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On the linearized Whitham–Broer–Kaup system on bounded domains

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We consider the system of partial differential equations

$$\begin{cases} \eta_t - \alpha u_{xxx} - \beta \eta_{xx} = 0 \\ u_t + \eta_x + \beta u_{xx} = 0 \end{cases}$$

on bounded domains, known in the literature as the Whitham–Broer–Kaup system. The well-posedness of the problem, under suitable boundary conditions, is addressed, and it is shown to depend on the sign of the number

$$\varkappa = \alpha - \beta^2.$$

In particular, existence and uniqueness occur if and only if $\varkappa > 0$. In which case, an explicit representation for the solutions is given. Nonetheless, for the case $\varkappa \leqslant 0$ we have uniqueness in the class of strong solutions, and sufficient conditions to guarantee exponential instability are provided.

Keywords: Whitham–Broer–Kaup system; dispersive equations; spectrum; linear semigroups

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1. Introduction

1.1. The model system

Given the parameters $\alpha \neq 0$ and $\beta \in \mathbb{R}$, we consider the system of linear partial differential equations

$$\begin{cases} \eta_t - \alpha u_{xxx} - \beta \eta_{xx} = 0, \\ u_t + \eta_x + \beta u_{xx} = 0, \end{cases} \tag{1.1}$$

ruling the evolution of the variables,

$$\eta = \eta(x,t) : [0,\pi] \times \mathbb{R}^+ \to \mathbb{R} \quad \text{and} \quad u = u(x,t) : [0,\pi] \times \mathbb{R}^+ \to \mathbb{R},$$

subject to the boundary conditions

$$\eta(0,t) = \eta(\pi,t) = 0 \quad \text{and} \quad u_x(0,t) = u_x(\pi,t) = 0. \tag{1.2}$$

The system is supplemented with the initial conditions

$$\eta(x,0) = \phi(x) \quad \text{and} \quad u(x,0) = \psi(x), \tag{1.3}$$

where ϕ and ψ are assigned functions. In this presentation, the interval $[0, \pi]$ is clearly fungible, and could be replaced by any interval $[a, b]$.

System (1.1) arises in the theoretical treatment of hydrodynamics, where it appears as the linearization of a renowned model for the propagation of shallow water waves. In this context, η represents the height surface of the wave, while u is the speed of propagation. Although the general model was first introduced by Kuperschmidt in 1985 [**13**] as a generalization of the Boussinesq system, a number of particular instances were already well known in the literature. For this reason, (1.1) is often referred to as the Whitham–Broer–Kaup (WBK) system, after the names of the ones who first derived and analysed it for specific values of α and β (see [**7**, **12**, **18**]).

Nowadays, the topic is deserving some attention: there is a wide recent literature on both the linear and the nonlinear versions of (1.1). Great part of the studies have been devoted to the search of solitons and travelling wave solutions (see, e.g., [**4**, **14**, **16**, **19**–**21**] and references therein), but some effort has also been directed towards numerical investigations [**2**, **3**, **8**, **9**]. In spite of this number of contributions, however, very few papers are concerned with the well-posedness of the general system, and with the decay properties (if any) of the general solution. One recent example are the works [**5**, **6**] where, among other results, the authors show that system (1.1) is ill-posed when $\alpha \leqslant \beta^2$.

Without exception, all of these articles share the common assumption that the variables involved are defined on the whole real line $\mathbb{R}$. Indeed, the analysis on unbounded domains is quite common in the theory of hydrodynamics, for it allows to investigate many interesting phenomena, ranging from the existence of solitons to the dispersive properties of the model. Notwithstanding, albeit being certainly useful for applicative purposes, the unboundedness hypothesis is a modelling simplification, that does not comply with the real (and structurally bounded) world. For this reason, it is of paramount importance to provide a solid mathematical justification of the model by thoroughly investigating the physical case of bounded domains. This is exactly the purpose of this work. Indeed, our aim is to describe the WBK system's properties in this novel setting. Which thing poses a main problem, since the Fourier transform is no longer applicable. Another issue is the correct choice of the boundary conditions. Letting aside the undeniable interest from the mathematical physics point of view, a strong theoretical background for the bounded case is also pivotal for numerical applications.

1.2. The results

We perform a complete analysis of (1.1)–(1.3), in dependence of the structural parameters α and β. Defining the number

$$\varkappa = \alpha - \beta^2,$$

we show that the problem is well-posed if and only if

$$\varkappa > 0.$$

At first glance, this might seem surprising. While the first leading equation, when $\beta > 0$ and in absence of the coupling, is of diffusion type, the second one resembles a backward heat equation, which is known to be ill-posed. Or the other way around if $\beta < 0$. This seeming contradiction can be resolved upon a closer look to system (1.1). Indeed, although the nature of the model looks parabolic, the system actually conceals a wave-like (hence hyperbolic) structure. To uncover it, one can simply take the time derivative of the two equations. After some simplifications, one arrives at

$$\begin{cases} \eta_{tt} + \varkappa \eta_{xxxx} = 0, \\ u_{tt} + \varkappa u_{xxxx} = 0, \end{cases}$$

that is, two uncoupled wave equations, except with the Bilaplacian instead of the usual Laplace-Dirichlet operator [**5**].

When the system is well-posed, we are actually able to compute the exact solution to (1.1)–(1.3), whose corresponding energy turns out to be constant in time, in compliance with the wave-like nature of (1.1).

Nevertheless, for the ill-posed problem corresponding to $\varkappa \leqslant 0$, we can still prove uniqueness within the class of strong solutions, as well as exponential instability, by means of logarithmic convexity arguments. A particular situation occurs in the limit case $\varkappa = 0$, where, although the problem is generally ill-posed, we have existence and uniqueness of strong solutions.

