

ESSENTIAL COMMUTATIVITY OF SOME INTEGRAL AND COMPOSITION OPERATORS

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Abstract

In general, multiplication of operators is not essentially commutative in an algebra generated by integral-type operators and composition operators. In this paper, we characterize the essential commutativity of the integral operators and composition operators from a mixed-norm space to a Bloch-type space, and give a complete description of the universal set of integral operators. Corresponding results for boundedness and compactness are also obtained.

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1. Introduction

Let dA denote Lebesgue measure on \mathbb{D} , normalized so that $A(\mathbb{D}) = 1$, where \mathbb{D} is the open unit disk in the complex space \mathbb{C} . Let $H(\mathbb{D})$ be the space of all analytic functions on \mathbb{D} , and $S(\mathbb{D})$ the collection of all analytic self mappings of \mathbb{D} . For $0 < p, q < \infty$ and $-1 < \gamma < \infty$, the mixed-norm space $H_{p,q,\gamma} = H_{p,q,\gamma}(\mathbb{D})$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{H_{p,q,\gamma}} = \left(\int_0^1 M_p^q(f, r)(1-r)^\gamma dr \right)^{1/q} < \infty,$$

where

$$M_p(f, r) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}.$$

For $\alpha > 0$, recall that the α -Bloch space $\mathcal{B}^\alpha = \mathcal{B}^\alpha(\mathbb{D})$ is the space of all $f \in H(\mathbb{D})$ such that

$$b_\alpha(f) = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

The expression $b_\alpha(f)$ defines a semi-norm, while the natural norm is given by $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + b_\alpha(f)$. This norm makes \mathcal{B}^α into a Banach space (see, for example [14]). For

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$\alpha \leq 0$, we still use \mathcal{B}^α to stand for the space of all $f \in H(\mathbb{D})$ such that $b_\alpha(f) < +\infty$. By the maximum modulus theorem, it is easy to show that $\mathcal{B}^\alpha = \mathbb{C}$ for $\alpha < 0$.

Every $\varphi \in S(\mathbb{D})$ induces a composition operator C_φ defined by $C_\varphi f = f \circ \varphi$ for $f \in H(\mathbb{D})$. Readers interested in this operator can refer to the papers [7–9, 12, 13] and the references therein.

If $g \in H(\mathbb{D})$, the integral operators J_g and I_g are defined by

$$J_g f(z) = \int_0^z f(\zeta)g'(\zeta) d\zeta \quad \text{and} \quad I_g f(z) = \int_0^z f'(\zeta)g(\zeta) d\zeta,$$

where $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$. These operators are close companions of the multiplication operator $M_g f(z) = g(z)f(z)$. To see this, note that integration by parts gives

$$M_g f = f(0)g(0) + J_g f + I_g f.$$

The boundedness and compactness of J_g and I_g have recently been characterized on certain spaces of analytic functions. For example, the boundedness of J_g on Hardy spaces, Bergman spaces, BMOA space (the space of analytical functions with bounded mean oscillation), and the Möbius invariant space \mathcal{Q}_p , are characterized in [1, 2, 6, 10], respectively.

Next, we write $T_{\varphi,g}$ and $S_{\varphi,g}$ for the operators $C_\varphi J_g - J_g C_\varphi$ and $C_\varphi I_g - I_g C_\varphi$, respectively. These operators are from $H(p, q, \gamma)$ to \mathcal{B}^α . Generally speaking, it is clear that $C_\varphi J_g \neq J_g C_\varphi$, but it is interesting to consider when

$$V_g C_\varphi (H_{p,q,\gamma} \rightarrow H_{p,q,\gamma}) \equiv C_\varphi (\mathcal{B}^\alpha \rightarrow \mathcal{B}^\alpha) V_g \quad \text{mod } \mathcal{K}, \quad (1.1)$$

where \mathcal{K} denotes the collection of all compact operators from $H_{p,q,\gamma}$ to \mathcal{B}^α , and V_g is J_g or I_g . If C_φ and V_g satisfy (1.1), we say they are essentially commutative.

