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On multiparameter CAR (canonical anticommutation relation) flows

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Abstract. Let P be a pointed, closed convex cone in $\mathbb{R}^d$. We prove that for two pure isometric representations $V^{(1)}$ and $V^{(2)}$ of P, the associated CAR flows $\beta^{V^{(1)}}$ and $\beta^{V^{(2)}}$ are cocycle conjugate if and only if $V^{(1)}$ and $V^{(2)}$ are unitarily equivalent. We also give a complete description of pure isometric representations of P with commuting range projections that give rise to type I CAR flows. We show that such an isometric representation is completely reducible with each irreducible component being a pullback of the shift semigroup $\{S_t\}_{t\geq 0}$ on $L^2[0,\infty)$. We also compute the index and the gauge group of the associated CAR flows and show that the action of the gauge group on the set of normalized units need not be transitive.

1 Introduction

Broadly speaking, the subject of irreversible non-commutative dynamics is concerned with the study of dynamical systems where instead of a group action on a non-commutative space (Hilbert spaces, C^*-algebras, von Neumann algebras), we have a semigroup action. The operator algebraic aspects of such irreversible dynamical systems have received the attention of many authors over the years. Some of the topics that were investigated in detail and continue to be a source for much research are semigroup C^*-algebras [11], semi-crossed products of non self-adjoint algebras [13], dilation theory of semigroups of contractions and CP-semigroups [28], E_0-semigroups and product systems [8]. This paper comes under the topic of E_0-semigroups and product systems.

An E_0-semigroup (CP-semigroup) over a semigroup P on $B(\mathcal{H})$ is a semigroup $\alpha := \{\alpha_x\}_{x\in P}$ of unital, normal *-endomorphisms (CP-maps) of $B(\mathcal{H})$. If P has a topology, we require the semigroup α to satisfy an appropriate continuity hypothesis. The study of such semigroups, when $P = [0,\infty)$, has a long history which dates back to Powers' works [19, 20]. Arveson wrote several influential papers on the subject [4–7] and also authored [8] which is the standard reference for the subject. More on Arveson's contribution and by others to the subject of E_0-semigroups can be found in a survey article [15] by Izumi.

In the last 15 years, several papers [1–3, 17, 25–27, 30, 32, 36] appeared where semigroups of endomorphisms/CP-maps and product systems, over more general monoids, were considered. Moreover, it was demonstrated that significant differences

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show up in the multiparameter case. A few features that are in stark contrast to the 1-parameter case are listed below.

(1) A CP-semigroup over $\mathbb{N}^3$ need not have a dilation to an E_0-semigroup [29].
(2) CCR and CAR flows need not be cocycle conjugate in the multiparameter case [33].
(3) In the multiparameter case, decomposable product systems need not be spatial [18, 36].

These contrasting phenomena make the multiparameter theory interesting, and the authors believe that multiparameter E_0-semigroups are objects worthy of investigation. Here, we study a class of E_0-semigroups called CAR flows and classify a subclass of them.

In view of the bijective correspondence between the class of product systems and the class of E_0-semigroups [9, 17, 31], we explain the problem studied and the results obtained in the language of product systems. A product system is a measurable field of separable Hilbert spaces $E := \{E(x)\}_{x\in P}$ endowed with a multiplication that is compatible with the measurable structure.

The two simplest product systems, whose definitions we recall, are the ones associated with CCR and CAR flows. Let $\mathcal{H}$ be a separable Hilbert space. (All Hilbert spaces considered in this paper are tacitly assumed to be separable.) We denote the symmetric Fock space of $\mathcal{H}$ by $\Gamma_s(\mathcal{H})$ and the antisymmetric Fock space of $\mathcal{H}$ by $\Gamma_a(\mathcal{H})$. Throughout this paper, the letter P stands for a closed, convex cone in $\mathbb{R}^d$ that is pointed, i.e., $P \cap -P = \{0\}$ and spanning, i.e., $P - P = \mathbb{R}^d$. Let $V := \{V_x\}_{x\in P}$ be a strongly continuous semigroup of isometries on $\mathcal{H}$. We call such a semigroup of isometries an isometric representation of P on $\mathcal{H}$. We assume V is pure, i.e., $\bigcap_{x\in P} V_x\mathcal{H} = \{0\}$.

Consider the field of Hilbert spaces $F^V := \{F^V(x)\}_{x\in P}$, $F^V(x) := \Gamma_s(Ker(V_x^*))$ for $x \in P$. We impose a measurable structure on F^V as follows. For every $x \in P$, we can view $F^V(x)$ as a subspace of $\Gamma_s(\mathcal{H})$, as the embedding $Ker(V_x^*) \subset \mathcal{H}$ induces an embedding of $\Gamma_s(Ker(V_x^*))$ in $\Gamma_s(\mathcal{H})$. Let Γ be the set of all maps $t : P \to \Gamma_s(\mathcal{H})$ such that

(a) the map t is weakly measurable, and
(b) for $x \in P$, $t(x) \in F^V(x)$.

Then, F^V is a measurable field of Hilbert spaces with Γ being the space of measurable sections. Define a product rule on F^V by

$$e(\xi)e(\eta) = e(\xi + V_x\eta) \tag{1.1}$$

for $\xi \in Ker(V_x^*)$ and $\eta \in Ker(V_y^*)$. Here, $\{e(\xi) : \xi \in Ker(V_x^*)\}$ denotes the collection of exponential vectors in the symmetric Fock space $\Gamma_s(Ker(V_x^*))$. Then, the multiplication is compatible with the measurable structure making F^V a product system. The product system F^V is called the product system of the CCR flow associated with V. The corresponding E_0-semigroup α^V is called the CCR flow associated with V. Concrete examples of multiparameter CCR flows and their intrinsic properties like index, gauge group, type were analyzed in [2, 3] and in [23].

For $x \in P$, let Ω_x be the vacuum vector of $\Gamma_a(Ker(V_x^*))$. Consider the field of Hilbert spaces $E^V := \{E^V(x)\}_{x\in P}$, where $E^V(x) := \Gamma_a(Ker(V_x^*))$ for $x \in P$. The measurable structure on E^V is defined as in the CCR case. Define a multiplication on E^V as follows:

$$\xi \cdot \eta = V_x\eta_1 \wedge V_x\eta_2 \ldots \wedge V_x\eta_n \wedge \xi_1 \wedge \xi_2 \wedge \ldots \wedge \xi_m \tag{1.2}$$

for $\xi = \xi_1 \wedge \xi_2 \cdots \wedge \xi_m \in \Gamma_a(Ker(V_x^*)$, $\eta = \eta_1 \wedge \eta_2 \wedge \ldots \wedge \eta_n \in \Gamma_a(Ker(V_y^*))$ and $x, y \in P$. For $m = 0$ or $n = 0$, (1.2) is interpreted as follows:

$$\Omega_x \cdot \Omega_y = \Omega_{x+y},$$
$$\Omega_x \cdot \eta = V_x\eta_1 \wedge V_x\eta_2 \wedge \ldots \wedge V_x\eta_n,$$
$$\xi \cdot \Omega_y = \xi_1 \wedge \xi_2 \wedge \ldots \wedge \xi_m.$$

Then, E^V is a product system and is called the product system of the CAR flow associated with V. The corresponding E_0-semigroup β^V is called the CAR flow associated with V.

It is known that in the 1-parameter case, i.e., when $P = [0, \infty)$, E^V and F^V are isomorphic [21]. Then, it follows from the work of Arveson [4] that 1-parameter CAR flows are classified by a single numerical invariant called index that takes values in $\{0, 1, 2, \ldots\} \cup \{\infty\}$. In particular, the map

$$V \to E^V$$

is injective. Also, 1-parameter CAR flows are type I. Here, we take up the multiparameter case, and we completely classify type I CAR flows associated with isometric representations with commuting range projections.

Before we state our results, we mention here that, in the multiparameter case, the study of CCR flows and CAR flows are not the same thing. For, it was demonstrated in [33] that E^V and F^V need not be isomorphic. In [1], Arjunan studied the decomposability of the product system E^V when V is the shift semigroup associated with a free and transitive action of P. He showed that for a large class of isometric representations V, E^V fails to be decomposable, and hence not isomorphic to the product system of a CCR flow.

Our first result concerning CAR flows is given below.

Theorem 1.1 *Let $V^{(1)}$ and $V^{(2)}$ be pure isometric representations of P on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. The product systems $E^{V^{(1)}}$ and $E^{V^{(2)}}$ are isomorphic if and only if $V^{(1)}$ and $V^{(2)}$ are unitarily equivalent, i.e., there exists a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that for every $a \in P$,*

$$UV_a^{(1)}U^* = V_a^{(2)}.$$

The above theorem was statedin [33, Proposition 4.7]. However, the proof given there is incorrect. The author argues that in view of [33, Proposition 4.1], it suffices to prove that the gauge group of a CAR flow acts transitively on the set of normalized units, and he gives an incorrect proof of this assertion. In fact, the last assertion is false. We show by counterexamples that the gauge group of a CAR flow need not act transitively on the set of normalized units.