Our approach heavily relies on the theory of linear semigroups. Specifically, having called $\boldsymbol{u}(t) = (\eta(t), u(t))$, we rephrase our system as the abstract ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}(t) = \mathbb{A}\boldsymbol{u}(t)$$

on a suitable Hilbert space $\mathcal{H}$. The well-posedness of the system strongly depends on the knowledge of the spectrum $\sigma(\mathbb{A})$ of $\mathbb{A}$, of which a detailed description is given,

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showing in particular that it is purely punctual whenever $\varkappa \neq 0$. Besides, when $\varkappa > 0$ the (normalized) eigenfunctions of $\mathbb{A}$ form a complete orthonormal basis of $\mathcal{H}$. This fact is quite remarkable, as the operator $\mathbb{A}$ is not selfadjoint. Incidentally, this is what allows us to write down the exact solutions to the system.

2. Preliminaries

2.1. Functional setting and notation

In what follows, the symbols $\langle\cdot,\,\cdot\rangle$ and $\|\cdot\|$ stand for the inner product and norm on the real Hilbert space of square summable functions $H = L^2(0,\,\pi)$. Besides, we denote the standard Sobolev spaces $H^j(0,\,\pi)$ and $H^1_0(0,\,\pi)$ simply by H^j and H^1_0. Finally, we define the Hilbert subspace H_* of H of zero-mean functions, i.e.,

$$H_* = \Big\{ v \in H : \int_0^\pi v(x)\,\mathrm{d}x = 0 \Big\},$$

and we set

$$H^1_* = H^1 \cap H_*.$$

2.2. Some operator theoretical issues

We recall some basic facts on the functional calculus of the Laplace-Dirichlet operator

$$A = -\partial_{xx} \text{ with domain } \mathfrak{D}(A) = H^2 \cap H^1_0,$$

that will be used in the sequel (see, e.g., [17]). It is well known that the spectrum $\sigma(A)$ of A is purely punctual. Being defined on the interval $[0,\,\pi]$, the eigenvalues of A are given by

$$\lambda_n = n^2, \quad n = 1, 2, 3, \ldots.$$

Besides, each λ_n is simple, and the corresponding (normalized) eigenfunction is

$$w_n(x) = \sqrt{\frac{2}{\pi}}\,\sin nx. \tag{2.1}$$

By the spectral theorem, the set $\{w_n\}_{n=1}^\infty$ forms a complete orthonormal basis of H. For every measurable function $F : \sigma(A) \to \mathbb{C}$, it is possible to define via the functional calculus the operator $F(A) : H \to H$ as

$$F(A)w = \sum_{n=1}^\infty F(n^2)\langle w, w_n\rangle, \tag{2.2}$$

with domain

$$\mathfrak{D}(F(A)) = \Big\{ w \in H : \sum_{n=1}^\infty |F(n^2)|^2 |\langle w, w_n\rangle|^2 < \infty \Big\}.$$

In particular, $F(A)$ is a bounded operator on H if and only if F is a bounded function on $\sigma(A)$. Finally, given two measurable functions G, F on $\sigma(A)$, the operator

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$G(A)F(A)$ acts as

$$G(A)F(A)w = \sum_{n=1}^{\infty} G(n^2)F(n^2)\langle w, w_n\rangle,$$

wherever is defined.

REMARK 2.1. We will also consider the complexification of A (that we keep denoting by A), namely, the operator on the complex Hilbert space $H \oplus iH$ acting as

$$A(u + iv) = -u_{xx} - iv_{xx},$$

with u, v real-valued functions.

3. The problem in abstract form

We introduce the phase space of our problem, namely, the product Hilbert space

$$\mathcal{H} = H \times H^1_*,$$

endowed with the scalar product

$$\langle(\eta, u), (\eta', u')\rangle_{\mathcal{H}} = \langle\eta, \eta'\rangle + \langle u_x, u'_x\rangle,$$

and norm

$$\|(\eta, u)\|^2_{\mathcal{H}} = \|\eta\|^2 + \|u_x\|^2.$$

REMARK 3.1. The norm in H^1_* does not contain the term $\|u\|$ due to the Poincaré inequality, which holds for zero-mean functions.

Defining the state vector

$$\boldsymbol{u}(t) = (\eta(t), u(t)),$$

we view (1.1)–(1.2) as the ODE in $\mathcal{H}$

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = \mathbb{A}\boldsymbol{u}(t).$$

Here, $\mathbb{A}$ is the linear operator on $\mathcal{H}$ acting as

$$\mathbb{A}\begin{pmatrix}\eta \\ u\end{pmatrix} = \begin{pmatrix}\alpha u_{xxx} + \beta\eta_{xx} & -\eta_x - \beta u_{xx}\end{pmatrix},$$

with (dense) domain

$$\mathfrak{D}(\mathbb{A}) = \left\{(\eta, u) \in \mathcal{H} \,\middle|\, \begin{array}{c}\alpha u_x + \beta\eta \in H^2 \cap H^1_0 \\ \eta + \beta u_x \in H^2 \cap H^1_0\end{array}\right\}.$$

As typically occurs for differential operators, it can be easily seen that $\mathbb{A}$ is a closed operator. Besides, observe that if $(\eta, u) \in \mathfrak{D}(\mathbb{A})$, then

$$\alpha u_{xxx} + \beta\eta_{xx} \in H \quad \text{and} \quad -\eta_x - \beta u_{xx} \in H^1_*.$$

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REMARK 3.2. When $\varkappa \neq 0$ the domain of $\mathbb{A}$ can be equivalently written as

$$\mathfrak{D}(\mathbb{A}) = \left\{ (\eta, u) \in \mathcal{H} \,\middle|\, \begin{array}{c} \eta \in H^2 \cap H_0^1 \\ u \in H^3 \\ u_x \in H_0^1 \end{array} \right\}.$$

REMARK 3.3. Instead, when $\varkappa = 0$,

$$\alpha u_x + \beta\eta = \beta(\eta + \beta u_x).$$

Accordingly, the two conditions in the definition of $\mathfrak{D}(\mathbb{A})$ coincide, and this does not allow to recover (1.2) any longer. Indeed, although in the previous literature the boundary conditions are always given in the form (1.2), even when $\varkappa = 0$, we believe that more correctly they should read

$$\begin{aligned} \alpha u_x(0,t) + \beta\eta(0,t) = \alpha u_x(\pi,t) + \beta\eta(\pi,t) = 0, \\ \eta(0,t) + \beta u_x(0,t) = \eta(\pi,t) + \beta u_x(\pi,t) = 0. \end{aligned} \tag{3.1}$$

In which case, (1.2) and (3.1) are the same if and only if $\varkappa \neq 0$.

