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THE FAILURE PROBABILITY OF COMPONENTS IN THREE-STATE NETWORKS WITH APPLICATIONS TO AGE REPLACEMENT POLICY

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Abstract

In this paper we investigate the stochastic properties of the number of failed components of a three-state network. We consider a network made up of n components which is designed for a specific purpose according to the performance of its components. The network starts operating at time $t = 0$ and it is assumed that, at any time $t > 0$, it can be in one of states *up*, *partial performance*, or *down*. We further suppose that the state of the network is inspected at two time instants t_1 and t_2 ($t_1 < t_2$). Using the notion of the two-dimensional signature, the probability of the number of failed components of the network is calculated, at t_1 and t_2, under several scenarios about the states of the network. Stochastic and ageing properties of the proposed failure probabilities are studied under different conditions. We present some optimal age replacement policies to show applications of the proposed criteria. Several illustrative examples are also provided.

Keywords: Signature matrix; multi-state system; bivariate increasing failure rate; totally positive of order two; stochastic order

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Secondary 90B25

1. Introduction

Nowadays, networks (systems), such as communication networks and computer networks, play an important role in various areas of science and technology. A network is a series of points (nodes) interconnected by communication paths (links) which allow nodes to exchange data through the links. The networks can be modeled mathematically as a graph $G(V; E)$, where V denotes the collection of nodes and E denotes the collection of links connecting the selected pairs of nodes. In the simplest case, a network has two states: *up* or *down*. However, in a general case, depending on how the states of a network are defined, the network may have several states. A network with several states is called a multi-state network. Multi-state networks have extensive applications in various areas of reliability and other disciplines. From a mathematical point of view, the states of a multi-state network can be denoted by $K = 0, 1, \ldots, M$, where $K = 0$ is used to show the complete failure of the network and $K = M$ is used to show the perfect functioning of the network. There is an extensive literature on the reliability and stochastic properties of multi-state networks and systems under different conditions. Among others, Lisnianski and Levitin [23] studied the tools for the reliability assessment and optimization of systems having several states. Huang *et al.* [17] and Zuo and

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Tian [34] proposed generalizations for the multi-state k-out-of-n systems and presented some algorithms for assessing the reliability of the system. Tian *et al.* [30] presented reliability bounds for the multi-state k-out-of-n systems. Zhao and Cui [33] evaluated the distribution of states in a generalized multi-state k-out-of-n system. Eryilmaz [6] investigated the mean residual lifetime of multi-state k-out-of-n systems. Eryilmaz and Xie [11] considered three-state k-out-of-n systems made up of independent and nonidentical components and studied marginal and joint survival functions for the lifetimes of two different k-out-of-n systems.

Among various approaches introduced to explore the reliability and ageing properties of the networks, one approach is based on the concept of the so-called *signature* (or *D-spectrum*) (see, for example, [31] and [32] and the references therein for a review of different approaches). The concept of the signature, which is a topological invariant of the network design, has proven useful in the analysis of the network performance particularly for comparisons between different network structures. Consider a network (system) which includes n components where we assume that the component lifetimes are independent and identically distributed (i.i.d.) random variables $X_1, X_2, \ldots, X_n$ with a common continuous distribution function F. Assuming that $T = \phi(X_1, \ldots, X_n)$ denotes the network lifetime, the signature vector associated to the network is a probability vector $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$, in which the ith element is defined as

$$s_i = \mathbb{P}(T = X_{i:n}), \qquad i = 1, 2, \ldots, n,$$

where $X_{i:n}$ is the ith ordered random variable among $X_1, \ldots, X_n$. For more details on signatures and their applications in the study of reliability of systems, see, for example, [21], [25], [27], and [28]. The notion of the signature has been extended to single-step multi-state networks by Gertsbakh and Shpungin [13]. Recall that a single-step network is a network such that the failure of one component changes the network state at most by one. Throughout the paper we are dealing with a single-step network consisting of n links where we assume that the network has three states. When the network is in the *up* state (perfect functioning), we show its state by $K = 2$, when the network is in *partial* performance, we use $K = 1$, and with $K = 0$, we mean that the network is *down*. Further, in the sequel the nodes are assumed to be absolutely reliable and whenever we assume that the components of a network fail, we mean that the links of the network fail. Under the assumption that the network components have i.i.d. lifetimes, we denote by $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ the ordered lifetimes of the components. Assume that the network starts to operate at time $t = 0$ where it is in state $K = 2$. Let the random variable T_1 denote the time that the network enters from state $K = 2$ into state $K = 1$ and the random variable T denote the network lifetime, i.e. the first time that the network moves into state $K = 0$. Let us consider, in a three-state network, a probability matrix $\mathcal{S}$ with elements defined as

$$s_{i,j} = \mathbb{P}(T_1 = X_{i:n}, T = X_{j:n}) = \frac{n_{i,j}}{n!}, \qquad 1 \le i < j \le n,$$

where $n_{i,j}$ represents the number of permutations in which the ith and the jth component failures change the network state from $K = 2$ to $K = 1$ and from $K = 1$ to $K = 0$, respectively. Then the matrix $\mathcal{S}$ is known as the signature matrix and $s_{i,j}$ is called the two-dimensional signature (see [13]). It should be noted that the calculation of the two-dimensional signature is only dependent on the network structure and does not depend on the distribution of the component lifetimes. Recently, Gertsbakh *et al.* [15] obtained several results on the two-dimensional signature. Levitin *et al.* [20] used the multi-dimensional signature to evaluate the expected damage associated with disintegrating networks that arise from an intentional attack. Ashrafi and Asadi [2], [3] have studied several properties of the lifetimes of three-state networks using the signature matrix $\mathcal{S}$ under different conditions. Navarro *et al.* [26] introduced a new variant of the two-dimensional signature for two systems having shared components.

The objective of the present paper is to investigate the stochastic and ageing properties of the number of failed components of a single-step three-state network with signature matrix $\mathcal{S}$. To be more specific, under the assumption that the network lifetimes T_1 and T are in the given conditions, we are interested in calculating the probability of the number of components that have failed in an operating network. Computing the probability of the number of failed components in an operating network is not only important for engineers and network designers to plan maintenance policies but is also important in designing optimized networks. Recent literature has dealt with the number of failed (or working) components in two-state systems. Asadi and Berred [1] studied the distribution of the number of failed components in a binary coherent system. Eryilmaz [7] studied the probability and expectation of the number of working components in a consecutive k-out-of-n system when the system is operating at time t. Eryilmaz [8] explored the properties of the expected number of working components in weighted k-out-of-n systems. Eryilma [9] and Kelkinnama *et al.* [19] investigated the probability of the number of failed components in a binary system having exchangeable components. Ling and Li [22] studied the effect of the random environment on the number of working components of a binary system having heterogeneous components. Eryilmaz [10] considered a three-state system where the components of the system are also assumed to have three states. The cited author calculated the mean number of components that are in the up state or the partial performance state after the failure of the system.

The rest of the paper is organized as follows. In Section 2 we consider a three-state network described as above. We assume that the network is inspected at two time instants t_1 and t_2 $(t_1 < t_2)$. We further assume that at inspection time instants t_1 and t_2, it is known that $T_1 \in A_1(t_1, t_2)$ and $T \in A_2(t_1, t_2)$, where A_1 and A_2 are subsets of $[0, \infty)$ depending on the values of t_1 and t_2. Under these conditions, we obtain the probability that at time t_1 there are k failed components and at time t_2 there are l failed components in the network. Under several choices for $A_1(t_1, t_2)$ and $A_2(t_1, t_2)$, the proposed conditional probabilities are calculated in terms of the signature matrix and the common distribution function of the component lifetimes. Some stochastic and ageing properties of the proposed probabilities are studied in this section. In Section 3, under some age replacement policies, two applications of the proposed conditional probabilities are presented. In Section 4 we summarize the achievements of this paper and present some concluding remarks. Throughout the paper, several illustrative examples are also provided.

2. The probability of the number of failed components

Suppose that $X_1, \ldots, X_n$ denote the component lifetimes of a network made up of n components, where the X_i's are assumed to be i.i.d. with a common continuous distribution function $F(x)$. Suppose that the network is inspected at time instants t_1 and t_2 $(t_1 < t_2)$. Let us assume that the operator of the network has some information about the states of the network at time instants t_1 and t_2. For instance, the operator may realize that $T_1 \in A_1 = A_1(t_1, t_2)$ and $T \in A_2 = A_2(t_1, t_2)$, where A_1 and A_2 are subsets of $[0, \infty)$. Let $N(t)$ denote the number of failed components in $[0, t]$. Then we are interested in calculating the conditional probabilities in the following form:

$$p_{k,l}(A_1, A_2) = \mathbb{P}(N(t_1) = k, N(t_2) = l \mid T_1 \in A_1, T \in A_2), \qquad 0 \le k \le l \le n.$$

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Among the many different special cases that can be considered for A_1 and A_2, we consider the following cases.

