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Optimal insurance control for insurers with jump-diffusion risk processes

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Abstract
In this paper, we model the surplus process as a compound Poisson process perturbed by diffusion and allow the insurer to ask its customers for input to minimize the distance from some prescribed target path and the total discounted cost on a fixed interval. The problem is reduced to a version of a linear quadratic regulator under jump-diffusion processes. It is treated using three methods: dynamic programming, completion of square and the stochastic maximum principle. The analytic solutions to the optimal control and the corresponding optimal value function are obtained.

Keywords
Compound Poisson process; Diffusion; Hamilton-Jacobi-Bellman equation; Completion of square; Stochastic maximum principle

Mathematics Subject Classification (2000)
93E20; 91B30

1. Introduction

In recent years, stochastic control theory has gained significant interest in the insurance literature. This is because the insurance company can control the surplus process such that a certain objective function is minimized (maximized). In particular, there are three main criteria: maximizing the discounted total dividend, minimizing the probability of ruin and maximizing the exponential utility. The corresponding modern continuous-time approach was pioneered by Browne (1995) and Asmussen & Taksar (1997), who applied classical stochastic control methods to reduce the optimization problem to a matter of solving a Hamilton-Jacobi-Bellman (HJB) equation. Browne (1995) found the optimal investment strategies to minimize the probability of ruin and to maximize the exponential utility function under the model of Brownian motion with drift. For the same model, Asmussen & Taksar (1997) obtained the optimal dividend strategy. Since their pioneering work, many attempts have been made to solve the optimization problem in a framework that allows more controls. Examples where the optimal dividend problem was treated under the model of diffusion are Paulsen & Gjessing (1997), Paulsen (2003), Asmussen *et al.* (2000), Hϕjgaard & Taksar (1999, 2001, 2004) and Choulli *et al.* (2003). For the same model, Schmidli (2001).

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Taksar & Markussen (2003), Promislow & Young (2005) and Bai & Guo (2008) considered the problem of minimizing the probability of ruin. All of these under the diffusion model gave the closed-form solution. However, in the classical risk model, since the corresponding HJB equation contains the integro term and differential term simultaneously, it is more difficult to solve. For when the objective function is an exponential utility function, Yang & Zhang (2005) gave the closed-form solution for the jump-diffusion model, whereas when the objective function is mean variance, Bai & Zhang (2008) gave the optimal solution for the classical risk model. For other objective functions, especially the ruin probability, only the existence of a solution to the HJB equation was proved, and a verification theorem was given. Among them are Hipp & Plum (2000), Hipp & Taksar (2000), Schmidli (2001, 2002), and Hipp & Vogt (2003).

In this paper, the surplus is modeled as a compound Poisson process perturbed by diffusion. Assume that the insurer is allowed to ask its customers for input to minimize the distance from some prescribed target path and the total discounted cost on a fixed interval. Then, the objective is to find the amount of the input at every time (the optimal control) such that the distance from some prescribed target path and the total discounted cost are minimized and to calculate the minimizing value (the optimal value function).

For the above optimization problem, we first use a dynamic programming approach to solve it. By changing the HJB equation to an ordinary partial differential equation, the analytic solutions to the optimal control and the optimal value function are obtained. Then, it is treated again by the completion of square and stochastic maximum principle. This is different from the dynamic programming approach in that two methods lead to a stochastic differential equation for the optimal control process and not a nonlinear partial differential equation for the optimal value function. Solving the stochastic differential equation yields the optimal control. Then, the optimal value function is obtained by two different methods again.

By comparing three methods, it can be found that (1) the dynamic programming is the best method for solving the optimal solution in this paper and that (2) the dynamic programming is limited for the Markov process, and the two other methods, the completion of square and stochastic maximum principle, are not. Therefore, the process given in this paper to solve for the optimal solution has the inspiring effect of using these two methods to solve the optimal control problems in Non-Markov risk processes (for example, the classical risk model with fractional Brownian motion perturbation).

The paper is organized as follows. In section 2, the model assumptions are formulated. The control and the objective function are introduced. In section 3, the control problem is solved. The problem is divided into three parts. In subsection 3.1, the dynamic programming approach is used. The optimal value function and the optimal control are obtained by the solution and the minimizing function of the HJB equation. In subsection 3.2, stochastic differential equations for the optimal control process are first obtained by the completion of square approach. Solving the equation results in the optimal control. Then, the optimal value function is obtained by its definition. In subsection 3.4, the same stochastic differential equations are obtained using the Hamiltonian system. Then, to obtain the optimal value function, we give a lemma that complements the results given by Framstad *et al.* (2004). Combining these results, the expression of the optimal value function is again obtained.

2. The Model

Consider the following classical surplus process perturbed by diffusion

$$X_t = ct - \sum_{i=1}^{N_t} Y_i + \sigma B_t, \quad t \geq 0, \tag{2.1}$$

where c is the rate at which the premiums are received. $\{N_t; t \geq 0\}$ is a Poisson process with parameter β, denoting the total number of claims with claim times $T_i(i=1, 2 \ldots)$. Y_1, Y_2, … , independent of $\{N_t; t \geq 0\}$, are positive i.i.d. random variables with a common distribution function (df)$F(x)$, the moment $\mu_j = \int_0^\infty x^j F(dx)$, for $j=1, 2$. $\{B_t; t \geq 0\}$ is a standard Wiener process that is independent of the aggregate claim process $\sum_{i=1}^{N_t} Y_i$, and σ is the dispersion parameter.

In addition to the premium income, we here assume that the company also receives interest on its reserves with interest force δ_t. The interest is assumed to be a deterministic function of time. Thus, the surplus at time t, without control, is given by the dynamics

$$\begin{cases} dX_t = (\delta_t X_t + c)dt - d\sum_{i=1}^{N_t} Y_i + \sigma dB_t, & t \in [s, T], \\ X_s = x, \end{cases} \tag{2.2}$$

where T is a fixed time and s and x denote the initial time and initial surplus, respectively.

