J. Austral. Math. Soc. **20** (Series A) (1975), 225–229.

GROWTH SEQUENCES OF FINITE GROUPS II

Dedicated to Philip Hall on the occasion of his 70th birthday

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(Received 11 December 1973)

Communicated by G. E. Wall

1. Introduction and preliminaries

The aim of this paper is to give solutions of some of the problems raised in Wiegold (1974). The main content is the proof of Theorems A and B following, which are improvements of Theorems 4.5 and 3.4 respectively of Wiegold (1974). All unexplained notation will be as in that paper, which would perhaps best be viewed as a preliminary to this one.

THEOREM A. *Let G be a finite non-trivial perfect group, s the size of the smallest simple image of G, and c any real number greater than 1. Then $d(G^n) \leqq c \log_s n$ for all sufficiently large n.*

This theorem is fairly nearly best possible, as can be seen from §4 of Wiegold (1974).

THEOREM B. *Let G be a finite imperfect group, and set $d(G) = \alpha$, $d(G/G') = \beta$.*

(i) *If $\beta \geqq 2$, then $d(G^n) = \beta n$ for $n \geqq \dfrac{\alpha - 1}{\beta - 1}$.*

(ii) *If $\beta = 1$, then $d(G^n) = n$ for $n \geqq 2\alpha + 1$.*

Examples in §4 show that these bounds are somewhere near the truth. More precisely: Lemma 3.3 of Wiegold (1974) shows that if n' is any integer such that $d(G^{n'}) = \beta n'$, then $d(G^n) = \beta n$ for $n \geqq n'$; so that the settling-down of the growth sequence to its eventual linear behaviour is sudden and permanent. The examples referred to are of groups whose growth sequences have not settled down to linearity until n excesess about $(1/\beta)(\alpha + \log \alpha)$ or so. In fact I believe that these examples are close to the right answer, though this would be very much harder to establish than Theorem B, which is comparatively easy. For an application of Theorem B to presentations of direct products of finite groups, see Cossey, Gruenberg and Kovács (1974).

For finite soluble groups, Theorem B can be improved as follows (see also Gaschütz (1959):

THEOREM B′. *Let G be a finite non-trivial soluble group, and set $d(G) = \alpha$, $d(G/G') = \beta$. Then $d(G^n) = \beta n$ for $n \geqq \alpha/\beta$.*

These theorems depend on the following result, which is a consequence of Satz 3 of Gaschütz (1955). I am grateful to L. G. Kovács and M. F. Newman for pointing out this approach to me; it is considerably easier and more efficient than an *ad hoc* approach I had adopted to prove a result like Theorem B. This *ad hoc* method has the minor advantage that it applies to finitely generated (infinite) soluble groups as well; however, as the main interest is in finite groups, I shall not go into details.

LEMMA 1.1. *Let G be a finite group, $R(G)$ the intersection of the maximal normal subgroups of G, set $G/R(G) = G_*$, and let r be any positive integer. Then $d(G^r) \leqq k$ if and only if $d(G) \leqq k$ and $d(G_*^r) \leqq k$.*

Clearly, $G^* \cong A \times B$, where A is a direct product of abelian simple groups with $d(A) = d(G/G') = \beta$, let us say as usual, and B is a direct product of non-abelian simple groups. Furthermore, $G_*^n \cong A^n \times B^n$, so that by Wiegold (1974), Lemma 5.1, for instance, $d(G_*^n) = \max(\beta n, d(B^n))$. We already know from Wiegold (1974), Theorem 4.5, that $d(B^n)$ is eventually roughly logarithmic in n, so that $d(G_*^n)$ is eventually βn whenever $n \geqq 1$; by Lemma 1.1, so therefore is $d(G^n)$. The main import of what follows is to study $d(B^n)$ in a little more detail.

Our final preliminary is well-known, though I can find no reference in the literature. The proof is routine.