(I) Assume that at time instants t_1 and t_2 the network is in states $K = 2$ and $K = 1$, respectively. In this case, $A_1 = (t_1, t_2)$ and $A_2 = (t_2, \infty)$. Then $p_{k,l}(A_1, A_2)$, denoted by $p_{k,l}(t_1, t_2)$, is given by

$$p_{k,l}(t_1, t_2) = \mathbb{P}(N(t_1) = k, N(t_2) = l \mid t_1 < T_1 < t_2, T > t_2), \qquad 0 \le k < l \le n-1.$$

(II) Assume that at time t_1 the network is in state $K = 2$, and at time t_2 it is functioning. In this case, $A_1 = (t_1, \infty)$ and $A_2 = (t_2, \infty)$, and $p_{k,l}(A_1, A_2)$, denoted by $q_{k,l}(t_1, t_2)$, is given by

$$q_{k,l}(t_1, t_2) = \mathbb{P}(N(t_1) = k, N(t_2) = l \mid T_1 > t_1, T > t_2), \qquad 0 \le k \le n-2,\ k \le l \le n-1.$$

(III) Suppose that at both time instants t_1 and t_2 the state of the network is $K = 1$. In this case, $A_1 = (0, t_1)$ and $A_2 = (t_2, \infty)$, and $p_{k,l}(A_1, A_2)$, denoted by $r_{k,l}(t_1, t_2)$, is given by

$$r_{k,l}(t_1, t_2) = \mathbb{P}(N(t_1) = k, N(t_2) = l \mid T_1 < t_1, T > t_2), \qquad 1 \le k \le l \le n-1.$$

In the following theorem, $p_{k,l}(t_1, t_2)$, $q_{k,l}(t_1, t_2)$, and $r_{k,l}(t_1, t_2)$ are computed in terms of the distribution function F and the signature matrix $\mathcal{S}$.

Theorem 1. *Consider a network including n i.i.d. components. Let T_1 denote the time that the network stays in state $K = 2$ and T denote the lifetime of the network. Assume that $F(x)$ denotes the common distribution of the component lifetimes and $\mathcal{S}$ denotes the signature matrix of the network.*

(i) *If $\alpha_{k,l} = \sum_{i=k+1}^{l} \sum_{j=l+1}^{n} s_{i,j}$ then, for $0 \le k < l \le n-1$,*

$$p_{k,l}(t_1, t_2) = \frac{\alpha_{k,l} c_{k,l,n} F^k(t_1)(F(t_2) - F(t_1))^{l-k} \bar{F}^{n-l}(t_2)}{\sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \alpha_{i,j} c_{i,j,n} F^i(t_1)(F(t_2) - F(t_1))^{j-i} \bar{F}^{n-j}(t_2)},$$

where $c_{k,l,n} = n!/k!\,(l-k)!\,(n-l)!$.

(ii) *If $\beta_{k,l} = \sum_{i=k+1}^{n-1} \sum_{j=\max\{i,l\}+1}^{n} s_{i,j}$ then, for $0 \le k \le n-2,\ k \le l \le n-1$,*

$$q_{k,l}(t_1, t_2) = \frac{\beta_{k,l} c_{k,l,n} F^k(t_1)(F(t_2) - F(t_1))^{l-k} \bar{F}^{n-l}(t_2)}{\sum_{i=0}^{n-2} \sum_{j=i}^{n-1} \beta_{i,j} c_{i,j,n} F^i(t_1)(F(t_2) - F(t_1))^{j-i} \bar{F}^{n-j}(t_2)}.$$

(iii) *If $\gamma_{k,l} = \sum_{i=1}^{k} \sum_{j=l+1}^{n} s_{i,j}$ then, for $1 \le k \le l \le n-1$,*

$$r_{k,l}(t_1, t_2) = \frac{\gamma_{k,l} c_{k,l,n} F^k(t_1)(F(t_2) - F(t_1))^{l-k} \bar{F}^{n-l}(t_2)}{\sum_{i=1}^{n-1} \sum_{j=i}^{n-1} \gamma_{i,j} c_{i,j,n} F^i(t_1)(F(t_2) - F(t_1))^{j-i} \bar{F}^{n-j}(t_2)}.$$

Proof. We prove part (i). The proofs of (ii) and (iii) are similar to (i) and, hence, are omitted. First, we should note that the event $(N(t) = r)$ is equivalent to $(X_{r:n} \le t < X_{r+1:n})$. Hence, we can write

$$p_{k,l}(t_1, t_2) = \frac{\mathbb{P}(T_1 > t_1, T_1 < t_2 < T, X_{k:n} \le t_1 < X_{k+1:n}, X_{l:n} \le t_2 < X_{l+1:n})}{\mathbb{P}(t_1 < T_1 < t_2, T > t_2)}.$$

We have

$$\begin{aligned}
&\mathbb{P}(T_1 > t_1, T_1 < t_2 < T, X_{k:n} \le t_1 < X_{k+1:n}, X_{l:n} \le t_2 < X_{l+1:n}) \\
&\quad = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathbb{P}(T_1 = X_{i:n}, T = X_{j:n}) \mathbb{P}(T_1 > t_1, T_1 < t_2 < T, X_{k:n} \le t_1 < X_{k+1:n}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad X_{l:n} \le t_2 < X_{l+1:n} \mid T_1 = X_{i:n}, T = X_{j:n}) \\
&\quad = \sum_{i=k+1}^{l} \sum_{j=l+1}^{n} s_{i,j} \mathbb{P}(X_{i:n} > t_1, X_{i:n} < t_2 < X_{j:n}, X_{k:n} \le t_1 < X_{k+1:n}, \\
&\qquad\qquad\qquad X_{l:n} \le t_2 < X_{l+1:n}) \\
&\quad = \sum_{i=k+1}^{l} \sum_{j=l+1}^{n} s_{i,j} \mathbb{P}(X_{k:n} \le t_1 < X_{k+1:n}, X_{l:n} \le t_2 < X_{l+1:n}) \\
&\quad = c_{k,l,n} F^k(t_1)(F(t_2) - F(t_1))^{l-k} \bar{F}^{n-l}(t_2) \sum_{i=k+1}^{l} \sum_{j=l+1}^{n} s_{i,j} \\
&\quad = \alpha_{k,l} c_{k,l,n} F^k(t_1)(F(t_2) - F(t_1))^{l-k} \bar{F}^{n-l}(t_2), \qquad (1)
\end{aligned}$$

where the second equality is obtained from the fact that the event $\{T_1 = X_{i:n}, T = X_{j:n}\}$ does not depend on the distribution of the component lifetimes and it only depends on the network structure. On the other hand, it can be shown that, for $0 \le t_1 < t_2$,

$$\mathbb{P}(X_{i:n} > t_1, X_{i:n} < t_2 < X_{j:n}) = \sum_{k=0}^{i-1} \sum_{l=i}^{j-1} c_{k,l,n} F^k(t_1)(F(t_2) - F(t_1))^{l-k} \bar{F}^{n-l}(t_2). \qquad (2)$$

Hence, from (2) and similar to the steps we used to obtain (1), it can be shown that

$$\mathbb{P}(T_1 > t_1, T_1 < t_2 < T) = \sum_{k=0}^{n-2} \sum_{l=k+1}^{n-1} \alpha_{k,l} c_{k,l,n} F^k(t_1)(F(t_2) - F(t_1))^{l-k} \bar{F}^{n-l}(t_2).$$

Thus, the proof is complete. □

Remark 1. Alternative forms to represent the conditional probabilities in the above theorem are, respectively, as follows:

$$p_{k,l}(t_1, t_2) = \frac{\alpha_{k,l} c_{k,l,n} \varphi^k(t_1, t_2) \xi^l(t_1, t_2)}{\sum_{k=0}^{n-2} \sum_{l=k+1}^{n-1} \alpha_{k,l} c_{k,l,n} \varphi^k(t_1, t_2) \xi^l(t_1, t_2)}, \qquad 0 \le k < l \le n-1,$$

$$q_{k,l}(t_1, t_2) = \frac{\beta_{k,l} c_{k,l,n} \varphi^k(t_1, t_2) \xi^l(t_1, t_2)}{\sum_{k=0}^{n-1} \sum_{l=k}^{n-1} \beta_{k,l} c_{k,l,n} \varphi^k(t_1, t_2) \xi^l(t_1, t_2)}, \qquad 0 \le k \le l \le n-1,$$

$$r_{k,l}(t_1, t_2) = \frac{\gamma_{k,l} c_{k,l,n} \varphi^k(t_1, t_2) \xi^l(t_1, t_2)}{\sum_{k=1}^{n-1} \sum_{l=k}^{n-1} \gamma_{k,l} c_{k,l,n} \varphi^k(t_1, t_2) \xi^l(t_1, t_2)}, \qquad 1 \le k \le l \le n-1,$$

where $\varphi(t_1, t_2) = F(t_1)/(F(t_2) - F(t_1))$ and $\xi(t_1, t_2) = (\bar{F}(t_1) - \bar{F}(t_2))/\bar{F}(t_2)$.

In order to obtain the main results, we introduce the matrices $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ for which the nonzero elements are, respectively, defined as

$$\alpha_{k,l} = \sum_{i=k+1}^{l} \sum_{j=l+1}^{n} s_{i,j}, \qquad 0 \le k < l \le n-1, \tag{3}$$

$$\beta_{k,l} = \sum_{i=k+1}^{n-1} \sum_{j=\max\{i,l\}+1}^{n} s_{i,j}, \qquad 0 \le k \le n-2,\ k \le l \le n-1, \tag{4}$$

$$\gamma_{k,l} = \sum_{i=1}^{k} \sum_{j=l+1}^{n} s_{i,j}, \qquad 1 \le k \le l \le n-1. \tag{5}$$

We also introduce probability matrices $\mathcal{P}(t_1, t_2)$, $\mathcal{Q}(t_1, t_2)$, and $\mathcal{R}(t_1, t_2)$ such that their elements are defined, respectively, as $p_{k,l}(t_1, t_2)$ for $0 \le k < l \le n-1$, $q_{k,l}(t_1, t_2)$ for $0 \le k \le n-2,\ k \le l \le n-1$, and $r_{k,l}(t_1, t_2)$ for $1 \le k \le l \le n-1$.

