So,

$$\frac{P^{(1)}\{i \text{ precedes } j\}}{P^{(1)}\{j \text{ precedes } i\}} = \frac{\sum_{B\subseteq N\backslash\{i,j\}}\left[p_{\{i\}\cup B} + \frac{1}{2}p_{\{i,j\}\cup B}\right]}{\sum_{B\subseteq N\backslash\{i,j\}}\left[p_{\{j\}\cup B} + \frac{1}{2}p_{\{i,j\}\cup B}\right]}.$$

On the other hand, by Theorem 3, Lemma 5, and Lemma 6 in the Appendix,

$$P^{(2)}\{i \text{ precedes } j\} = \sum_{\tau} T_{(i,j)}^{\tau} = \sum_{\tau} T_{(i,j),\tau} = Y_{(i,j)}^{N\backslash\{i,j\}} = T_{(i,j),\underbrace{(1),(2),\ldots,(n)}_{[\text{omitting } (i) \text{ and } (j)]}},$$

where $Y_{(i,j)}^{N\backslash\{i,j\}}$ is defined as in Lemma 5 in the Appendix, and τ runs over all permutations of the subset $N\backslash\{i,j\}$. Thus,

$$\frac{P^{(2)}\{i \text{ precedes } j\}}{P^{(2)}\{j \text{ precedes } i\}} = \frac{\sum_{B\subseteq N\backslash\{i,j\}} p_{\{i\}\cup B}}{\sum_{B\subseteq N\backslash\{i,j\}} p_{\{j\}\cup B}},$$

which is not less than

$$\frac{\sum_{B\subseteq N\backslash\{i,j\}}[p_{\{i\}\cup B} + \frac{1}{2}p_{\{i,j\}\cup B}]}{\sum_{B\subseteq N\backslash\{i,j\}}[p_{\{j\}\cup B} + \frac{1}{2}p_{\{i,j\}\cup B}]} = \frac{P^{(1)}\{i \text{ precedes } j\}}{P^{(1)}\{j \text{ precedes } i\}},$$

because

$$\sum_{B\subseteq N\backslash\{i,j\}} p_{\{i\}\cup B} \geq \sum_{B\subseteq N\backslash\{i,j\}} p_{\{j\}\cup B}. \qquad \blacksquare$$

Remark: Another reasonable definition for cost is

$$\mathrm{E}[\text{cost}|\pi] = \sum_{A\subseteq N} p_A \max_{a\in A} \pi^{-1}(a),$$

under which Theorem 5 no longer remains valid. For example, let $n = 3$, $p_{\{1\}} = 0.2$, $p_{\{2\}} = 0.01$, $p_{\{3\}} = 0.01$, $p_{\{1,2\}} = 0.5$, $p_{\{1,3\}} = 0.01$, $p_{\{2,3\}} = 0.26$, $p_{\{1,2,3\}} = 0.01$. Then the costs corresponding to move-to-front rules, preserving order and not preserving order, are 2.32568 and 2.33747, respectively. If we change $p_{\{1\}}$ to 0.1 and $p_{\{2,3\}}$ to 0.36, however, the costs will then be 2.45419 and 2.45315, respectively.

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APPENDIX

To prove Theorem 1 we need the notations

$$V_{\pi(t)}^{\{\pi(t+1),\ldots,\pi(n)\}} := \sum_{\mu} S_{(\pi(t),\mu)}$$

and

$$W_{\{\pi(t),\pi(t+1),\ldots,\pi(n)\}} := \sum_{\theta} S_{\theta},$$

where μ runs over all permutations of $\{\pi(t+1),\ldots,\pi(n)\}$, θ runs over all permutations of $\{\pi(t),\,\pi(t+1),\ldots,\pi(n)\}$ and $(\pi(t),\mu)$ represents a permutation of $\{\pi(t),\ldots,\pi(n)\}$ with $\pi(t)$ as its first element.