Theorem 1.1 says that the task of parametrizing/listing CAR flows is equivalent to the problem of parametrizing isometric representations of P. However, the process of

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inducing isometric representations [12, 23] of $\mathbb{N}^2$ to that of $\mathbb{R}_+^2$ allows us to conclude that the classification problem of isometric representations of $\mathbb{R}_+^2$ is at least as hard as describing the dual of $C^*(\mathbb{Z}_2 * \mathbb{Z})$ which is known to be pathological, i.e., it is not a standard Borel space. Thus, describing a good parameterization of all CAR flows is beyond the scope of this paper and the authors. Neverthless, we show that a suitable subclass, i.e., the class of type I CAR flows associated with isometric representations with commuting range projections can be completely classified and described in concrete terms.

Recall that an isometric representation $V = \{V_a\}_{a\in P}$ is said to have *commuting range projections* if $\{V_a V_a^* : a \in P\}$ is a commuting family of projections, and recall that a product system E is said to be type I if the only subsystem of E that contains all the units of E is E. A unit of E is a non-zero multiplicative section of E.

We fix notation to describe our results. Let $\mathbb{N}_\infty := \{1, 2, \ldots\} \cup \{\infty\}$. Let P^* be the dual cone of P, i.e.,

$$P^* := \{\lambda \in \mathbb{R}^d : \langle \lambda | x \rangle \geq 0 \text{ for } x \in P\}.$$

We denote by $S(P^*)$ the unit sphere of P^*, i.e.,

$$S(P^*) := \{\lambda \in P^* : \langle \lambda | \lambda \rangle = 1\}.$$

Let $\lambda \in S(P^*)$, let $k \in \mathbb{N}_\infty$, and let $\mathcal{K}$ be a Hilbert space of dimension k. Denote the one parameter shift semigroup on $\mathcal{H} := L^2([0,\infty), \mathcal{K})$ by $S^{(k)} := \{S_t^{(k)}\}_{t\geq 0}$. Recall that for $t \geq 0$, $S_t^{(k)}$ is the isometry on $\mathcal{H}$ defined by

$$S_t^{(k)} f(x) := \begin{cases} f(x-t) & \text{if } x - t \geq 0, \\ \\ 0 & \text{otherwise.} \end{cases}$$

For $a \in P$, let $S_a^{(\lambda,k)} := S_{\langle \lambda | a \rangle}^{(k)}$. Then, $S^{(\lambda,k)} := \{S_a^{(\lambda,k)}\}_{a\in P}$ is an isometric representation of P on $\mathcal{H}$. If $k = 1$, we denote $S^{(\lambda,k)}$ by S^λ.

For a non-empty countable subset I, an injective map $\lambda : I \to S(P^*)$ and a map $k : I \to \mathbb{N}_\infty$, set

$$S^{(\lambda,k)} := \bigoplus_{i\in I} S^{(\lambda_i,k_i)}.$$

Let $E^{(\lambda,k)}$ be the product system of the CAR flow associated with $S^{(\lambda,k)}$.

With the above notation, we have the following classification result which is the main result of this paper.

Theorem 1.2 *For every non-empty countable set I, an injective map $\lambda : I \to S(P^*)$ and a map $k : I \to \mathbb{N}_\infty$, the product system $E^{(\lambda,k)}$ is type I. Conversely, suppose V is an isometric representation of P with commuting range projections such that E^V is type I. Then, E^V is isomorphic to $E^{(\lambda,k)}$ (equivalently, V is unitarily equivalent to $S^{(\lambda,k)}$) for a non-empty countable set I, an injective map $\lambda : I \to S(P^*)$ and a map $k : I \to \mathbb{N}_\infty$. Moroever, the maps λ and k are unique up to conjugacy.*

We explain briefly the ideas involved in the the proof of the converse part. A certain "universal irreversible" dynamical system that encodes all pure semigroups of

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isometries with commuting range projections was constructed in [35] (see also [37]). The irreversible system is given by the pair (X_u, P) where

$$X_u := \{A \subset \mathbb{R}^d : A \neq \emptyset, A \neq \mathbb{R}^d, -P + A \subset A, 0 \in A, A \text{ is closed}\}.$$

The topology that we impose on X_u is the Fell topology. The semigroup P acts on X_u by translations.

The results of [34] allow us to conclude that if we focus on one unit $u := \{u_a\}_{a \in P}$ at a time, then we can assume that V is the shift semigroup V^μ on X_u associated with a translation invariant Radon measure μ on X_u. Making use of the equality

$$u_a u_b = u_b u_a$$

and by few computations, we show that the support of μ can be identified with $[0, \infty)$. Under this identification, the action of P on $[0, \infty)$ is then given by

$$[0, \infty) \times P \ni (x, a) \to x + \langle \lambda | a \rangle \in [0, \infty)$$

for a unique $\lambda \in S(P^*)$. Then, it is clear that V^μ is unitarily equivalent to S^λ. The proof of the converse part of Theorem 1.2 is completed by a Zorn's lemma argument.

We also compute the index and the gauge group, i.e., the group of automorphisms, of the product system $E^{(\lambda,k)}$. For $\ell \in \mathbb{N}_\infty$, the unitary group of a Hilbert space of dimension ℓ will be denoted by $U(\ell)$. We denote the gauge group of $E^{(\lambda,k)}$ by G.

Theorem 1.3 *Let I be a non-empty countable set, let $\lambda : I \to S(P^*)$ be an injective map, and let $k : I \to \mathbb{N}_\infty$ be a map. Then,*

$$Ind(E^{(\lambda,k)}) = \sum_{i \in I} k_i.$$

(1) *If $I = \{i\}$ is singleton, then $E^{(\lambda,k)}$ is isomorphic to the product system of the CCR flow associated with $S^{(\lambda_i,k_i)}$. In this case, the gauge group G acts transitively on the set of normalized units, and G is isomorphic to the gauge group of the CCR flow associated with $S^{(\lambda_i,k_i)}$.*

(2) *If the cardinality of I is at least two, then the gauge group G does not act transitively on the set of normalized units. In this case, G is isomorphic to $\mathbb{R}^d \times \prod_{i \in I} U(k_i)$.*

The organization of this paper is as follows.

After this introductory section, in Section 2, we collect a few definitions concerning product systems, additive decomposable vectors and units. To keep the paper self contained, we give an overview of the results derived in [16] and in [1] concerning the exponential map that plays a crucial role in our analysis. Theorem 1.1 is proved in Section 3. In Section 4, we prove Theorem 1.2. In Section 5, we prove Theorem 1.3. We also prove that the gauge group of a CAR flow need not act transitively on the set of normalized units.

2 Additive decomposable vectors and the exponential map

We will make extensive use of the exponential map defined initially for "addits" in [16] and later extended to "coherent sections of additive decomposable vectors" in [1].

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To keep the paper fairly self contained, we give a quick overview of the results of [16] and [1]. We start by first recalling the definition of product systems, units, and what it means for a product system to be type I.

Let P be a closed, convex cone in $\mathbb{R}^d$, where $d \geq 1$, which is spanning, i.e., $P - P = \mathbb{R}^d$ and pointed, i.e., $P \cap -P = \{0\}$. The letter P is reserved to denote such a cone for the rest of this paper. Let $P_\infty := P \cup \{\infty\}$. For $x, y \in \mathbb{R}^d$, we say $x \leq y$ if $y - x \in P$. For $x \in P$, set

$$[0, x] := \{y \in \mathbb{R}^d : 0 \leq y \leq x\}.$$

For $x = \infty$, we let $[0, x] = P$.

Let $E := \{E(x)\}_{x \in P}$ be a measurable field of non-zero separable Hilbert spaces together with an associative multiplication defined on the disjoint union $\coprod_{x \in P} E(x)$. Then, E together with the multiplication is called *a product system* if the following properties are satisfied.

(1) If $u \in E(x)$ and $v \in E(y)$, then $uv \in E(x + y)$.
(2) For $x, y \in P$, the map

$$E(x) \otimes E(y) \ni u \otimes v \to uv \in E(x + y)$$

is a unitary operator.
(3) For measurable sections r, s, t, the map

$$P \times P \ni (x, y) \to \langle r(x)s(y)|t(x + y)\rangle \in \mathbb{C}$$

is measurable.

Let $E := \{E(x)\}_{x \in P}$ be a product system. A measurable section $u : P \to \coprod_{x \in P} E(x)$ is called a *unit* if

(1) for $x \in P$, $u_x \neq 0$, and
(2) for $x, y \in P$, $u_x u_y = u_{x+y}$.

We denote the set of units of E by $\mathcal{U}_E$. We say that E is *spatial* if $\mathcal{U}_E$ is non-empty.