4. The spectrum of $\mathbb{A}$

In this section, we provide a complete characterization of the spectrum $\sigma(\mathbb{A})$ of the (complexification of the) operator $\mathbb{A}$. We first consider the case $\varkappa \neq 0$. To this end, we define the parameter

$$\varrho = \begin{cases} i\sqrt{\varkappa} & \text{if } \varkappa > 0, \\ \sqrt{-\varkappa} & \text{if } \varkappa < 0. \end{cases}$$

The following theorem holds.

THEOREM 4.1. *Let $\varkappa \neq 0$, and let λ be a complex number such that*

$$\lambda \neq \pm\varrho n^2, \quad n = 1, 2, 3, \ldots.$$

Then λ belongs to the resolvent set $\rho(\mathbb{A})$ of $\mathbb{A}$.

Proof. Let $\lambda \in \mathbb{C}$ be fixed. For $\boldsymbol{f} = (f, g) \in \mathcal{H}$, we look for the unique solution to the functional equation

$$(\mathbb{A} - \lambda)\boldsymbol{u} = \boldsymbol{f}. \tag{4.1}$$

In components, this reads

$$\begin{cases} \alpha u_{xxx} + \beta\eta_{xx} - \lambda\eta = f, \\ -\eta_x - \beta u_{xx} - \lambda u = g. \end{cases}$$

Setting $v = u_x$, and taking the derivative of the second equation, we get

$$\begin{cases} \alpha v_{xx} + \beta\eta_{xx} - \lambda\eta = f, \\ -\eta_{xx} - \beta v_{xx} - \lambda v = g_x. \end{cases} \tag{4.2}$$

We now multiply the first equation of (4.2) by β, and the second one by α. Adding the results, we obtain

$$-\varkappa\eta_{xx} - \lambda(\beta\eta + \alpha v) = h, \tag{4.3}$$

where

$$h = \beta f + \alpha g_x \in H.$$

At this point, we introduce the new variable

$$\xi = \beta\eta + \alpha v,$$

and we consider the system made by (minus) the first equation of (4.2) and (4.3), to wit,

$$\begin{cases} A\xi + \lambda\eta = -f, \\ \varkappa A\eta - \lambda\xi = h, \end{cases} \tag{4.4}$$

where A is the Laplace-Dirichlet operator discussed in Subsection 2.2. Then, from the first equation above we learn that

$$\xi = -\lambda A^{-1}\eta - A^{-1}f. \tag{4.5}$$

Substituting (4.5) into the second equation of (4.4), we end up with

$$F(A)\eta = q, \tag{4.6}$$

having set

$$F(s) = \frac{\varkappa s^2 + \lambda^2}{s},$$

and

$$q = h - \lambda A^{-1}f \in H.$$

By the functional calculus of A, we conclude that (4.6) has a unique solution given by

$$\eta = G(A)q,$$

where

$$G(s) = \frac{s}{\varkappa s^2 + \lambda^2},$$

if and only if G is bounded on the spectrum of A. More explicitly, recalling (2.2),

$$\eta = \sum_{n=1}^{\infty} \frac{n^2}{\varkappa n^4 + \lambda^2}\langle q, w_n\rangle.$$

Thus, we have a unique solution for every $q \in H$ if and only if

$$\lambda^2 \neq -\varkappa n^4,$$

that is, if and only if

$$\lambda \neq \pm \varrho n^2.$$

In which case, since when $s \to +\infty$

$$G(s) \sim \frac{1}{\varkappa s},$$

the solution η belongs to the space $H^2 \cap H_0^1$. By (4.5), this implies that $\xi \in H^2 \cap H_0^1$. Finally, from the very definition of ξ, we read that $u_x \in H_0^1$ and $u \in H^3$. Summarizing, for every $\lambda \neq \pm\rho n^2$ we have found a unique solution $(\eta, u) \in \mathfrak{D}(\mathbb{A})$ to the resolvent equation (4.1). Hence, each of these λ belongs to the resolvent set $\rho(\mathbb{A})$. □

To complete the analysis, we show that the remaining values of λ are elements of the spectrum of $\mathbb{A}$.

THEOREM 4.2. *Let $\varkappa \neq 0$. Then all the numbers*

$$\lambda_n^{\pm} = \pm\, \varrho n^2, \quad n = 1, 2, 3, \ldots$$

are eigenvalues of $\mathbb{A}$. Besides, they are all simple.

Proof. For every $n = 1, 2, 3, \ldots$, let us define the functions

$$\boldsymbol{u}_n^{\pm}(x) = \Big(\alpha \sin nx, \beta \pm \rho/n \cos nx\Big). \tag{4.7}$$

It is apparent that

$$\boldsymbol{u}_n^{\pm} \in \mathfrak{D}(\mathbb{A}),$$

and a straightforward calculation reveals that

$$\mathbb{A}\boldsymbol{u}_n^{\pm} = \lambda_n^{\pm}\boldsymbol{u}_n^{\pm}.$$

This tells that each $\lambda_n^{\pm}$ is an eigenvalue of $\mathbb{A}$, and $\boldsymbol{u}_n^{\pm}$ is a corresponding eigenfunction. We are left to show that any other eigenfunction of $\lambda_n^{\pm}$ is a multiple of $\boldsymbol{u}_n^{\pm}$. To this end, let n be fixed, and let $\boldsymbol{v} = (\eta, u)$ be an eigenfunction of λ_n^{+} (the argument for λ_n^{-} is the same). Hence,

$$\mathbb{A}\boldsymbol{v} = \lambda_n^{+}\boldsymbol{v}.$$

Exploiting the same change of variables of the previous proof, we arrive at the system

$$\begin{cases} A\xi = -\lambda_n^{+}\eta, \\ \varkappa A\eta = \lambda_n^{+}\xi. \end{cases}$$