LEMMA 3: *For $t = 1, 2, \ldots, n$ and for any permutation π,*

$$W_{\{\pi(t),\pi(t+1),\ldots,\pi(n)\}} = 1.$$

PROOF: First note that $W_{\{\pi(n)\}} = S_{(\pi(n))} = 1$. For the use of mathematical induction, let B run over all subsets of $\{\pi(1), \pi(2), \ldots, \pi(t-1)\}$ and $\mu \equiv (\mu(1), \ldots, \mu(n-t))$ run over all permutations of $\{\pi(t+1), \ldots, \pi(n)\}$ as above. Also, let C^j run over all subsets consisting of just j elements of $\{\pi(t+1), \ldots, \pi(n)\}$ and D^j run over all subsets consisting of just j elements of $\{\pi(t), \ldots, \pi(n)\}$. Then,

$$\begin{aligned} & V_{\pi(t)}^{\{\pi(t+1),\ldots,\pi(n)\}}[1 - Q(\{\pi(1), \pi(2), \ldots, \pi(t-1)\})] \\ &= \sum_{j=1}^{n-t+1} \frac{1}{j!} \sum_{\mu,B} p_{\{\pi(t)\}\cup\{\mu(1),\ldots,\mu(j-1)\}\cup B} S_{(\mu(j),\ldots,\mu(n-t))} \\ &= \sum_{j=1}^{n-t+1} \frac{1}{j} \sum_{C^{j-1},B} p_{\{\pi(t)\}\cup C^{j-1}\cup B} W_{\{\pi(t+1),\ldots,\pi(n)\}\setminus C^{j-1}} \\ &= \sum_{j=1}^{n-t+1} \frac{1}{j} \sum_{C^{j-1},B} p_{\{\pi(t)\}\cup C^{j-1}\cup B}, \end{aligned}$$

where the last equality comes from the assumption of induction. So,

$$\begin{aligned} & W_{\{\pi(t),\pi(t+1),\ldots,\pi(n)\}} \\ &= V_{\pi(t)}^{\{\pi(t+1),\ldots,\pi(n)\}} + V_{\pi(t+1)}^{\{\pi(t),\pi(t+2),\ldots,\pi(n)\}} + \cdots + V_{\pi(n)}^{\{\pi(t),\ldots,\pi(n-1)\}} \\ &= \frac{1}{1 - Q(\{\pi(1), \pi(2), \ldots, \pi(t-1)\})} \sum_{j=1}^{n-t+1} \frac{1}{j} \sum_{D^j,B} j p_{D^j\cup B} \\ &= 1. \end{aligned}$$

■

LEMMA 4: *For $1 \le k < m \le n$, let*

$$S_{(\pi(m),\ldots,\pi(n))}^{\{\pi(1),\ldots,\pi(k)\}}$$

be the sum of all S_ν's such that in the permutation ν of the set $\{\pi(1), \ldots, \pi(k), \pi(m), \ldots, \pi(n)\}$, $\pi(j)$ always precedes $\pi(j+1)$ for all $j = m, m+1, \ldots, n-1$. For convenience let also $S_{(\pi(m),\ldots,\pi(n))}^{\{\pi(1),\ldots,\pi(k)\}} := 1$, when $1 \le k < m = n+1$. Then,

$$S_{(\pi(m),\ldots,\pi(n))}^{\{\pi(1),\ldots,\pi(k)\}} = S_{(\pi(m),\ldots,\pi(n))} \qquad \text{for } 1 \le k < m \le n+1.$$

PROOF: By the convention in Section 2.1 that $S_{(\pi(t+j),\pi(t+j+1),\ldots,\pi(n))} = 1$ for $j = n-t+1$, the lemma is true for $1 \le k < m = n+1$.

For $k = 1$, $m = n$,

$$\begin{aligned} S_{(\pi(n))}^{\{\pi(1)\}} &= S_{(\pi(1),\pi(n))} + S_{(\pi(n),\pi(1))} \\ &= W_{\{\pi(1),\pi(n)\}} \\ &= 1 \qquad\qquad \text{(by Lemma 3)} \\ &= S_{(\pi(n))}. \end{aligned}$$