LEMMA 1.2. *Let $S_1, S_2, \cdots, S_r$ be pairwise non-isomorphic simple groups (abelian or not, infinite or not), let $m_1, m_2, \cdots, m_r$ be positive integers, and let F be a group containing normal subgroups $N_1, N_2, \cdots, N_r$ such that $F/N_i \cong S_i^{m_i}$ for each $i = 1, 2, \cdots, r$. Then $F / \bigcap_{i=1}^{r} N_i \cong S_1^{m_1} \times S_2^{m_2} \times \cdots \times S_r^{m_r}$.*

2. Proof of Theorem A

Let G be a finite perfect group and $S_1, S_2, \cdots, S_r$ the different simple images of G, so that $G_* \cong S_1^{m_1} \times S_2^{m_2} \times \cdots \times S_r^{m_r}$ (with G_* as in Lemma 1.1) for suitable positive integers $m_1, m_2, \cdots, m_r$. For each positive integer n, $G_*^n \cong S_1^{m_1 n} \times S_2^{m_2 n} \times \cdots \times S_r^{m_r n}$, and so by a simple application of Lemma 1.2 with F free, we get:

LEMMA 2.1. *For every n, $d(G_*^n) = \max\{d(S_i^{m_i n}) \colon 1 \leqq i \leqq r\}$.*

The next result is an immediate consequence of Wiegold (1974; 4.4).

LEMMA 2.2. *For every finite non-abelian simple group S, and every real number $e > 1$, $d(S^n) \leqq e \log_{|S|} n$ for sufficiently large n.*

Theorem A is now easy to prove. With the notation and assumptions set up in the statement of the theorem and in this section, we know from Lemma 1.1 that we need only to show that $d(G_*^n) \leqq c \log_s n$ for large enough n; so by Lemma 2.2, we have only to show that $d(S_i^{m_i n}) \leqq c \log_s n$ for large enough n. But this is clear since, firstly, for $e > 1$ we have $d(S_i^{m_i n}) \leqq e(\log_{|S_i|} n + \log_{|S_i|} m_i)$ for large enough n (Lemma 2.2), and secondly, $|S_i| \geqq s$ for each i and the m_i are constants depending only on G.

3. Proof of Theorem B

Here everything depends on the following lemma, which is perhaps of minor independent interest.

LEMMA 3.1. *Let X be any finitely generated group, and Y any finite perfect group. Then $d(X \times Y) \leqq \max(d(X), d(Y)) + 1$.*

PROOF. It is a simple consequence of Wiegold (1974; Lemma 3.1) that there exists an element y of Y such that $[y, Y] = Y$. (This is slightly stronger than an unpublished theorem of P. M. Neumann — which would do for our purposes — stating that Y is the normal closure of a single element.) Let $r = \max(d(X), d(Y))$. Then, supplementing a minimal generating set of one or other of these groups with dummy generators (if necessary), we get a generating set $x_1, x_2, \cdots, x_r$ for X and a generating set $y_1, y_2, \cdots, y_r$ for Y. The following elements generate $X \times Y$:

$$(x_1, y_1), (x_2, y_2), \cdots, (x_r, y_r), (1, y).$$

This is because commutating the element $(1, y)$ with a suitable word in the preceding elements produces every element of the form $(1, [y, g])$, g an element of Y, and thus gives the whole of the second component of the direct product.

The bound given here is best possible. For example, if A denotes the alternating group of degree 5, then $d(A) = d(A^{19}) = 2$, whereas $d(A^{20}) = 3$. There are numerous other examples.

COROLLARY 3.2. *For every finite perfect group B and every positive integer n, $d(B^n) \leqq d(B) + n - 1$.*

This is straightforward. It is also asymptotically a very weak result, since we know that eventually $d(B^n)$ behaves roughly logarithmically in n. Its virtue lies in the fact that it is valid for all n, and it is strong enough to give part (i) of Theorem B. On the other hand, its weakness is one of the main reasons for thinking that Theorem B is not quite all that it might be.

We can now prove part (i) of Theorem B. It is an unsatisfactory feature of the article that we need a quite different argument to deal with part (ii). Other than searching for a substantial improvement of Corollary 3.2, which I have done without success, I see no possibility of eradicating this difficulty.

Let then G be a finite imperfect group with $d(G) = \alpha$, $d(G/G') = \beta \geqq 2$. Then, with G_* as in Lemma 1.1, $G_* \cong A \times B$ where A is an abelian group with $d(A) = \beta$, and B is a direct product of non-abelian simple groups. Then $d(G_*) = \max(\beta, d(B)) = \gamma$ say, where $\gamma \leqq \alpha$. Moreover $d(G_*^n) = \max(\beta n, d(B^n))$, and by Corollary 3.2, $d(G_*^n) = \beta n$ provided that $\beta n \geqq \gamma + n - 1$, that, is provided $n \geqq (\gamma - 1)/(\beta - 1)$. This is certainly true if $n \geqq (\alpha - 1)/(\beta - 1)$; and to complete the proof of the theorem, we need only observe that Lemma 1.1 applies with $k = \beta n$, since for $n \geqq (\alpha - 1)/(\beta - 1)$ we have $d(G_*^n) = \beta n \geqq \beta((\alpha - 1)/(\beta - 1)) \geqq \alpha$, and therefore $d(G^n) \leqq \beta n$. But $d(G^n) \geqq d(G_*^n)$ since G_* is a homomorphic image of G, and so $d(G^n) = \beta n$ for $n \geqq (\alpha - 1)/(\beta - 1)$, as required.