Then,

$$\xi = -\lambda_n^{+} A^{-1}\eta,$$

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and substituting this expression into the second equation above, we get

$$A^2\eta = n^4\eta.$$

Therefore, either $\eta = 0$, in which case $\xi = 0$, or η is an eigenfunction of the operator A^2 relative to the eigenvalue n^4. In the latter case, since the eigenvalues of A^2 are all simple (a straightforward consequence of the functional calculus of A), η is unique up to a multiplicative constant. More precisely, $\eta = w_n$, with w_n given by (2.1). But then ξ is unique as well (up to the same constant), and in turn so is u. In conclusion, $\boldsymbol{v} = \boldsymbol{u}_n^+$, up to a multiplicative constant, meaning that λ_n^+ is a simple eigenvalue. □

REMARK 4.3. The proof of the invertibility of $\mathbb{A}$, is actually simpler. Indeed, it amounts to consider system (4.4) with $\lambda = 0$. At that point, noting that the domain of $\mathbb{A}$ is compactly embedded into $\mathcal{H}$, one could conclude that the spectrum of $\mathbb{A}$ is made by eigenvalues only (and each with finite multiplicity). This is a direct consequence of a celebrated theorem of Kato [**11**, Th. 6.29], which states that the spectrum of a closed operator with compact inverse is purely punctual.

We finally turn to the case $\varkappa = 0$. Here the picture is much different.

THEOREM 4.4. *Let $\varkappa = 0$. Then the spectrum of $\mathbb{A}$ is the whole complex plane $\mathbb{C}$. Furthermore, 0 is the only eigenvalue of $\mathbb{A}$.*

Proof. Let $\lambda \in \mathbb{C}$ such that $\lambda \neq 0$ be fixed. We consider once again the resolvent equation

$$(\mathbb{A} - \lambda)\boldsymbol{u} = \boldsymbol{f},$$

for $\boldsymbol{f} = (f, g) \in \mathcal{H}$. We show that for some $\boldsymbol{f} \in \mathcal{H}$ the equation above has no solution in $\mathfrak{D}(\mathbb{A})$. By the same computations of the proof of theorem 4.2, we arrive at the system

$$\begin{cases} A\xi + \lambda\eta = -f, \\ -\lambda\xi = h, \end{cases} \tag{4.8}$$

with

$$h = \beta f + \alpha g_x \in H.$$

From (4.8) we immediately obtain

$$\xi = -\frac{h}{\lambda}.$$

In particular, this implies that ξ attains the same regularity of h. Thus, taking h in H but not more regular (e.g., $h \notin H^2$), the resolvent equation will have no solution in $\mathfrak{D}(\mathbb{A})$. On the other hand, if $\boldsymbol{f} = (0, 0)$, the unique solution is clearly the null one. Accordingly, λ is not an eigenvalue.

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Instead, when $\lambda = 0$, let $u \neq 0$ be any function such that $u \in H^3$ and $u_x \in H^1_0$, and define

$$\eta = -\beta u_x.$$

It is then apparent that $\eta \in H^2 \cap H^1_0$. Besides, since $\varkappa = 0$ implies $\alpha = \beta^2$, a quick check reveals that

$$\mathbb{A}\boldsymbol{u} = 0,$$

so that 0 is an eigenvalue, as claimed. □

Remark 4.5. When $\lambda \neq 0$, if f and h have enough regularity, system (4.8) has actually the unique solution

$$\eta = \frac{1}{\lambda^2}(Ah - \lambda f) \quad \text{and} \quad \xi = -\frac{h}{\lambda}.$$

This tells that the resolvent equation is solvable for a dense set of functions $\boldsymbol{f}$. In other words, the range of $(\mathbb{A} - \lambda)$ is dense in $\mathcal{H}$. Accordingly, every $\lambda \neq 0$ belongs to the continuous spectrum of $\mathbb{A}$.

5. The ill-posed problem

By the Hille–Yosida theorem (see, e.g., [**10**, **15**]), if a closed operator $\mathbb{A}$ is the infinitesimal generator of a strongly continuous semigroup of linear operators, then it has a finite spectral bound, that is,

$$\sigma_0 = \sup\left\{\Re\lambda : \lambda \in \sigma(\mathbb{A})\right\} < \infty.$$

But we know from the previous § 4 that this is never the case when $\varkappa \leqslant 0$. Thus, if $\varkappa \leqslant 0$ we do not have a well-posedness result for (1.1)–(1.3), at least not for every possible choice of the initial data $(\phi, \psi) \in \mathcal{H}$. Nevertheless, for the case $\varkappa < 0$ we can prove a uniqueness result for those solutions that belong to the domain of $\mathbb{A}$ for all times, generally called strong solutions.