Since the two occurrences of C_φ in (1.1) act on different spaces, they are in fact different operators. To describe this type of essential commutation more precisely, we say that C_φ can be *transferred* from $H(p, q, \gamma)$ to \mathcal{B}^α by *intertwining* V_g . When it is convenient to de-emphasize the analytic function spaces, we say that C_φ is V_g -transferable.

What properties does a nonconstant g have, while (1.1) holds for all $\varphi \in S(\mathbb{D})$? An interesting problem is to determine a class of universal symbol g of V_g which ensures that all bounded composition operators are V_g -transferable. Let X and Y be Banach spaces of analytic functions on \mathbb{D} , and let $V_g : X \rightarrow Y$ be a bounded integral-type operator. The collection of composition operators C_φ which are bounded both on X and Y is denoted by $\mathcal{B}(X, Y)$.

DEFINITION 1.1. We denote by $\Omega_{\text{co}}(V_g)$ the collection of all g which satisfy the following two conditions:

- (1) V_g is bounded;
- (2) every $C_\varphi \in \mathcal{B}(X, Y)$ is V_g -transferable.

We call $\Omega_{\text{co}}(V_g)$ the *universal set* of V_g .

Here the subscript ‘co’ stands for composition operator, while the reference to V_g indicates that the universal set is a collection of the symbols of V_g .

In this paper, we investigate the boundedness and compactness of the operators $C_\varphi J_g - J_g C_\varphi$ and $C_\varphi I_g - I_g C_\varphi$ from $H(p, q, \gamma)$ to \mathcal{B}^α . We also describe the universal sets $\Omega_{\text{co}}(J_g)$ and $\Omega_{\text{co}}(I_g)$. Even on the unit disk \mathbb{D} of \mathbb{C} , some properties are not easily derived, and we need some new methods and calculation techniques.

Throughout this paper, C denotes a positive constant, the exact value of which will vary from one appearance to the next.

2. Some lemmas

To begin our discussion, we state a couple of lemmas which will be used in the proofs of the main results.

The following lemma is the crucial criterion for compactness, and is established by an easy modification of [3, Proof of Proposition 3.11].

LEMMA 2.1. *Suppose that $0 < p, q < \infty$, $\alpha > 0$ and $-1 < \gamma < \infty$. Suppose further that $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. Then $T_{\varphi, g}$ (respectively $S_{\varphi, g}$) is a compact operator from $H_{p, q, \gamma}$ to \mathcal{B}^α if and only if $T_{\varphi, g}$ (respectively $S_{\varphi, g}$) is bounded. Also, for any bounded sequence $\{f_k\}_{k \in \mathbb{N}}$ in $H_{p, q, \gamma}$ which converges to zero uniformly on compact subsets of \mathbb{D} as $k \rightarrow \infty$, we have $\|T_{\varphi, g} f_k\|_{\mathcal{B}^\alpha} \rightarrow 0$ (respectively $\|S_{\varphi, g} f_k\|_{\mathcal{B}^\alpha} \rightarrow 0$) as $k \rightarrow \infty$.*

We set $\beta = (1/p) + ((\gamma + 1)/q)$.

LEMMA 2.2 [5, Lemma 1]. *Assume that $0 < p, q < \infty$, $-1 < \gamma < \infty$ and $f \in H_{p, q, \gamma}$. Then there exists a positive constant C , independent of f , such that*

$$|f^{(n)}(z)| \leq C \frac{\|f\|_{H_{p, q, \gamma}}}{(1 - |z|^2)^{\beta+n}}.$$

The next lemma holds by direct calculation, and we omit the proof here.

LEMMA 2.3. *Assume that $0 < p, q < \infty$, $\alpha > 0$, $-1 < \gamma < \infty$ and $g \in H(\mathbb{D})$. Then $J_g : H(p, q, \gamma) \rightarrow \mathcal{B}^\alpha$ is bounded if and only if*

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha-\beta} |g'(z)| < \infty,$$

and I_g is bounded if and only if

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha-\beta-1} |g(z)| < \infty.$$

LEMMA 2.4 ([4], or [11, Theorem 1]). Assume that $\varphi \in S(\mathbb{D})$. Then for $\alpha > 0$, C_φ is bounded on \mathcal{B}^α if and only if

$$\sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^\alpha} |\varphi'(z)| < \infty.$$

LEMMA 2.5. Suppose that $\varphi \in S(\mathbb{D})$ and $\varphi(0) = 0$. Then C_φ is bounded on $H_{p,q,\gamma}$.