For the remainder of this paper, we work on a complete probability space (Ω, F, P) on which the process $\{X_t, 0 \leq t \leq T\}$ is defined. The information at time t is given by the complete filtration $\{F_t^s\}$ generated by $\{X_t, s \leq t \leq T\}$.

A strategy α is described by a stochastic process $\{u_t^\alpha, s \leq t \leq T\}$, where u_t^α represents the input in time interval $(t, t+dt)$. When applying the strategy α, we let $\{X_t^\alpha\}$ denote the controlled risk process. The dynamic for X_t^α is then given by

$$\begin{cases} dX_t^\alpha = \left(\delta_t X_t^\alpha + c + u_t^\alpha\right)dt - d\sum_{i=1}^{N_t} Y_i + \sigma dB_t, & t \in [s, T] \\ X_s = x, \end{cases} \tag{2.3}$$

The strategy α is said to be admissible if u_t^α is F_t^s-progressively measurable and such that stochastic differential equation (2.3) has a unique solution. In this case, we call the process $\{u_t^\alpha\}$ the control process or simply the control. We denote by Π the set of all admissible strategies.

For a given admissible strategy α, we define the objective function V_α by

$$V_\alpha(s,x) = \frac{1}{2} E\left[\int_s^T \left(q_t\left(X_t^\alpha - A_t\right)^2 + e^{-\lambda t}\left(u_t^\alpha\right)^2\right)dr + q_T\left(X_T^\alpha - A_T\right)^2\right],$$
$$\forall (s,x) \in [0,T) \times R. \tag{2.4}$$

In (2.4), q_t and A_t are both continuous functions on the interval $[0, T)$, and λ denotes a discount rate. A_t represents the prescribed target path, and q_t represents the prescribed proportion. In particular, when $q_t = 1$, this choice of objective function ensures the minimization of the distance from some prescribed target path A_t and simultaneously minimized the total discounted cost over the interval $[s, T]$.

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The objective is to find the optimal value function

$$V(s,x)=\min_{\alpha\in\Pi} V_\alpha(s,x), \quad \forall (s,x)\in[0,T)\times R, \tag{2.5}$$

and to find an optimal control α^* such that

$$V(s,x)=V_{\alpha^*}(s,x)=\min_{\alpha\in\Pi} V_\alpha(s,x), \quad \forall (s,x)\in[0,T)\times R.$$

Let $C^{1,2}$ denote the space of $\phi(r,x)$ such that ϕ and its partial derivatives ϕ_r, ϕ_x, ϕ_{xx} are continuous on $[0,T]\times R$. Let $C^{1,2}_{pc}$ denote the space of $\phi(r,x)$ such that $\phi\in C^{1,2}$ and satisfies a polynomial growth condition, i.e., there exist constants K and n such that, for all $(r,x)\in R^+\times R$, $|\phi(r,x)|\leq K(1+|x|^n)$. Moreover, $\phi(r,x)$ satisfies

$$E\left[\int_s^T\int_0^\infty \left|\phi(r,X_r^\alpha)-\phi(r,X_r^\alpha-z)\right| F(dz)dr\right]<\infty \tag{2.6}$$

for any control α. As will be seen in Theorem 3.1, the polynomial growth condition mainly ensures that the term of the stochastic integral over Brownian motion is a martingale (see Fleming & Soner (1993), P135), while (2.6) ensures that the term of the stochastic integral over the compensated Poisson point process is a martingale (see Brémaud (1981), P235).

Let $L^2_F(s,T;R)$ denotes the set of all $\{F^s_t\}_{t\geq s}$-adapted R-valued processes $Y(\cdot)$ such that $E\int_s^T|Y(r)|^2dr<\infty$.

3. Solution of the Control Problem

We now present an analytic solution of the control problem. The problem is treated in three ways. One way is through the dynamic programming approach, which is traditionally used to solve the optimal control problem for the case whereby the controlled process has the Markov property. The second method is through a completion of squares method, inspired by the recent work of Frangos *et al.* (2008) on the same linear quadratic problem under a fractional Brownian motion. The third method is through the application of a stochastic maximum principle for jump diffusion. This method was proposed for general control problems by Framstad *et al.* (2004).

3.1. The dynamic programming method

From standard arguments, we know that if the optimal value function $V\in C^{1,2}$, then V satisfies the following Hamilton-Jacobi-Bellman (HJB) equation

$$\min_{u\in R}\left\{V_t+(\delta_t x+c+u)V_x+\frac{1}{2}\sigma^2V_{xx}+\beta EV(t,x-Y)-\beta V(t,x)+\frac{1}{2}q_t(x-A_t)^2+\frac{1}{2}e^{-\lambda t}u^2\right\}=0,$$
$$\forall (t,x)\in[0,T)\times R, \tag{3.1}$$

with the terminal value

$$V(T,x)=\frac{1}{2}q_T(x-A_T)^2, \tag{3.2}$$

where, for notational convenience, we replace s by t in (2.5), and Y is a generic random variable that has the same distribution as $Y_i (i=1,2\ldots)$.

Note that in many cases, the optimal value function may fail to be sufficiently smooth to satisfy the HJB equation (3.1) in the classical sense. However, it still satisfies (3.1) in the viscosity sense (see Fleming & Soner (1993). The following verification theorem shows that the classical solution to the HJB equation yields the solution to the optimization problem.

Theorem 3.1. *Assume that $W \in C^{1,2}_{pc}$ satisfies (3.1)–(3.2). Then, the value function V given by (2.5) and W coincide. Furthermore, let $u^*(t, x)$ be such that*

$$V_t + (\delta_t x + c + u^*(t,x))V_x + \frac{1}{2}\sigma^2 V_{xx} + \beta EV(t, x-Y) - \beta V(t,x) + \frac{1}{2}q_t(x-A_t)^2 + \frac{1}{2}e^{-\lambda t}u^{*^2}(t,x) = 0$$

for all $(t, x) \in [0, T] \times R$. Then, the Markov control strategy α^* of the from $u_t^* = u^*\left(t, X_t^{u^*}\right)$ is optimal. Specifically, $W(t,x) = V(t,x) = V_{\alpha^*}(t,x)$.

