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THE WELFARE COST OF EXCESS VOLATILITY IN INCOMPLETE MARKETS WITH SUNSPOTS

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In an incomplete markets economy with sunspots, the Pareto-criterion cannot rank sunspot equilibria of different levels of excess price-level volatility. Therefore, I propose a measure of excess volatility cost in terms of a period-0 endowment good. Ex-ante endowment subsidies are provided, in theory, to each consumer, so that the resulting equilibrium allocation of the higher volatility is Pareto-equivalent to the original benchmark equilibrium with a lower volatility level. The aggregate volatility cost is computed as the sum of all consumers' subsidies. Focusing on local analysis that considers small variations around a given volatility level, I show that the aggregate cost strictly increases in volatility even though each individual cost does not necessarily have this property.

Keywords: Excess Volatility, Market Incompleteness, Price-Level Volatility, Sunspots, Welfare Cost

1. INTRODUCTION

I introduce a measure of price-level volatility cost in terms of consumption goods in an incomplete markets economy with sunspots. The cost is, in theory, the endowment compensations that induce the equilibrium with high price-level volatility to be Pareto-equivalent to a benchmark equilibrium with low volatility.[1] The compensation is ex-ante in the sense that it affects the equilibrium prices and allocations. It is well known that some consumers can be better off with increased excess volatility. However, this paper shows that the proposed measure of the cost strictly increases with higher price volatility. The idea of using the unit of consumption good to measure volatility cost has been originated from the business cycle literature with a representative agent. The new contribution of this paper is applying the same exercise in the incomplete markets where there is heterogeneity. These findings also provide a justification for sunspot-stabilizing policies, which are commonly introduced in the literature.[2] From the main result of this paper, we conclude that stabilizing policies can minimize the volatility cost, even though

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they do not necessarily Pareto-improve the economy. The main content of this paper focuses on the local analysis in which an infinitesimal variation in price volatility is considered in order to avoid the singularity issue.

Market incompleteness with sunspots provides theoretical explanations for excess price-level volatility, which were shown initially by Cass (1989, 1992).[3] Excess volatility has a negative impact on the welfare of risk-averse consumers in a partial equilibrium model. However, in a general equilibrium model, it was shown that some consumers are better off under sunspots.[4] The excess volatility from sunspots affects the equilibrium asset prices, and the change in asset prices can benefit some consumers. In the case where the beneficial asset price effect dominates the negative excess-volatility effect, the consumer can be better off with sunspots. The main question in this paper is whether increased excess volatility is harmful to the economy as a whole. Because it is known that excess price volatility can benefit some consumers, we need a new measurement to encompass the total impact of the volatility on the entire economy. The measurement is the endowment transfer that induces the equilibrium allocation of higher price volatility to be Pareto-equivalent to the benchmark equilibrium with lower volatility.

This paper compares equilibria with different levels of excess volatility but with the same economic fundamentals. In the comparison between the benchmark equilibrium with lower volatility and the other equilibrium under higher volatility, we first compute each consumer's utility level with the lower-volatility equilibrium. We assume that an ex-ante tax-subsidy (transfer) implemented in period 0 can be applied to the equilibrium with higher price volatility. The transfer is ex-ante in the sense that it affects the equilibrium asset price, i.e., the value of money. Along the continuum of equilibria in an economy with tax-subsidy plans, we focus on an equilibrium with the higher price volatility. We compute the amount of subsidies applied to each consumer that would result in the same utility levels as the benchmark equilibrium with a lower volatility. The aggregate cost of increased price-volatility is measured as the sum of all consumers' subsidies. Each individual's subsidy can be negative or positive, but we show that the aggregate subsidy strictly increases with higher price-level volatility.[5]

The measure of welfare losses from excess volatility in this paper is similar to that in Lucas (1987, 2003) where the volatility cost is measured in terms of consumption goods. This paper applies the measuring exercise in an incomplete financial market economy where there are heterogeneous consumers. With this heterogeneity, welfare by increased or decreased volatility is not ranked by a Paretian criterion due to the general equilibrium effect. However, in the typical macroeconomic model with a representative agent, there is no general equilibrium effect so the welfare can simply be ranked by the magnitude of intrinsic shocks such as total factor productivity shocks. This paper shows that even though there are heterogeneous general equilibrium effects in an incomplete markets economy, the aggregation of such effects is equal to zero by the market clearing condition so the aggregate welfare can be ranked by excess volatility.

The proposed measure of welfare cost in terms of the consumption good might facilitate the quantification of the welfare loss from excess volatility. Previous sunspot literature has focused mainly on the proof of nonoptimality of sunspot equilibria, but this paper tries to quantify the extent to which sunspot-driven volatility affects welfare. The main methodologies introduced in this paper could be used to quantify the sunspot effect across various sunspot applications, such as business cycles and bank runs.[6]

This paper is organized as follows. In Section 2, I introduce the general setting of the model. Section 3 presents the measure of price-level volatility. Section 4 conducts the local analysis, which shows how the equilibrium is parameterized by the tax-subsidy plan and price-level volatility. The main result of this paper is presented in Section 5. Finally, concluding remarks are presented in Section 6.