PROOF. By the Littlewood subordination theorem [3, Theorem 2.22],

$$\begin{aligned} M_p(C_\varphi f, r) &= \left(\frac{1}{2\pi} \int_0^{2\pi} |f(\varphi(re^{i\theta}))|^p d\theta \right)^{1/p} \\ &\leq \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} = M_p(f, r). \end{aligned}$$

Thus

$$\begin{aligned} \|C_\varphi f\|_{H_{p,q,\gamma}} &= \left(\int_0^1 M_p^q(C_\varphi f, r)(1-r)^\gamma dr \right)^{1/q} \\ &\leq \left(\int_0^1 M_p^q(f, r)(1-r)^\gamma dr \right)^{1/q} = \|f\|_{H_{p,q,\gamma}}. \end{aligned}$$

This completes the proof. □

3. Essential commutativity of C_φ and J_g from $H(p, q, \gamma)$ to \mathcal{B}^α

In this section, we characterize essential commutativity of the integral operator J_g and composition operators from $H_{p,q,\gamma}$ to \mathcal{B}^α .

THEOREM 3.1. Assume that $0 < p, q < \infty$, $\alpha > 0$ and $-1 < \gamma < \infty$. Assume further that $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. Then $T_{\varphi,g} = C_\varphi J_g - J_g C_\varphi$ is a bounded operator from $H_{p,q,\gamma}$ to \mathcal{B}^α if and only if

$$\sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^\beta} |(g \circ \varphi)'(z) - g'(z)| < \infty. \tag{3.1}$$

PROOF. Suppose that (3.1) is true. Then for any $f \in H(p, q, \gamma)$,

$$\begin{aligned} T_{\varphi,g} f(z) &= C_\varphi \left(\int_0^z f(w)g'(w) dw \right) - J_g(f \circ \varphi)(z) \\ &= \int_0^{\varphi(z)} f(w)g'(w) dw - \int_0^z f(\varphi(w))g'(w) dw. \end{aligned}$$

It follows from Lemma 2.2 that

$$\begin{aligned} \|T_{\varphi,g} f\|_{\mathcal{B}^\alpha} &= |T_{\varphi,g} f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |(g \circ \varphi)'(z) - g'(z)| |f(\varphi(z))| \\ &\leq C \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\alpha |(g \circ \varphi)'(z) - g'(z)|}{(1 - |\varphi(z)|^2)^\beta} \|f\|_{H_{p,q,\gamma}} \\ &\leq C \|f\|_{H_{p,q,\gamma}}, \end{aligned}$$

where the last inequality follows from (3.1). This means that $T_{\varphi,g} = C_\varphi J_g - J_g C_\varphi$ is a bounded operator from $H_{p,q,\gamma}$ to \mathcal{B}^α .

Conversely, suppose that $T_{\varphi,g} : H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha$ is bounded, so that there exists a constant C such that $\|T_{\varphi,g} f\|_{\mathcal{B}^\alpha} \leq C \|f\|_{H_{p,q,\gamma}}$ for any $f \in H_{p,q,\gamma}$.

Now, for any fixed $w \in \mathbb{D}$, we choose the test function

$$f_w(z) = \left(\frac{1 - |w|^2}{(1 - \bar{w}z)^2} \right)^\beta, \quad z \in \mathbb{D}. \tag{3.2}$$

It is straightforward to show that $f_w \in H_{p,q,\gamma}$ and $\sup_{w \in \mathbb{D}} \|f_w\|_{H_{p,q,\gamma}} < C$.