Remark 2. Throughout this paper, it is clear that the calculation of the conditional probabilities and verification of the conditions in the theorems depend on the calculations of the two-dimensional signature. It should be mentioned that the computation of the two-dimensional signature, in networks with a large number of components, is rather involved and, hence, a challenging problem. In recent years, attempts have been made in the literature to propose alternative methods for assessing the two-dimensional signature based on computational algorithms or approaches such as decomposition of the systems to subsystems. For different computational methods of the one-dimensional signature, we refer the reader to, for example, [5], [12], [14], and [24]. A computational algorithm for the two-dimensional signature was discussed by Gertsbakh and Shpungin [13]. The authors have proposed a Monte Carlo procedure to approximate the two-dimensional signature in an n-component network (see also [20]). Da and Hu [4] proposed an efficient method for computing the two-dimensional signature for the n-component systems consisting of independent modules. Using computational approaches mentioned above, one can calculate the quantities related to the two-dimensional signature $s_{i,j}$ such as $\alpha_{i.j}$, $\beta_{i,j}$, and $\gamma_{i,j}$.

Let us consider the following example.

Example 1. Consider the bridge network with the structure shown in Figure 1. This network consists of four nodes, s, a, b, t and we assume the nodes s and t are terminals. The network includes five links, 1, 2, 3, 4, 5 that are subject to failure and each link has capacity one.

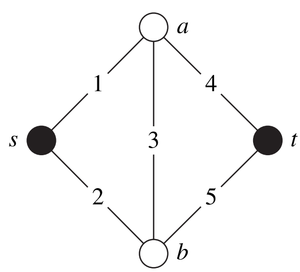

FIGURE 1: The bridge network.

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We define the states of the network as the maximal flow that can be transferred from s to t. Obviously, the network is in state $K = 0$ if there is no connection between s and t. We assume that it is in state $K = 1$ if a link among the links 1, 2, 4, and 5 fails, and is in state $K = 2$ if either all five links function or link 3 fails and the other links function. Hence, the positive elements of the signature matrix associated to this network are as follows (for details of the calculations, see [2]):

$$s_{1,2} = \tfrac{1}{5}, \qquad s_{1,3} = \tfrac{7}{15}, \qquad s_{1,4} = \tfrac{2}{15}, \qquad s_{2,3} = \tfrac{2}{15}, \qquad s_{2,4} = \tfrac{1}{15}.$$

Hence, it can be shown that the nonzero elements of $\mathcal{A}$ are given as

$$\alpha_{0,1} = \tfrac{4}{5}, \qquad \alpha_{0,2} = \tfrac{4}{5}, \qquad \alpha_{0,3} = \tfrac{1}{5}, \qquad \alpha_{1,2} = \tfrac{1}{5}, \qquad \alpha_{1,3} = \tfrac{1}{15}.$$

Let the link lifetimes be independent exponential random variables with mean 1. Then we have $\varphi(t_1, t_2) = (1 - \mathrm{e}^{-t_1})/(\mathrm{e}^{-t_1} - \mathrm{e}^{-t_2})$ and $\xi(t_1, t_2) = \mathrm{e}^{-(t_1 - t_2)} - 1$ and, hence, the matrix of failure probabilities $\mathcal{P} = (p_{k,l}(t_1, t_2))$, $k = 0, 1, 2, 3$, $l = 1, 2, 3, 4$, can be written as

$$\begin{pmatrix} \dfrac{2}{c(t_1,t_2)} & \dfrac{4\xi(t_1,t_2)}{c(t_1,t_2)} & \dfrac{\xi^2(t_1,t_2)}{c(t_1,t_2)} & 0 \\ 0 & \dfrac{2\varphi(t_1,t_2)\xi(t_1,t_2)}{c(t_1,t_2)} & \dfrac{\varphi(t_1,t_2)\xi^2(t_1,t_2)}{c(t_1,t_2)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$c(t_1, t_2) = 2 + 4\xi(t_1, t_2) + \xi^2(t_1, t_2) + 2\varphi(t_1, t_2)\xi(t_1, t_2) + \varphi(t_1, t_2)\xi^2(t_1, t_2).$$

In Figures 2 and 3 we present the plots of $p_{1,2}(t_1, t_2)$ and $p_{1,3}(t_1, t_2)$, respectively. From Figure 2, we see that $p_{1,2}(t_1, t_2)$ is increasing in t_1 and decreasing in t_2. Also, in Figure 3, we see that $p_{1,3}(t_1, t_2)$ as a function of t_1 has a maximum and as a function of t_2 is increasing.

In the following we give results that compare the probabilities of the number of failed components in two networks. Before that, we need to state the following definitions. Definition 1(i) is a discrete version of the totally positive order presented in [18]. Also, in the following definition, for any x and y, we employ the notation $x \wedge y$ for $\min\{x, y\}$ and the notation $x \vee y$ for $\max\{x, y\}$. For more details about stochastic orderings, see [29].

Definition 1. (i) Let $P = (p_{ij})$ and $Q = (q_{ij})$, $i = 1, \dots, n$, $j = 1, \dots, m$, be two nonnegative matrices. We say that P is less than Q in the totally positive order (denoted by $P \leq_{\mathrm{TP}} Q$) if, for every $i_1, i_2 = 1, \dots, n$, $j_1, j_2 = 1, \dots, m$,

$$p_{i_1,j_1} q_{i_2,j_2} \leq p_{(i_1,j_1)\wedge(i_2,j_2)} q_{(i_1,j_1)\vee(i_2,j_2)},$$

where $(i_1, j_1) \wedge (i_2, j_2) = (i_1 \wedge i_2, j_1 \wedge j_2)$ and $(i_1, j_1) \vee (i_2, j_2) = (i_1 \vee i_2, j_1 \vee j_2)$.

(ii) Let A and B be two subsets on $(-\infty, \infty)$ and $\mathcal{K}$ be a nonnegative function defined on $A \times B$. We say that $\mathcal{K}$ is totally positive of order 2 (TP_2) if, for all $a_1 < a_2$, $b_1 < b_2$, ($a_i \in A$, $b_i \in B$, $i = 1, 2$),

$$\mathcal{K}(a_2, b_2)\mathcal{K}(a_1, b_1) \geq \mathcal{K}(a_1, b_2)\mathcal{K}(a_2, b_1).$$

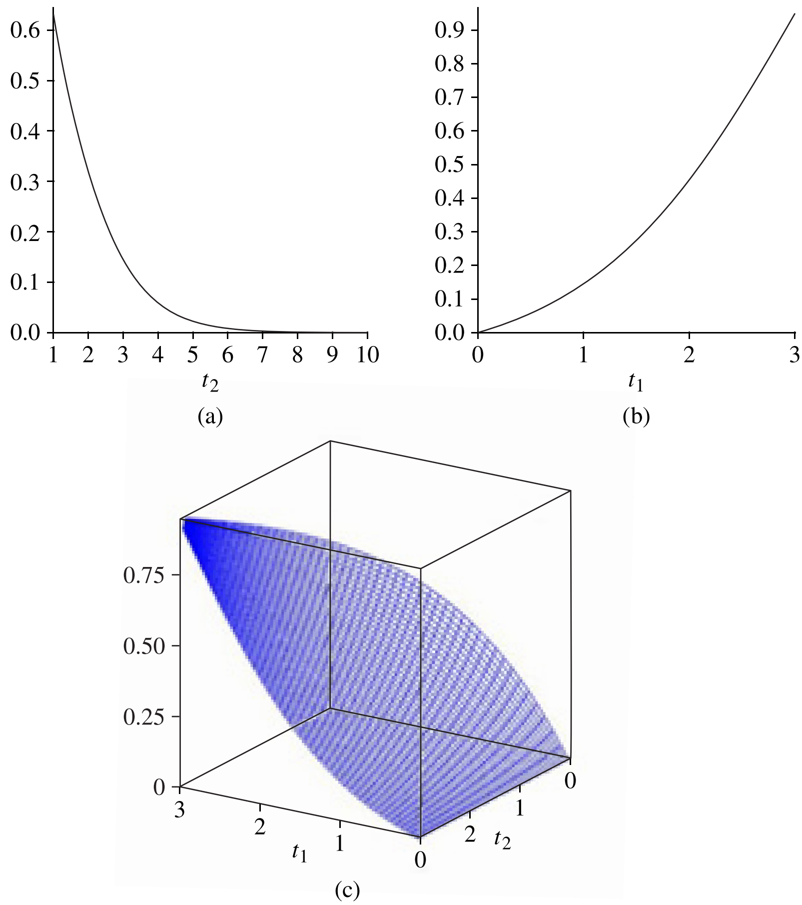

FIGURE 2: The plots of (a) $p_{12}(1, t_2)$, (b) $p_{12}(t_1, 3)$, and (c) $p_{12}(t_1, t_2)$.

If, in Definition 1(i), we assume that $P = (p_{ij})$ and $Q = (q_{ij})$ are probability matrices, then P is said to be less than Q in the likelihood ratio order, denoted by $P \leq_{\mathrm{lr}} Q$.

Definition 2. Let the random variable X, respectively Y, have distribution function F, respectively G, survival functions $\bar{F}$, respectively $\bar{G}$, and density function f, respectively g.

(i) We say that F is less than G in the hazard rate order, denoted by $F \leq_{\mathrm{hr}} G$, if $\bar{G}(x)/\bar{F}(x)$ is an increasing function of x.