Proof. Let α be an arbitrary control. Then, by applying the Itô formula

$$\begin{aligned}
W\left(T, X_T^\alpha\right) &= W(t,x) + \int_s^T \left[W_r\left(r, X_r^\alpha\right) + \left(\delta_r X_r^\alpha + c + u_r^\alpha\right)W_x\left(r, X_r^\alpha\right) + \frac{1}{2}\sigma^2 W_{xx}\left(r, X_r^\alpha\right)\right]dr \\
&\quad + \sigma\int_s^T W_x\left(r, X_r^\alpha\right)dB_r + \sum_{i=N_t+1}^{N_T}\left(W\left(T_i, X_{T_i}^\alpha\right) - W\left(T_i, X_{T_i-}^\alpha\right)\right) \\
&\geq W(t,x) + \int_s^T \left[\beta W\left(r, X_r^\alpha\right) - \beta EW\left(r, X_r^\alpha - Y\right) - \frac{1}{2}q_r\left(X_r^\alpha - A_r\right)^2 - \frac{1}{2}e^{-\lambda r}u_\alpha^2(r)\right]dr \\
&\quad + \sigma\int_s^T W_x\left(r, X_r^\alpha\right)dB_r + \sum_{i=N_t+1}^{N_T}\left(W\left(T_i, X_{T_i}^\alpha\right) - W\left(T_i, X_{T_i-}^\alpha\right)\right)
\end{aligned}$$

since $W(t, x)$ satisfies (3.1). The terminal value (3.2) implies that $W\left(T, X_T^\alpha\right) = \frac{1}{2}q_T\left(X_T^\alpha - A_T\right)^2$. Then, rearranging yields

$$\begin{aligned}
&\frac{1}{2}\left[q_T\left(X_T^\alpha - A_T\right)^2 + \int_s^T \left(q_r(X_r^\alpha - A_r)^2 + e^{-\lambda r}(u_r^\alpha)^2\right)dr\right] \\
&\geq W(t,x) + \sigma\int_s^T W_x\left(r, X_r^\alpha\right)dB_r + \sum_{i=N_t+1}^{N_T}\left(W\left(T_i, X_{T_i}^\alpha\right) - W\left(T_i, X_{T_i-}^\alpha\right)\right) \\
&\quad + \beta\int_s^T \left[W\left(r, X_r^\alpha\right) - EW\left(r, X_r^\alpha - Y\right)\right]dr. \qquad (3.3)
\end{aligned}$$

Since the compound Poisson process jumps only finitely in any finite interval, the second integral does not change if r is replaced by $r-$. Thus, by $W \in C^{1,2}_{pc}$, we have that

$$\sigma\int_0^t \left(W_x(r, X_r^\alpha)dB_r + \sum_{i=1}^{N_t}\left(W\left(T_i, X_{T_i}^\alpha\right) - W\left(T_i, X_{T_i-}^\alpha\right)\right) + \beta\int_0^t \left[W\left(r, X_r^\alpha\right) - W\left(r, X_r^\alpha - Y\right)\right]dr$$

is a martingale. Taking expectations on both sides of inequality (3.3), it follows that $V_\alpha(t, x) \geq W(t, x)$, which implies $V(t, x) \geq W(t, x)$. For the optimal control α^*, the inequality becomes an equality, that is, $V_{\alpha^*}(t,x) = W(t,x)$. Thus, $V(t, x) \leq W(t, x)$, which completes the proof.

We see from Theorem 3.1 that if the classical solution $W \in C^{1,2}_{pc}$ to (3.1)-(3.2) can be found, then we have the (unique) optimal value function $V(t, x)$ and the corresponding optimal control $\{\alpha^*\}$. In other

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words, for the above optimal problem, we need to solve the nonlinear partial differential equation (3.1) and find the value $u^*(t, x)$ that minimizes the function

$$W_t+(\delta_t x+c+u)W_x+\frac{1}{2}\sigma^2 W_{xx}+\beta EW(t,x-Y)-\beta W(t,x)+\frac{1}{2}q_t(x-A_t)^2+\frac{1}{2}e^{-\lambda t}u^2. \tag{3.4}$$

Theorem 3.2. *Define*

$$W(t,x)=\frac{1}{2}\pi(t)x^2+(g(t)-q_T A_T)x+f(t), \tag{3.5}$$

where $\pi(\cdot)$, $g(\cdot)$ satisfy

$$\begin{cases} \pi'(t)-\pi^2(t)e^{\lambda t}+2\delta_t\pi(t)+q_t=0, \\ g'(t)+g(t)\left(\delta_t-\pi(t)e^{\lambda t}\right)+\left(q_T A_T e^{\lambda t}+c-\beta\mu_1\right)\pi(t)-\delta_t q_T A_T-q_t A_t=0, \\ \pi(T)=q_T, \\ g(T)=0. \end{cases} \tag{3.6}$$

f(t) satisfies

$$f'(t)=\frac{1}{2}(g(t)-q_T A_T)^2 e^{\lambda t}-(c-\beta\mu_1)(g(t)-q_T A_T)-\frac{1}{2}\beta\pi(t)\mu_2-\frac{1}{2}q_t A_t^2-\frac{1}{2}\sigma^2\pi(t) \tag{3.7}$$

with boundary condition

$$f(T)=\frac{1}{2}q_T A_T^2. \tag{3.8}$$

Then $W(t,x)\in C^{1,2}_{pc}$ is a solution of the HJB equation (3.1). The corresponding minimizing function is given by

$$u(t,x)=(-\pi(t)x-g(t)+q_T A_T)e^{\lambda t} \tag{3.9}$$

with terminal condition

$$u(T,x)=-q_T(x-A_T)e^{\lambda T}.$$

Proof. By direct calculation, we obtain that

$$W_x=\pi(t)x+(g(t)-q_T A_T), \quad W_t=\frac{1}{2}x^2\pi'(t)+g'(t)x+f'(t), \quad W_{xx}=\pi(t). \tag{3.10}$$