Therefore

$$\begin{aligned} \infty &> C \|T_{\varphi,g}\|_{H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha} \geq \|T_{\varphi,g} f_{\varphi(\lambda)}\|_{\mathcal{B}^\alpha} \\ &\geq \sup_{\lambda \in \mathbb{D}} (1 - |\lambda|^2)^\alpha |(g \circ \varphi)'(\lambda) - g'(\lambda)| |f_{\varphi(\lambda)}(\varphi(\lambda))| \\ &= \sup_{\lambda \in \mathbb{D}} (1 - |\lambda|^2)^\alpha |(g \circ \varphi)'(\lambda) - g'(\lambda)| \frac{1}{(1 - |\varphi(\lambda)|^2)^\beta}, \end{aligned}$$

from which the desired result (3.1) holds. This completes the proof. □

THEOREM 3.2. *Assume that $0 < p, q < \infty, \alpha > 0$ and $-1 < \gamma < \infty$. Assume further that $\varphi \in S(\mathbb{D}), g \in H(\mathbb{D})$ and $T_{\varphi,g} = C_\varphi J_g - J_g C_\varphi : H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha$. Then C_φ and J_g are essentially commutative if and only if $T_{\varphi,g}$ is bounded and*

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^\beta} |(g \circ \varphi)'(z) - g'(z)| = 0. \tag{3.3}$$

PROOF. Suppose first that C_φ and J_g are essentially commutative. Then $T_{\varphi,g} = C_\varphi J_g - J_g C_\varphi$ is compact, hence bounded. Let $\{z_k\}_{k \in \mathbb{N}}$ be a sequence in \mathbb{D} such that $|\varphi(z_k)| \rightarrow 1$ as $k \rightarrow \infty$. We consider the test functions defined by

$$f_k(z) = \left(\frac{1 - |\varphi(z_k)|^2}{(1 - \varphi(z_k)z)^2} \right)^\beta. \tag{3.4}$$

Direct computation shows that $f_k \in H_{p,q,\gamma}$ with $\sup_k \|f_k\|_{H_{p,q,\gamma}} < C$ and $\{f_k\}$ converges to 0 uniformly on compact subsets of \mathbb{D} as $k \rightarrow \infty$. Since $T_{\varphi,g} = C_\varphi J_g - J_g C_\varphi$ is compact, it follows from Lemma 2.1 that

$$\lim_{k \rightarrow \infty} \|T_{\varphi,g} f_k\|_{\mathcal{B}^\alpha} = 0. \tag{3.5}$$

Therefore

$$\begin{aligned} \|T_{\varphi,g} f_k\|_{\mathcal{B}^\alpha} &\geq (1 - |z_k|^2)^\alpha |(g \circ \varphi)'(z_k) - g'(z_k)| |f_k(\varphi(z_k))| \\ &= (1 - |z_k|^2)^\alpha |(g \circ \varphi)'(z_k) - g'(z_k)| \frac{1}{(1 - |\varphi(z_k)|^2)^\beta}. \end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6), the condition (3.3) follows.

Conversely, suppose that $T_{\varphi,g}$ is bounded and (3.3) holds. By Theorem 3.1, the boundedness of $T_{\varphi,g}$ implies that

$$(1 - |z|^2)^\alpha |(g \circ \varphi)'(z) - g'(z)| \leq C \tag{3.7}$$

for any $z \in \mathbb{D}$.

Let $\{f_k\}_{k \in \mathbb{N}}$ be any bounded sequence in $H_{p,q,\gamma}$ which converges to zero uniformly on compact subsets of \mathbb{D} as $k \rightarrow \infty$, with $\sup_k \|f_k\|_{H_{p,q,\gamma}} \leq M$.

It follows from (3.3) that for any $\varepsilon > 0$, there exists $\delta > 0$, with $\delta < |\varphi(z)| < 1$, such that

$$\frac{(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^\beta} |(g \circ \varphi)'(z) - g'(z)| < \frac{\varepsilon}{M}. \tag{3.8}$$

Let $E = \{z \in \mathbb{D} : |\varphi(z)| \leq \delta\}$, and let $K = \{w = \varphi(z) : z \in E\}$, a compact subset of \mathbb{D} . By Theorem 3.1, the boundedness of $T_{\varphi,g}$ implies (3.1), and it follows from (3.7) and (3.8) that

$$\begin{aligned} \|T_{\varphi,g}f_k\|_{\mathcal{B}^\alpha} &\leq |T_{\varphi,g}f_k(0)| + \sup_{z \in E} (1 - |z|^2)^\alpha |(g \circ \varphi)'(z) - g'(z)| |f_k(\varphi(z))| \\ &\quad + \sup_{z \in \mathbb{D} \setminus E} (1 - |z|^2)^\alpha |(g \circ \varphi)'(z) - g'(z)| \frac{\|f_k\|_{H_{p,q,\gamma}}}{(1 - |\varphi(z)|^2)^\beta} \\ &\leq |T_{\varphi,g}f_k(0)| + C \sup_{w \in K} |f_k(w)| + \varepsilon. \end{aligned} \tag{3.9}$$