(ii) We say that F is less than G in the reversed hazard rate order, denoted by $F \leq_{\mathrm{rh}} G$, if $G(x)/F(x)$ is an increasing function of x.

(iii) We say that F is less than G in the likelihood ratio order, denoted by $F \leq_{\mathrm{lr}} G$, if $g(x)/f(x)$ is an increasing function of x.

Theorem 2. *Consider two networks each including n i.i.d. components, where the component lifetimes of the two networks have the same distribution function. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the corresponding signature matrices and the matrices $\mathcal{A}_i$, $\mathcal{B}_i$, and $\mathcal{C}_i$, $i = 1, 2$, have the corresponding elements as defined in (3), (4), and (5), $i = 1, 2$, respectively. Suppose that*

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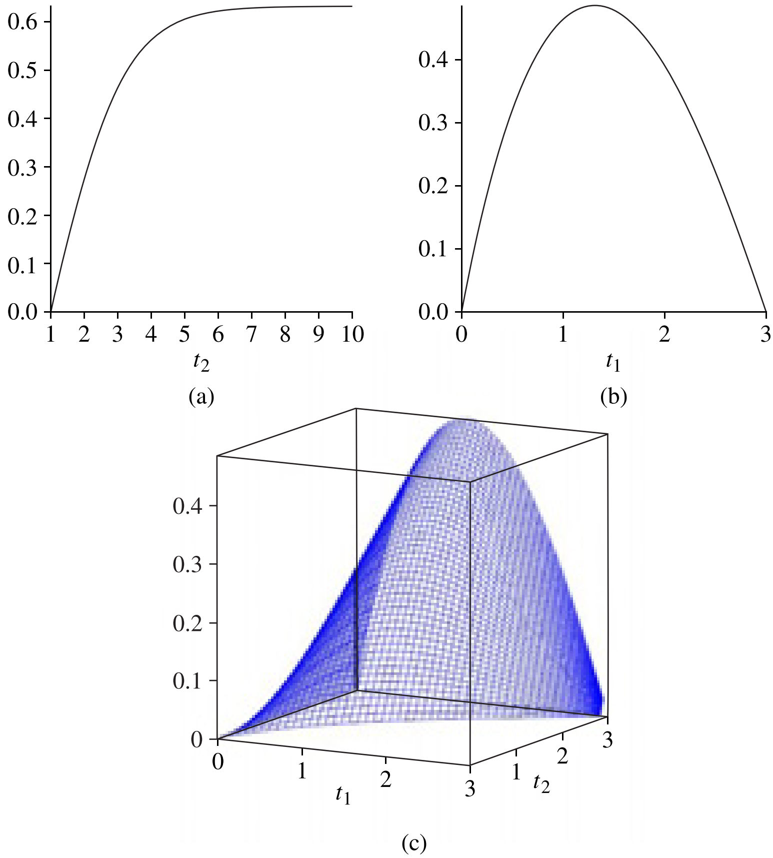

FIGURE 3: The plots of (a) $p_{13}(1, t_2)$, (b) $p_{13}(t_1, 3)$, and (c) $p_{13}(t_1, t_2)$.

$\mathcal{P}_i(t_1, t_2)$, $\mathcal{Q}_i(t_1, t_2)$, and $\mathcal{R}_i(t_1, t_2)$ are the probability matrices corresponding to $\mathcal{A}_i$, $\mathcal{B}_i$, and $\mathcal{C}_i$, $i = 1, 2$, respectively.

(i) *If $\mathcal{A}_1 \le_{\mathrm{TP}} \mathcal{A}_2$ then $\mathcal{P}_1(t_1, t_2) \le_{\mathrm{lr}} \mathcal{P}_2(t_1, t_2)$.*

(ii) *If $\mathcal{B}_1 \le_{\mathrm{TP}} \mathcal{B}_2$ then $\mathcal{Q}_1(t_1, t_2) \le_{\mathrm{lr}} \mathcal{Q}_2(t_1, t_2)$.*

(iii) *If $\mathcal{C}_1 \le_{\mathrm{TP}} \mathcal{C}_2$ then $\mathcal{R}_1(t_1, t_2) \le_{\mathrm{lr}} \mathcal{R}_2(t_1, t_2)$.*

Proof. We prove part (i). The proofs of (ii) and (iii) are similar and, hence, are omitted. Suppose that $\mathcal{P}_i(t_1, t_2)$ has the elements $p_{i,k,l}(t_1, t_2)$, $i = 1, 2$. To prove the result, we need to show that, for every $k_1, k_2 \in \{0, \dots, n-2\}$, $l_1, l_2 \in \{1, \dots, n-1\}$,

$$p_{1,k_1,l_1}(t_1, t_2)\, p_{2,k_2,l_2}(t_1, t_2) \le p_{1,k_1\wedge k_2,l_1\wedge l_2}(t_1, t_2)\, p_{2,k_1\vee k_2,l_1\vee l_2}(t_1, t_2).$$

It is equivalent to show that

$$\alpha_{1,k_1,l_1} c_{k_1,l_1,n} \alpha_{2,k_2,l_2} c_{k_2,l_2,n} \le \alpha_{1,k_1\wedge k_2,l_1\wedge l_2} c_{k_1\wedge k_2,l_1\wedge l_2,n} \alpha_{2,k_1\vee k_2,l_1\vee l_2} c_{k_1\vee k_2,l_1\vee l_2,n}.$$

This inequality holds from the fact that $c_{k,l,n}$ is TP in k and l and $\mathcal{A}_1 \le_{\mathrm{TP}} \mathcal{A}_2$. □

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Example 2. Consider again the network shown in Figure 1 and assume that links are subject to failure. Let us define the following two cases for the network states.

(i) First, assume that the network states are as described in Example 1. In this case, we have derived the nonzero elements of $\mathcal{A}$ in that example. It can be also shown that the nonzero elements of $\mathcal{B}$ and $\mathcal{C}$ are, respectively, given as

$$\beta_{0,0} = 1, \qquad \beta_{0,1} = 1, \qquad \beta_{0,2} = \tfrac{4}{5}, \qquad \beta_{0,3} = \tfrac{1}{5},$$
$$\beta_{1,1} = \tfrac{1}{5}, \qquad \beta_{1,2} = \tfrac{1}{5}, \qquad \beta_{1,3} = \tfrac{1}{15},$$
$$\gamma_{1,1} = \tfrac{4}{5}, \qquad \gamma_{1,2} = \tfrac{3}{5}, \qquad \gamma_{1,3} = \tfrac{2}{15}, \qquad \gamma_{2,2} = \tfrac{4}{5}, \qquad \gamma_{2,3} = \tfrac{1}{5}, \qquad \gamma_{3,3} = \tfrac{1}{5}.$$

(ii) Let us now assume that the nodes s, t, and b are terminals. In this case, we suppose that the network remains in state $K = 2$ if the three terminals are connected, it is in state $K = 1$ if two terminals among three are connected, and, finally, it is in state $K = 0$ if the three terminals are disconnected. For instance, when links 4 and 5 fail, terminals s and b still have a connection and, hence, the network state is $K = 1$. The positive elements of the signature matrix for such a network are

$$s^*_{2,4} = \tfrac{2}{15}, \qquad s^*_{2,5} = \tfrac{1}{15}, \qquad s^*_{3,4} = \tfrac{7}{15}, \qquad s^*_{3,5} = \tfrac{7}{30}, \qquad s^*_{4,5} = \tfrac{1}{10}.$$

It can be shown that the nonzero elements of $\mathcal{A}^*$, $\mathcal{B}^*$, and $\mathcal{C}^*$ are, respectively, given as

$$\alpha^*_{0,2} = \tfrac{1}{5}, \qquad \alpha^*_{0,3} = \tfrac{9}{10}, \qquad \alpha^*_{0,4} = \tfrac{2}{5},$$
$$\alpha^*_{1,2} = \tfrac{1}{5}, \qquad \alpha^*_{1,3} = \tfrac{9}{10}, \qquad \alpha^*_{1,4} = \tfrac{2}{5},$$
$$\alpha^*_{2,3} = \tfrac{7}{10}, \qquad \alpha^*_{2,4} = \tfrac{1}{3}, \qquad \alpha^*_{3,4} = \tfrac{1}{10},$$
$$\beta^*_{0,0} = \beta^*_{0,1} = \beta^*_{0,2} = \beta^*_{0,3} = \beta^*_{1,1} = \beta^*_{1,2} = \beta^*_{1,3} = 1, \qquad \beta^*_{0,4} = \beta^*_{1,4} = \tfrac{2}{5},$$
$$\beta^*_{2,2} = \beta^*_{2,3} = \tfrac{4}{5}, \qquad \beta^*_{2,4} = \tfrac{1}{3}, \qquad \beta^*_{3,3} = \beta^*_{3,4} = \tfrac{1}{10},$$
$$\gamma^*_{2,2} = \tfrac{1}{5}, \qquad \gamma^*_{2,3} = \tfrac{1}{5}, \qquad \gamma^*_{2,4} = \tfrac{1}{15}, \qquad \gamma^*_{3,3} = \tfrac{9}{10}, \qquad \gamma^*_{3,4} = \tfrac{3}{10}, \qquad \gamma^*_{4,4} = \tfrac{2}{5}.$$

Let $\mathcal{P}(t_1, t_2)$, $\mathcal{Q}(t_1, t_2)$, and $\mathcal{R}(t_1, t_2)$ be the probability matrices of the number of failed components of the network presented in (i) and $\mathcal{P}^*(t_1, t_2)$, $\mathcal{Q}^*(t_1, t_2)$, and $\mathcal{R}^*(t_1, t_2)$ be the probability matrices corresponding to the network described in (ii). Then, we have shown using MATLAB® software that $\mathcal{A} \leq_{\mathrm{TP}} \mathcal{A}^*$, $\mathcal{B} \leq_{\mathrm{TP}} \mathcal{B}^*$, and $\mathcal{C} \leq_{\mathrm{TP}} \mathcal{C}^*$. Thus, from Theorem 2, we conclude that $\mathcal{P}(t_1, t_2) \leq_{\mathrm{lr}} \mathcal{P}^*(t_1, t_2)$, $\mathcal{Q}(t_1, t_2) \leq_{\mathrm{lr}} \mathcal{Q}^*(t_1, t_2)$, and $\mathcal{R}(t_1, t_2) \leq_{\mathrm{lr}} \mathcal{R}^*(t_1, t_2)$.