Differentiating with respect to u in (3.4) and setting the derivative to be zero result in

$$u(t,x)=-W_x(t,x)e^{\lambda t}.$$

Thus, we have

$$\begin{aligned} &\min_{u\in R}\left\{\left[W_t+(\delta_t x+c+u)W_x+\frac{1}{2}\sigma^2 W_{xx}+\frac{1}{2}e^{-\lambda t}u^2\right](t,x)+\frac{1}{2}q_t(x-A_t)^2+\beta EW(t,x-Y)-\beta W(t,x)\right\} \\ &=\left[W_t+\left(\delta_t x+c-W_x e^{\lambda t}\right)W_x+\frac{1}{2}\sigma^2 W_{xx}+\frac{1}{2}e^{\lambda t}W_x^2\right](t,x)+\beta EW(t,x-Y)-\beta W(t,x) \\ &+\frac{1}{2}q_t(x-A_t)^2. \end{aligned} \tag{3.11}$$

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Plugging (3.10) into (3.11) we obtain

$$\min_{u\in R}\left\{\left[W_t+(\delta_t x+c+u)W_x+\frac{1}{2}\sigma^2W_{xx}+\frac{1}{2}e^{-\lambda t}u^2\right](t,x)+\frac{1}{2}q_t(x-A_t)^2+\beta EW(t,x-Y)-\beta W(t,x)\right\}$$

$$=\frac{1}{2}\pi'(t)x^2+g'(t)x+f'(t)+(\delta_t x+c)[\pi(t)x+g(t)-q_TA_T]-\frac{1}{2}e^{\lambda t}[\pi(t)x+g(t)-q_TA_T]^2+\frac{1}{2}\sigma^2\pi(t)$$

$$+\beta E\left[\frac{1}{2}\pi(t)(x-Y)^2+(g(t)-q_TA_T)(x-Y)\right]-\beta\left[\frac{1}{2}\pi(t)x^2+(g(t)-q_TA_T)x\right]. \quad (3.12)$$

Notice that $E[Y]=\mu_1$ and $E[Y^2]=\mu_2$. Inserting (3.6), (3.7) and (3.8) into (3.12), we obtain that

$$\min_{u\in R}\left\{W_t+(\delta_t x+c+u)W_x+\frac{1}{2}\sigma^2W_{xx}+\beta EW(t,x-Y)-\beta W(t,x)+\frac{1}{2}q_t(x-A_t)^2+\frac{1}{2}e^{-\lambda t}u^2\right\}(t,x)=0.$$

It is obvious that $W(T, x)$ satisfies (3.2). Thus, we deduce that $W(t, x)$ is solution of (3.1)–(3.2) and the optimal control is given by (3.9).

Remark 3.1. In particular, let $\delta_t=q_t=0$ for $t<T$. In this case, we can obtain the solution of equation (3.6):

$$\begin{cases}\pi(t)=\frac{\lambda}{\frac{\lambda}{q_T}-e^{\lambda t}+e^{\lambda T}},\\ g(t)=\frac{\lambda(c-\beta\mu_1)(t-T)+q_TA_T\left(e^{\lambda t}-e^{\lambda T}\right)}{e^{\lambda t}-e^{\lambda T}-\frac{\lambda}{q_T}}.\end{cases} \quad (3.13)$$

3.2. The completion of squares method

Now, we show that the optimal control can be given by the solution of a forward backward stochastic differential equation. The approach is similar to that of Frangos *et al.* (2008).

Theorem 3.3 *The optimal control α^* is given by $u_t^*=-p_te^{\lambda t}$, where p_t is the solution of the following backward stochastic differential equation:*

$$\begin{cases}dX_t^*=\left(\delta_tX_t^*+c-e^{\lambda t}p_t\right)dt-d\sum_{i=1}^{N_t}Y_i+\sigma dB_t,\\ dp_t=\left(-\delta_tp_t-q_t\left(X_t^*-A_t\right)\right)dt-\eta_t\left(d\sum_{i=1}^{N_t}Y_i-\beta\mu_1dt\right)+\sigma\gamma_tdB_t, \quad t\in[s,T],\\ X_s=x,\\ p_T=q_T(X_T^*-A_T),\end{cases} \quad (3.14)$$

for $(s, x)\in[0, T]\times R$. Here, X_t^* denotes the resulting process controlled by $\{u_t^*\}$, and η_t and γ_t are two continuous processes such that

$$E\left[\int_s^T\sigma^2\gamma_t^2\left(X_t^*-X_t^\alpha\right)^2dt\right]<\infty, \quad (3.15)$$

$$E\left[\int_s^T|\beta\mu_1\eta_t \mid X_t^*-X_t^\alpha|dt\right]<\infty, \quad (3.16)$$

for any control α.

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Proof. Let α be an arbitrary control and recall the definition of V_α, as given in (2.4). The objective function V_α may not be continuously differentiable. Consider

$$V_\alpha(s,x)-V_{\alpha^*}(s,x)=\frac{1}{2}E\Big[q_T\Big(\big(X_T^\alpha-A_T\big)^2-\big(X_T^*-A_T\big)^2\Big)$$
$$+\int_s^T\Big(q_t\Big(\big(X_t^\alpha-A_t\big)^2-\big(X_t^*-A_t\big)^2\Big)+e^{-\lambda t}\Big(\big(u_t^\alpha\big)^2-\big(u_t^*\big)^2\Big)\Big)dt\Big].$$