Proposition 5.1. *Let $\varkappa < 0$. If (η, u) is a strong solution to (1.1)–(1.3) then it is unique.*

Proof. In light of the linearity of the problem, it is sufficient to show that the only strong solution corresponding to null initial data is the trivial one. To this end, let (η, u) be any strong solution to (1.1)–(1.3) with $(\phi, \psi) = (0, 0)$. Defining

$$\zeta(t) = \int_0^t \eta(s)\,\mathrm{d}s \quad \text{and} \quad v(t) = \int_0^t u(s)\,\mathrm{d}s,$$

and taking into account the null initial conditions, integrating (1.1) in time we get

$$\begin{cases} \zeta_t - \alpha v_{xxx} - \beta\zeta_{xx} = 0, \\ v_t + \zeta_x + \beta v_{xx} = 0, \end{cases} \tag{5.1}$$

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where $\zeta_t = \eta$ and $v_t = u$. Taking the derivative in space of the second equation of (5.1), after some straightforward manipulations we obtain

$$\zeta_{xx} = -\frac{1}{\varkappa}\big(\beta\eta + \alpha u_x\big),$$

which tells that

$$\zeta_{xx} \in H^2 \cap H^1_0.$$

Let us now introduce the functional

$$\mathsf{F}(t) = \frac{1}{2}\Big[\|\zeta_t(t)\|^2 + \varkappa\|\zeta_{xx}(t)\|^2\Big].$$

Considering the first equation of system (1.1), written in terms of ζ and v, along with the first equation of (5.1), we end up with

$$\zeta_{tt} + \varkappa\zeta_{xxxx} = 0. \tag{5.2}$$

A simple computation yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{F} = 0,$$

hence

$$\mathsf{F}(t) = \mathsf{F}(0) = 0. \tag{5.3}$$

At this point, we introduce the further functional

$$\mathsf{G}(t) = \frac{1}{2}\|\zeta(t)\|^2.$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{G} = \langle\zeta_t, \zeta\rangle.$$

Moreover, by (5.2) together with the fact that $\zeta_{xx} \in H^2 \cap H^1_0$, and exploiting (5.3),

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathsf{G}(t) = -\varkappa\langle\zeta_{xxxx}, \zeta\rangle + \|\zeta_t\|^2 = 2\|\zeta_t\|^2.$$

By the Cauchy–Schwarz inequality,

$$\mathsf{G}\ddot{\mathsf{G}} - \dot{\mathsf{G}}^2 \geqslant 0,$$

where we used the *dot* to denote the time derivative. This implies that G is a log-convex function of time. Accordingly, for any fixed interval $[0, T]$, we must have that

$$\mathsf{G}(t) \leqslant \mathsf{G}(0)^{1-\frac{t}{T}}\mathsf{G}(T)^{t/T}, \quad \forall\, t \in [0, T].$$

Since $\mathsf{G}(0) = 0$, we conclude that $\mathsf{G}(t)$ is zero for every t, implying that $\zeta = 0$. But this in turn implies that $\eta = 0$. Once we know that $\eta = 0$, the first equation of (1.1) simply becomes

$$u_{xxx} = 0,$$

which, as $u_{xxx} \in H^2 \cap H^1_0$, readily yields $u_x = 0$. Since $u \in H^1_*$, the latter equality implies that $u = 0$ as well. □

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Under certain conditions on the initial data, the unique strong solution, whenever exists, blows up exponentially fast as time goes to infinity.

PROPOSITION 5.2. *Let $\varkappa < 0$ and let $(\eta,\, u)$ be a strong solution to (1.1)–(1.3), for some initial data $(\phi,\, \psi)$. If either*

$$\|\beta\phi_{xx} + \alpha\psi_{xxx}\|^2 + \varkappa\|\phi_{xx}\|^2 < 0, \tag{5.4}$$

or

$$\|\phi_{xx} + \beta\psi_{xxx}\|^2 + \varkappa\|\psi_{xxx}\|^2 < 0, \tag{5.5}$$

then it follows that

$$\|(\eta(t), u(t))\|^2_{\mathcal{H}} \geqslant ce^{\nu t},$$

for some $c > 0$ and $\nu > 0$, both depending on $(\phi,\, \psi)$.

Proof. Let $(\eta,\, u)$ be a strong solution. Once we know from proposition 5.1 that it is unique, it is standard matter to approximate it with solutions as regular as needed. Accordingly, along this proof, we can work with functions regular enough to withstand the forthcoming calculations. Let us assume (5.4). We first define the counterpart of the functional F of the previous proof, that is,

$$\mathsf{F}_0(t) = \frac{1}{2}\Big[\|\eta_t(t)\|^2 + \varkappa\|\eta_{xx}(t)\|^2\Big].$$

Observe that $\mathsf{F}_0(t) = \mathsf{F}_0(0)$ for every t and

$$\mathsf{F}_0(0) = \frac{1}{2}\Big[\|\beta\phi_{xx} + \alpha\psi_{xxx}\|^2 + \varkappa\|\phi_{xx}\|^2\Big] < 0.$$

Besides, let

$$\mathsf{G}_0(t) = \frac{1}{2}\|\eta(t)\|^2 + \omega(t + t_0)^2,$$

where ω and t_0 are two strictly positive constants to be chosen later. By the same computations of the previous proof, just replacing $(\zeta,\, v)$ with $(\eta,\, u)$, we arrive at

$$\mathsf{G}_0\ddot{\mathsf{G}}_0 - \dot{\mathsf{G}}_0^2 \geqslant -2\big[\omega + \mathsf{F}_0(0)\big]\mathsf{G}_0^2.$$

Hence, taking $\omega = -\mathsf{F}_0(0)$, we get

$$\mathsf{G}_0\ddot{\mathsf{G}}_0 - \big(\dot{\mathsf{G}}_0\big)^2 \geqslant 0,$$

and a simple calculations yields (see, e.g., [**1**])

$$\mathsf{G}_0(t) \geqslant \mathsf{G}_0(0)e^{\dot{\mathsf{G}}_0(0)/\mathsf{G}_0(0)t}.$$

Choosing t_0 such that $\dot{\mathsf{G}}_0(0) > 0$ we have obtained that the norm of η (hence the norm of the solution) blows up exponentially fast. If instead (5.5) holds, in a completely similar way we can prove the exponential blow up of $\|u_x\|$. $\square$

REMARK 5.3. The norm of the solution blows up exponentially even in the case $\mathsf{F}_0(0) = 0$, provided that

$$\langle \phi, \beta\phi_{xx} + \alpha\psi_{xxx} \rangle > 0.$$

Indeed, recasting the proof of the proposition above with $\omega = 0$, we still end up with the final exponential estimate, and the condition above is precisely equivalent to $\dot{\mathsf{G}}_0(0) > 0$.