Note that ε is an arbitrary positive number. In view of the assumption that $f_k \rightarrow 0$ as $k \rightarrow \infty$ on compact subsets of \mathbb{D} , on letting $k \rightarrow \infty$ in (3.9), we have $\|T_{\varphi,g}f_k\|_{\mathcal{B}^\alpha} \rightarrow 0$. It follows from Lemma 2.1 that $T_{\varphi,g}$ is compact, and so C_φ and J_g are essentially commutative. The proof of the theorem is complete. \square

THEOREM 3.3. *Assume that $0 < p, q < \infty, \alpha > 0, -1 < \gamma < \infty, \varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. Assume further that $J_g : H(p, q, \gamma) \rightarrow \mathcal{B}^\alpha$ is bounded. Then J_g is essentially commutative with any $C_\varphi \in \mathcal{B}(H_{p,q,\gamma}, \mathcal{B}^\alpha)$ if and only if*

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^{\alpha-\beta} |g'(z)| = 0, \tag{3.10}$$

that is, $\Omega_{\text{co}}(J_g) = \mathcal{B}^{\alpha-\beta}$.

PROOF. We first prove necessity. Let $\varphi(z) = e^{i\theta}z$ for any $\theta \in [0, 2\pi]$. Then by Lemmas 2.4 and 2.5, it is easy to see that C_φ is bounded both on $H_{p,q,\gamma}$ and \mathcal{B}^α , so that $C_\varphi \in \mathcal{B}(H_{p,q,\gamma}, \mathcal{B}^\alpha)$. From the assumptions we know that J_g and C_φ are essentially commutative, and it follows from Theorem 3.2 that

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^{\alpha-\beta} |e^{i\theta}g'(e^{i\theta}z) - g'(z)| = 0. \tag{3.11}$$

It is necessary to bound from above the left-hand side of (3.11). We have

$$\begin{aligned} (1 - |z|^2)^{\alpha-\beta} |e^{i\theta}g'(e^{i\theta}z) - g'(z)| &\leq (1 - |z|^2)^{\alpha-\beta} |e^{i\theta}g'(e^{i\theta}z)| + (1 - |z|^2)^{\alpha-\beta} |g'(z)| \\ &= (1 - |e^{i\theta}z|^2)^{\alpha-\beta} |g'(e^{i\theta}z)| + (1 - |z|^2)^{\alpha-\beta} |g'(z)| \\ &\leq 2\|g\|_{\mathcal{B}^{\alpha-\beta}}. \end{aligned}$$

By Lemma 2.2, the boundedness of $J_g : H(p, q, \gamma) \rightarrow \mathcal{B}^\alpha$ implies that $\|g\|_{\mathcal{B}^{\alpha-\beta}}$ is finite, and so the left-hand side of (3.11) is bounded independently of θ . We write $g(z) = \sum_{n=0}^\infty a_n z^n$, and integrate the left-hand side of (3.11) with respect to θ from 0 to 2π . By the dominating convergence theorem, we deduce that

$$\begin{aligned} 0 &= \int_0^{2\pi} \lim_{|z| \rightarrow 1} (1 - |z|^2)^{\alpha-\beta} |e^{i\theta} g'(e^{i\theta} z) - g'(z)| \, d\theta \\ &= \lim_{|z| \rightarrow 1} \int_0^{2\pi} (1 - |z|^2)^{\alpha-\beta} \left| \sum_{n=1}^\infty n a_n z^{n-1} (e^{in\theta} - 1) \right| \, d\theta \\ &\geq \lim_{|z| \rightarrow 1} (1 - |z|^2)^{\alpha-\beta} \left| \sum_{n=1}^\infty n a_n z^{n-1} \int_0^{2\pi} (e^{in\theta} - 1) \, d\theta \right| \geq \lim_{|z| \rightarrow 1} (1 - |z|^2)^{\alpha-\beta} |g'(z)|. \end{aligned}$$

Hence (3.10) holds, and so $g \in \mathcal{B}_0^{\alpha-\beta}$.