Using the equality

$$y^2-y^{*^2}=(y-y^*)^2+2(y-y^*)y^*$$

results in $V_\alpha(s,x)-V_{\alpha^*}(s,x)=I_1+I_2$, where

$$I_1=\frac{1}{2}E\left[q_T\big(X_T^\alpha-X_T^*\big)^2+\int_s^T\Big(q_t\big(X_t^\alpha-X_t^*\big)^2+e^{-\lambda t}\big(u_t^\alpha-u_t^*\big)^2\Big)dt\right]\geq 0,$$

$$I_2=E\left[q_T\big(X_T^\alpha-X_T^*\big)\big(X_T^*-A_T\big)+\int_s^T\big(q_t\big(X_t^\alpha-X_t^*\big)\big(X_t^*-A_t\big)-p_t\big(u_t^\alpha-u_t^*\big)\big)dt\right],$$

in which we have used $u_t^*=-p_te^{\lambda t}$. Considering that X_t^α and X_t^* solve equation (2.3), we can obtain

$$d\big(X_t^\alpha-X_t^*\big)=\big(\delta_t\big(X_t^\alpha-X_t^*\big)+\big(u_t^\alpha-u_t^*\big)\big)dt. \tag{3.17}$$

Substituting $u_t^\alpha-u_t^*$ of (3.17) into I_2 yields

$$I_2=E\Big[q_T\big(X_T^\alpha-X_T^*\big)\big(X_T^*-A_T\big)+\int_s^T\big(q_t\big(X_t^\alpha-X_t^*\big)\big(X_t^*-A_t\big)+\delta_tp_t\big(X_t^\alpha-X_t^*\big)\big)dt-\int_s^T p_td\big(X_t^\alpha-X_t^*\big)\Big].$$

In view of (3.14), p_t satisfies

$$dp_t=\big(-\delta_tp_t-q_t\big(X_t^*-A_t\big)\big)dt-\eta_t\left(d\sum_{i=1}^{N_t}Y_i-\beta\mu_1dt\right)+\sigma\gamma_tdB_t.$$

Then, I_2 evolves as

$$I_2=E\Big[q_T\big(X_T^\alpha-X_T^*\big)\big(X_T^*-A_T\big)-\int_s^T p_td\big(X_t^\alpha-X_t^*\big)-\int_s^T\big(X_t^\alpha-X_t^*\big)dp_t$$
$$-\int_s^T\big(X_t^\alpha-X_t^*\big)\eta_t\left(d\sum_{i=1}^{N_t}Y_i-\beta\mu_1dt\right)+\int_s^T\sigma\big(X_t^\alpha-X_t^*\big)\gamma_tdB_t].$$

Since (3.15) and (3.16) implies

$$-\int_s^r\big(X_t^\alpha-X_t^*\big)\eta_t\left(d\sum_{i=1}^{N_t}Y_i-\beta\mu_1dt\right)+\int_s^r\sigma\big(X_t^\alpha-X_t^*\big)\gamma_tdB_t$$

is a martingale, I_2 becomes

$$I_2=E\Big[q_T\big(X_T^\alpha-X_T^*\big)\big(X_T^*-A_T\big)-\int_s^T p_td\big(X_t^\alpha-X_t^*\big)-\int_s^T\big(X_t^\alpha-X_t^*\big)dp_t\Big].$$

Applying the Itô formula to $\big(X_t^\alpha-X_t^*\big)p_t$ results in

$$d\big(X_t^\alpha-X_t^*\big)p_t=p_{t-}d\big(X_t^\alpha-X_t^*\big)+\big(X_{t-}^\alpha-X_{t-}^*\big)dp_t+d\Big\langle X^\alpha-X^*,p\Big\rangle_t.$$

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An analysis similar to that in Theorem 3.1 shows that the first term does not change if $t-$ is replaced by t. Simultaneously, that $X_t^\alpha - X_t^*$ is a continuous finite variation process implies that $t-$ in the second term can also be replaced by t and that the last term is equal to zero. Thus, we have

$$I_2 = E\left[q_T\left(X_T^\alpha - X_T^*\right)\left(X_T^* - A_T\right) - \int_s^T dp_t\left(X_t^\alpha - X_t^*\right)\right] = 0,$$

where the second equality follows from the boundary condition $p_T = q_T\left(X_T^* - A_T\right)$. We therefore conclude that $V_\alpha(s,x) \geq V_{\alpha^*}(s,x)$ for $\alpha \in \Pi$, which proves that α^* is optimal.

We now give the solution of equation (3.14), which provides the optimal control and coincides with the result obtained in the above subsection.

Theorem 3.4. *The solution of equation (3.14) has the form*

$$\begin{cases} p_t = \pi(t)X_t^* + g(t) - q_T A_T, \\ \eta_t = \pi(t), \quad \gamma_t = \pi(t), \quad t \in [s,T], \end{cases} \tag{3.18}$$

where the deterministic functions $\pi(t)$ and $g(t)$ are the solutions of the ordinary differential equation (3.6).

Proof. Assume that q_T, A_T are deterministic. Then,

$$dp_t = \pi'(t)X_t^* dt + \pi(t)dX_t^* + g'(t)dt. \tag{3.19}$$

Substituting (3.18) into (3.14), we have

$$dp_t = \left(-\delta_t\left(\pi(t)X_t^* + g(t) - q_T A_T\right) - q_t\left(X_t^* - A_t\right)\right)dt - \eta_t\left(d\sum_{i=1}^{N_t} Y_i - \beta\mu_1 dt\right) + \sigma\gamma_t dB_t, \tag{3.20}$$

and

$$dX_t^* = \left(\delta_t X_t^* + c - e^{\lambda t}\left(\pi(t)X_t^* + g(t) - q_T A_T\right)\right)dt - d\sum_{i=1}^{N_t} Y_i + \sigma dB_t.$$

Then, (3.19) becomes

$$\begin{aligned} dp_t = {} & \pi'(t)X_t^* dt + \pi(t)\left(\delta_t X_t^* + c\right)dt - \pi(t)e^{\lambda t}\left(\pi(t)X_t^* + g(t) - q_T A_T\right)dt \\ & -\pi(t)d\sum_{i=1}^{N_t} Y_i + \sigma\pi(t)dB_t + g'(t)dt. \end{aligned} \tag{3.21}$$