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Now suppose

$$S^{\{\pi(1)\}}_{(\pi(m+i),\dots,\pi(n))} = S_{(\pi(m+1),\dots,\pi(n))}$$

is true for some m and all i such that $1 \le i \le n - m + 1$. Then,

$$\begin{aligned}
S^{\{\pi(1)\}}_{(\pi(m),\dots,\pi(n))} &= S_{(\pi(1),\pi(m),\dots,\pi(n))} + S_{(\pi(m),\pi(1),\pi(m+1),\dots,\pi(n))} + \cdots + S_{(\pi(m),\dots,\pi(n),\pi(1))} \\
&= \frac{1}{1 - Q(\{\pi(2),\dots,\pi(m-1)\})} \\
&\quad \times \left[\sum_{j=1}^{n-m+2} \frac{S_{(\pi(m+j-1),\dots,\pi(n))}}{j!} \sum_E jp_{\{\pi(1),\pi(m),\dots,\pi(m+j-2)\}\cup E} \right. \\
&\quad\quad \left. + \sum_{j=1}^{n-m+1} \frac{S^{\{\pi(1)\}}_{(\pi(m+j),\dots,\pi(n))}}{j!} \sum_E p_{\{\pi(m),\dots,\pi(m+j-1)\}\cup E} \right] \\
&= \frac{1}{1 - Q(\{\pi(2),\dots,\pi(m-1)\})} \\
&\quad \times \left[\sum_{j=0}^{n-m+1} \frac{S_{(\pi(m+j),\dots,\pi(n))}}{j!} \sum_E p_{\{\pi(1),\pi(m),\dots,\pi(m+j-1)\}\cup E} \right. \\
&\quad\quad \left. + \sum_{j=1}^{n-m+1} \frac{S_{(\pi(m+j),\dots,\pi(n))}}{j!} \sum_E p_{\{\pi(m),\dots,\pi(m+j-1)\}\cup E} \right] \\
&= \frac{1}{1 - Q(\{\pi(2),\dots,\pi(m-1)\})} \\
&\quad \times \left[\sum_{j=1}^{n-m+1} \frac{S_{(\pi(m+j),\dots,\pi(n))}}{j!} \sum_F p_{\{\pi(m),\dots,\pi(m+j-1)\}\cup F} \right. \\
&\quad\quad \left. + S_{(\pi(m),\dots,\pi(n))} \sum_E p_{\{\pi(1)\}\cup E} \right] \\
&= \frac{1}{1 - Q(\{\pi(2),\dots,\pi(m-1)\})} \\
&\quad \times \left[(1 - Q(\{\pi(1),\dots,\pi(m-1)\}))S_{(\pi(m),\dots,\pi(n))} \right. \\
&\quad\quad \left. + S_{(\pi(m),\dots,\pi(n))} \sum_E p_{\{\pi(1)\}\cup E} \right] \\
&= S_{(\pi(m),\dots,\pi(n))},
\end{aligned}$$

where the third equality is from the assumption of induction, with E and F running over all subsets of $\{\pi(2),\dots,\pi(m-1)\}$ and $\{\pi(1),\dots,\pi(m-1)\}$, respectively. Up to now we already have

$$S^{\{\pi(1)\}}_{(\pi(m),\dots,\pi(n))} = S_{(\pi(m),\dots,\pi(n))}$$

for $m = 2, 3, \dots, n, n+1$.

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Next, we assume that for some k and all $m > k$,

$$S^{\{\pi(1),\dots,\pi(k)\}}_{(\pi(m),\dots,\pi(n))} = S_{(\pi(m),\dots,\pi(n))}.$$

Thus, if $k < m - 1$,

$$\begin{aligned} S^{\{\pi(1),\dots,\pi(k+1)\}}_{(\pi(m),\dots,\pi(n))} &= S^{\{\pi(1),\dots,\pi(k)\}}_{(\pi(k+1),\pi(m),\dots,\pi(n))} + S^{\{\pi(1),\dots,\pi(k)\}}_{(\pi(m),\pi(k+1),\pi(m+1),\dots,\pi(n))} + \cdots + S^{\{\pi(1),\dots,\pi(k)\}}_{(\pi(m),\dots,\pi(n),\pi(k+1))} \\ &= S_{(\pi(k+1),\pi(m),\dots,\pi(n))} + S_{(\pi(m),\pi(k+1),\pi(m+1),\dots,\pi(n))} + \cdots + S_{(\pi(m),\dots,\pi(n),\pi(k+1))} \\ &= S^{\{\pi(k+1)\}}_{(\pi(m),\dots,\pi(n))} \\ &= S_{(\pi(m),\dots,\pi(n))} \end{aligned}$$

and the proof of Lemma 4 is completed. ■

Proof of Theorem 1: By Lemma 3, we know immediately that the sum of all $S_{(\pi(1),\pi(2),\dots,\pi(n))}$'s is 1. Now,