Turning to sufficiency, suppose that (3.10) holds. Corollary 2.40 in [3] states that $|\varphi(z)| \leq (|z| + |\varphi(0)|)/(1 + |z| |\varphi(0)|)$, and so we have

$$\frac{1 - |z|}{1 - |\varphi(z)|} \leq \frac{1 + |z| |\varphi(0)|}{1 - |\varphi(0)|}. \tag{3.12}$$

By Lemmas 2.4, 2.5 and (3.12),

$$\begin{aligned} &\lim_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^\beta} |(g \circ \varphi)'(z) - g'(z)| \\ &\leq \lim_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^\alpha} |\varphi'(z)| (1 - |\varphi(z)|^2)^{\alpha-\beta} |g'(\varphi(z))| \\ &\quad + \lim_{|\varphi(z)| \rightarrow 1} \frac{(2(1 - |z|))^\alpha}{(1 - |\varphi(z)|)^\alpha} (1 - |z|^2)^{\alpha-\beta} |g'(z)| \\ &\leq C \lim_{|\varphi(z)| \rightarrow 1} (1 - |\varphi(z)|^2)^{\alpha-\beta} |g'(\varphi(z))| \\ &\quad + 2^\alpha \lim_{|\varphi(z)| \rightarrow 1} \left(\frac{1 + |z| |\varphi(0)|}{1 - |\varphi(0)|} \right)^\alpha (1 - |z|^2)^{\alpha-\beta} |g'(z)| \\ &\leq C \lim_{|w| \rightarrow 1} (1 - |w|^2)^{\alpha-\beta} |g'(w)| + C \lim_{|z| \rightarrow 1} (1 - |z|^2)^{\alpha-\beta} |g'(z)| = 0. \end{aligned}$$

By Theorem 3.2, J_g is essentially commutative with any $C_\varphi \in \mathcal{B}(H_{p,q,\gamma}, \mathcal{B}^\alpha)$, so that $g \in \Omega_{\text{co}}(J_g)$. This completes the proof. \square

4. Essential commutativity of I_g and C_φ from $H_{p,q,\gamma}$ to \mathcal{B}^α

We characterize essential commutativity of the integral operator I_g and composition operators from $H_{p,q,\gamma}$ to \mathcal{B}^α in this section.

THEOREM 4.1. Assume that $0 < p, q < \infty, \alpha > 0$ and $-1 < \gamma < \infty$. Assume further that $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. Then $S_{\varphi,g} = C_{\varphi}I_g - I_gC_{\varphi}$ is a bounded operator from $H(p, q, \gamma)$ to \mathcal{B}^{α} if and only if

$$\sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^{\alpha}}{(1 - |\varphi(z)|^2)^{\beta+1}} |\varphi'(z)| |g(\varphi(z)) - g(z)| < \infty. \tag{4.1}$$

PROOF. First, suppose that $S_{\varphi,g} : H_{p,q,\gamma} \rightarrow \mathcal{B}^{\alpha}$ is bounded, so that there exists a constant C such that $\|S_{\varphi,g}f\|_{\mathcal{B}^{\alpha}} \leq C\|f\|_{H_{p,q,\gamma}}$ for all $f \in H_{p,q,\gamma}$. Note that

$$(1 - |z|^2)^{\alpha} |(S_{\varphi,g}f)'(z)| = (1 - |z|^2)^{\alpha} |\varphi'(z)| |g(\varphi(z)) - g(z)| |f'(\varphi(z))|.$$

Taking $f(z) = z$, we deduce that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha} |\varphi'(z)| |g(\varphi(z)) - g(z)| < \infty. \tag{4.2}$$

We now choose the same test function (3.2), and let $\lambda \in \mathbb{D}$. Then one can show that

$$|f'_{\varphi(\lambda)}(\varphi(\lambda))| = 2 \left(\frac{1}{p} + \frac{\gamma + 1}{q} \right) \frac{|\varphi(\lambda)|}{(1 - |\varphi(\lambda)|^2)^{\beta+1}}.$$