Thus, by comparing the coefficient of $X_t^* dt$ in (3.20) and (3.21), we have

$$\pi'(t) - \pi^2(t)e^{\lambda t} + 2\delta_t\pi(t) + q_t = 0.$$

Taking the coefficients of $\sum_{i=1}^{N_t} Y_i$ and dB_t yields

$$\eta_t = \pi(t), \quad \gamma_t = \pi(t).$$

From the terms with dt, we have

$$g'(t) + g(t)\left(\delta_t - \pi(t)e^{\lambda t}\right) + \left(q_T A_T e^{\lambda t} + c - \beta\mu_1\right)\pi(t) - \delta_t q_T A_T - q_t A_t = 0.$$

The condition $p_T = q_T(X_T^* - A_T)$ implies that we have the following final conditions:

$$\pi(T) = q_T,$$

$$g(T) = 0.$$

The proof is complete.

We now calculate the optimal value function $V(s, x)$, that is, the corresponding objective function to the optimal control u_t^*. First, by definition, we have

$$2V(s,x) = E\left[\int_s^T \left(q_t(X_t^* - A_t)^2 + e^{-\lambda t} u_t^{*2}\right)dt + q_T\left(X_T^* - A_T\right)^2\right]. \tag{3.22}$$

Substituting $p_T = q_T(X_T^* - A_T)$ into the last term and applying the Itô formula to the resulting term yield

$$\begin{aligned} p_T(X_T^* - A_T) &= p_s(x - A_s) + \int_s^T p_{t-} d(X_t^* - A_t) + \int_s^T (X_{t-}^* - A_t) dp_t + \int_s^T d\langle p, X^* - A\rangle_t, \\ &= p_s(x - A_s) + \int_s^T p_{t-} dX_t^* + \int_s^T X_{t-}^* dp_t - p_T A_T + p_s A_s + \int_s^T (\beta\mu_2\pi(t) + \sigma^2\pi(t))dt. \end{aligned}$$

Then, we denote $Ep_T(X_T^* - A_T) = J_1 + J_2$, where

$$\begin{aligned} J_1 &= p_s x - p_T A_T + \int_s^T (\beta\mu_2\pi(t) + \sigma^2\pi(t))dt \\ &= (\pi(s)x + g(s) - q_T A_T)x - p_T A_T + \int_s^T (\beta\mu_2\pi(t) + \sigma^2\pi(t))dt, \end{aligned}$$

and

$$J_2 = E\left[\int_s^T p_{t-} dX_t^* + \int_s^T X_{t-}^* dp_t\right].$$

Plugging (3.14) into the right-hand side of the above equality and using the martingale property result in

$$J_2 = E\left[\int_s^T (p_{t-}(\delta_t X_t^* + c - e^{\lambda t} p_t - \beta\mu_1) - X_{t-}^* \delta_t p_t - X_{t-}^* q_t(X_t^* - A_t))dt\right].$$

Note that J_2 does not change if $t-$ is replaced by t. Thus, we have

$$J_2 = E\left[\int_s^T (p_t(\delta_t X_t^* + c - e^{\lambda t} p_t - \beta\mu_1) - X_t^* \delta_t p_t - X_t^* q_t(X_t^* - A_t))dt\right].$$

Rearranging yields

$$J_2 = E\left[\int_s^T (p_t(c - \beta\mu_1) - A_t q_t(X_t^* - A_t))dt - \int_s^T \left(q_t(X_t^* - A_t)^2 + e^{\lambda t} p_t^2\right)dt\right].$$

In addition, Theorem 3.3 shows that $e^{2\lambda t}p_t^2=u_t^{*^2}$. Thus, all the above reasoning yields

$$2V(s,x)=(\pi(s)x+g(s)-q_TA_T)x+\int_s^T(\beta\pi(t)\mu_2+\sigma^2\pi(t))dt$$

$$+E\left[\int_s^T(p_t(c-\beta\mu_1)-A_tq_t(X_t^*-A_t))dt-p_TA_T\right]$$

$$=(\pi(s)x+g(s)-q_TA_T)x+\int_s^T(\beta\pi(t)\mu_2+\sigma^2\pi(t))dt$$

$$+E\left[\int_s^T((\pi(t)X_t^*+g(t)-q_TA_T)(c-\beta\mu_1)-A_tq_t(X_t^*-A_t))dt-p_TA_T\right]. \qquad (3.23)$$

In the following, we use properties of the function $g(x)$ to cancel the stochastic term of (3.23). First, the boundary condition $g(T)=0$ yields $g(T)(X_T^*-A_T)=0$. On the other hand, applying the Itô formula to it results in

$$g(T)(X_T^*-A_T)=g(s)(x-A_s)+\int_s^Tg(t-)d(X_t^*-A_t)+\int_s^T(X_{t-}^*-A_t)dg(t)$$

$$=g(s)x+\int_s^Tg(t-)dX_t^*+\int_s^Tg'(t)X_{t-}^*dt. \qquad (3.24)$$

Replacing dX_t^* by the first equality in (3.14) and g'(t) by the second equality in (3.6) yields

$$Eg(T)(X_T^*-A_T)=g(s)x+E\left[\int_s^Tg(t-)(\delta_tX_{t-}^*+c-e^{\lambda t}p_t-\beta\mu_1)dt\right.$$

$$+\int_s^T(-X_{t-}^*g(t)(\delta_t-\pi(t)e^{\lambda t})-X_{t-}^*(q_TA_Te^{\lambda t}+c-\beta\mu_1)\pi(t)$$

$$\left.+X_{t-}^*\delta_tq_TA_T+X_{t-}^*q_tA_t)dt\right]. \qquad (3.25)$$

Replacing $t-$ by t and adding (3.25) to (3.23) result in

$$2V(s,x)=\pi(s)x^2+2(g(s)-q_TA_T)x$$

$$+E\left[\int_s^T(2(c-\beta\mu_1)(g(t)-q_TA_T)+\beta\pi(t)\mu_2+q_tA_t^2+\sigma^2\pi(t))dt-p_TA_T+q_TA_Tx\right.$$

$$\left.+\int_s^T(-g(t)e^{\lambda t}p_t+(g(t)e^{\lambda t}-q_TA_T)e^{\lambda t})\pi(t)X_t^*+X_t^*\delta_tq_TA_T)dt\right].$$