Our analysis is much more precise in the case $\varkappa = 0$, where we do have existence and uniqueness of strong solutions.

PROPOSITION 5.4. *Let $\varkappa = 0$. Then, system (1.1)–(1.3) has a unique solution for any $(\phi, \psi) \in \mathfrak{D}(\mathbb{A})$. Besides, such a solution is a strong one.*

Proof. Since $\alpha = \beta^2$, system (1.1) becomes

$$\begin{cases} \eta_t - \beta^2 u_{xxx} - \beta\eta_{xx} = 0, \\ u_t + \eta_x + \beta u_{xx} = 0. \end{cases} \tag{5.6}$$

Let $(\phi, \psi) \in \mathfrak{D}(\mathbb{A})$. Then it is straightforward to check that the functions

$$\eta(t) = \phi + \beta(\phi_{xx} + \beta\psi_{xxx})t \quad \text{and} \quad u(t) = \psi - (\phi_x + \beta\psi_{xx})t \tag{5.7}$$

are (strong) solutions to (5.6) with initial conditions (1.3). To prove the uniqueness, let (η, u) be a solution to (5.6) with initial conditions (1.3). Then, adding the first equation to the spatial derivative of the second equation multiplied by β, we see at once that

$$\eta_t + \beta u_{tx} = 0.$$

This gives

$$\eta = -\beta u_x + p,$$

where $p : [0, \pi] \to \mathbb{R}$ is the time-independent function

$$p(x) = \phi(x) + \beta\psi_x(x).$$

Substituting η into the second equation, we get

$$u_t = -p_x,$$

implying that

$$u(t) = \psi - p_x t = \psi - (\phi_x + \beta\psi_{xx})t.$$

In turn, this yields

$$\eta(t) = \phi + \beta(\phi_{xx} + \beta\psi_{xxx})t,$$

which establishes the desired uniqueness. □

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6. The well-posed problem

We now turn to the case $\varkappa > 0$. In this situation, we show that problem (1.1)–(1.2) is well-posed. Specifically, we have the following result.

THEOREM 6.1. *Let $\varkappa > 0$. Then the operator $\mathbb{A}$ is the infinitesimal generator of a strongly continuous bounded semigroup*

$$S(t) = e^{t\mathbb{A}} : \mathcal{H} \to \mathcal{H}.$$

Accordingly, for every initial datum $\boldsymbol{\phi} = (\phi, \psi) \in \mathcal{H}$, there exists a unique solution

$$\boldsymbol{u}(t) = (\eta(t), u(t)) = S(t)\boldsymbol{\phi} \in \mathcal{C}([0,\infty), \mathcal{H})$$

to (1.1)-(1.3)*. Moreover, there exists an equivalent norm on $\mathcal{H}$ for which $S(t)$ is a actually a contraction semigroup.*

In order to prove the theorem, the first step is to devise the correct norm. To this end, let us introduce the functional $|\cdot|_{\mathcal{H}}$ on $\mathcal{H}$, acting as

$$|(\eta, u)|_{\mathcal{H}}^2 = \|\eta\|^2 + 2\beta\langle \eta, u_x\rangle + \alpha\|u_x\|^2. \tag{6.1}$$

This object turns out to be an equivalent norm on $\mathcal{H}$. This is a consequence of the following lemma.

LEMMA 6.2. *Let $\varkappa > 0$. Then there exist a constant $c > 1$ such that*

$$\frac{1}{c}\|(\eta, u)\|_{\mathcal{H}} \leqslant |(\eta, u)|_{\mathcal{H}} \leqslant c\|(\eta, u)\|_{\mathcal{H}}.$$

Proof. The second inequality is straightforward. Concerning the first one, notice that by the Young inequality we have

$$2\beta|\langle \eta, u_x\rangle| \leqslant \varepsilon\|\eta\|^2 + \frac{\beta^2}{\varepsilon}\|u_x\|^2.$$

Therefore,

$$|(\eta, u)|_{\mathcal{H}}^2 \geqslant (1-\varepsilon)\|\eta\|^2 + \Big(\alpha - \frac{\beta^2}{\varepsilon}\Big)\|u_x\|^2.$$

Exploiting the fact that $\varkappa > 0$, we can choose ε close to 1 so that

$$1 - \varepsilon > 0 \quad \text{and} \quad \alpha - \frac{\beta^2}{\varepsilon} > 0,$$

and the proof is finished. □

REMARK 6.3. For $\boldsymbol{u} = (\eta, u)$ and $\boldsymbol{u}' = (\eta', u')$, the norm $|\cdot|_{\mathcal{H}}$ defined in (6.1) is induced by the scalar product

$$(\boldsymbol{u}, \boldsymbol{u}')_{\mathcal{H}} = \langle \eta, \eta'\rangle + \beta\langle \eta, u_x'\rangle + \beta\langle u_x, \eta'\rangle + \alpha\langle u_x, u_x'\rangle. \tag{6.2}$$

We are ready to prove theorem 6.1.

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Proof of Theorem 6.1. Consider the space $\mathcal{H}$ endowed with the equivalent norm $|\cdot|_{\mathcal{H}}$. The result then follows from an application of the classical Lumer–Phillips theorem (see [**10**, **15**]), which states that a densely defined linear operator $\mathbb{A}$ is the infinitesimal generator of a contraction semigroup $S(t) = e^{t\mathbb{A}}$ if and only if

(i) $\mathbb{A}$ is dissipative, that is,

$$(\mathbb{A}\boldsymbol{u}, \boldsymbol{u})_{\mathcal{H}} \leqslant 0, \quad \forall\ \boldsymbol{u} \in \mathfrak{D}(\mathbb{A}).$$

(ii) The operator $\mathbb{A} - \mathbb{I}$ is onto, where $\mathbb{I}$ is the identity operator on $\mathcal{H}$.