Let $F := \{F(x)\}_{x \in P}$ be a field of non-zero Hilbert spaces such that, for $x \in P$, $F(x) \subset E(x)$. We say F is a *subsystem of* E if for every $x, y \in P$,

$$F(x + y) = \overline{span\{uv : u \in F(x), v \in F(y)\}}.$$

Let $u \in \mathcal{U}_E$. We say a subsystem F contains u if $u_x \in F(x)$ for every $x \in P$. We say F contains $\mathcal{U}_E$ if F contains u for every $u \in \mathcal{U}_E$. The product system E is said to be type I if E is spatial and the only subsystem of E that contains $\mathcal{U}_E$ is E.

We denote the gauge group of E, i.e., the group of automorphisms of E, by G_E. For $u \in \mathcal{U}_E$ and $\Psi \in G_E$, let $\Psi.u \in \mathcal{U}_E$ be given by

$$(\Psi.u)_x = \Psi_x(u_x)$$

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for $x \in P$. A unit $u = \{u_x\}_{x \in P}$ of E is said to be *normalized* if $\|u_x\| = 1$ for each $x \in P$. The set of normalized units of E is denoted by $\mathcal{U}_E^n$. We say that the gauge group acts transitively on the set of normalized units if the action of G_E on $\mathcal{U}_E^n$ given by

$$G_E \times \mathcal{U}_E^n \ni (\Psi, u) \to \Psi.u \in \mathcal{U}_E^n$$

is transitive.

In [16], Margetts and Srinivasan introduced the notion of addits of a 1-parameter product system and constructed an exponential map that, after suitable normalization, sets up a bijective correspondence between addits and units. The notion of addits was also considered independentely by Bhat, Lindsay, and Mukherjee in [22]. In [33], Srinivasan introduced the concept of additive decomposable vectors. Imitating the techniques of [16], Arjunan in [1] showed that there is a bijective correspondence between the set of additive decomposable vectors and the set of decomposable vectors. Since this bijection and the exponential map play a key role in what follows, we summarize the main resuls of [33] and [1].

Let $E = \{E(x)\}_{x \in P}$ be a spatial product system over P with a reference unit $e := \{e_x\}_{x \in P}$ that is normalized, i.e., $\|e_x\| = 1$ for $x \in P$. Such a pair (E, e) was called a pointed product system in [22]. The reference unit e is fixed until further mention.

Definition 2.1 [33] Let $x \in P$, and let $b \in E(x)$. We say that b is an additive decomposable vector if $b \perp e_x$ and for $y \leq x$, there exists $b_y \in E(y)$ and $b(y, x) \in E(y - x)$ (that are necessarily unique) such that

(1) $b_y \perp e_y$, $b(y, x) \perp e_{x-y}$, and
(2) $b = b_y e_{x-y} + e_y b(y, x)$.

For $x \in P$, let

$$\mathcal{A}_e(x) := \{b \in E(x) : b \text{ is additive decomposable}\}.$$

Let $x \in P_\infty$, and let $\{b_y\}_{y \in [0,x]}$ be a family of additive decomposable vectors such that $b_y \in \mathcal{A}_e(y)$ for every $y \in [0, x]$. We call such a family a coherent section of additive decomposable vectors if for every $y, z \in [0, x]$ with $y \leq z$, there exists, a necessarily unique, $b(y, z) \in E(z - y)$ such that

$$b_z = b_y e_{z-y} + e_y b(y, z).$$

A coherent section of additive decomposable vectors $\{b_y\}_{y \in P}$ is called *an addit* if $b(y, z) = b_{z-y}$ whenever $y \leq z$.

Suppose $\{b_y\}_{y \in [0,x]}$ is a coherent section of additive decomposable vectors. It is clear that the collection $\{b(y, z) : y, z \in [0, x], y \leq z\}$ satisfies the following propagator equation: for $y_1 \leq y_2 \leq y_3$,

$$b(y_1, y_3) = b(y_1, y_2) e_{y_3 - y_2} + e_{y_2 - y_1} b(y_2, y_3).$$

Let $x \in P$. Given $b \in A_e(x)$, it follows from [33, Lemma 3.2] that there exists a unique coherent section of additive decomposable vectors $\{b_y\}_{y \in [0,x]}$ such that $b_x = b$.

Next, we recall the definition of decomposable vectors. Let $x \in P$, and let $u \in E(x)$ be a non-zero vector. We say that u is *decomposable* if whenever $y \le x$, there exists $v \in E(y)$ and $w \in E(x-y)$ such that $u = vw$. For $x \in P$, let

$$D_e(x) := \{u \in E(x) : u \text{ is decomposable and} \langle u|e_x\rangle = 1\}.$$

Let $x \in P_\infty$, and let $\{u_y\}_{y\in[0,x]}$ be a family of decomposable vectors such that $u_y \in D_e(y)$ for every $y \in [0,x]$. We say that $\{u_y\}_{y\in[0,x]}$ is a left coherent section of decomposable vectors if for $y, z \in [0,x]$ with $y \le z$, there exists a unique $u(y,z) \in D_e(z-y)$ such that $u_z = u_y u(y,z)$.

Next, we recall the definition of the exponential map, in the 1-parameter setting, that sets up a bijective correspondence between $\mathcal{A}_e(\cdot)$ and $D_e(\cdot)$. Let $E := \{E(t)\}_{t\ge 0}$ be a 1-parameter product system with a reference unit $\{e_t\}_{t\ge 0}$ that is normalized. Suppose $t \ge 0$. Let $b \in \mathcal{A}_e(t)$ be given. Let $\{b_s\}_{s\in[0,t]}$ be the coherent section of additive decomposable vectors such that $b_t = b$.

For every $n \in \{0,1,2,\dots\}$, define a section $x^{(n)} : [0,t] \to \coprod_{s\in[0,t]} E(s)$ inductively as follows: for $s \in [0,t]$, set $x_s^{(0)} := e_s$ and $x_s^{(1)} := b_s$, and for $n \ge 2$, let

$$x_s^{(n)} := \int_0^s x_r^{(n-1)} db_r.$$

Then, we set

$$Exp(b) := \sum_{n=0}^{\infty} x_t^{(n)}.$$

It was proved in [1, Proposition 3] that the series $\sum_{n=1}^{\infty} x_t^{(n)}$ is norm convergent in $E(t)$. The integral $\int_0^s x_r^{(n-1)} db_r$ is called Itô integral whose definition is recalled below for the reader's benefit.

Let $n \ge 2$, and let $s \in [0,t]$ be given. For every $k \ge 1$, partition $[0,s]$ into k intervals of length $\frac{s}{k}$. For $i = 0,1,2,\dots,k-1$, set $r_i^{(k)} := \frac{is}{k}$. Define

$$S_k := \sum_{i=0}^{k-1} x^{(n-1)}_{r_i^{(k)}} b(r_i^{(k)}, r_{i+1}^{(k)}) e_{s-r_{i+1}^{(k)}} \tag{2.1}$$

Note that $S_k \in E(s)$ for every $k \ge 1$. Moreover, the sequence $(S_k)_k$ converges in norm whose limit we denote by $\int_0^s x_r^{(n-1)} db_r$. The norm convergence of $(S_k)_k$ was shown in [16] when $\{b_s\}_{s\ge 0}$ is an addit (see [16, Proposition 5.4]). It was observed in [1] that the same proof works if we replace an addit by a coherent section of additive decomposable vectors (see [1, Proposition 2]).

Proposition 2.1 [16], [1] *The exponential map satisfies the following key properties.*

(1) *Let $t \ge 0$. For $b \in \mathcal{A}_e(t)$, $Exp(b) \in \mathcal{D}_e(t)$. Moreover, the map*

$$\mathcal{A}_e(t) \ni b \to Exp(b) \in D_e(t)$$

is a bijection. For $b_1, b_2 \in \mathcal{A}_e(t)$,

$$\langle Exp(b_1)|Exp(b_2)\rangle = exp(\langle b_1|b_2\rangle).$$

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For $b \in A_e(t)$, let $\{b_s\}_{s\in[0,t]}$ be the unique coherent section of additive decomposable vectors such that $b_t = b$. Then,

$$Exp(b_r)Exp(b(r,s)) = Exp(b_s). \tag{2.2}$$

for $0 \leq r \leq s \leq t$.

(2) *If $\{b_t\}_{t\in[0,\infty)}$ is an addit, then $\{Exp(b_t)\}_{t\in[0,\infty)}$ is a unit. Conversely, if $\{u_t\}_{t\in[0,\infty)}$ is a unit and $u_t \in D_e(t)$ for every t, then there exists an addit $\{b_t\}_{t\in[0,\infty)}$ such that $u_t = Exp(b_t)$ for every $t \geq 0$.*

For a proof of the above proposition, we refer the reader to [16] and [1].