Replacing $\pi(t)X_t^*$ by (3.18) to the right-hand side and rearranging yields

$$2V(s,x)=\pi(s)x^2+2(g(s)-q_TA_T)x$$

$$-\int_s^T\Big((g(t)-q_TA_T)^2e^{\lambda t}-2(c-\beta\mu_1)(g(t)-q_TA_T)-\beta\pi(t)\mu_2-q_tA_t^2-\sigma^2\pi(t)\Big)dt+J_3,$$

where

$$J_3=-E[q_T(X_T^*-A_T)A_T]+q_TA_Tx+q_TA_TE\left[\int_s^T((c-\beta\mu_1)-e^{\lambda t}p_t+X_t^*\delta_t)dt\right].$$

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In view of the first equality of (3.14), we have

$$J_3 = -E\left[q_T\left(X_T^* - A_T\right)A_T\right] + q_T A_T x + q_T A_T E\left[\int_s^T dX_t^*\right] = q_T A_T^2.$$

So

$$2V(s,x) = \pi(s)x^2 + 2(g(s) - q_T A_T)x$$
$$-\int_s^T \left[(g(t) - q_T A_T)^2 e^{\lambda t} - 2(c - \beta\mu_1)(g(t) - q_T A_T) - \beta\pi(t)\mu_2 - q_t A_t^2 - \sigma^2\pi(t)\right]dt + q_T A_T^2,$$

which coincides with (3.5), (3.7) with the boundary condition (3.8).

3.3. The maximum principle

This subsection employs the maximum principle to solve the problem. According to Framstad *et al.* (2004), the Hamiltonian H:[0, T] $\times R^4 \times R \to R$ for the above problem becomes

$$H(t,x,u,p,Q,r) = -\frac{1}{2}e^{-\lambda t}u^2 - \frac{1}{2}q_t(x - A_t)^2 + (\delta_t x + u + c)p + \sigma Q$$
$$+ \int_R (-zr(t,z) - zp + xr(t,z))\beta F(dz), \qquad (3.26)$$

where $\mathscr{R}$ is the set of functions r:$R^2 \to R$ such that the integral in (3.26) converges. The adjoint equation (corresponding to the pair (X, u)) in the unknown adapted process $p(t) \in R$, $Q(t) \in R$ and $r(t, z) \in R$ is the backward stochastic differential equation (BSDE)

$$dp(t) = -\frac{\partial H}{\partial x}H(t, X(t), u(t), p(t), Q(t), r(t,\cdot))dt + Q(t)dB(t) + \int_R r(t_-, z)N(dt, dz)$$
$$= -(\delta_t p_t - q_t(X_t - A_t))dt + Q(t)dB_t + \int_R r(t_-, z)N(dt, dz) - \int_R r(t,z)\beta F(dz)dt, \qquad (3.27)$$

with terminal condition

$$p(T) = -q_T(x - A_T), \qquad (3.28)$$

where $N(t, z)$ is a Poisson random measures with Lévy measures $\beta F(dz)$.

By Framstad *et al.* (2004, Theorem 2.1), (X^*, u^*) is an optimal pair if it satisfies

$$H\left(t, X_t^*, u_t^*, p(t), Q(t), r(t,\cdot)\right) = \max_{u \in R} H\left(t, X_t^*, u, p(t), Q(t), r(t,\cdot)\right)$$

for all $t \in [s, T]$ and that

$$\widehat{H}(x) := \max_{u \in R} H(t, x, u, p(t), Q(t), r(t,\cdot)) \qquad (3.29)$$

exists and is a concave function of x for all $t \in [s, T]$, where ($p(t)$, $Q(t)$, $r(t, z)$) is a solution of the corresponding (X^*, u^*) to adjoint equation (3.27). We take $\gamma(t, z) = \eta_t z$ and $Q(t) = -\sigma\gamma_t$; then,

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(3.27)-(3.28) become

$$\begin{cases} dp_t = \left(-\delta_t p_t + q_t\left(X_t^* - A_t\right)\right)dt + \eta_t\left(d\sum_{i=1}^{N_t} Y_i - \beta\mu_1 dt\right) - \sigma\gamma_t dB_t, \\ p_T = -q_T(X_T - A_T). \end{cases} \tag{3.30}$$

All the above statements yield that the optimal control u_t^* is

$$u_t^* = p_t e^{\lambda t}, \tag{3.31}$$

in which p_t is the solution of the following equation:

$$\begin{cases} dX_t^* = \left(\delta_t X_t^* + c + e^{\lambda t} p_t\right)dt - d\sum_{i=1}^{N_t} Y_i + \sigma dB_t, \\ dp_t = \left(-\delta_t p_t + q_t(X_t^* - A_t)\right)dt + \eta_t\left(d\sum_{i=1}^{N_t} Y_i - \beta\mu_1 dt\right) - \sigma\gamma_t dB_t, \\ X_s^* = x, \\ p_T = -q_T(X_T - A_T). \end{cases} \tag{3.32}$$

Note that the optimal control given by (3.31)–(3.32) is equal to that given by Theorem 3.3. Thus, Theorem 3.3 is again proven. The solution of equation (3.32) has been given by Theorem 3.4.