Indeed, after theorem 4.1, we know that $1 \in \rho(\mathbb{A})$, which implies (ii). We are left to prove (i). A direct computation yields

$$\begin{aligned}(\mathbb{A}\boldsymbol{u}, \boldsymbol{u})_{\mathcal{H}} &= \langle \alpha u_{xxx} + \beta\eta_{xx}, \eta\rangle + \beta\langle \alpha u_{xxx} + \beta\eta_{xx}, u_x\rangle \\ &\quad - \beta\langle \eta, \eta_{xx} + \beta u_{xxx}\rangle - \alpha\langle \eta_{xx} + \beta u_{xxx}, u_x\rangle.\end{aligned}$$

Accordingly, by an integration by parts, along with the fact that $\eta,\ u_x \in H_0^1$,

$$(\mathbb{A}\boldsymbol{u}, \boldsymbol{u})_{\mathcal{H}} = 0, \tag{6.3}$$

for every choice of $(\eta,\ u) \in \mathfrak{D}(\mathbb{A})$. □

Remark 6.4. It is apparent from the proof that the operator $-\mathbb{A}$ fulfils the hypotheses of the Lumer–Phillips Theorem as well. Consequently, $S(t)$ is actually a strongly continuous group of bounded operators (see, e.g., [**15**]).

The energy corresponding to the solution (1.1)–(1.2) with initial datum $\boldsymbol{\phi} = (\phi,\ \psi)$ is classically defined by

$$\mathsf{E}(t) = \frac{1}{2}|S(t)\boldsymbol{\phi}|^2_{\mathcal{H}} = \frac{1}{2}\Big[\|\eta(t)\|^2 + 2\beta\langle \eta(t), u_x(t)\rangle + \alpha\|u_x(t)\|^2\Big].$$

Such an energy turns out to be conserved.

Corollary 6.5. *For every initial datum $\boldsymbol{\phi} \in \mathcal{H}$, the corresponding energy $\mathsf{E}(t)$ is constant.*

Proof. Assume first $\boldsymbol{\phi} \in \mathfrak{D}(\mathbb{A})$. Since $\mathbb{A}$ is the infinitesimal generator of $S(t)$, we know that $\boldsymbol{u}(t) \in \mathfrak{D}(\mathbb{A})$ for every $t \geqslant 0$, and

$$\frac{\mathrm{d}}{\mathrm{d}t} S(t)\boldsymbol{\phi} = \mathbb{A}S(t)\boldsymbol{\phi}.$$

Therefore, exploiting (6.3),

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{E}(t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|S(t)\boldsymbol{\phi}|^2_{\mathcal{H}} = (\mathbb{A}S(t)\boldsymbol{\phi}, S(t)\boldsymbol{\phi}) = 0.$$

Accordingly, the equality

$$\mathsf{E}(t) = \mathsf{E}(0)$$

holds for every $t \geqslant 0$. By density, and due to the continuity properties of the semigroup, this equality remains valid for every $\boldsymbol{\phi} \in \mathcal{H}$. □

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7. The explicit representation of the solutions

When the problem is well-posed, it is actually possible to provide an explicit representation for the solution to (1.1)–(1.3).

THEOREM 7.1. *Let $\varkappa = \alpha - \beta^2 > 0$. For any $(\phi, \psi) \in \mathcal{H}$ and any $n = 1, 2, 3, \ldots$, define the sequences*

$$A_n = \int_0^\pi \phi(x) \sin nx \, \mathrm{d}x \quad \textit{and} \quad B_n = \int_0^\pi \psi_x(x) \sin nx \, \mathrm{d}x.$$

Then the solution (η, u) to problem (1.1)–(1.2) with initial data (1.3) has the explicit representation

$$\eta(t) = \frac{2}{\pi\sqrt{\varkappa}} \sum_{n=1}^{\infty} p_n(t) \sin nx,$$

and

$$u(t) = -\frac{2}{\pi\sqrt{\varkappa}} \sum_{n=1}^{\infty} q_n(t) \frac{\cos nx}{n},$$

where

$$p_n(t) = \sqrt{\varkappa}\, A_n \cos(\sqrt{\varkappa}\, n^2 t) - \big(\beta A_n + \alpha B_n\big) \sin(\sqrt{\varkappa}\, n^2 t),$$

and

$$q_n(t) = \sqrt{\varkappa}\, B_n \cos(\sqrt{\varkappa}\, n^2 t) + \big(A_n + \beta B_n\big) \sin(\sqrt{\varkappa}\, n^2 t).$$

In order to prove the theorem, we work in the complexification of the Hilbert space $\mathcal{H}$, endowed with the (complex) scalar product $(\cdot, \cdot)_{\mathcal{H}}$ given by (6.2). Hence, the norm of $\boldsymbol{u} = (\eta, u)$ now reads

$$|\boldsymbol{u}|^2_{\mathcal{H}} = \|\eta\|^2 + 2\beta\, \Re\langle \eta, u_x\rangle + \alpha \|u_x\|^2.$$

Let us consider suitable multiples of the eigenfunctions $\boldsymbol{u}_n^{\pm}$ given by (4.7), corresponding to the eigenvalues $\lambda_n^{\pm} = \pm i\sqrt{\varkappa}\, n^2$ of $\mathbb{A}$, namely,

$$\boldsymbol{w}_n^{\pm}(x) = \frac{1}{\sqrt{\alpha\varkappa\pi}} \Big(\alpha \sin nx, \frac{\beta \pm i\sqrt{\varkappa}}{n} \cos nx\Big).$$

The following holds.