Definition 2.2 Let $E := \{E(x)\}_{x\in P}$ be a spatial product system over P with a normalized reference unit $\{e_x\}_{x\in P}$. For $x \in P$, let $\widetilde{E} := \{\widetilde{E}(tx)\}_{t\geq 0}$ be the spatial 1-product system with reference unit $\{\widetilde{e}_{tx}\}_{t\geq 0}$. We denote the exponential map of $\widetilde{E}$ by Exp_x. In particular, if $b \in \mathcal{A}_e(x)$, then $Exp_x(b)$ is a well defined vector in $E(x)$. If $P = [0,\infty)$, then for $s \geq 0$, we omit the subscript "s" from Exp_s and simply denote it by Exp.

Suppose V is a pure isometric representation of P on a separable Hilbert space $\mathcal{H}$, and let $E^V := \{E^V(x)\}_{x\in P}$ be the product system of the CAR flow β^V. Recall that, for $x \in P$, $E^V(x) = \Gamma_a(Ker(V_x^*))$ and the multiplication rule is given by (1.2). For $x \in P$, let Ω_x be the vacuum vector of $\Gamma_a(Ker(V_x^*))$. Then, $\Omega := \{\Omega_x\}_{x\in P}$ is a unit which we call the vacuum unit. We always consider the *vacuum unit as the reference unit* of E^V while considering the exponential map. The set of additive decomposable vectors was determined by Srinivasan in [33], and we summarize the results in the next remark. A couple of definitions are in order before we make the remark.

For $x \in P$, we denote the range projection of V_x by E_x.

Definition 2.3 A map $\xi : P \to \mathcal{H}$ is called an additive cocycle if for $x \in P$, $E_x\xi_x = 0$, and for $x, y \in P$,

$$\xi_{x+y} = \xi_x + V_x\xi_y.$$

The set of additive cocycles is denoted by $\mathcal{A}(V)$.

Let $u := \{u_x\}_{x\in P}$ be a unit of E^V. We say that u is an *exponential unit* if $u_x \in \mathcal{D}_\Omega(x)$ for every $x \in P$, i.e., $\langle u_x|\Omega_x\rangle = 1$ for $x \in P$.

Remark 2.2 With the foregoing notation, we have the following.

(1) Let $x \in P$. Then, it follows from [33, Proposition 4.1] that $\mathcal{A}_\Omega(x) = Ker(V_x^*)$. It is easy to see that $Ker(V_x^*) \subset \mathcal{A}_\Omega(x)$. To see this suppose $x \in P$ and $\xi \in Ker(V_x^*)$. Let $y \in P$ be such that $y \leq x$. Set $\xi_y := E_y^\perp\xi$ and $\xi(y,x) := V_y^*\xi$. Then, it is clear from the definition of the multiplication rule that

$$\xi_y \cdot \Omega_{x-y} + \Omega_y \cdot \xi(y,x) = \xi. \tag{2.3}$$

Suppose $P = [0,\infty)$, $t \geq 0$ and $\xi \in Ker(V_t^*)$. It follows from (2.2) that

$$Exp(E_s^\perp\xi)Exp(V_s^*\xi) = Exp(\xi)$$

for $s \leq t$. Equivalently, if $P = [0,\infty)$, then for $s,t \geq 0$, $\xi \in Ker(V_s^*)$ and $\eta \in Ker(V_t^*)$,

$$Exp(\xi)Exp(\eta) = Exp(\xi + V_s\eta). \tag{2.4}$$

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(2) Let $\xi : P \to \mathcal{H}$ be a map such that $\xi_x \in Ker(V_x^*)$ for every $x \in P$. It is clear from the definition of an addit, (2.3) and (1.2) that $\{\xi_x\}_{x\in P}$ is an addit if and only if ξ is an additive cocycle for V.

(3) Let u be an exponential unit of E^V. Let $x \in P$. Note that $\{u_{tx}\}_{t\geq 0}$ is a unit for the one parameter product system $\{E(tx)\}_{t\geq 0}$. It follows from Proposition 2.1 that there exists a unique $\xi_x \in Ker(V_x^*)$ such that $u_x = Exp_x(\xi_x)$. It follows from the definition of Exp_x that the 1-particle vector of u_x is ξ_x. Comparing the 1-particle vectors of the equation

$$Exp_{x+y}(\xi_{x+y}) = u_{x+y} = u_x \cdot u_y = Exp_x(\xi_x) \cdot Exp_y(\xi_y)$$

we see that for $x, y \in P$, $\xi_{x+y} = \xi_x + V_x \xi_y$, i.e., $\xi := \{\xi_x\}_{x\in P}$ is an additive cocycle for V.

Thus, if u is an exponential unit, then u is of the form $\{Exp_x(\xi_x)\}_{x\in P}$ for a unique $\xi \in \mathcal{A}(V)$. However, it is not necessary that $Exp(\xi) := \{Exp_x(\xi_x)\}_{x\in P}$ is a unit if $\xi \in \mathcal{A}(V)$ (see Proposition 4.16).

3 Injectivity of the CAR functor

In this section, we prove Theorem 1.1. Suppose V is a pure isometric representation of $[0, \infty)$. Let $F := \{F(t)\}_{t\geq 0}$ be the product system of the associated CCR flow α^V. Recall that for $t \geq 0$, $F(t) := \Gamma_s(Ker(V_t^*))$ and the multiplication rule is given by

$$e(\xi)e(\eta) = e(\xi + V_s\eta) \tag{3.1}$$

for $\xi \in Ker(V_s^*)$ and $\eta \in Ker(V_t^*)$. Here, $\{e(\xi) : \xi \in Ker(V_s^*)\}$ denotes the set of exponential vectors. The vacuum unit of F is denoted by $\Omega := \{\Omega_t\}_{t\geq 0}$. We use the same letter Ω to denote the vacuum unit of both CCR and CAR flows. Let E denote the product system of the CAR flow β^V with the reference unit Ω. For $t \geq 0$, let $D^E(t)$ and $D^F(t)$ denote the decomposable vectors in $E(t)$ and $F(t)$, respectively.

For $t \geq 0$, let

$$D_\Omega^E(t) := \{u \in D^E(t) : \langle u|\Omega_t\rangle = 1\}$$

and $D_\Omega^F(t)$ is defined similarly. From Remark 2.2 and Proposition 2.1, we have

$$D_\Omega^E(t) = \{Exp(\xi) : \xi \in Ker(V_t^*)\},$$

and from [36, Proposition 2.2], we have

$$D_\Omega^F(t) = \{e(\xi) : \xi \in Ker(V_t^*\}.$$

It is known that E and F are isomorphic, a fact first proved by Robinson and Powers in [21]. For our purposes, we need the following coordinate free isomorphism.

Lemma 3.1 *For each $t \geq 0$, the map $\Psi_t : D_\Omega^E(t) \to D_\Omega^F(t)$ defined by*

$$D_\Omega^E(t) \ni Exp(\xi) \to e(\xi) \in D_\Omega^F(t)$$

is a bijection. Moreover,

$$\Psi_s Exp(\xi).\Psi_t Exp(\eta) = \Psi_{s+t}(Exp(\xi).Exp(\eta))$$

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for $s, t \geq 0$, $\xi \in Ker(V_s^*)$ *and* $\eta \in Ker(V_t^*)$. *The map* Ψ_t *extends uniquely to a unitary (again denoted by* Ψ_t*)* $\Psi_t : E(t) \to F(t)$ *for* $t \geq 0$. *The field of maps* $\Psi := \{\Psi_t\}_{t\geq 0}$ *is an isomorphism from E onto F.*

Proof From [8, Corollary 6.8.3], $D_\Omega^E(t)$ is total in $E(t)$, and $D_\Omega^F(t)$ is total in $F(t)$ for $t \geq 0$. Also, by Proposition 2.1,

$$\langle Exp(\eta)|Exp(\xi)\rangle = e^{\langle \eta|\xi\rangle} = \langle e(\eta)|e(\xi)\rangle$$

for $\eta, \xi \in Ker(V_t^*)$ for $t \geq 0$. Hence, Ψ_t can be extended to a unitary operator, again denoted Ψ_t, $\Psi_t : E(t) \to F(t)$, for $t \geq 0$. It follows from (2.4) and (3.1) that $\Psi := \{\Psi_t\}_{t\geq 0} : E \to F$ is an isomorphism. Hence the proof. ∎

Suppose $V^{(1)}$ and $V^{(2)}$ are pure isometric representations of $[0, \infty)$ on $\mathcal{H}$ and $\mathcal{K}$, respectively. Denote the product systems of the respective CAR flows by E_1 and E_2. Let

$$\mathscr{U}(V^{(1)}, V^{(2)}) := \{U : \mathcal{H} \to \mathcal{K} : U \text{ is a unitary}, UV_t^{(1)} = V_t^{(2)}U, UV_t^{(1)*} = V_t^{(2)*}U \text{for} t \geq 0\}.$$

The proof of the next lemma is essentially an application of Lemma 3.1 and the gauge group computation of CCR flows due to Arveson.