2. THE MODEL

There are two periods labeled by the superscripts $t = 0, 1$. In period 1, there are two sunspots states, $\theta = \alpha, \beta$, which have positive probabilities $0 < \mu^\alpha < 1$ and $\mu^\beta = 1 - \mu^\alpha$. There are H consumers, labeled by the subscripts $h \in H = \{1, 2, \ldots, H\}$. Consumer h's consumption allocation is $x_h = (x_h^0, x_h^\alpha, x_h^\beta) \in \boldsymbol{R}^3_{++}$, which corresponds to price $p = (p^0, p^\alpha, p^\beta) \in \boldsymbol{R}^3_{++}$, where $p^0 = 1$. Consumer h's endowment is $e_h = (e_h^0, e_h^1) \in \boldsymbol{R}^2_{++}$. Consumer h's preference is

$$u_h(x_h) = \mu^\alpha v_h\left(x_h^0, x_h^\alpha\right) + \mu^\beta v_h\left(x_h^0, x_h^\beta\right),$$

where the subutility function $v_h(\cdot)$ is strictly increasing, strictly concave, twice-continuously differentiable, and satisfies the von Neumann–Morgenstern expected utility hypothesis. We assume that the closure of indifference curves are contained in $\boldsymbol{R}^2_{++}$.[7] We also assume that the initial endowment is not Pareto-optimal (i.e., there is an incentive for at least two of the consumers to trade).

In a monetary market, m_h denotes consumer h's money holdings. In the monetary equilibrium, there are some positive spot prices $p \gg 0$ and associated money holdings $\{m_h\}_{h=1}^H$ such that each household chooses (x_h, m_h) in the optimization problem:

$$\max \quad u_h(x_h)$$

$$\text{subject to} \quad \begin{cases} p^0 x_h^0 + m_h \leq p^0 e_h^0 \\ p^\alpha x_h^\alpha \leq p^\alpha e_h^1 + m_h \\ p^\beta x_h^\beta \leq p^\beta e_h^1 + m_h \end{cases} \tag{1}$$

$$\text{and} \quad x_h \in \boldsymbol{R}^3_{++},$$

and the market-clearing conditions are

$$\sum\nolimits_{h\in H} x_h^0 = \sum\nolimits_{h\in H} e_h^0, \ \sum\nolimits_{h\in H} x_h^\alpha = \sum\nolimits_{h\in H} x_h^\beta = \sum\nolimits_{h\in H} e_h^1$$

and

$$\sum_{h \in H} m_h = 0.$$

3. THE EQUILIBRIUM

Cass (1989, 1992) and Siconolfi (1991) have shown that in a general model of sunspots with incomplete markets, the set of equilibrium allocations takes on a continuum. The economy has two states at period 1, but one financial asset. Therefore, the equilibrium set has one degree of indeterminacy. We define a relative price for price-level between two states. The relative price is defined as

$$P = p^{\beta}/p^{\alpha}. \tag{2}$$

Assuming that $P \geq 1$, there is a one-to-one relationship between the relative standard deviation σ of the price level and P[8]:

$$\sigma = \frac{\sqrt{\mu^{\alpha}\mu^{\beta}}}{\mu^{\alpha} + \mu^{\beta}P}(P-1) \geq 0. \tag{3}$$

Then, the continuum of equilibria can be parameterized by the measure of volatility σ, where $0 \leq \sigma < \sqrt{\mu^{\alpha}/\mu^{\beta}}$. The equilibrium with $\sigma = 0$ is a nonsunspot equilibrium, but the equilibrium with $\sigma > 0$ is a sunspot equilibrium

In Section 2, the monetary market is defined with nominal assets (money). Here, we consider an economy with real assets, which results in the same equilibrium allocations as those of the monetary market for a given volatility level σ. The real returns of money, denoted as $\left(r^{\alpha}, r^{\beta}\right)$, have an inverse relationship with price levels (i.e., $r^{\alpha}/r^{\beta} = p^{\beta}/p^{\alpha}$). With the real return of $\left(r^{\alpha}, r^{\beta}\right)$, we can define the real asset economy, where consumer h's asset holding is b_h and the price of the asset is q. If b_h units of the asset are held at the end of the first period, $b_h r^s$ units of consumption good would be delivered in the second period. The economy with the real asset (b_h) results in the same equilibrium allocations as those of the original monetary economy with m_h.