$$\begin{aligned} S_{(\pi(1),\pi(2),\dots,\pi(n))} &= \frac{1}{1-p_\varnothing}\sum_{j=1}^{n} \frac{S_{(\pi(j+1),\dots,\pi(n))}\,p_{\{\pi(1),\dots,\pi(j)\}}}{j!} \\ &= \frac{1}{1-p_\varnothing}\left[\sum_{j=1}^{n-1} \frac{p_{\{\pi(1),\dots,\pi(j)\}}}{j!} S^{\{\pi(1),\dots,\pi(j)\}}_{(\pi(j+1),\dots,\pi(n))} + \frac{p_N}{n!}\right] \qquad \text{(by Lemma 4)}, \end{aligned}$$

that is,

$$S_{(\pi(1),\pi(2),\dots,\pi(n))} = p_\varnothing S_{(\pi(1),\pi(2),\dots,\pi(n))} + \sum_{j=1}^{n-1} \frac{p_{\{\pi(1),\dots,\pi(j)\}}}{j!} S^{\{\pi(1),\dots,\pi(j)\}}_{(\pi(j+1),\dots,\pi(n))} + \frac{p_N}{n!}.$$

Note that, in order to reach state π in one step from some permutation, only sets of records $\{\pi(1),\dots,\pi(j)\}$, $j = 0,1,\dots,n$ should be accessed. When $j = 0$, that is, no record is accessed, the Markov chain will stay in the state π in one step provided that its current state is π. When $j = n$, that is, all records are accessed at the same time, the chain may arrive at the state π in one step (with probability $p_N/n!$) no matter which the current state is. When $j = 1,2,\dots,n-1$, $S^{\{\pi(1),\dots,\pi(j)\}}_{(\pi(j+1),\dots,\pi(n))}$ is just the sum of all S_ν's such that, starting from the state ν, the chain may arrive at the state π in one step (with probability $p_{\{\pi(1),\dots,\pi(j)\}}/j!$). So, Theorem 1 is proved. ■

We now proceed to prove Theorem 3.

Lemma 5: *Let $\{A_1,\dots,A_m, E_1,\dots,E_k\}$ be a partition of N (note that one of m and k may be zero). Let α_i be a permutation of A_i, $i = 1,\dots,m$, $E_i = \{e_i(1), e_i(2),\dots,e_i(g_i)\}$, $i = 1,\dots,k$. Define*

$$Y^{E_1,\dots,E_k}_{\alpha_1,\dots,\alpha_m} := \sum_{\epsilon_1,\dots,\epsilon_k} T_{\alpha_1,\dots,\alpha_m,\epsilon_1,\dots,\epsilon_k},$$

where ϵ_i runs over all permutations of E_i, $i = 1,\dots,k$.

Then,

$$Y^{E_1,\dots,E_k}_{\alpha_1,\dots,\alpha_m} = T_{\alpha_1,\dots,\alpha_m,(e_1(1)),\dots,(e_1(g_1)),\dots,(e_k(1)),\dots,(e_k(g_k))}.$$

PROOF: Obviously, the conclusion is true when each E_i contains not more than one element, $i = 1,\dots,k$.

In general, by the definition of $T_{\alpha_1,\dots,\alpha_m,\epsilon_1,\dots,\epsilon_k}$,

$$Y^{E_1,\dots,E_k}_{\alpha_1,\dots,\alpha_m} = \frac{1}{1 - R(\alpha_1,\dots,\alpha_m,E_1,\dots,E_k)} \times \sum_{\beta_1,\dots,\beta_m,F_1,\dots,F_k} p_{(\cup_{i=1}^m B_i)\cup(\cup_{j=1}^k F_j)} Y^{F_1,E_1\setminus F_1,\dots,F_k,E_k\setminus F_k}_{\beta_1,\beta_1',\dots,\beta_m,\beta_m'},$$

where the summation runs over all β_i's as defined in Eq. (1) ($i = 1,\dots,m$) and all F_j's being subsets of E_i ($j = 1,\dots,k$) with the restriction that there exists *either* at least one i such that $B_i \neq A_i$ and $B_i \neq \varnothing$ *or* at least one j such that $F_j \neq E_j$ and $F_j \neq \varnothing$.