Lemma 3.2 *Suppose* $\Psi : E_1 \to E_2$ *is an isomorphism of product systems. Then, there exists* $U \in \mathscr{U}(V^{(1)}, V^{(2)})$, $\xi = \{\xi_t\}_{t\geq 0} \in \mathcal{A}(V^{(2)})$, *and* $\lambda \in \mathbb{R}$ *such that*

$$\Psi_t(Exp(\eta)) = e^{i\lambda t} e^{\frac{-\|\xi_t\|^2}{2} - \langle U\eta|\xi_t\rangle} Exp(U\eta + \xi_t)$$

for $\eta \in Ker(V_t^{(1)*})$, $t \geq 0$.

Proof We denote by F_1 and F_2 the product systems of the CCR flows associated with $V^{(1)}$ and $V^{(2)}$ respectively. For $i = 1, 2$, let $\Theta_i = \{\Theta_i(t)\}_{t\geq 0}$ be the isomorphism from E_i onto F_i given by

$$\Theta_i(t)Exp(\eta) = e(\eta)$$

for $\eta \in Ker(V_t^{(i)*})$ and $t \geq 0$. The isomorphism Θ_i for $i = 1, 2$ is guaranteed by Lemma 3.1. Then, $\Delta = \Theta_2 \circ \Psi \circ \Theta_1^{-1}$ is an isomorphism from F_1 onto F_2.

It follows from [8, Corollary 2.6.10] that $V^{(1)}$ and $V^{(2)}$ are unitarily equivalent. Suppose $W : \mathcal{H} \to \mathcal{K}$ is a unitary such that $WV_t^{(1)} = V_t^{(2)}W$ for $t \geq 0$. For $t \geq 0$, let $\Lambda_t : F_1(t) \to F_2(t)$ be the unitary operator such that

$$\Lambda_t exp(\xi) = exp(W\xi)$$

for $\xi \in Ker(V_t^{(1)*})$. Clearly, $\Lambda := \{\Lambda_t\}_{t\geq 0} : F_1 \to F_2$ is an isomorphism.

Now consider the map $T := \Lambda^{-1} \circ \Delta$. The map T is an automorphism of F_1. By [8, Theorem 3.8.4], there exists $U \in \mathscr{U}(V^{(1)}, V^{(1)})$, an additive cocycle $\xi = \{\xi_t\}_{t\geq 0}$ for $V^{(1)}$ and $\lambda \in \mathbb{R}$ such that

$$T_t e(\eta) = e^{i\lambda t} e^{-\frac{\|\xi_t\|^2}{2} - \langle U\eta|\xi_t\rangle} e(U\eta + \xi_t)$$

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for $\eta \in Ker(V_t^{(1)*})$. For $t \geq 0$, set $\widetilde{\xi}_t := W\xi_t$ and $\widetilde{U} := WU$. Then, $\widetilde{U} \in \mathscr{U}(V^{(1)}, V^{(2)})$ and $\widetilde{\xi} := \{\widetilde{\xi}_t\}_{t\geq 0}$ is an additive cocycle for $V^{(2)}$.

Since $\Delta := \Lambda \circ T$, it follows that for for $t \geq 0$ and $\eta \in Ker(V_t^{(1)*})$,

$$\begin{aligned}\Delta_t e(\eta) &= e^{i\lambda t} e^{-\frac{\|\xi_t\|^2}{2} - \langle U\eta|\xi_t\rangle} e(WU\eta + W\xi_t)\\ &= e^{i\lambda t} e^{-\frac{\|W\xi_t\|^2}{2} - \langle WU\eta|W\xi_t\rangle} e(WU\eta + \widetilde{\xi}_t)\\ &= e^{i\lambda t} e^{-\frac{\|\widetilde{\xi}_t\|^2}{2} - \langle \widetilde{U}\eta|\widetilde{\xi}_t\rangle} e(\widetilde{U}\eta + \widetilde{\xi}_t).\end{aligned}$$

Since $\Psi = \Theta_2^{-1} \circ \Delta \circ \Theta_1$, it is immediate that

$$\Psi_t Exp(\eta) = e^{i\lambda t} e^{\frac{-\|\widetilde{\xi}_t\|^2}{2} - \langle \widetilde{U}\eta|\widetilde{\xi}_t\rangle} Exp(\widetilde{U}\eta + \widetilde{\xi}_t)$$

for $\eta \in Ker(V_t^{(1)*})$ and $t \geq 0$. Hence the proof. ∎

Let P be a closed convex cone in $\mathbb{R}^d$ which is spanning and pointed. Let $V^{(1)}, V^{(2)}$ be two pure isometric representations of P on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. We denote the product systems of the corresponding CAR flows by $E^{(1)}$ and $E^{(2)}$, respectively.

Theorem 3.3 *The product systems $E^{(1)}$ and $E^{(2)}$ are isomorphic if and only if $V^{(1)}$ and $V^{(2)}$ are unitarily equivalent.*

Proof Suppose $E^{(1)}$ and $E^{(2)}$ are isomorphic. Let $\Psi = \{\Psi_a\}_{a\in P} : E^{(1)} \to E^{(2)}$ be an isomorphism. Let $a \in Int(P)$. Consider the one parameter product system $E_a^{(1)} := \{E_a^{(1)}(t)\}_{t\geq 0}$, where for $t \geq 0$, $E_a^{(1)}(t) := E^{(1)}(ta)$. Similarly, consider the one parameter product system $E_a^{(2)} := \{E_a^{(2)}(t)\}_{t\geq 0}$, where $E_a^{(2)}(t) := E^{(2)}(ta)$ for $t \geq 0$. Then, $\{\Psi_{ta}\}_{t\geq 0} : E_a^{(1)} \to E_a^{(2)}$ is an isomorphism.

By Lemma 3.2, there exists a unitary $U_a : \mathcal{H}_1 \to \mathcal{H}_2$ intertwining $\{V_{ta}^{(1)}\}_{t\geq 0}$ and $\{V_{ta}^{(2)}\}_{t\geq 0}$, an additive cocycle $\{\xi_t^a\}_{t\geq 0}$ of $\{V_{ta}^{(2)}\}_{t\geq 0}$ and $\lambda_a \in \mathbb{R}$ such that Ψ_{ta} is of the form

$$\Psi_{ta} Exp_a(\xi) = e^{i\lambda_a t} e^{-\frac{\|\xi_t^a\|^2}{2}} e^{-\langle U_a\xi|\xi_t^a\rangle} Exp_a(U_a\xi + \xi_t^a), \tag{3.2}$$

for $\xi \in Ker(V_{ta}^{(1)*})$ and $t > 0$. We will denote ξ_1^a by ξ_a. Hence,

$$\Psi_a Exp_a(\xi) = e^{i\lambda_a} e^{-\frac{\|\xi_s\|^2}{2}} e^{-\langle U_a\xi|\xi_a\rangle} Exp_a(U_a\xi + \xi_a) \tag{3.3}$$

for $a \in Int(P)$ and $\xi \in Ker(V_a^{(1)*})$.

Let $a, b \in Int(P)$. Since Ψ is an isomorphism, we have

$$\Psi_a Exp_a(\xi).\Psi_b Exp_b(\eta) = \Psi_{a+b}(Exp_a(\xi).Exp_b(\eta)),$$

for $\xi \in Ker(V_a^{(1)*})$ and $\eta \in Ker(V_b^{(1)*})$.

In particular,

$$\Psi_{a+b}(Exp_a(0).Exp_b(0)) = \Psi_a Exp_a(0).\Psi_b Exp_b(0),$$

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i.e.,

$$\Psi_{a+b}Exp_{a+b}(0) = e^{i(\lambda_a+\lambda_b)}e^{-\frac{(\|\xi_a\|^2+\|\xi_b\|^2)}{2}}Exp_a(\xi_a).Exp_b(\xi_b)$$

$$e^{i\lambda_{a+b}}e^{-\frac{\|\xi_{a+b}\|^2}{2}}Exp_{a+b}(\xi_{a+b}) = e^{i(\lambda_a+\lambda_b)}e^{-\frac{(\|\xi_a\|^2+\|\xi_b\|^2)}{2}}Exp_a(\xi_a).Exp_b(\xi_b). \tag{3.4}$$

Comparing the 0-particle vectors in LHS and RHS of (3.4), we have

$$e^{i\lambda_{a+b}-\frac{\|\xi_{a+b}\|^2}{2}} = e^{i(\lambda_a+\lambda_b)-\frac{(\|\xi_a\|^2+\|\xi_b\|^2)}{2}}. \tag{3.5}$$

Now, comparing the 1-particle vectors of LHS and RHS in (3.4), we have

$$\xi_{a+b} = \xi_a + V_a^{(2)}\xi_b. \tag{3.6}$$

Let $\xi \in Ker(V_a^{(1)*})$. Consider the equation

$$\Psi_{a+b}\xi = \Psi_a\xi.\Psi_bExp_b(0). \tag{3.7}$$

Considering the exponential map in the one parameter product system $E_a^{(1)}$, we have

$$Exp_a(\xi) = \sum_{n=0}^{\infty} x_1^{(n)}$$

where for $r \in (0,1]$, $x_r^{(0)} = \Omega_{ra}$, $x_r^{(1)} = (1 - V_{ra}^{(1)}V_{ra}^{(1)*})\xi$ and $x_r^{(n)} = \int_0^r x_t^{(n-1)}d\xi_t$. Here, for $t \in (0,1]$, $\xi_t = (1 - V_{ta}^{(1)}V_{ta}^{(1)*})\xi$.