For convenience of computation, we fix the expected value of the real return as 1 (i.e., $\mu^{\alpha}r^{\alpha} + \mu^{\beta}r^{\beta} = 1$). Then, r^{α} and r^{β} can be parameterized by σ:

$$r^{\alpha} = \frac{\sqrt{\mu^{\alpha}\mu^{\beta}} + \sigma\mu^{\alpha}}{\sqrt{\mu^{\alpha}\mu^{\beta}} + \sigma\left(\mu^{\alpha} - \mu^{\beta}\right)} \tag{4}$$

and

$$r^{\beta} = \frac{\sqrt{\mu^{\alpha}\mu^{\beta}} - \sigma\mu^{\beta}}{\sqrt{\mu^{\alpha}\mu^{\beta}} + \sigma\left(\mu^{\alpha} - \mu^{\beta}\right)}. \tag{5}$$

We restrict that the volatility level σ should be smaller than $\sqrt{\mu^{\alpha}/\mu^{\beta}}$, which guarantees that the asset return in state β (r^{β}) is positive. The return in state α (r^{α}) is positive for any value of σ because we assume that $p^{\beta} \geq p^{\alpha}$ [see equation (2)].

Solving the budget constraint in b_h, we can re-write the maximization problem in (1) as follows:

$$\max_{b_h} \sum_{\theta=\alpha,\beta} \mu^s v_h \left(e_h^0 - qb_h, e_h^1 + r^s b_h\right). \tag{6}$$

We compare the equilibria that have different levels of excess volatility in a given economy. The benchmark equilibrium has a lower equilibrium price-level volatility, σ. We denote the equilibrium utility levels of all consumers at given σ as $U(\sigma) = (u_1, u_2, \ldots, u_h)$. It is known that the equilibrium allocations with the volatility level σ' are not necessarily Pareto-inferior to those with σ. For many examples, we do not have the following condition: $U(\sigma) \gg U(\sigma')$, where $\sigma < \sigma'$.[9] We consider ex-ante endowment subsidies that are provided, in theory, to each consumer. The subsidy-adjusted endowment of the economy is given as $(e_h^0 + s_h, e_h^1, e_h^1)_{h=1}^H$ instead of $(e_h^0, e_h^1, e_h^1)_{h=1}^H$, where s_h is the subsidy provided to consumer h in period 0. The subsidy is ex-ante in the sense that it affects equilibrium prices and allocations. Let us denote the equilibrium utilities with subsidies as $U\left(\sigma, \{s_h\}_{h=1}^H\right)$. The welfare cost of increased volatility from the volatility level σ to σ' is defined as

$$\sum_{h\in H} s_h, \tag{7}$$

where

$$U\left(\sigma, 0^H\right) = U\left(\sigma', \{s_h\}_{h=1}^H\right). \tag{8}$$

0^H in $U\left(\sigma, 0^H\right)$ means that there is no tax-subsidy plan implemented at the equilibrium with the lower volatility level σ. However, the tax-subsidy plan $\{s_h\}_{h=1}^H$ is applied to the equilibrium at higher volatility level σ'.

This paper first shows that (a) for given σ and σ', there exists a tax-subsidy plan $\{s_h\}_{h=1}^H$ that induces the resulting equilibrium with volatility level σ' to be Pareto-equivalent to the original equilibrium with volatility level σ and (b) the defined aggregate cost $\sum_{h\in H} s_h$ is strictly increasing in volatility σ. In this paper, we focus mainly on the local analysis that considers the infinitesimal change in price volatility, i.e., $\left(\sigma' - \sigma\right) \to 0$ to avoid a singular equilibrium that possibly exists between the two volatility levels σ and σ'.

4. THE LOCAL ANALYSIS

This section introduces how the welfare can be parameterized by both tax-subsidy plans and changes in price volatility. It is assumed that tax-subsidy plans from outside the economy can be implemented in period 0. The tax-subsidy plan is perfectly anticipated so that it affects equilibrium prices and allocations. The subsidy-adjusted endowment is given by

$$(e_h^0 + s_h, e_h^1, e_h^1)_{h=1}^H. \tag{9}$$

In equation (9), s_h would be negative if consumer h becomes better off with sunspots. However, in the local analysis that considers an infinitesimal change in the volatility level σ, the change in s_h is also small. Therefore, with the assumption that $e_h^0 > 0$, $e_h^0 + s_h$ would remain positive.

Consumer h's problem is to choose $b_h \in \boldsymbol{R}$ to maximize the following expected utility:

$$\max_{b_h} \sum_{\theta=\alpha,\beta} \mu^s v_h \left(e_h^0 + s_h - q b_h, e_h^1 + r^s b_h\right). \tag{10}$$