By mathematical induction,

$$\begin{aligned}
&[1 - R(\alpha_1,\dots,\alpha_m,E_1,\dots,E_k)]Y^{E_1,\dots,E_k}_{\alpha_1,\dots,\alpha_m} \\
&= \sum_{\beta_1,\dots,\beta_m:\exists i \text{ such that } B_i\neq A_i \text{ and } B_i\neq\varnothing} T_{\beta_1,\beta_1'\dots,\beta_m,\beta_m',(e_1(1)),\dots,(e_k(g_k))} \sum_{F\subseteq\cup_{j=1}^k E_j} p_{(\cup_{i=1}^m B_i)\cup F} \\
&\quad + T_{\alpha_1,\dots,\alpha_m,(e_1(1)),\dots,(e_k(g_k))} \sum_{B:\forall i,\, B\cap A_i = A_i \text{ or } \varnothing} \ \sum_{F\subset\cup_{j=1}^k E_j,\, F\neq\cup_{j=1}^k E_j,\, F\neq\varnothing} p_{B\cup F} \\
&= [1 - R(\alpha_1,\dots,\alpha_m,(e_1(1)),\dots,(e_k(g_k)))]T_{\alpha_1,\dots,\alpha_m,(e_1(1)),\dots,(e_k(g_k))} \\
&\quad + \{T_{\alpha_1,\dots,\alpha_m,(e_1(1)),\dots,(e_k(g_k))}[R(\alpha_1,\dots,\alpha_m,(e_1(1)),\dots,(e_k(g_k))) \\
&\qquad - R(\alpha_1,\dots,\alpha_m,E_1,\dots,E_k)]\} \\
&= [1 - R(\alpha_1,\dots,\alpha_m,E_1,\dots,E_k)]T_{\alpha_1,\dots,\alpha_m,(e_1(1)),\dots,(e_k(g_k))}.
\end{aligned}$$

■

LEMMA 6: *Suppose*

$$\{A_1,A_2,\dots,A_m,C_1,C_2,\dots,C_m\}$$

is a partition of N, α_i *is a permutation of* A_i, γ_i *is a permutation of* C_i, $i = 1,2,\dots,m$. *Assume that*

$$\alpha_i \equiv (a_1^i,a_2^i,\dots,a_{n(1)}^i) \quad \textit{and} \quad \gamma_i \equiv (c_1^i,c_2^i,\dots,c_{u(i)}^i).$$

Let

$$T^{\gamma_1,\gamma_2,\dots,\gamma_m}_{\alpha_1,\alpha_2,\dots,\alpha_m} := \sum_{\zeta_1,\zeta_2,\dots,\zeta_m} T_{\zeta_1,\zeta_2,\dots,\zeta_m}$$

where ζ_i *runs over all permutations of* $A_i \cup C_i$ *with the relative orders of the elements of* A_i *and those of* C_i *remaining the same as in* α_i *and* γ_i, *respectively.*

Then,

$$T^{\gamma_1,\gamma_2,\dots,\gamma_m}_{\alpha_1,\alpha_2,\dots,\alpha_m} = T_{\alpha_1,\alpha_2,\dots,\alpha_m,\gamma_1,\gamma_2,\dots,\gamma_m}.$$

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PROOF: When all n_i's and u_i's are 0 or 1,

$$T^{\gamma_1,\gamma_2,\ldots,\gamma_m}_{\alpha_1,\alpha_2,\ldots,\alpha_m} = Y^{A_1\cup C_1,\ldots,A_m\cup C_m} = 1$$

by Lemma 5. Generally,

$$\begin{aligned}T^{\gamma_1,\gamma_2,\ldots,\gamma_m}_{\alpha_1,\alpha_2,\ldots,\alpha_m} &= \sum_{\zeta_1,\zeta_2,\ldots,\zeta_m} T_{\zeta_1,\zeta_2,\ldots,\zeta_m}\\ &= \frac{1}{1-R(A_1\cup C_1,\ldots,A_m\cup C_m)}\sum p_{\cup_{i=1}^m(B_i\cup D_i)} T^{\delta_1,\delta_1',\ldots,\delta_m,\delta_m'}_{\beta_1,\beta_1',\ldots,\beta_m,\beta_m'},\end{aligned}$$

where β_i, B_i, and β_i' are defined as in Eq. (1),

$$\begin{aligned}\delta_i &= (c^i_1,\ldots,c^i_{v(i)}),\\ D_i &= \{c^i_1,\ldots,c^i_{v(i)}\},\\ \delta_i' &= (c^i_{v(i)+1},\ldots,c^i_{u(i)}),\end{aligned}$$

and the summation runs over all β_i's and all δ_i's with $0 \le l(i) \le n(i)$, $0 \le v(i) \le u(i)$, and under the restriction

$$\exists i \text{ such that } 0 < l(i) + v(i) < n(i) + u(i). \qquad \textbf{(3)}$$