In the following theorem, under some conditions on the distribution functions of the component lifetimes of two networks, we compare the probabilities of the number of failed components of the networks.

Theorem 3. *Consider two networks each including n i.i.d. components with distribution functions F_1 and F_2, respectively. Assume that the two networks have the same structure and $\mathcal{P}_i(t_1, t_2)$, $\mathcal{Q}_i(t_1, t_2)$, and $\mathcal{R}_i(t_1, t_2)$ are the probability matrices corresponding to F_i, $i = 1, 2$. Let $F_1 \leq_{\mathrm{lr}} F_2$.*

(i) *If $\alpha_{k,l}$ is* TP_2 *in k and l then $\mathcal{P}_1(t_1, t_2) \geq_{\mathrm{lr}} \mathcal{P}_2(t_1, t_2)$.*

(ii) *If $\beta_{k,l}$ is* TP_2 *in k and l then $\mathcal{Q}_1(t_1, t_2) \geq_{\mathrm{lr}} \mathcal{Q}_2(t_1, t_2)$.*

(iii) *If $\gamma_{k,l}$ is* TP_2 *in k and l then $\mathcal{R}_1(t_1, t_2) \geq_{\mathrm{lr}} \mathcal{R}_2(t_1, t_2)$.*

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Proof. We only prove the result in (i). Parts (ii) and (iii) can be proved similarly. We have shown in Remark 1 that the elements of matrix $\mathcal{P}_i(t_1, t_2)$ can be written as

$$p_{i,k,l}(t_1, t_2) = \frac{\alpha_{k,l} c_{k,l,n} \varphi_i^k(t_1, t_2) \xi_i^l(t_1, t_2)}{\sum_{k=0}^{n-2} \sum_{l=k+1}^{n-1} \alpha_{k,l} c_{k,l,n} \varphi_i^k(t_1, t_2) \xi_i^l(t_1, t_2)}, \qquad 0 \le k < l \le n-1,\ 0 \le t_1 < t_2,$$

where $\varphi_i(t_1, t_2) = F_i(t_1)/(F_i(t_2) - F_i(t_1))$ and $\xi_i(t_1, t_2) = (\bar{F}_i(t_1) - \bar{F}_i(t_2))/\bar{F}_i(t_2),\ i = 1, 2$. From Theorem 1.C.1 of [29], if $F_1 \le_{\text{lr}} F_2$ then $F_1 \le_{\text{hr}} F_2$ and $F_1 \le_{\text{rh}} F_2$. It can be seen that if $F_1 \le_{\text{hr}} F_2$ then $\xi_1(t_1, t_2) \ge \xi_2(t_1, t_2)$ and if $F_1 \le_{\text{rh}} F_2$ then $\varphi_1(t_1, t_2) \ge \varphi_2(t_1, t_2)$, where $t_1 < t_2$. Hence, for any $k_1, k_2, l_1, l_2 \in \{1, \dots, n\}$ and $t_1 < t_2$, we have

$$\begin{aligned} &\varphi_1^{k_1}(t_1, t_2)\xi_1^{l_1}(t_1, t_2)\varphi_2^{k_2}(t_1, t_2)\xi_2^{l_2}(t_1, t_2) \\ &\qquad \le \varphi_1^{(k_1 \vee k_2)}(t_1, t_2)\xi_1^{(l_1 \vee l_2)}(t_1, t_2)\varphi_2^{(k_1 \wedge k_2)}(t_1, t_2)\xi_2^{(l_1 \wedge l_2)}(t_1, t_2). \end{aligned} \tag{6}$$

Thus, from the fact that $c_{k,l,n}$ is TP_2 in k and l and the assumption that $\alpha_{k,l}$ is TP_2 in k and l, we conclude, from (6), that, for every $k_1, k_2 \in \{0, \dots, n-2\}$, $l_1, l_2 \in \{1, \dots, n-1\}$,

$$p_{1,k_1,l_1}(t_1, t_2)p_{2,k_2,l_2}(t_1, t_2) \le p_{1,k_1 \vee k_2, l_1 \vee l_2}(t_1, t_2)p_{2,k_1 \wedge k_2, l_1 \wedge l_2}(t_1, t_2), \qquad t_1 < t_2.$$

Thus, the proof is completed. □

In the following example we provide an application of the above theorem.

Example 3. Consider again Example 2. Suppose that two networks have the same structures as Example 2(ii). Let the link lifetimes of the networks have exponential distributions with survival functions

$$\bar{F}_1(t) = \mathrm{e}^{-\lambda_1 t}, \quad t > 0,\ \lambda_1 > 0, \qquad \bar{F}_2(t) = \mathrm{e}^{-\lambda_2 t}, \quad t > 0,\ \lambda_2 > 0.$$

We see that if $\lambda_2 < \lambda_1$ then $F_1 \le_{\text{lr}} F_2$. Using MATLAB, it can be shown that $\alpha_{k,l}$, $\beta_{k,l}$, and $\gamma_{k,l}$ are TP_2 in k and l. Then we conclude from Theorem 3 that for the bridge network, under the given conditions, $\mathcal{P}_1(t_1, t_2) \ge_{\text{lr}} \mathcal{P}_2(t_1, t_2)$, $\mathcal{Q}_1(t_1, t_2) \ge_{\text{lr}} \mathcal{Q}_2(t_1, t_2)$, and $\mathcal{R}_1(t_1, t_2) \ge_{\text{lr}} \mathcal{R}_2(t_1, t_2)$.

Harris [16] defined the notion of increasing failure rate (IFR) for the bivariate random vectors in the continuous setting. In the following definition, a discrete version of the concept of bivariate increasing failure rate (BIFR) is given.

Definition 3. Let $p_{i,j}$, $i, j = 0, 1, \dots$, be the bivariate mass function with survival function $\bar{P}_{i,j}$. We say that $p_{i,j}$ is BIFR if $\bar{P}_{i,j}$ is TP_2 in i and j and $\bar{P}_{i+m,\, j+m}/\bar{P}_{i,j}$ is decreasing in i and j for any $m = 1, 2, \dots$.

In order to prove our next theorem, we need the following lemma.

Lemma 1. *Let $\omega(t_1, t_2) = F(t_1)/\bar{F}(t_2)$, $0 < t_1 < t_2$, and $\alpha_{i,j}$ be as defined in (3). Suppose that*

$$h(i, j) = \alpha_{i,j} c_{i,j,n} \omega^i(t_1, t_2)(\omega(t_2, t_2) - \omega(t_1, t_2))^{j-i},$$

$g(k, l) = \sum_{j=l}^{n-1} h(k, j)$, $l > k$, and $g^*(k, l) = \sum_{i=k}^{l-1} h(i, l)$. If $\alpha_{k,l}$ is TP_2 in k and l and $\alpha_{k+1,\, l+1}/\alpha_{k,l}$ is decreasing in k and l, where $0 \le k < l \le n-1$, then $g(k+1, l+1)/g(k, l)$ is decreasing in k and $g^*(k+1, l+1)/g^*(k, l)$ is decreasing in l.

Proof. From the assumption that $\alpha_{k,l}$ is TP_2 in k and l, it can be seen that $\alpha_{k+1,l}/\alpha_{k,l}$ is increasing in l. Also, the assumptions that $\alpha_{k,l}$ is TP_2 in k and l and $\alpha_{k+1,l+1}/\alpha_{k,l}$ is decreasing in k imply that $\alpha_{k+1,l}/\alpha_{k,l}$ is decreasing in k. We have

$$\begin{aligned}
&g(k+1,l+1)g(k+1,l) - g(k+2,l+1)g(k,l) \\
&\quad = \sum_{j=l+1}^{n-1}\sum_{i=l}^{n-1}[h(k+1,j)h(k+1,i) - h(k+2,j)h(k,i)] \\
&\quad = \sum_{j=l}^{n-2}\sum_{i=l}^{n-1}[h(k+1,j+1)h(k+1,i) - h(k+2,j+1)h(k,i)] \\
&\quad = \sum_{j=l}^{n-2}[h(k+1,j+1)h(k+1,n-1) - h(k+2,j+1)h(k,n-1)] \\
&\qquad + \sum_{j=l}^{n-2}[h(k+1,j+1)h(k+1,j) - h(k+2,j+1)h(k,j)] \\
&\qquad + \sum_{j=l}^{n-3}\sum_{i=j+1}^{n-2}[h(k+1,j+1)h(k+1,i) + h(k+1,i+1)h(k+1,j) \\
&\qquad\qquad - h(k+2,j+1)h(k,i) - h(k+2,i+1)h(k,j)]. \qquad (7)
\end{aligned}$$