For part (ii), we use the *ad hoc* methods mentioned in the introduction. Let G be a finite imperfect group with G/G' cyclic and $d(G) \geqq \alpha$. By Wiegold (1974), Lemma 3.1, there is an element a of G generating $G \bmod G'$ and such that $[a, G] = G'$. It therefore follows that there are elements $b_1, b_2, \cdots, b_\alpha$ of G' such that $G = \langle a, b_1, \cdots, b_\alpha \rangle$. By Wiegold (1974; Lemma 3.3) all we need to do is to show that $d(G^{2\alpha+1}) = 2\alpha + 1$; to do this we actually exhibit generators. Write elements of $G^{2\alpha+1}$ as strings of length $2\alpha + 1$; then it is a matter of not too difficult but tedious calculation to show that the $2\alpha + 1$ "cyclic rearrangements" of the element

$$(a, 1, b_1, 1, b_2, 1, \cdots, 1, b_\alpha)$$

generate $G^{2\alpha+1}$. We shall omit the proof because of its length and routine nature.

The proof of Theorem B′ follows the same lines as that if Theorem B.

4. Examples

The examples to be exhibited here are just those of Wiegold (1974; § 5); all we do is to look at them from a slightly different angle. The reader is reminded that all unexplained notation is to be found in Wiegold (1974).

Let K be a finite non-abelian simple group, so that for $\alpha \geqq 2$ the group $K^{h(\alpha, K)}$ is α-generator. For neatness we write $h(n, K) = h(n)$ for each n, and agree that logarithms are taken to the base $|K|$. Set $G = K^{h(\alpha)} \times A$, where A is an abelian group on β generators, so that $d(G/G') = \beta$.

Now let c be any positive real number less than 1, let k be any positive real number, and θ the integer part of $(1/\beta)(\alpha + c \log \alpha) + k$. Then $d(G^\theta) = \beta\theta$ if and only if $d(K^{h(\alpha)\theta}) \leqq \beta\theta$; and a careful look at § 4 of Wiegold (1974) shows that this is so if and only if

$$h(\alpha)\theta \leqq h(\beta\theta);$$

by Wiegold (1974); this is equivalent to

$$\theta \sum \mu(H) |H|^{\alpha} \leqq \sum \mu(H) |H|^{\beta\theta}, \tag{4.1}$$

where μ is the Möbius function of K and the sum is taken over all subgroups of K. As in Wiegold (1974), write $\varepsilon(m)$ for $\sum_{H \neq K} \mu(H) |K : H|^{-m}$. Then 4.1 is equivalent to

$$\theta(1 + \varepsilon(\alpha)) \leqq |K|^{\beta\theta - \alpha}(1 + \varepsilon(\beta\theta)) \tag{4.2}$$

By definition of θ, the right hand side of this inequality is not more than $|K|^{\beta k} \alpha^c$, while the left hand side is greater than $((1/\beta)(\alpha + c \log \alpha) + k - 1)(1 + \varepsilon(\alpha))$. Now fix K, c, k, β. Then $\beta\theta \to \infty$ as $\alpha \to \infty$, and by Wiegold (1974; §4), $\varepsilon(m) \to 0$ as $m \to \infty$; thus, as $c < 1$, it follows that 4.2 fails for almost all α, and therefore that $d(G^\theta) > \beta\theta$ for almost all α. Summing up, we can state:

THEOREM 4.1. *Given any positive integer $\beta \geqq 1$ and any positive real numbers k, c with $c < 1$, there exists an integer α' such that for all $\alpha \geqq \alpha'$ there exists a finite group G with $d(G) = \alpha$, $d(G/G') = \beta$ and $d(G^n) > \beta n$ for $n \leqq [(1/\beta)(\alpha + c \log_s \alpha) + k]$, where s is the size of the smallest simple non-abelian image of G.*

Although the examples given here seem unnatural, Lemma 1.1 shows how they are prototypical for the growth sequences of imperfect groups.

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