Let $s \in \mathbb{R}$. Suppose

$$Exp_a(s\xi) = \sum_{n=0}^{\infty} y_1^{(n)},$$

where the summands are obtained via Itô integration. A moment's reflection on the definition of the Itô integral reveals that

$$y_1^{(n)} = s^n x_1^{(n)}$$

for $n = 0, 1, 2, \cdots$.

Note that

$$\Psi_aExp_a(t\xi) = \sum_{n=0}^{\infty}\Psi_a y_1^{(n)} = \sum_{n=0}^{\infty} t^n\Psi_a x_1^{(n)} \tag{3.8}$$

for $t \in \mathbb{R}$. Since the power series in (3.8) is norm convergent for every $t \in \mathbb{R}$, it may be differentiated term by term, and the derivative is given by

$$\frac{d}{dt}\Psi_aExp_a(t\xi) = \sum_{n=1}^{\infty} nt^{n-1}\Psi_a(x_1^{(n)}).$$

Hence,

$$\Psi_a(\xi) = \frac{d}{dt}\Big|_{t=0}\Psi_aExp_a(t\xi). \tag{3.9}$$

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From (3.3) we have,

$$\begin{aligned}
&\frac{d}{dt}\Psi_a Exp_a(t\xi)\\
&= \frac{d}{dt} e^{i\lambda_a} e^{-\frac{||\xi_a||^2}{2}} e^{-t\langle U_a\xi|\xi_a\rangle} Exp_a(tU_a\xi + \xi_a)\\
&= e^{i\lambda_a} e^{-\frac{||\xi_a||^2}{2}} \Big(Exp_a(tU_a\xi + \xi_a)\frac{d}{dt} e^{-t\langle U_a\xi|\xi_a\rangle} + e^{-t\langle U_a\xi|\xi_a\rangle}\frac{d}{dt} Exp_a(tU_a\xi + \xi_a)\Big)\\
&= e^{i\lambda_a - \frac{||\xi_a||^2}{2}} e^{-t\langle U_a\xi|\xi_a\rangle}\Big(-\langle U_a\xi|\xi_a\rangle Exp_a(tU_a\xi + \xi_a) + U_a\xi + q_a(t,\xi)\Big)
\end{aligned}$$

where the projection of $q_a(t, \xi)$ onto 0-particle and 1-particle space is zero. By (3.9),

$$\Psi_a(\xi) = \frac{d}{dt}\Big|_{t=0}\Psi_a Exp_a(t\xi) = e^{i\lambda_a - \frac{||\xi_a||^2}{2}}(-\langle U_a\xi|\xi_a\rangle Exp_a(\xi_a) + U_a\xi + q_a(0,\xi)).$$

Similarly,

$$\Psi_{a+b}(\xi) = e^{i\lambda_{a+b} - \frac{||\xi_{a+b}||^2}{2}}(-\langle U_{a+b}\xi|\xi_{a+b}\rangle Exp_{a+b}(\xi_{a+b}) + U_{a+b}\xi + q_{a+b}(0,\xi))$$

where the projection of $q_{a+b}(0, \xi)$ onto 0-particle and 1-particle space is zero.

Therefore, (3.7) implies,

$$\begin{aligned}
&e^{i(\lambda_a+\lambda_b) - \frac{(||\xi_a||^2 + ||\xi_b||^2)}{2}}\Big(-\langle U_a\xi|\xi_a\rangle Exp_a(\xi_a) + U_a\xi + q_a(0,\xi)\Big).Exp_b(\xi_b)\\
&\quad = e^{i\lambda_{a+b} - \frac{||\xi_{a+b}||^2}{2}}\Big(-\langle U_{a+b}\xi|\xi_{a+b}\rangle Exp_{a+b}(\xi_{a+b}) + U_{a+b}\xi + q_{a+b}(0,\xi)\Big).
\end{aligned}$$

Using (3.5), the above equation reduces to

$$\begin{aligned}
&\Big(-\langle U_a\xi|\xi_a\rangle Exp_a(\xi_a) + U_a\xi + q_a(0,\xi)\Big).Exp_b(\xi_b)\\
&\quad = -\langle U_{a+b}\xi|\xi_{a+b}\rangle Exp_{a+b}(\xi_{a+b}) + U_{a+b}\xi + q_{a+b}(0,\xi).
\end{aligned} \tag{3.10}$$

By equating the 0-particle vectors in the above equation, we see that

$$\langle U_a\xi|\xi_a\rangle = \langle U_{a+b}\xi|\xi_{a+b}\rangle. \tag{3.11}$$

Equating the 1-particle vectors in (3.10), we have

$$-\langle U_a\xi|\xi_a\rangle(\xi_a + V_a^{(2)}\xi_b) + U_a\xi = -\langle U_{a+b}\xi|\xi_{a+b}\rangle\xi_{a+b} + U_{a+b}\xi.$$

Hence, by (3.6) and (3.11),

$$U_a\xi = U_{a+b}\xi.$$

Thus, if $\xi \in Ker(V_a^{(1)*})$ and $\xi \in Ker(V_b^{(1)*})$ for $a, b \in Int(P)$, $U_a\xi = U_{a+b}\xi = U_b\xi$. Since $\bigcup_{a\in Int(P)} Ker(V_a^{(1)*})$ is dense in $\mathcal{H}_1$, it follows that there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$U\xi = U_a\xi$$

if $\xi \in Ker(V_a^{(1)*})$.

Similarly, considering the equation

$$\Psi_{a+b}(Exp_a(0).\eta) = \Psi_a(Exp_a(0)).\Psi_b\eta$$

for $\eta \in Ker(V_b^{(1)*})$ and comparing 0 and 1-particle vectors as before, it is not difficult to see that for $a, b \in Int(P)$ and $\eta \in Ker(V_b^{(1)*})$

$$UV_a^{(1)}\eta = V_a^{(2)}U\eta.$$

Since $\bigcup_{b\in Int(P)} Ker(V_b^{(1)*})$ is dense in $\mathcal{H}_1$, it follows that $UV_a^{(1)} = V_a^{(2)}U$ for every $a \in Int(P)$. As $Int(P)$ is dense in P, U intertwines $V^{(1)}$ and $V^{(2)}$. This completes the proof. ∎

4 Type I CAR flows associated with isometric representations with commuting range projections

In this section, we characterize pure isometric representations of P with commuting range projections that give rise to type I CAR flows. Recall that a pure isometric representation V of P is said to have *commuting range projections* if $\{V_aV_a^* : a \in P\}$ is a commuting family of projections. First, we recall the dynamical system that encodes all pure isometric representations with commuting range projections. Let $\mathcal{C}(\mathbb{R}^d)$ denote the set of closed subsets of $\mathbb{R}^d$ which we equip with the Fell topology.

Let (A_n) be a sequence in $\mathcal{C}(\mathbb{R}^d)$, and let $A \in \mathcal{C}(\mathbb{R}^d)$. Then, $(A_n) \to A$ if and only if $\limsup A_n = \liminf A_n = A$, where

$$\begin{aligned}\liminf A_n &:= \{x \in \mathbb{R}^d : \text{for every} n, \text{there exists } x_n \in A_n \text{ such that } (x_n) \to x\}\\ \limsup A_n &:= \{x \in \mathbb{R}^d : \text{there exists a subsequence } (n_k) \text{ and } x_{n_k} \in A_{n_k}\\ &\qquad \text{such that } (x_{n_k})_k \to x\}.\end{aligned}$$

Let

$$\begin{aligned}Y_u &:= \{A \in \mathcal{C}(\mathbb{R}^d) : -P + A \subset A, A \neq \emptyset, A \neq \mathbb{R}^d\},\\ X_u &:= \{A \in Y_u : 0 \in A\}.\end{aligned}$$

The space Y_u is locally compact, Hausdorff and second countable on which the group $\mathbb{R}^d$ acts. The action is given by the map

$$\mathbb{R}^d \times Y_u \ni (x, A) \to A + x \in Y_u.$$

Note that $X_u + P \subset X_u$ and $Y_u = \bigcup_{a\in P}(X_u - a)$.