First, we show that the first summation in (7) is nonnegative. Note that $\alpha_{k+1,l}/\alpha_{k,l}$ and $c_{k+1,l,n}/c_{k,l,n}$ are decreasing in k and increasing in l. Therefore, we have

$$\begin{aligned}
h_1^*(k,j,n) &= \alpha_{k+1,j+1}c_{k+1,j+1,n}\alpha_{k+1,n-1}c_{k+1,n-1,n} - \alpha_{k+2,j+1}c_{k+2,j+1,n}\alpha_{k,n-1}c_{k,n-1,n} \\
&\geq 0,
\end{aligned}$$

which implies that, for $0 < t_1 < t_2$,

$$\begin{aligned}
&h(k+1,j+1)h(k+1,n-1) - h(k+2,j+1)h(k,n-1) \\
&\quad = h_1^*(k,j,n)\omega^{2k+2}(t_1,t_2)(\omega(t_2,t_2) - \omega(t_1,t_2))^{j+n-2k-2} \\
&\quad \geq 0.
\end{aligned}$$

Now we show that the second summation in (7) is nonnegative. Note that $c_{k+1,j+1,n}/c_{k,j,n}$ is decreasing in k. Thus, from the assumption that $\alpha_{k+1,j+1}/\alpha_{k,j}$ is decreasing in k, we have

$$h_2^*(k,j,n) = \alpha_{k+1,j+1}c_{k+1,j+1,n}\alpha_{k+1,j}c_{k+1,j,n} - \alpha_{k+2,j+1}c_{k+2,j+1,n}\alpha_{k,j}c_{k,j,n} \geq 0,$$

and, hence, for $0 < t_1 < t_2$,

$$\begin{aligned}
&h(k+1,j+1)h(k+1,j) - h(k+2,j+1)h(k,j) \\
&\quad = h_2^*(k,j,n)\omega^{2k+2}(t_1,t_2)(\omega(t_2,t_2) - \omega(t_1,t_2))^{2j-2k-1} \\
&\quad \geq 0.
\end{aligned}$$

Finally, we show that the last summation in (7) is nonnegative. In order to do so, let

$$\begin{aligned}
a &= h(k+2,j+1)h(k,i), & b &= h(k+2,i+1)h(k,j), \\
c &= h(k+1,i+1)h(k+1,j), & d &= h(k+1,j+1)h(k+1,i).
\end{aligned}$$

It can be shown that $d \geq a$ because $\alpha_{k+1,j}/\alpha_{k,j}$ and $c_{k+1,j,n}/c_{k,j,n}$ are decreasing in k and increasing in j. Also, since $\alpha_{k+1,\,j+1}/\alpha_{k,j}$ and $c_{k+1,\,j+1,n}/c_{k,j,n}$ are decreasing in k and j, it can be seen that $d \geq b$ and $cd \geq ab$. Therefore, we can write $d + c - a - b = (1/d)[(d-a)(d-b) + cd - ab] \geq 0$, which implies that

$$\begin{aligned} &h(k+1, j+1)h(k+1, i) + h(k+1, i+1)h(k+1, j) - h(k+2, j+1)h(k, i) \\ &\quad - h(k+2, i+1)h(k, j) \\ &\quad \geq 0. \end{aligned}$$

Hence, we conclude that

$$g(k+1, l+1)g(k+1, l) \geq g(k+2, l+1)g(k, l),$$

which implies that $g(k+1, l+1)/g(k, l)$ is decreasing in k. The proof for the case that $g^*(k+1, l+1)/g^*(k, l)$ is decreasing in l is the same as the above and, hence, is omitted. Thus, the proof is completed. □

Using Lemma 1, we can prove the following result.

Theorem 4. *Let $\alpha_{k,l}$, $\beta_{k,l}$, and $\gamma_{k,l}$ be defined as in (3), (4), and (5), respectively.*

(i) *If $\alpha_{k,l}$ is TP_2 in k, l and $\alpha_{k+1,\,l+1}/\alpha_{k,l}$ is decreasing in k and l, then $p_{k,l}(t_1, t_2)$, $0 \leq k < l \leq n-1$, is BIFR.*

(ii) *If $\beta_{k,l}$ is TP_2 in k, l and $\beta_{k+1,\,l+1}/\beta_{k,l}$ is decreasing in k and l, then $q_{k,l}(t_1, t_2)$, $0 \leq k \leq n-2$, $k \leq l \leq n-1$, is BIFR.*

(iii) *If $\gamma_{k,l}$ is TP_2 in k, l and $\gamma_{k+1,l+1}/\gamma_{k,l}$ is decreasing in k and l, then $r_{k,l}(t_1, t_2)$, $1 \leq k \leq l \leq n-1$, is BIFR.*

Proof. We prove (i). Parts (ii) and (iii) can be proved similarly and, hence, are omitted. Consider $h(i, j)$ and $g(i, j)$ as defined in Lemma 1 and $\bar{P}_{k,l}(t_1, t_2)$ as the survival function of $p_{k,l}(t_1, t_2)$, which is given by

$$\bar{P}_{k,l}(t_1, t_2) = \frac{\sum_{i=k+1}^{n-2} \sum_{j=\max\{l,i\}+1}^{n-1} h(i, j)}{\sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} h(i, j)}.$$

In order to prove the theorem, it is enough to prove that $\bar{P}_{k+1,\,l+1}(t_1, t_2)/\bar{P}_{k,l}(t_1, t_2)$ is decreasing in k, l for every $k, l \in \{0, 1, \ldots, n-2\}$. First, we assume that $l \geq k+1$ and show that $\bar{P}_{k+1,l+1}(t_1, t_2)/\bar{P}_{k,l}(t_1, t_2)$ is decreasing in k. That is, for $l \geq k+1$, we show that

$$\bar{P}_{k+1,\,l+1}(t_1, t_2)\bar{P}_{k+1,\,l}(t_1, t_2) - \bar{P}_{k+2,\,l+1}(t_1, t_2)\bar{P}_{k,l}(t_1, t_2) \geq 0. \tag{8}$$

Let $m = \max\{i+1, l+2\}$ and $m^* = \max\{r, l\}$. It is equivalent to show that

$$\sum_{i=k+2}^{n-2} \sum_{j=m}^{n-1} \sum_{r=k+2}^{n-2} \sum_{s=m^*+1}^{n-1} h(i, j)h(r, s) - \sum_{i=k+3}^{n-2} \sum_{j=m}^{n-1} \sum_{r=k+1}^{n-2} \sum_{s=m^*+1}^{n-1} h(i, j)h(r, s) \geq 0.$$

On the other hand, we have

$$\sum_{i=k+2}^{n-2}\sum_{j=m}^{n-1}\sum_{r=k+2}^{n-2}\sum_{s=m^*+1}^{n-1} h(i,j)h(r,s) - \sum_{i=k+3}^{n-2}\sum_{j=m}^{n-1}\sum_{r=k+1}^{n-2}\sum_{s=m^*+1}^{n-1} h(i,j)h(r,s)$$

$$= \sum_{j=l+1}^{n-2}\sum_{r=k+2}^{n-2}\sum_{s=m^*+1}^{n-1} h(k+2, j+1)h(r,s)$$

$$- \sum_{j=l+1}^{n-1}\sum_{r=k+2}^{n-3}\sum_{s=m^*+1}^{n-2} h(r+1, s+1)h(k+1, j)$$

$$= g(k+2, l+2)h(n-2, n-1)$$

$$+ \sum_{r=l+1}^{n-3}\sum_{s=r+1}^{n-2}\sum_{j=l+1}^{r} [h(k+2, j+1)h(r,s) - h(r+1, s+1)h(k+1, j)] \tag{9}$$

$$+ \sum_{r=k+2}^{n-3} [g(k+2, m^*+2)g(r, m^*+1) - g(r+1, m^*+2)g(k+1, m^*+1)] \tag{10}$$

$$+ \sum_{r=k+2}^{n-3} [g(k+2, l+2) - g(k+2, m^*+2)]h(r, n-1). \tag{11}$$

It can be shown that $h(i+1, j+1)/h(i,j)$ is decreasing in i and j because $\alpha_{i+1,j+1}/\alpha_{i,j}$ and $c_{i+1,j+1,n}/c_{i,j,n}$ are decreasing in i and j. Thus, (9) is nonnegative. From Lemma 1, $g(k+1, l+1)/g(k,l)$ is decreasing in k, which implies that (10) is nonnegative. Also, (11) is nonnegative because $g(k,l)$ is decreasing in l. Therefore, we have the inequality in (8), i.e. $\bar{P}_{k+1,l+1}(t_1,t_2)/\bar{P}_{k,l}(t_1,t_2)$ is decreasing in k. Using similar steps, one can show that when $l \le k$, $\bar{P}_{k+1,l+1}(t_1,t_2)/\bar{P}_{k,l}(t_1,t_2)$ is also decreasing. We omit the proof that $\bar{P}_{k+1,l+1}(t_1,t_2)/\bar{P}_{k,l}(t_1,t_2)$ is decreasing in l as it is the same. Finally, it can be shown that if $\alpha_{k,l}$ is TP_2 in k and l then $p_{k,l}(t_1,t_2)$ is TP_2 in k and l, which, in turn, implies that $\bar{P}_{k,l}(t_1,t_2)$ is TP_2 in k and l. This completes the proof of theorem. □

Example 4. Gertsbakh and Shpungin [13] considered a network with five nodes and ten links shown in Figure 4. The authors assumed that the links are subject to failure and defined the states of the network as follows. If all nodes are in connection, the network is in state $K=2$, if nodes are separated into two disjoint sets, it is in state $K=1$, and if the nodes are divided into at least three disjoint sets, it is in state $K=0$.