PROPOSITION 7.2. *For $n = 1, 2, 3, \ldots$, the functions $\boldsymbol{w}_n^{\pm}$ form an orthonormal basis of (the complexification of) $\mathcal{H}$, with respect to the scalar product $(\cdot, \cdot)_{\mathcal{H}}$.*

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Proof. By direct calculations, one can check the relations

$$(\boldsymbol{w}_n^+, \boldsymbol{w}_m^-)_{\mathcal{H}} = 0, \quad \forall\ n, m,$$

and

$$(\boldsymbol{w}_n^\pm, \boldsymbol{w}_m^\pm)_{\mathcal{H}} = \delta_{nm}.$$

We are left to show that the system is complete. Let $\boldsymbol{f} = (f,\, g) \in \mathcal{H}$. The two components of $\boldsymbol{f}$ can be decomposed as

$$f = \sum_{n=1}^{\infty} a_n \sin nx \quad \text{and} \quad g = \sum_{n=1}^{\infty} b_n \frac{\cos nx}{n}.$$

It is then sufficient to verify that each vector

$$\boldsymbol{f}_n = \left(a_n \sin nx, b_n \frac{\cos nx}{n}\right)$$

can be expressed as a linear combination of $\boldsymbol{w}_n^+$ and $\boldsymbol{w}_n^-$. That is, we are looking for constants c_n and d_n such that

$$\boldsymbol{f}_n = c_n \boldsymbol{w}_n^+ + d_n \boldsymbol{w}_n^-,$$

which is equivalent to solve

$$\begin{cases} a_n = \dfrac{1}{\sqrt{\alpha\varkappa\pi}}\big[\alpha c_n + \alpha d_n\big], \\ b_n = \dfrac{1}{\sqrt{\alpha\varkappa\pi}}\big[(\beta + i\sqrt{\varkappa})c_n + (\beta - i\sqrt{\varkappa})d_n\big]. \end{cases}$$

But for every n, this system has a unique solution, due to the fact that $\varkappa \neq 0$. $\square$

Proof of Theorem 7.1. On account of proposition 7.2, we look for a solution $\boldsymbol{u}(t) = (\eta(t),\, u(t))$ to (the complexification of) (1.1)–(1.3) of the form

$$\boldsymbol{u}(t) = \sum_{n=1}^{\infty} a_n^+(t)\boldsymbol{w}_n^+ + \sum_{n=1}^{\infty} a_n^-(t)\boldsymbol{w}_n^-.$$

Then, it is readily seen that

$$\boldsymbol{u}(t) = \sum_{n=1}^{\infty} a_n^+(0)e^{i\sqrt{\varkappa}\,n^2 t}\boldsymbol{w}_n^+ + \sum_{n=1}^{\infty} a_n^-(0)e^{-i\sqrt{\varkappa}\,n^2 t}\boldsymbol{w}_n^-, \tag{7.1}$$

where the initial values $a_n^\pm(0)$ are deduced from the initial conditions (1.3) via the formula

$$a_n^\pm(0) = ((\phi, \psi), \boldsymbol{w}_n^\pm)_{\mathcal{H}}.$$

More explicitly,

$$a_n^\pm(0) = \frac{1}{\sqrt{\alpha\varkappa\pi}}\big[\varkappa A_n \pm i\sqrt{\varkappa}\,(\beta A_n + \alpha B_n)\big].$$

At this point, assuming ϕ and ψ to be real functions, we find the explicit form of the (real) solution $\eta(t)$ and $u(t)$ as in the statement of the theorem. $\square$

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The explicit representation of the solution allows us to recover the conservation of the energy predicted by corollary 6.5. Indeed, the energy $\mathsf{E}(t)$ corresponding to the initial datum (ϕ, ψ) reads

$$\mathsf{E}(t) = \frac{1}{\pi\varkappa}\sum_{n=1}^{\infty}\Big[p_n^2(t) + 2\beta p_n(t)q_n(t) + \alpha q_n^2(t)\Big].$$

Observe that for every fixed n, the equality

$$p_n^2(t) + 2\beta p_n(t)q_n(t) + \alpha q_n^2(t) = \varkappa\Big[A_n^2 + 2\beta A_n B_n + \alpha B_n^2\Big]$$

holds for every $t \geqslant 0$. This means that

$$\mathsf{E}(t) = \frac{1}{\pi}\sum_{n=1}^{\infty}\Big[A_n^2 + 2\beta A_n B_n + \alpha B_n^2\Big] = \mathsf{E}(0),$$

that is, the energy is constant in time, as expected.

Remark 7.3. In fact, the representation above for (η, u) is valid for all times $t \in \mathbb{R}$.

8. Analysis of the case $\alpha = 0$

Although in the whole paper we have assumed $\alpha \neq 0$, for the sake of completeness we finally discuss the degenerate case $\alpha = 0$. We shall distinguish two cases.

8.1. The case $\beta \neq 0$

System (1.1) becomes

$$\begin{cases} \eta_t - \beta\eta_{xx} = 0, \\ u_t + \eta_x + \beta u_{xx} = 0. \end{cases}$$

Since $\varkappa = -\beta^2 < 0$, in light of theorem 6.1 well-posedness is not expected. This is exactly what happens. Clearly, the first equation can only have a solution if $\beta > 0$, for otherwise it is a backward heat equation, which is known to be ill-posed. However, if $\beta > 0$, once η is deduced from the first equation, we end up with a nonhomogeneous backward heat equation for u, once again ill-posed.

8.2. The case $\beta = 0$

We have the even simpler system

$$\begin{cases} \eta_t = 0, \\ u_t + \eta_x = 0. \end{cases}$$

If there exists a solution in some weak sense, with initial data (ϕ, ψ), it must be of the form

$$\eta = \phi \quad \text{and} \quad u = \psi - \phi_x t.$$

Then it is readily seen that it is not possible to find a solution for all initial data in $\mathcal{H}$. Just choose $\phi \in H$ but not more regular. Nevertheless, the problem turns out

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to be well-posed if we change the underlying phase space. For instance, considering the space

$$\mathcal{V} = H_0^1 \times H,$$

we see that for every initial datum $(\phi, \psi) \in \mathcal{V}$, the pair (η, u) defined above is indeed the unique solution of the system in $\mathcal{V}$.

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