The objective function in (10) is strictly concave in b_h, and the optimal choice is characterized by a solution to the first-order condition:

$$\begin{aligned} -q \sum_{\theta=\alpha,\beta} \mu^s \frac{\partial}{\partial x^0} v_h \left(e_h^0 + s_h - q b_h, e_h^1 + r^s b_h\right) \\ + \sum_{\theta=\alpha,\beta} \mu^s r^s \frac{\partial}{\partial x^1} v_h \left(e_h^0 + s_h - q b_h, e_h^1 + r^s b_h\right) = 0, \end{aligned} \tag{11}$$

where $\frac{\partial}{\partial x^0} v_h$ and $\frac{\partial}{\partial x^1} v_h$ are derivatives with respect to the first-period consumption and the second-period consumption, respectively. Because v_h is strictly concave, the solution exists and is unique for any $q \in \boldsymbol{R}_{++}$. The unique solution depends on the volatility σ, the subsidy s, and the price of the real asset q. Let $b_h(\sigma, s, q)$ be the unique solution to equation (11). Let

$$b(\sigma, s, q) = \sum_{h \in H} b_h(\sigma, s, q) \tag{12}$$

be the market excess demand function for the asset, which is twice continuously differentiable. The equilibrium equation can be expressed as

$$b(\sigma, s, q) = 0. \tag{13}$$

It is known that, generically in endowments, there are finitely many (or unique) equilibria for any given real returns (r^α, r^β) [see Cass (1989)]. Because (r^α, r^β) has a one-to-one relationship with σ and the subsidy-adjusted endowment can be characterized by the subsidy plan (s), the C^2 function $\widehat{q}$ can be defined locally around a regular point $(\overline{\sigma}, \overline{s})$. This paper assumes that the function $\widehat{q}$ is well defined around $(\overline{\sigma}, \overline{s})$, which means that the endowments are chosen in such a way that a finite number of sunspot equilibria exist for a given $\overline{\sigma}$. We state the regularity assumption below.

Regularity assumption: For a given $(\sigma, s) = (\overline{\sigma}, \overline{s})$, $\overline{q}$ is the equilibrium price of the real asset such that $b(\overline{\sigma}, \overline{s}, \overline{q}) = 0$. We assume that the equilibrium equation of the price effect is nonsingular, i.e., $\partial b(\overline{\sigma}, \overline{s}, \overline{q})/\partial q \neq 0$.

By the regularity assumption, we can solve for q as a smooth function of (σ, s) locally around a regular point, $(\overline{\sigma}, \overline{s})$, so there is a C^2 function $\widehat{q}(\sigma, s)$ defined on a neighborhood $\Lambda \subset \boldsymbol{R}_+ \times \boldsymbol{R}^H$ around $(\overline{\sigma}, \overline{s})$ such that

$$b(\sigma, s, \widehat{q}(\sigma, s)) = 0. \tag{14}$$

Then, we can define $\widehat{b}_h$ and $\widehat{u}_h$ from equation (14) by the following rule:

$$\widehat{b}_h(\sigma, s) := b_h(\sigma, s, \widehat{q}(\sigma, s)), \tag{15}$$

$$\widehat{u}_h(\sigma, s) := \sum_{\theta=\alpha,\beta} \mu^s v_h\left(e_h^0 + s_h - \widehat{q}(\sigma, s)\widehat{b}_h(\sigma, s), e_h^1 + r^s(\sigma)\widehat{b}_h(\sigma, s)\right). \tag{16}$$

The next step is to introduce a smooth function $S(\sigma) = [S_h(\sigma)]_{h=1}^H$, which maps points around $\overline{\sigma}$ to transfers around $\overline{s} \in \boldsymbol{R}^H$. If $(\overline{\sigma}, \overline{s})$ is a regular point, the composition functions $\widehat{q}(\sigma, S(\sigma))$, $\widehat{b}_h(\sigma, S(\sigma))$, and $\widehat{u}_h(\sigma, S(\sigma))$ are smooth around $\overline{\sigma}$ for any smooth function $S(\sigma)$ satisfying $S(\overline{\sigma}) = 0$.

$\widehat{u}_h(\sigma, S(\sigma))$ is the corresponding utility level in equilibrium given σ and $S(\sigma)$, whereas $\widehat{u}_h(\sigma, 0^H)$ represents the utility levels associated with volatility level σ without any tax-subsidy plans.

Differentiating $\widehat{u}_h(\sigma, S(\sigma))$ with respect to σ, we have

$$\begin{aligned}\frac{\partial \widehat{u}_h(\sigma, S(\sigma))}{\partial \sigma} &= \left(S_h'(\sigma) - \frac{\partial \widehat{q}(\sigma, S(\sigma))}{\partial \sigma}\widehat{b}_h(\sigma, S(\sigma))\right) \sum_{\theta=\alpha,\beta} \mu^s \frac{\partial v_h\left(\widehat{x}_h^0, \widehat{x}_h^s\right)}{\partial x^0} \\ &+ \sum_{\theta=\alpha,\beta} \mu^s \frac{\partial v_h\left(\widehat{x}_h^0, \widehat{x}_h^s\right)}{\partial x^1}\left(\frac{\partial r^s(\sigma)}{\partial \sigma}\widehat{b}_h(\sigma, S(\sigma))\right) \\ &+ \frac{\partial \widehat{b}_h(\sigma, S(\sigma))}{\partial \sigma} \sum_{\theta=\alpha,\beta}\left(\mu^s \frac{\partial v_h\left(\widehat{x}_h^0, \widehat{x}_h^s\right)}{\partial x^1} r^s(\sigma) - \mu^s \frac{\partial v_h\left(\widehat{x}_h^0, \widehat{x}_h^s\right)}{\partial x^0}\widehat{q}(\sigma, S(\sigma))\right),\end{aligned} \tag{17}$$