They estimated the positive elements of the signature matrix ($\mathcal{S}$) as

$$s_{4,7}=0.0047,\quad s_{4,8}=0.0194,\quad s_{5,7}=0.0191,\quad s_{5,8}=0.0751,$$
$$s_{6,7}=0.0596,\quad s_{6,8}=0.227,\quad s_{7,8}=0.5951.$$

It can be shown that the nonzero elements of the estimated matrix $\mathcal{A}$ are as follows. For $k=0,1,2,3$,

$$\alpha_{k,4}=0.0241,\quad \alpha_{k,5}=0.1183,\quad \alpha_{k,6}=0.4049,\quad \alpha_{k,7}=0.9166,$$
$$\alpha_{4,5}=0.0942,\quad \alpha_{4,6}=0.3808,\quad \alpha_{4,7}=0.8972,$$
$$\alpha_{5,6}=0.2866,\quad \alpha_{5,7}=0.8221,\quad \alpha_{6,7}=0.5951.$$

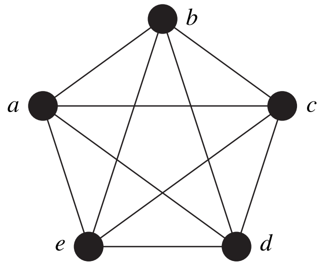


FIGURE 4: Network with five terminals.

Also, the positive elements of the estimated matrix $\mathcal{B}$ are as follows: $\beta_{k,l} = 1$ for $k = 0, 1, 2, 3, l = 0, \ldots, 6, l \geq k$, and

$$\beta_{k,7} = 0.9166, \quad k = 0, 1, 2, 3, \qquad \beta_{4,l} = 0.9759, \quad l = 4, 5, 6,$$
$$\beta_{4,7} = 0.8972, \qquad \beta_{5,5} = \beta_{5,6} = 0.8817, \qquad \beta_{5,7} = 0.8221, \qquad \beta_{6,6} = \beta_{6,7} = 0.5951.$$

Using MATLAB, it can be seen that $\alpha_{k,l}$ and $\beta_{k,l}$ are TP_2 in k and l. Also, it follows that $\alpha_{k+1,l+1}/\alpha_{k,l}$ is decreasing in k and l, where $k = 0, 1, \ldots, 5$, and $l = 4, 5, 6, 7, l > k$. On the other hand, $\beta_{k+1,l+1}/\beta_{k,l}$ is decreasing in k and l, where $k = 0, 1, \ldots, 5$, and $l = 0, 1, \ldots, 6, l \geq k$. Thus, from Theorem 4, $p_{k,l}(t_1, t_2)$ and $q_{k,l}(t_1, t_2)$ are BIFR.

3. Optimal age replacement problems

In this section we present two optimal age replacement policies in order to provide some illustrative examples as applications of the conditional probabilities given in Section 2.

Policy 1. In the first policy, we deal with a single-step three-state n-component network described in Section 2. We assume that an operator has inspected the network at time t_1 and he/she has realized that the network is in the state $K = 1$. Let the operator consider another inspection time t_2 after t_1. It is clear that at t_2 the network would be either in state $K = 1$ or it has already failed before t_2. If the network has failed before t_2, the operator decides to replace all components of the network by new ones. If the network is in state $K = 1$ at time t_2, then the operator decides just to replace the failed components by new ones. Now, an interesting problem is to find the optimum replacement time t_2^* that minimizes the mean cost per unit of time. It is clear that t_2^* depends on t_1. We define a cost function as follows. Let c_1 be the cost of replacement of a component by a new one, c_2 be the cost of inspection of a nonfailed component, and c_3 be the cost of network failure. Suppose also that $\psi(t_1, t_2) = \mathbb{E}(N(t_2) \mid T_1 < t_1, T > t_2)$. Then it can be seen from Theorem 1(iii) that

$$\psi(t_1, t_2) = \frac{\sum_{l=1}^{n-1} \sum_{k=1}^{l} l c_{k,l,n} \gamma_{k,l} F^k(t_1)(F(t_2) - F(t_1))^{l-k} \bar{F}^{n-l}(t_2)}{\sum_{i=1}^{n-1} \sum_{j=i}^{n} c_{i,j,n} \gamma_{i,j} F^i(t_1)(F(t_2) - F(t_1))^{j-i} \bar{F}^{n-j}(t_2)}, \qquad 0 < t_1 < t_2.$$

If the network has failed before the inspection time t_2 then the total cost is $(nc_1 + c_3)$. If, at time t_2, it is in state $K = 1$ then the total expected cost is $(\psi(t_1, t_2)c_1 + (n - \psi(t_1, t_2))c_2)$. Hence, the expected cost per unit of time can be written as

$$\begin{aligned} \eta_1(t_1, t_2) &= \frac{\mathbb{P}(T < t_2 \mid T_1 < t_1 < T)(nc_1 + c_3) + \mathbb{P}(T > t_2 \mid T_1 < t_1 < T)(\psi(t_1, t_2)c_1 + (n - \psi(t_1, t_2))c_2)}{\mathbb{E}(\min\{T, t_2\} \mid T_1 < t_1 < T)} \\ &= \frac{\mathbb{P}(T_1 < t_1, t_1 < T < t_2)(nc_1 + c_3) + \mathbb{P}(T_1 < t_1, T > t_2)(\psi(t_1, t_2)c_1 + (n - \psi(t_1, t_2))c_2)}{\mathbb{P}(T_1 < t_1 < T)\mathbb{E}(\min\{T, t_2\} \mid T_1 < t_1 < T)}, \end{aligned} \qquad (12)$$

where

$$\mathbb{E}(\min\{T, t_2\} \mid T_1 < t_1 < T) = t_1 + \frac{\int_{t_1}^{t_2} \mathbb{P}(T_1 < t_1, y < T < t_2)\,\mathrm{d}y + (t_2 - t_1)\mathbb{P}(T_1 < t_1, T > t_2)}{\mathbb{P}(T_1 < t_1 < T)}.$$

It should be noted that the event $\{T_1 = X_{i:n}, T = X_{j:n}\}$ does not depend on the distribution function of the component lifetimes and only depends on the network structure. Hence, by applying the law of total probability, the probabilities in the cost function in (12) can be written as

$$\begin{aligned}\mathbb{P}(T_1 < t_1, t_1 < T < t_2) &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathbb{P}(T_1 = X_{i:n}, T = X_{j:n})\mathbb{P}(T_1 < t_1, t_1 < T < t_2 \mid T_1 = X_{i:n}, T = X_{j:n}) \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} s_{i,j}\mathbb{P}(X_{i:n} < t_1, t_1 < X_{j:n} < t_2).\end{aligned}$$

Using the same argument, it can be seen that

$$\mathbb{P}(T_1 < t_1, T > t_2) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} s_{i,j}\mathbb{P}(X_{i:n} < t_1, X_{j:n} > t_2)$$

and

$$\mathbb{P}(T_1 < t_1 < T) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} s_{i,j}\mathbb{P}(X_{i:n} < t_1 < X_{j:n}).$$

In order to give the next example, we need the following remark.

Remark 3. Suppose that the network is inspected at time t and it is observed that the network is in state $K = 1$. In this situation, one might be interested in the following probability:

$$p_i^*(t) = \mathbb{P}(N(t) = i \mid T_1 < t < T).$$

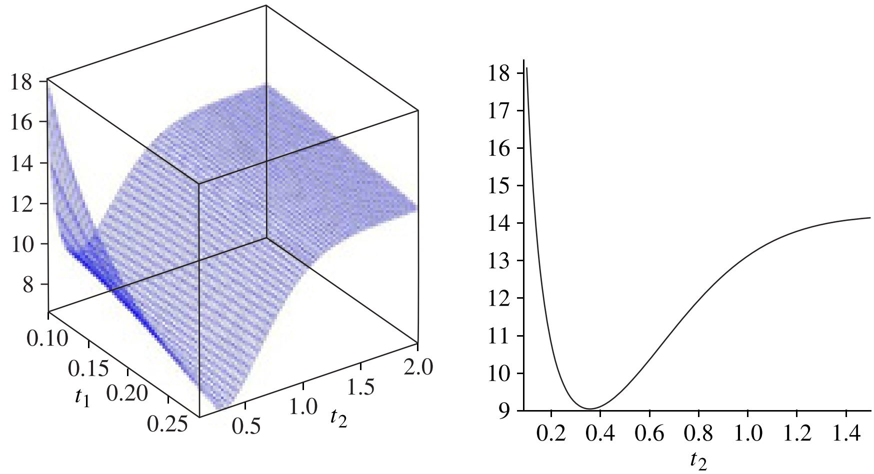
Consider $p_{k,l}(t_1, t_2)$ as defined in Theorem 1(i). Then, using the fact that $p_i^*(t) = p_{0,l}(0, t)$, it can be seen that

$$p_i^*(t) = \frac{a_i \binom{n}{i} \phi^i(t)}{\sum_{j=1}^{n-1} a_j \binom{n}{j} \phi^j(t)}, \qquad i = 1, \dots, n-1, \tag{13}$$

where $\phi(t) = F(t)/\bar{F}(t)$ and $a_r = \sum_{i=1}^{r} \sum_{j=r+1}^{n} s_{i,j}$. Asadi and Berred [1] studied several properties of type $p_i^*(t)$. If one decides to replace the failed components by new ones at time t, then the expected cost per unit of time can be written as

$$\eta_2(t) = \frac{\mathbb{E}(N(t) \mid T_1 < t < T)(c_1 - c_2) + nc_2}{t},$$

where $\mathbb{E}(N(t) \mid T_1 < t < T) = \sum_{i=1}^{n-1} i p_i^*(t)$ and $p_i^*(t)$ is defined in (13). It can be seen that if, in policy 1, $t_1 = t_2$ then $\eta_1(t_1, t_2) = \eta_2(t_1)$.