Therefore

$$\begin{aligned} \infty &> C \|S_{\varphi,g}\|_{H_{p,q,\gamma} \rightarrow \mathcal{B}^{\alpha}} \geq \|S_{\varphi,g}f_{\varphi(\lambda)}\|_{\mathcal{B}^{\alpha}} \\ &\geq 2 \left(\frac{1}{p} + \frac{\gamma + 1}{q} \right) \frac{(1 - |\lambda|^2)^{\alpha} |\varphi(\lambda)| |\varphi'(\lambda)|}{(1 - |\varphi(\lambda)|^2)^{\beta+1}} |g(\varphi(\lambda)) - g(\lambda)|, \end{aligned}$$

from which we obtain

$$\begin{aligned} &\sup_{|\varphi(\lambda)| > 1/2} \frac{(1 - |\lambda|^2)^{\alpha} |\varphi'(\lambda)|}{(1 - |\varphi(\lambda)|^2)^{\beta+1}} |g(\varphi(\lambda)) - g(\lambda)| \\ &\leq \sup_{|\varphi(\lambda)| > 1/2} 2 \frac{(1 - |\lambda|^2)^{\alpha} |\varphi(\lambda)| |\varphi'(\lambda)|}{(1 - |\varphi(\lambda)|^2)^{\beta+1}} |g(\varphi(\lambda)) - g(\lambda)| < \infty. \end{aligned} \tag{4.3}$$

It follows from (4.2) that

$$\begin{aligned} &\sup_{|\varphi(\lambda)| \leq 1/2} \frac{(1 - |\lambda|^2)^{\alpha} |\varphi'(\lambda)|}{(1 - |\varphi(\lambda)|^2)^{\beta+1}} |g(\varphi(\lambda)) - g(\lambda)| \\ &\leq \sup_{|\varphi(\lambda)| \leq 1/2} \left(\frac{4}{3} \right)^{\beta+1} (1 - |\lambda|^2)^{\alpha} |\varphi'(\lambda)| |g(\varphi(\lambda)) - g(\lambda)| < \infty. \end{aligned} \tag{4.4}$$

Combining (4.3) and (4.4), the desired result (4.1) follows.

Conversely, suppose that the condition (4.1) holds. Then for any $f \in H_{p,q,\gamma}$, we deduce from Lemma 2.2 that

$$\begin{aligned} \|S_{\varphi,g}f\|_{\mathcal{B}^{\alpha}} &= |S_{\varphi,g}f(0)| + \sup_{z \in D} (1 - |z|^2)^{\alpha} |\varphi'(z)| |g(\varphi(z)) - g(z)| |f'(\varphi(z))| \\ &\leq C \sup_{z \in D} \frac{(1 - |z|^2)^{\alpha} |\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\beta+1}} |g(\varphi(z)) - g(z)| \|f\|_{H_{p,q,\gamma}} \\ &\leq C \|f\|_{H_{p,q,\gamma}}. \end{aligned}$$

Hence $S_{\varphi,g} = C_{\varphi}I_g - I_gC_{\varphi}$ is bounded from $H_{p,q,\gamma}$ to \mathcal{B}^{α} . This completes the proof. \square

THEOREM 4.2. *Assume that $0 < p, q < \infty, \alpha > 0, -1 < \gamma < \infty, \varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. Assume further that $S_{\varphi,g} = C_\varphi I_g - I_g C_\varphi : H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha$. Then C_φ and I_g are essentially commutative if and only if $S_{\varphi,g}$ is bounded and*

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^{\beta+1}} |\varphi'(z)| |g(\varphi(z)) - g(z)| = 0. \tag{4.5}$$

PROOF. We first prove sufficiency. For any bounded sequence $\{f_k\}_{k \in \mathbb{N}}$ in $H_{p,q,\gamma}$ which converges to zero uniformly on compact subsets of \mathbb{D} as $k \rightarrow \infty$, we have $\sup_k \|f_k\|_{H_{p,q,\gamma}} \leq M$. From (4.5), there exists $\delta > 0, \delta < |\varphi(z)| < 1$, such that for any $\varepsilon > 0$,

$$\frac{(1 - |z|^2)^\alpha |\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\beta+1}} |g(\varphi(z)) - g(z)| < \frac{\varepsilon}{M}. \tag{4.6}$$