where $S'(\sigma) = \partial S(\sigma)/\partial \sigma$, $\widehat{x}_h^0 := e_h^0 + S_h(\sigma) - \widehat{q}(\sigma, S(\sigma))\widehat{b}_h(\sigma, S(\sigma))$ and $\widehat{x}_h^s := e_h^1 + r^s(\sigma)\widehat{b}_h(\sigma, S(\sigma))$.

By the first-order condition from equation (11), we can define $\lambda_h(\sigma, S(\sigma))$ as

$$\begin{aligned}\lambda_h(\sigma, S(\sigma)) &= \sum_{\theta=\alpha,\beta} \mu^s \frac{\partial v_h\left(\widehat{x}_h^0, \widehat{x}_h^s\right)}{\partial x^0} \\ &= \frac{1}{\widehat{q}(\sigma, S(\sigma))} \sum_{\theta=\alpha,\beta} \mu^s r^s(\sigma) \frac{\partial v_h\left(\widehat{x}_h^0, \widehat{x}_h^s\right)}{\partial x^1},\end{aligned} \tag{18}$$

which is strictly positive.

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With equation (18), we can simplify equation (17) as

$$\frac{1}{\lambda_h(\sigma, S(\sigma))}\frac{\partial \widehat{u}_h(\sigma, S(\sigma))}{\partial \sigma} = \underbrace{\frac{\widehat{b}_h(\sigma, S(\sigma))}{\lambda_h(\sigma, S(\sigma))}\sum_{\theta=\alpha,\beta}\mu^s\frac{\partial r^s}{\partial \sigma}\frac{\partial v_h\left(\widehat{x}_h^0, \widehat{x}_h^s\right)}{\partial x^1}}_{\text{Risk effect}} \underbrace{-\frac{\partial \widehat{q}(\sigma, S(\sigma))}{\partial \sigma}\widehat{b}_h(\sigma, S(\sigma))}_{\text{General equilibrium effect}} + \underbrace{S_h'(\sigma)}_{\text{Tax-subsidy effect}}. \tag{19}$$

Equation (19) can be decomposed into the risk effect, the general equilibrium effect, and the tax-subsidy effect. The risk effect can be interpreted as a pure sunspot effect in the sense that the term is not affected directly by the change in the asset price. The general equilibrium effect is from the changes in the equilibrium asset price, $\partial\widehat{q}(\sigma, S(\sigma))/\partial\sigma$. Both risk and general equilibrium effects are affected by the tax-subsidy plan $S(\sigma)$, but this paper will show that for any smooth function $S(\sigma)$, (a) the risk effect for any consumer $h \in H$ is strictly negative and (b) the aggregate general equilibrium effect $\sum_{h\in H} -\frac{\partial\widehat{q}(\sigma,S(\sigma))}{\partial\sigma}\widehat{b}_h(\sigma, S(\sigma))$ is always zero. The two properties are crucial in proving the main proposition in this paper. In the following lemma, we show that the risk effect is strictly negative

LEMMA 1. *The risk effect is negative, that is,*

$$\frac{\widehat{b}_h(\sigma, S(\sigma))}{\lambda_h(\sigma, S(\sigma))}\sum_{\theta=\alpha,\beta}\mu^s\frac{\partial r^s}{\partial \sigma}\frac{\partial v_h\left(\widehat{x}_h^0, \widehat{x}_h^s\right)}{\partial x^1} < 0. \tag{20}$$