(a) (b)

FIGURE 5: (a) The plot of $\eta_1(t_1, t_2)$ and (b) the plot of $\eta_1(0.1, t_2)$ for $c_1 = 1$, $c_2 = 0.2$, $c_3 = 5$.

In the next example, the optimal time t_2^* is obtained for different costs.

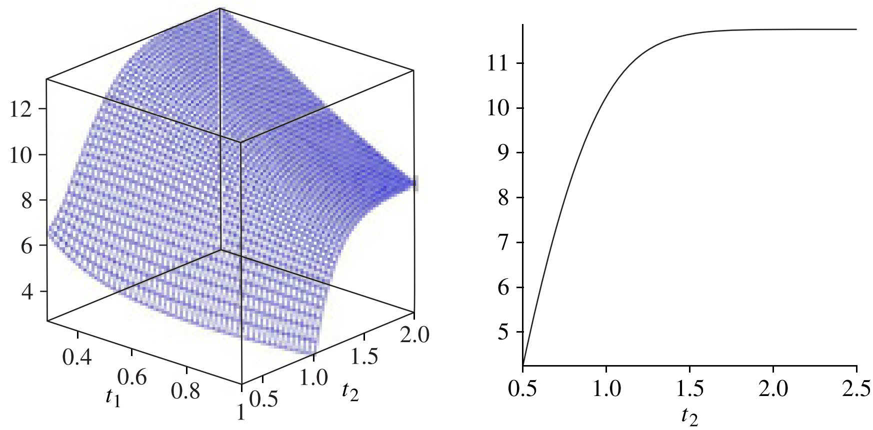
Example 5. Consider again the network in Example 1. Assume that the link lifetimes are independent having Weibull distribution with reliability function $\bar{F}(t) = \mathrm{e}^{-t^2}$. In Table 1 we present the optimal time t_2^* that minimizes the expected cost per unit of time and $\eta_1(t_1, t_2^*)$ for several time instants t_1 and different costs. We see that if the costs c_1 or c_2 increase then the optimal time t_2^* increases. Also, it can be seen that when c_3 increases then t_2^* decreases.

Suppose that $c_1 = 1$, $c_2 = 0.2$, and $c_3 = 5$. It can be seen from Figure 5(a) that when $t_1 < 0.29$, the plot of $\eta_1(t_1, t_2)$ as a function of t_2 has a minimum after time t_1 (see also Figure 5(b)). However, when $t_1 \geq 0.29$ the plot of $\eta_1(t_1, t_2)$ is increasing in t_2 where its minimum occurs at time t_1, i.e. $t_2^* = t_1$ (see also Figures 6(a) and 6(b)).

It can be shown that $\mathbb{E}(T_1) = 0.443\,113$. It can be seen from Table 1, for several costs that are presented, if $t_1 = 0.443\,113$ then $t_2^* = 0.443\,113$.

TABLE 1: Optimal values of replacement at time t_2.

t_1	t_2^*	$\eta_1(t_1, t_2^*)$	t_1	t_2^*	$\eta_1(t_1, t_2^*)$
$c_1 = 1, c_2 = 0.2, c_3 = 5$			$c_1 = 1.5, c_2 = 0.2, c_3 = 5$		
0.100 000	0.357 475	9.041 675 245	0.100 000	0.359 082 6	11.659 186 830
0.200 000	0.330 775	8.101 239 037	0.200 000	0.331 330 0	10.476 724 580
0.443 113	0.443 113	4.657 975 252	0.443 113	0.443 113 0	6.158 734 352
1.000 000	1.000 000	2.664 435 858	1.000 000	1.000 000 0	3.704 708 269
$c_1 = 1, c_2 = 0.3, c_3 = 5$			$c_1 = 1, c_2 = 0.2, c_3 = 10$		
0.100 000	0.401 617 65	9.896 697 384	0.100 000	0.290 752 0	10.710 265 700
0.200 000	0.378 427 50	9.063 875 726	0.200 000	0.256 627 3	8.973 165 093
0.443 113	0.443 113 00	5.486 203 777	0.443 113	0.443 113 0	4.657 975 252
1.000 000	1.000 000 00	2.956 381 375	1.000 000	1.000 000 0	2.664 435 858



(a) (b)

FIGURE 6: (a) The plot of $\eta_1(t_1, t_2)$ and (b) the plot of $\eta_1(0.5, t_2)$ for $c_1 = 1$, $c_2 = 0.2$, $c_3 = 5$.

In the following, we consider another optimal age replacement problem.

Policy 2. Assume that the network inspection time is t. Clearly, at time t the state of the network is either $K = 2$, $K = 1$, or it has failed before t. If the network has failed before t, the operator decides to replace all components by new ones. If it is in state $K = 1$ then just the failed components are replaced and if it is in state $K = 2$, the operator does not replace any components. Let c_1 be the cost of replacing a component by a new component, c_2 be the cost of inspection of a nonfailed component, c_3 be the cost of the network failure, and c_4 be the cost of inspection of the network when it is in state $K = 2$. Then the mean cost per unit of time can be written as

$$\eta_3(t) = \frac{\mathbb{P}(T < t)(nc_1 + c_3) + (\mathbb{P}(T > t) - \mathbb{P}(T_1 > t))(\psi_2(t)c_1 + (n - \psi_2(t))c_2) + \mathbb{P}(T_1 > t)c_4}{\mathbb{E}(\min\{t, T\})},$$

where $\psi_2(t) = \mathbb{E}(N(t) \mid T_1 < t < T)$ and $\mathbb{E}(\min\{t, T\}) = \int_0^t \mathbb{P}(T > x)\,\mathrm{d}x$. It can be seen that

$$\mathbb{P}(T_1 > t) = \sum_{i=1}^{n} s_i^{(1)} \mathbb{P}(X_{i:n} > t), \qquad \mathbb{P}(T > t) = \sum_{i=1}^{n} s_i^{(2)} \mathbb{P}(X_{i:n} > t),$$

where $\boldsymbol{s}^{(1)} = (s_1^{(1)}, \dots, s_n^{(1)})$ and $\boldsymbol{s}^{(2)} = (s_1^{(2)}, \dots, s_n^{(2)})$ are the marginal signature vectors corresponding to signature matrix $\mathcal{S}$, i.e. $s_i^{(1)} = \mathbb{P}(T_1 = X_{i:n})$ and $s_i^{(2)} = \mathbb{P}(T = X_{i:n})$. As an application of this policy, we have the following example.

Example 6. Consider again the network described in Example 2(ii). Let the link lifetimes have Weibull distribution with reliability function $\bar{F}(t) = \mathrm{e}^{-t^2}$. In Table 2 we present the optimal time t^* that minimizes the expected cost per unit of time and $\eta_3(t^*)$ for different costs. From Table 2, we can see that the optimal time t^* is a decreasing function of costs c_1, c_2, or c_3. Also, when $c_1 = 1$, $c_2 = 0.2$, and $c_3 = 5$, we see that when c_4 increases then t^* increases.

TABLE 2: Optimal values of replacement at time t^*.

c_4	t^*	$\eta_3(t^*)$	c_3	t^*	$\eta_3(t^*)$
$c_1 = 1, c_2 = 0.2, c_3 = 5$			$c_1 = 1, c_2 = 0.2, c_4 = 1$		
0.5	0.414 339	1.580 615 480	5	0.508 938	2.566 309 515
1.0	0.508 938	2.566 309 515	7	0.499 970	2.592 069 484
1.5	0.583 638	3.337 885 659	10	0.488 905	2.626 482 095
$c_1 = 1, c_3 = 5, c_4 = 1$			$c_2 = 0.2, c_3 = 5, c_4 = 1$		
0.2	0.508 938	2.566 309 515	1.0	0.508 938	2.566 309 515
0.5	0.488 236	2.721 760 332	1.5	0.459 862	2.850 504 647
0.7	0.475 294	2.818 013 962	2.0	0.427 409	3.068 872 716

4. Conclusions

In this paper we have dealt with a single-step three-state network with n components (links). The states of the network are considered as *up* ($K = 2$), *partial* performance ($K = 1$), and *down* ($K = 0$). The network starts to operate at time $t = 0$ where it is in state $K = 2$. The network is assumed to stay for a random time T_1 in state $K = 2$ and then moves to state $K = 1$. The lifetime of the network is denoted by a random variable T. We have assumed that the network is inspected at two time instants t_1 and t_2 ($t_1 < t_2$). Under different conditions on the state lifetimes T_1 and T, we have calculated the probabilities of the number of components that have failed in the network in terms of the signature matrix $\mathcal{S}$ and the common distribution of component lifetimes. The calculated probabilities have been compared, in terms of likelihood ratio order, for two different networks where their corresponding signature matrices are ordered in terms of totally positive order. Conditions on the signature matrix under which the calculated probabilities are bivariate increasing failure rates have been studied. Under some age replacement policies, two applications of the proposed conditional probabilities have been presented. Throughout the paper, we have investigated the three-state networks. However, we should mention that extension of the results to general networks with higher states is an interesting and, of course, a challenging problem which is under consideration by the authors as a topic of future research.

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