Let $E = \{z \in \mathbb{D} : |\varphi(z)| \leq \delta\}$, and let $K = \{w = \varphi(z) : z \in E\}$, a compact subset of \mathbb{D} . By the boundedness of $S_{\varphi,g}$, it follows from (4.2) and (4.6) that

$$\begin{aligned} \|S_{\varphi,g} f_k\|_{\mathcal{B}^\alpha} &\leq |S_{\varphi,g} f_k(0)| + \sup_{z \in E} (1 - |z|^2)^\alpha |\varphi'(z)| |g(\varphi(z)) - g(z)| |f'_k(\varphi(z))| \\ &\quad + \sup_{z \in \mathbb{D} \setminus E} \frac{(1 - |z|^2)^\alpha |\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\beta+1}} |g(\varphi(z)) - g(z)| \|f_k\|_{H_{p,q,\gamma}} \\ &\leq |S_{\varphi,g} f_k(0)| + C \sup_{w \in K} |f'_k(w)| + \varepsilon. \end{aligned} \tag{4.7}$$

The assumption that $f_k \rightarrow 0$ as $k \rightarrow \infty$ on compact subsets of \mathbb{D} , along with Cauchy’s estimate, gives that $f'_k \rightarrow 0$ as $k \rightarrow \infty$ on compact subsets of \mathbb{D} . Letting $k \rightarrow \infty$ in (4.7), and noting that ε is an arbitrary positive number, we have $\|S_{\varphi,g} f_k\|_{\mathcal{B}^\alpha} \rightarrow 0$. Then it follows from Lemma 2.1 that C_φ and I_g are essentially commutative.

We now turn to necessity. First, suppose that C_φ and I_g are essentially commutative. Then $S_{\varphi,g} = C_\varphi I_g - I_g C_\varphi$ is compact, hence bounded. Let $\{z_k\}_{k \in \mathbb{N}}$ be a sequence in \mathbb{D} such that $|\varphi(z_k)| \rightarrow 1$ as $k \rightarrow \infty$, and consider the sequence $\{f_k\}_{k \in \mathbb{N}}$ defined by (3.4) as in the proof of Theorem 3.2. Then one can show that

$$|f'_k(\varphi(z_k))| = 2 \left(\frac{1}{p} + \frac{\gamma + 1}{q} \right) \frac{|\varphi(z_k)|}{(1 - |\varphi(z_k)|^2)^{\beta+1}}.$$

Thus

$$\begin{aligned} \|S_{\varphi,g} f_k\|_{\mathcal{B}^\alpha} &\geq (1 - |z_k|^2)^{1-\alpha} |\varphi'(z_k)| |g(\varphi(z_k)) - g(z_k)| |f'_k(\varphi(z_k))| \\ &= 2 \left(\frac{1}{p} + \frac{\gamma + 1}{q} \right) \frac{|\varphi(z_k)| (1 - |z_k|^2)^{1-\alpha} |\varphi'(z_k)| |g(\varphi(z_k)) - g(z_k)|}{(1 - |\varphi(z_k)|^2)^{\beta+1}}. \end{aligned} \tag{4.8}$$

Since $S_{\varphi,g} = C_\varphi I_g - I_g C_\varphi$ is compact, by Lemma 2.1 we have $\lim_{k \rightarrow \infty} \|S_{\varphi,g} f_k\|_{\mathcal{B}^\alpha} = 0$. Combining this with (4.8), and noting that the right-hand side of (4.8) tends to zero as $k \rightarrow \infty$, the desired result (4.5) follows. The proof is complete. \square

THEOREM 4.3. Assume that $0 < p, q < \infty$, $0 < \alpha < 1$, $-1 < \gamma < \infty$, $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. Assume further that $I_g : H(p, q, \gamma) \rightarrow \mathcal{B}^\alpha$ is bounded. Then I_g is essentially commutative with any $C_\varphi \in \mathcal{B}(H_{p,q,\gamma}, \mathcal{B}^\alpha)$ if and only if

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^{\alpha-\beta-1} |g(z)| = 0,$$

that is, $\Omega_{\text{co}}(I_g) = \mathcal{B}_0^{\alpha-\beta-1}$.

PROOF. The proof is similar to that of Theorem 3.3, and we omit the details. \square

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