Note that the partition $\{B_1, A_1\backslash B_1, \ldots, B_m, A_m\backslash B_m, D_1, C_1\backslash D_1, \ldots, D_m, C_m\backslash D_m\}$ is finer than $\{A_1, A_2, \ldots, A_m, C_1, C_2, \ldots, C_m\}$. So, by mathematical induction,

$$\begin{aligned}&[1-R(A_1\cup C_1,\ldots,A_m\cup C_m)]T^{\gamma_1,\gamma_2,\ldots,\gamma_m}_{\alpha_1,\alpha_2,\ldots,\alpha_m}\\ &\quad= \sum_{\exists i \text{ such that } 0<l(i)<n(i) \text{ or } 0<v(i)<u(i)} p_{\cup_{i=1}^m(B_i\cup D_i)} T_{\beta_1,\beta_1',\ldots,\beta_m,\beta_m',\delta_1,\delta_1',\ldots,\delta_m,\delta_m'}\\ &\qquad+ \sum_{\text{satisfying Eq. (3) and } \forall i,\, l(i)=0 \text{ or } n(i),\, v(i)=0 \text{ or } u(i)} p_{\cup_{i=1}^m(B_i\cup D_i)} T_{\alpha_1,\ldots,\alpha_m,\gamma_1,\ldots,\gamma_m}\\ &\quad= [1-R(A_1,\ldots,A_m,C_1,\ldots,C_m)]T_{\alpha_1,\ldots,\alpha_m,\gamma_1,\ldots,\gamma_m}\\ &\qquad+ [R(A_1,\ldots,A_m,C_1,\ldots,C_m) - R(A_1\cup C_1,\ldots,A_m\cup C_m)]T_{\alpha_1,\ldots,\alpha_m,\gamma_1,\ldots,\gamma_m}\\ &\quad= [1-R(A_1\cup C_1, A_m\cup C_m)]T_{\alpha_1,\ldots,\alpha_m,\gamma_1,\ldots,\gamma_m}.\end{aligned}$$

■

PROOF OF THEOREM 3: By Lemma 5 (letting $k = 1$ and all α_i's be empty), we know that $\sum_\pi T_\pi = 1$ where π runs over all permutations of N. Moreover,

$$\begin{aligned}&T_{(\pi(1),\pi(2),\ldots,\pi(n))}\\ &\quad= \frac{1}{1-p_\varnothing-p_N}\sum_{j=1}^{n-1} p_{\{\pi(1),\pi(2),\ldots,\pi(j)\}} T_{(\pi(1),\pi(2),\ldots,\pi(j)),(\pi(j+1),\pi(j+2),\ldots,\pi(n))}\\ &\quad= \frac{1}{1-p_\varnothing-p_N}\sum_{j=1}^{n-1} p_{\{\pi(1),\pi(2),\ldots,\pi(j)\}} T^{(\pi(1),\pi(2),\ldots,\pi(j))}_{(\pi(j+1),\pi(j+2),\ldots,\pi(n))} \quad \text{(by Lemma 6)}\end{aligned}$$

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that is,

$$T_{(\pi(1),\pi(2),\ldots,\pi(n))} = \sum_{j=1}^{n-1} p_{\{\pi(1),\pi(2),\ldots,\pi(j)\}} T^{(\pi(1),\pi(2),\ldots,\pi(j))}_{(\pi(j+1),\pi(j+2),\ldots,\pi(n))} + (p_{\varnothing} + p_N) T_{(\pi(1),\pi(2),\ldots,\pi(n))}.$$

Similar to the proof of Theorem 1, note that, in order to reach state π in one step from some permutation, only sets of records $\{\pi(1),\ldots,\pi(j)\}$, $j = 0,1,\ldots,n$, should be accessed. When $j = 0$ or n, that is, none or all of the records are accessed, the Markov chain will stay in the state π in one step provided that its current state is π. When $j = 1,2,\ldots,n-1$, $T^{(\pi(1),\ldots,\pi(j))}_{(\pi(j+1),\ldots,\pi(n))}$ is just the sum of all T_ζ's such that, starting from the state ζ, the chain may arrive at the state π in one step. Thus, Theorem 3 is proved. ■