Remark 4.1 In the 1-dimensional case, the space Y_u has a very simple description. Let $P = [0, \infty)$. Suppose $A \in Y_u$. The fact that $A - [0, \infty) \subset A$ implies that A is an interval that is not bounded below. Since $A \neq \mathbb{R}$, and since A is closed, it follows that $A = (-\infty, a]$ for a unique $a \in \mathbb{R}$. It is not difficult to show that the map

$$\mathbb{R} \ni a \to (-\infty, a] \in Y_u$$

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is an $\mathbb{R}$-equivariant homeomorphism. Here, the action of $\mathbb{R}$ on $\mathbb{R}$ is by translations. With this identification of Y_u with $\mathbb{R}$, we have $X_u = [0,\infty)$.

It was proved in [37] that there is a bijective correspondence between the class of pure isometric representations of P with commuting range projections and the class of covariant representations of the dynamical system $(Y_u, \mathbb{R}^d)$. For us, the most important examples of isometric representations are the "shift semigroups on X_u" that arise from invariant measures on Y_u.

Let μ be a non-zero Radon measure on Y_u which is also invariant under the above mentioned action of $\mathbb{R}^d$. Define an isometric representation V^μ of P on $L^2(X_u,\mu)$ by

$$V_a^\mu f(A) := \begin{cases} f(A-a) & \text{if } A-a \in X_u, \\ 0 & \text{if } A-a \notin X_u. \end{cases}$$

for $a \in P$. Clearly, V^μ has commuting range projections.

It follows from [34, Lemmas 3.8 and 3.9] that the set $X_u\backslash(X_u+a)$ has compact closure for $a \in P$ and has positive measure if $a \in Int(P)$. It is now clear that $\{1_{X_u\backslash(X_u+a)}\}_{a\in P}$ is a non-zero additive cocycle for V^μ.

Recall that P^* stands for the dual cone of P, and $S(P^*) := \{\lambda \in P^* : ||\lambda|| = 1\}$. Suppose $\lambda \in S(P^*)$. Define

$$H^\lambda := \{x \in \mathbb{R}^d : \langle \lambda | x \rangle \le 0\}.$$

Define

$$Y_u^\lambda := \{H^\lambda + t\lambda : t \in \mathbb{R}\} = \{H^\lambda + z : z \in \mathbb{R}^d\},$$

and

$$X_u^\lambda := \{H^\lambda + t\lambda : t \ge 0\} = Y_u^\lambda \cap X_u.$$

Note that X_u^λ, $Y_u^\lambda \subset Y_u$ are closed. Also, observe that Y_u^λ is $\mathbb{R}^d$-invariant and $X_u^\lambda + P \subset P$.

Let $S := \{S_t\}_{t\ge 0}$ be the one parameter shift semigroup on $L^2[0,\infty)$ defined by

$$S_t f(x) = f(x-t)1_{[0,\infty)}(x-t).$$

For $\lambda \in S(P^*)$, define the isometric representation S^λ of P on $L^2[0,\infty)$ by

$$S_a^\lambda = S_{\langle \lambda | a \rangle}$$

for $a \in P$.

Let μ be a non-zero $\mathbb{R}^d$-invariant Radon measure on Y_u. Denote the product system of the CAR flow associated with V^μ by E^μ. For $a \in P$, let $\eta_a := 1_{X_u \backslash X_u + a}$, let $\eta := \{\eta_a\}_{a\in P}$, and let $Exp(\eta) := \{Exp_a(\eta_a)\}_{a\in P}$.

Lemma 4.2 *Suppose $supp(\mu) = Y_u^\lambda$ for some $\lambda \in S(P^*)$. Then,*

(i) *V^μ is unitarily equivalent to S^λ, and*
(ii) *$Exp(\eta)$ is a unit for E^μ.*

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Proof We first show that V^{μ} is unitarily equivalent to S^{λ}. The reader may verify that the map $\mathcal{F}: Y_u^{\lambda} \to \mathbb{R}$ defined by

$$\mathcal{F}(H^{\lambda} + t\lambda) = t$$

is a homeomorphism, and $\mathcal{F}(X_u^{\lambda}) = [0, \infty)$. Also,

$$\mathcal{F}((H_{\lambda} + t\lambda) + a) = t + \langle \lambda | a \rangle$$

for $t \in \mathbb{R}$ and $a \in \mathbb{R}^d$. Since μ is an invariant measure, the push forward measure $\mathcal{F}_*\mu$ is the Lebesgue measure, denoted by m. Define a unitary operator $U: L^2([0, \infty), m) \to L^2(X_u^{\lambda}, \mu)$ by

$$Uf = f \circ \mathcal{F}.$$

It is routine to verify that U intertwines S^{λ} and V^{μ}. Note that $U^{-1}(1_{X_u \backslash X_u + a}) = 1_{(0, \langle \lambda | a \rangle)}$ for $a \in P$. By abusing notation, we may assume that $V^{\mu} = S^{\lambda}$ and $\eta_a = 1_{(0, \langle \lambda | a \rangle)}$ for $a \in P$.

Let Exp denote the exponential map of the one parameter product system $\widetilde{E}$ of the CAR flow associated with $\{S_t\}_{t \geq 0}$. Let $u_t = Exp(1_{(0,t)})$ for $t \geq 0$. It follows from Remark 2.2 and Propostion 2.1 that $u = \{u_t\}_{t \geq 0}$ is a unit for $\widetilde{E}$. Note that $Exp_a(\eta_a) = u_{\langle \lambda | a \rangle}$ for $a \in P$. Observe that

$$Exp_a(\eta_a).Exp_b(\eta_b) = u_{\langle \lambda | a \rangle}.u_{\langle \lambda | b \rangle} = u_{\langle \lambda | a+b \rangle} = Exp_{a+b}(\eta_{a+b}).$$

for $a, b \in P$, i.e., $Exp(\eta)$ is a unit. ∎

Remark 4.3 In the 1-dimensional case, it is clear from Remark 4.1 that, up to a scalar multiple, there is a unique invariant Radon measure on Y_u. This is not true in the higher dimensional case. Nor is it true that such a measure is supported on Y_u^{λ} for some $\lambda \in S(P^*)$.

Consider $P = \mathbb{R}_+^2$ as an example. Let $A \in Y_u$, and let $G_A := \{z \in \mathbb{R}^2 : A + z = A\}$. Denote the map

$$\mathbb{R}^2/G_A \ni z \to A - z \in Y_u$$

by T. As T is equivariant, the push-forward measure $T_*\lambda$ is an invariant measure on Y_u and is supported on the orbit of A. Here, λ is the Haar measure on $\mathbb{R}^2/G_A$. However, $T_*\lambda$ need not be a Radon measure. For example, let $A = -\mathbb{R}_+^2$. In this case, $G_A = \{0\}$ and

$$T^{-1}(X_u \backslash (X_u + (a, b)) = -\mathbb{R}_+^2 \backslash (-\mathbb{R}_+^2 + (a, b))$$

which has infinite measure if $a \geq 0$ and $b \geq 0$.

Let $\phi = \sum_{n \in \mathbb{Z}} -n 1_{[n, n+1)}$. Note that the graph of ϕ is an infinite staircase. Let

$$A := \{(x, y) : y \leq \phi(x)\}.$$

In this case, $G_A = \mathbb{Z}(1, -1)$. Let $\pi: \mathbb{R}^2 \to \mathbb{R}^2/G_A$ be the quotient map. By drawing a "few pictures", it is immediate to see that $T^{-1}(X_u \backslash (X_u + (a, b))) = \pi(A \backslash (A + (a, b)))$ has compact closure. For example, mod G_A, $A \backslash (A + (1, 1))$ is just a bounded square. In this case, $T_*\lambda$ is a Radon measure. In fact, given $A \in Y_u$, if $G_A \neq 0$, then it can proved,

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by appealing to a few results from [24], that $T_*\lambda$ is a Radon measure. Since this fact is not needed and since we don't make use of the fact that there are enough invariant measures on Y_u in this paper, we omit details and proofs.

Next, we next explain that a typical element of Y_u can be described by a continuous function. We exploit this description to make computations. This is akin to introducing coordinates to compute. Let us first consider the case of the quadrant, i.e., let $P = \mathbb{R}^2_+ = [0,\infty) \times [0,\infty)$. Let $A \in Y_u$ be given. For each $x \in \mathbb{R}$, consider the x-section, i.e.,

$$A_x := \{y \in \mathbb{R} : (x,y) \in A\}.$$

It is possible that A_x is empty. Since $A - (0,t) \subset A$ for $t > 0$, it follows that A_x is an interval and is not bounded below if $A_x \neq \emptyset$. Since A is closed, A_x is closed. Hence, there exists a unique $\phi(x) \in [-\infty,\infty]$ such that $A_x = [-\infty,\phi(x)] \cap \mathbb{R}$. Since $A = \bigcup_{x \in \mathbb{R}} (\{x\} \times A_x)$, it follows that A is the closed region below the graph of ϕ, i.e.,

$$A := \{(x,y) \in \mathbb{R}^2 : y \leq \phi(x)\}.$$

The fact that $A - (s,0) \subset A$ implies that ϕ is decreasing. If $A = -\mathbb{R}^2_+$, then ϕ is given by

$$\phi(x) := \begin{cases} 0 & \text{if } x \leq 0, \\ -\infty & \text{if } x > 0. \end{cases}$$

In this case, ϕ is not continuous, and ϕ takes values in the extended real line. However, we can ensure that ϕ is continuous and ϕ takes only finite values by changing the coordinate system. This is the content of the next lemma which is [1, Lemma 4.3].