Proof. The risk effect can be expressed as

$$\begin{aligned}
&\frac{\widehat{b}_h(\sigma, S(\sigma))}{\lambda_h(\sigma, S(\sigma))}\sum_{\theta=\alpha,\beta}\mu^s\frac{\partial r^s}{\partial \sigma}\frac{\partial v_h\left(\widehat{x}_h^0, \widehat{x}_h^s\right)}{\partial x^1} \\
&= \frac{\widehat{b}_h(\sigma, S(\sigma))}{\lambda_h(\sigma, S(\sigma))}\left\{\begin{array}{l}\mu^\alpha\frac{\partial v_h\left(\widehat{x}_h^0,\widehat{x}_h^\alpha\right)}{\partial x^1}\frac{\mu^\beta\sqrt{\mu^\alpha\mu^\beta}}{\left\{\sqrt{\mu^\alpha\mu^\beta}+\sigma\left(\mu^\alpha-\mu^\beta\right)\right\}^2} \\ +\mu^\beta\frac{\partial v_h\left(\widehat{x}_h^0,\widehat{x}_h^\beta\right)}{\partial x^1}\frac{-\mu^\alpha\sqrt{\mu^\alpha\mu^\beta}}{\left\{\sqrt{\mu^\alpha\mu^\beta}+\sigma\left(\mu^\alpha-\mu^\beta\right)\right\}^2}\end{array}\right\} \qquad (21) \\
&= \frac{\left(\mu^\alpha\mu^\beta\right)^{3/2}}{\left\{\sqrt{\mu^\alpha\mu^\beta}+\sigma\left(\mu^\alpha-\mu^\beta\right)\right\}^2}\frac{\widehat{b}_h(\sigma, S(\sigma))}{\lambda_h(\sigma, S(\sigma))}\left(\frac{\partial v_h\left(\widehat{x}_h^0, \widehat{x}_h^\alpha\right)}{\partial x^1} - \frac{\partial v_h\left(\widehat{x}_h^0, \widehat{x}_h^\beta\right)}{\partial x^1}\right).
\end{aligned}$$

If $\widehat{b}_h(\sigma, S(\sigma)) > 0$, we have $x_h^\alpha > x_h^\beta$. Because v_h is strictly concave, we have $\frac{\partial v_h\left(x_h^0,x_h^\alpha\right)}{\partial x^1} < \frac{\partial v_h\left(x_h^0,x_h^\beta\right)}{\partial x^1}$. Therefore, the risk effect (21) is negative. If $\widehat{b}_h(\sigma, S(\sigma)) <$

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0, we have $x_h^\alpha < x_h^\beta$. Because v_h is strictly concave, we have $\frac{\partial v_h(x_h^0, x_h^\alpha)}{\partial x^1} > \frac{\partial v_h\left(x_h^0, x_h^\beta\right)}{\partial x^1}$. Therefore, the risk effect (21) is negative. ■

Lemma 1 indicates that on the equilibrium path from lower volatility to higher volatility, the risk effect is always negative. However, the general equilibrium effect can be negative or positive. If the positive general equilibrium effect outweighs the negative risk effect, consumer's utility would increase during increased price volatility.

5. THE COST OF INCREASED VOLATILITY

In Section 4, we showed that the equilibrium utility level can be parameterized by the volatility level σ and the tax-subsidy plan $\{s_h\}_{h=1}^H$. Even though both σ and $\{s_h\}_{h=1}^H$ affect equilibrium prices and allocations, we have shown that the risk effect of increased price volatility is always negative. In this section, we will show that the sum for all consumers' the general equilibrium effects of increased volatility is always zero for any given tax-subsidy plan $\{s_h\}_{h=1}^H$. The general equilibrium effect is due to the change in asset price (i.e., the changes in the expected return of money). When the asset price increases (decreases) with an increased volatility level, lenders (borrowers) could be better off through the asset trading. However, because the net supply of the asset is zero, the total gain or loss to the economy from the change in asset price is necessarily zero. Therefore, the aggregate cost of price-volatility is determined solely by the risk effects, not by the general equilibrium effects, even though the individual cost is determined by both, which will be shown in the following proposition.

PROPOSITION 1. *For a nonsingular equilibrium with $\overline{\sigma}$, there exists a smooth function $[S_h(\sigma)]_{h=1}^H$ such that*

$$S_h(\overline{\sigma}) = 0 \;\; and \;\; \frac{\partial \widehat{u}_h\,(\sigma, S(\sigma))}{\partial \sigma}|_{\sigma=\overline{\sigma}} = 0 \;\; for\ all\ h \in H,$$

and the smooth function $[S_h(\sigma)]_{h=1}^H$ satisfies the following:

$$\sum_{h \in H} \frac{\partial S_h(\sigma)}{\partial \sigma} > 0, \quad where\ \sigma = \overline{\sigma}.$$

Proof. From equation (19), we have

$$\frac{\partial \widehat{u}_h(\sigma, \sigma(\sigma))}{\partial \sigma} = \widehat{b}_h\,(\sigma, S(\sigma)) \sum_{\theta=\alpha,\beta} \mu^s \frac{\partial r^s}{\partial \sigma} \frac{\partial v_h\left(\widehat{x}_h^0, \widehat{x}_h^s\right)}{\partial x^1} \quad \mathbf{(22)}$$

$$+ \lambda_h\,(\sigma, S(\sigma)) \left\{ S_h'(\sigma) - \frac{\partial \widehat{q}\,(\sigma, S(\sigma))}{\partial \sigma} \widehat{b}_h\,(\sigma, S(\sigma)) \right\}.$$

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From equation (22), the equality $\frac{\partial \widehat{u}_h(\sigma, S(\sigma))}{\partial \sigma}|_{\sigma=\overline{\sigma}} = 0$ implies that

$$S_h'(\sigma) = \underbrace{\frac{\partial \widehat{q}\,(\sigma, S(\sigma))}{\partial \sigma} \widehat{b_h}\,(\sigma, S(\sigma))}_{\text{The general equilibrium effect}} \tag{23}$$

$$- \underbrace{\frac{\widehat{b_h}\,(\sigma, S(\sigma))}{\lambda_h\,(\sigma, S(\sigma))} \sum_{\theta=\alpha,\beta} \mu^s \frac{\partial r^s}{\partial \sigma} \frac{\partial v_h\left(\widehat{x}_h^0, \widehat{x}_h^s\right)}{\partial x^1}}_{\text{The risk effect}}$$

and $S_h(\sigma) = 0$ for all h, where $\sigma = \overline{\sigma}$.