We now seek the expression of the optimal value function V not by definition and the HJB equation but by the relations between the maximum principle and dynamic programming in the jump-diffusion case. By Framstad *et al.* (2004, equation 24a), we know that

$$p(t) = \frac{\partial V}{\partial x}\left(t, X_t^*\right). \tag{3.33}$$

In our case, this implies that

$$V(t,x) = \frac{1}{2}\pi(t)x^2 + (g(t) - q_T A_T)x + f(t), \tag{3.34}$$

where $f(t)$ is a suitable function. To determine $f(t)$, we give the following Lemma; it also complements the results given by Framstad *et al.* (2004).

Lemma 3.1 *Let (X^*, u^*) be an optimal pair. Suppose that the optimal value function $V \in C^{1,2}$. Then,*

$$\begin{aligned} V_t\left(t, X_t^*\right) &= G\left(t, X_t^*, u_t^*, -V_x\left(t, X_t^*\right), -V_{xx}\left(t, X_t^*\right)\right) \\ &= \max_{u \in R} G\left(t, X_t^*, u, -V_x\left(t, X_t^*\right), -V_{xx}\left(t, X_t^*\right)\right), \\ &\quad a.e.\, t \in [s,T], P-a.s. \end{aligned} \tag{3.35}$$

where G is defined by

$$\begin{aligned} G(t,x,u,p,P) :&= (\delta_t x + c + u)p + \frac{1}{2}\sigma^2 P - \beta E V(t, x - Y) + \beta V(t,x) \\ &\quad - \frac{1}{2}q_t(x - A_t)^2 - \frac{1}{2}e^{-\lambda t}u^2. \end{aligned} \tag{3.36}$$

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Proof. The previous analysis shows that the optimal control is

$$u_t^* = -(\pi(t)X_t^* + g(t) - q_T A_T)e^{-\lambda t} \tag{3.37}$$

It shows that the optimal control is Markovian, i.e., it depends only on the actual surplus and not on the history of the process. Thus, the resulting surplus process X_t^* still has the Markov property. Therefore, we have

$$\begin{aligned} V(t, X_t^*) &= \frac{1}{2} E\left[\int_t^T \left(q_r (X_r^* - A_r)^2 + e^{-\lambda r} u_r^{*^2}\right) dr + q_T (X_T^* - A_T)^2 \mid X_t^*\right] \\ &= \frac{1}{2} E\left[\int_t^T \left(q_r (X_r^* - A_r)^2 + e^{-\lambda r} u_r^{*^2}\right) dr + q_T (X_T^* - A_T)^2 \mid \mathscr{F}_t^s\right] \\ &\quad \forall t \in [s, T],\ P\text{-}a.s. \end{aligned} \tag{3.38}$$

Inspired by Yong & Zhou (1999, P251), we define

$$m_t := \frac{1}{2} E\left[\int_s^T \left(q_r (X_r^* - A_r)^2 + e^{-\lambda r} u_r^{*^2}\right) dr + q_T (X_T^* - A_T)^2 \mid \mathscr{F}_t^s\right]. \tag{3.39}$$

Clearly, $m(\cdot)$ is $\mathscr{F}_t^s$-adapted square-integrable martingale. Thus, by the martingale representation theorem (see Tang & Li (1994), Lemma 2.3), we have

$$\begin{aligned} m_t &= m_s + \int_s^t M_r dB_r + \int_R \int_s^t H(r,z) N(drdz) - \int_R \int_s^t H(r,z) \beta F(dz) dr \\ &= V(s,x) + \int_s^t M_r dB_r + \int_R \int_s^T H(r,z) N(drdz) - \int_R \int_s^T H(r,z) \beta F(dz) dr, \end{aligned} \tag{3.40}$$

where $M \in L_{\mathscr{F}}^2(s, T; R)$ and $H \in B_{\mathscr{F}}(s,\ T;\ R)$. Then, by (3.38) and (3.40),

$$\begin{aligned} V(t, X_t^*) &= m_t - \frac{1}{2} \int_s^t \left(q_r (X_r^* - A_r)^2 + e^{-\lambda r} u_r^{*^2}\right) dr \\ &= V(s,\ x) - \frac{1}{2} \int_s^t \left(q_r (X_r^* - A_r)^2 + e^{-\lambda r} u_r^{*^2}\right) dr \\ &\quad + \int_s^t M_r dB_r + \int_R \int_s^t H(r,z) N(drdz) - \int_R \int_s^T H(r,z) \beta F(dz) dr. \end{aligned} \tag{3.41}$$

On the other hand, applying the Itô formula to $V(t, X_t^*)$ yields

$$\begin{aligned} dV(t, X_t^*) &= \left\{ V_t(t, X_t^*) + (\delta_t X_t^* + c + u_t^*) V_x(t, X_t^*) + \frac{1}{2} \sigma^2 V_{xx}(t, X_t^*) \right\} dt \\ &\quad + \sigma V_x(t, X_t^*) dB_t + d \sum_{i=1}^{N_t} V\left(T_i, X_{T_i}^*\right) - V\left(T_i, X_{T_i -}^*\right). \end{aligned} \tag{3.42}$$

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Comparing (3.41) with (3.42) results in

$$\begin{cases} V_t\left(t, X_t^*\right) + \left(\delta_t X_t^* + c + u_t^*\right) V_x\left(t, X_t^*\right) \\ \quad + \frac{1}{2}\sigma^2 V_{xx}\left(t, X_t^*\right) + \beta \int_R \left(V\left(t, X_{t-}^* - z\right) - V\left(t, X_{t-}^*\right)\right) G(dz)dt \\ \quad = -\left(q_t\left(X_t^* - A_t\right)^2 + e^{-\lambda t} u_t^{*^2}\right) \\ \sigma V_x\left(t, X_t^*\right) = M_t \\ H(t,z) = V(t, X_{t-} - z) - V\left(t, X_{t-}^*\right) \end{cases} \tag{3.43}$$

This proves the first equality in (3.35). Since $V \in C^{1,2}$, it satisfies the HJB equation (3.1), which implies the second equality in (3.35).

Combining (3.34) and, (3.37) with the first equality of (3.35), we obtain (3.5) with *f*(*t*) given by (3.7)-(3.8) again.

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