From equation (23), $\sum_{i \in H} S_i'(\sigma)$ can be expressed as

$$\sum_{i \in H} S_i'(\sigma) = \frac{\partial \widehat{q}\,(\sigma, S(\sigma))}{\partial \sigma} \sum_{i \in H} \widehat{b_i}\,(\sigma, S(\sigma)) \tag{24}$$

$$- \sum_{i \in H} \frac{\widehat{b_i}\,(\sigma, S(\sigma))}{\lambda_i\,(\sigma, S(\sigma))} \sum_{\theta=\alpha,\beta} \mu^s \frac{\partial r^s}{\partial \sigma} \frac{\partial v_i\left(\widehat{x}_i^0, \widehat{x}_i^s\right)}{\partial x^1}.$$

By the asset market-clearing condition, $\sum_h b_h = 0$, and equation (24), we have

$$\sum_{i \in H} S_i'(\sigma) = -\sum_{i \in H} \frac{\widehat{b_i}\,(\sigma, S(\sigma))}{\lambda_i\,(\sigma, S(\sigma))} \sum_{\theta=\alpha,\beta} \mu^s \frac{\partial r^s}{\partial \sigma} \frac{\partial v_i\left(\widehat{x}_i^0, \widehat{x}_i^s\right)}{\partial x^1}, \tag{25}$$

$\sum_{i \in H} S_i'(\sigma)$ in (25) is strictly negative by Lemma 1. ■

Proposition 1 shows that corresponding to an infinitesimal increase in volatility σ, an infinitesimal variation in the aggregate cost of volatility $\sum_{i \in H} \frac{\partial S_i(\sigma)}{\partial \sigma}$ is strictly positive. The individual cost $\frac{\partial S_h(\sigma)}{\partial \sigma}$ in equation (23) can be decomposed into the risk effect and the general equilibrium effect. Due to the general equilibrium effect, the individual cost is not always positive. However, the proof of Proposition 1 shows that the sum of all consumers' general equilibrium effects is zero and, therefore, it is guaranteed that the aggregate cost is positive.

6. CONCLUSION

This paper proposes a measure of the cost of excess volatility in incomplete markets with sunspots. The cost is measured as the amount of endowment subsidies that induces the equilibrium of higher volatility to be Pareto-equivalent to the lower-volatility equilibrium of the benchmark equilibrium. The main result of this paper is that the individual cost is not positively correlated with price-level volatility necessarily, but the aggregate cost is positively correlated with price-level volatility.

Kang (2015) also compared two equilibria with lower and higher volatility levels in incomplete markets with sunspots. The tax-transfer plan is applied to lower volatility levels and showed that there exist tax-transfer plans that induce the equilibrium with lower volatility to be Pareto-superior to the equilibrium with higher volatility. The tax-transfer plan in Kang (2015) was balanced but is not balanced in this paper.

This paper assumed that there was no intrinsic uncertainty but the proposed measure of volatility cost can be applied to an economy with intrinsic uncertainty. Specifically, in an incomplete markets economy with intrinsic uncertainty and pure inside money, the optimal equilibrium results in price-level volatility. This means that there exists an "optimal" price volatility level in which Pareto-efficiency is achieved. It is highly possible that the proposed cost has its lowest value at the optimal volatility level in the economy with intrinsic uncertainty.

NOTES

1. By the same logic using Kaldor–Hicks criterion, the compensation does not actually occur and, thus, a more efficient outcome can, in theory, leave some consumers worse off.
2. For sunspots-stabilizing policies, see Cass and Shell (1983, Proposition 3) and Balasko (1983, Theorem 1), Mas-Colell (1992), Goenka and Préhac (2006), Antinolfi and Keister (1998), and Kajii (1997).
3. Without the existence of extrinsic uncertainty, market incompleteness has been conjectured to be one of the causes of excess price volatility in Shiller (1992), Constantinides and Duffie (1996), Calvet (2001), and Citanna and Schmedders (2005).
4. See Goenka and Prechac (2006), Kajii (2007), and Kang (2015) for a sunspots model with incomplete markets and see Bhattacharya et al. (1998) and Cozzi et al. (2016) for a sunspots model with restricted market participation.
5. This paper focuses on the local analysis that an infinitesimal change in the volatility level is considered. The main difficulty in extending the local result to a global result is the possibility of a singular equilibrium along the equilibrium path defined by price-volatility levels. This problem arises when there are multiple equilibria at some given price-volatility levels.
6. See Farmer and Guo (1994), Farmer and Woodford (1997), and Peck and Shell (2003).
7. This can be stated as limiting conditions, such that $\lim_{x^1\to 0}\partial v_h/\partial x^1 = \infty$, $\lim_{x^2\to 0}\partial v_h/\partial x^2 = \infty$, $\lim_{x^1\to\infty}\partial v_h/\partial x^1 = 0$, and $\lim_{x_2\to\infty}\partial v_h/\partial x^2 = 0$.
8. Equivalently, we have

$$P = \frac{\sqrt{\mu^\alpha \mu^\beta} + \sigma\mu^\alpha}{\sqrt{\mu^\alpha \mu^\beta} - \sigma\mu^\beta},$$

where $0 \le \sigma < \sqrt{\mu^\alpha/\mu^\beta}$.

9. See Goenka and Préhac (2006) and Kang (2015).

REFERENCES

Antinolfi, G. and T. Keister (1998) Options and sunspots in a simple monetary economy. *Economic Theory* 11(2), 295–315.

Balasko, Y. (1983) Extrinsic uncertainty revisited. *Journal of Economic Theory* 31(2), 203–210.

Bhattacharya, J., M. G. Guzman, and K. Shell (1998) Price level volatility: A simple model of money taxes and sunspots. *Journal of Economic Theory* 81(2), 401–430.

Calvet, L. E. (2001) Incomplete markets and volatility. *Journal of Economic Theory* 98(2), 295–338.

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Cass, D. (1989) Sunspots and incomplete fiancial markets: The leading example. In G. Feiwel (ed.), *The Economics of Imperfect Competition and Employment: Joan Robinson and Beyond*. London: Macmillan.

Cass, D. (1991) Incomplete fiancial market and indeterminancy of competitive equilibrium. In J. J. Laffont (ed.), *Advanced in Economic Theory, 6th World Congress.* London: Macmillan.

Cass, D. (1992) Sunspots and incomplete fiancial markets: The general case. *Economic Theory* 2(3), 341–358.

Cass, D. and K. Shell (1983) Do sunspots matter? *Journal of Political Economy* 91(2), 193–227.

Citanna, A. and K. Schmedders (2005) Excess price volatility and fiancial innovation. *Economic Theory* 26(3), 559–587.

Constantinides, G. M. and D. Duffie (1996) Asset pricing with heterogeneous consumers. *Journal of Political Economy* 104(2), 219–240.

Cozzi, G., A. Goenka, M. Kang, and K. Shell (2016) Winners and losers from price-level volatility: Money taxation and information frictions. In K. Nishimura, A. Venditti, and N. C. Yannelis (eds.), *Sunspots and Non-Linear Dynamics: Essays in Honor of Jean-Michel Grandmont*, Volume 31 of Studies in Economic Theory, pp. 387–402. Switzerland: Springer International Publishing.

Farmer, R. E. and J.-T. Guo (1994) Real business cycles and the animal spirits hypothesis. *Journal of Economic Theory* 63(1), 42–72.

Farmer, R. E. and M. Woodford (1997) Self-fulfilling prophecies and the business cycle. *Macroeconomic Dynamics* 1(04), 740–769.

Goenka, A. and C. Préchac (2006) Stabilizing sunspots. *Journal of Mathematical Economics* 42(4–5), 544–555.

Kajii, A. (1997) On the role of options in sunspot equilibria. *Econometrica* 65(4), 977–986.

Kajii, A. (2007) Welfare gains and losses in sunspot equilibria. *Japanese Economic Review* 58(3), 329–344.

Kang, M. (2015) Price-level volatility and welfare in incomplete markets with sunspots. *Journal of Mathematical Economics* 56, 58–66.

Lucas Jr, R. E. (1987) *Models of Business Cycles*, vol. 26. Oxford: Basil Blackwell.

Lucas Jr., R. E. (2003) Macroeconomic priorities. *American Economic Review* 93(1), 1–14.

Mas-Colell, A. (1992) Three observations on sunspots and asset redundancy. In P. Dasgupta, D. Gale, O. Hart, and E. Maskin (eds.), *Economic Analysis of Markets and Games: Essays in Honor of Frank Hahn*, pp. 465–474. Cambridge, MA: MIT Press.

Peck, J. and K. Shell (2003) Equilibrium bank runs. *Journal of Political Economy* 111(1), 103–123.

Shiller, R. J. (1992) *Market Volatility*. Cambridge, UK: MIT Press.

Siconolfi, P. (1991) Sunspot equilibria and incomplete fiancial markets. *Journal of Mathematical Economics* 20(3), 327–339.