We return to the general case now, and let $P \subset \mathbb{R}^d$ be a closed convex cone which is spanning and pointed. Let us first fix a few notation that we will use for the rest of this section. Let $\{v_1, v_2, \ldots, v_d\}$ be a basis for $\mathbb{R}^d$ such that $v_i \in Int(P)$ for $i \in \{1,2,\ldots,d\}$. Fix $i \in \{1,2,\ldots,d\}$. Let

$$Q_i := span\{v_j : j \in \{1,2,\ldots,d\}, j \neq i\}.$$

Let $f : Q_i \to \mathbb{R}$ be a map. Define

$$A_f := \left\{ \sum_{j=1}^{d} x_j v_j : x_i \leq f\left(\sum_{j \neq i} x_j v_j \right) \right\}.$$

The following assertion is [1, Lemma 4.3]. For completeness, we include a proof.

Lemma 4.4 *Let $i \in \{1,2,\ldots,d\}$, and let $A \in Y_u$. Then, there exists a unique continuous function $f : Q_i \to \mathbb{R}$ such that $A = A_f$.*

Proof Uniqueness is clear. Without loss of generality, we assume $i = d$ and write $Q_d = Q$. For $x \in Q$, let

$$A_x := \{t \in \mathbb{R} : x + tv_d \in A\}.$$

Since $\mathbb{R}^d \cong Q \times \mathbb{R}$, $A = \{x + tv_d : x \in Q, t \in A_x\}$. Since A is closed, A_x is closed for each $x \in Q$.

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Claim: For each $x \in Q$, $A_x \neq \emptyset$ and $A_x \neq \mathbb{R}$.

Note that since $-P + A \subset A$, $-Int(P) + A \subset Int(A)$ and $-P + Int(A) \subset Int(A)$. Let $B = \mathbb{R}^d \backslash Int(A)$. Then, B is a proper closed subset of $\mathbb{R}^d$ such that $P + B \subset B$. Also, $Int(B) = \mathbb{R}^d \backslash A$. It is clear that $\mathbb{R}^d \backslash A \subset Int(B)$ as A is closed. If $z \in \partial A$, then every neighbourhood of z intersects $Int(A)$. This is because $-Int(P) + A \subset Int(A)$ which in turn implies that $-tv_d + z \in Int(A)$ as $t \to 0+$. Thus, $Int(B) = \mathbb{R}^d \backslash A$.

Let $x \in Q$ be given. For $y \in \mathbb{R}^d$, there exists $n_0 \in \mathbb{N}$ such that $y - n_0 v_d \in -Int(P)$. For, $-Int(P)$ is an open cone, $v_d \in Int(P)$ and $\frac{y}{n} - v_d \in -Int(P)$ for large $n \in \mathbb{N}$. Fix $z_0 \in A$. Since $-Int(P) + z_0 \subset Int(A)$, given $y \in \mathbb{R}^d$, there exists a natural number $n_0 \in \mathbb{N}$ such that $y - n_0 v_d \in -Int(P) \subset Int(A) - z_0$. Setting $y = x - z_0$, we see that there exists $n_0 \in \mathbb{N}$ such that $x - n_0 v_d \in Int(A)$. Replacing A by B, P by $-P$ and v_d by $-v_d$ and arguing similarly, we see that there exists $n_1 \in \mathbb{N}$ such that $x + n_1 v_d \in Int(B) = \mathbb{R}^d \backslash A$. In particular, there exists $m, n \in \mathbb{Z}$ such that $m \in A_x$ and $n \in \mathbb{R} \backslash A_x$. This proves the claim.

As $-P + A \subset A$, note that if $x + tv_d \in A$ for $t \in \mathbb{R}$, then $x + sv_d \in A$ for every $s < t$. Thus, for each $x \in Q$, A_x is a closed interval which is not bounded below. Since $A_x \neq \mathbb{R}$, $A_x = (-\infty, f(x)]$, where

$$f(x) = \sup A_x = \sup\{t \in \mathbb{R} : x + tv_d \in A\} < \infty.$$

Hence, $A = A_f$.

Next, we prove that f is continuous. Let $x \in Q$ and let $\varepsilon > 0$ be given. Let $s_0, s_1 \in \mathbb{R}$ be such that $f(x) > s_0 > f(x) - \varepsilon$ and $f(x) + \varepsilon > s_1 > f(x)$. Since $x + f(x)v_d \in A$ and $-Int(P) + A \subset Int(A)$, $x + s_0 v_d \in Int(A)$. Also, $x + s_1 v_d \in \mathbb{R}^d \backslash A$. Let $\delta > 0$ be such that the open ball $B(x + s_0 v_d, \delta) \subset Int(A)$ and $B(x + s_1 v_d, \delta) \subset \mathbb{R}^d \backslash A$, respectively. Let $y \in Q$ be such that $|x - y| < \delta$. Since $|(x + s_0 v_d) - (y + s_0 v_d)| = |x - y| < \delta$, $y + s_0 v_d \in A$. Hence,

$$f(y) \geq s_0 > f(x) - \varepsilon$$

Similarly, since $|(x + s_1 v_d) - (y + s_1 v_d)| = |x - y| < \delta$, $y + s_1 v_d \in \mathbb{R}^d \backslash A$, i.e.,

$$f(y) < s_1 < f(x) + \varepsilon.$$

This implies $|f(x) - f(y)| < \varepsilon$. Hence, f is continuous. This completes the proof. ∎

Let $i \in \{1, 2, \ldots, d\}$. Let $f : Q_i \to \mathbb{R}$ be a function. Define

$$(\Psi_i(a)f)\left(\sum_{j \neq i} x_j v_j\right) = f\left(\sum_{j \neq i}(x_j - a_j)v_j\right) + a_i$$

for $a = \sum_{i=1}^d a_i v_i \in \mathbb{R}^d$. Let

$$\mathcal{F}_i := \{f : Q_i \to \mathbb{R} : f \text{ is continuous, and } \Psi_i(a)f \leq f \text{ for } a \in -P\}.$$

Remark 4.5 Let $P = [0, \infty) \times [0, \infty)$. Let $v_1 = (1, 1)$ and $v_2 = (2, 1)$. In this case, we identify Q_1 with $\mathbb{R}$. Let $e_1 = (1, 0) = v_2 - v_1$ and $e_2 = (0, 1) = 2v_1 - v_2$ be the standard basis. Note that for $A \subset \mathbb{R}^2$, $-P + A \subset A$ if and only if $A - se_1 \subset A$ and $A - se_2 \subset A$ for $s > 0$. Thus, for a continuous function $f : \mathbb{R} = Q_1 \to \mathbb{R}$, $f \in \mathcal{F}_1$ if and only if $A_f \in Y_u$ if

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and only if $\Psi_1(-se_1)f \leq f$ and $\Psi_1(-se_2)f \leq f$ for $s > 0$. Hence, $\mathcal{F}_1$ is given by

$$\begin{aligned}\mathcal{F}_1 = \{f : \mathbb{R} \to \mathbb{R} : & f \text{ is continuous}, \\ & f(x+s) + s \leq f(x) \leq f(x+s) + 2s \text{ for } x \in \mathbb{R}, s \geq 0\}.\end{aligned}$$

Similarly, $\mathcal{F}_2$ is given by

$$\begin{aligned}\mathcal{F}_2 = \{f : \mathbb{R} \to \mathbb{R} : & f \text{ is continuous}, \\ & f(x+2s) + s \leq f(x) \leq f(x+s) + s \text{ for } x \in \mathbb{R}, s \geq 0\}.\end{aligned}$$

Note that if $f \in \mathcal{F}_1, f$ is decreasing and is also Lipschitz.

Hereafter, the letter P stands for a closed, convex cone in $\mathbb{R}^d$ that is pointed and spanning. The notation introduced in this section will be used for the rest of this paper. Fix $i \in \{1, 2, \ldots, d\}$. Let $f : Q_i \to \mathbb{R}$ be continuous. Note that $a + A_f = A_{\Psi_i(a)f}$ for $a \in \mathbb{R}^d$. Hence $-P + A_f \subset A_f$ if and only if $\Psi_i(a)f \leq f$ for every $a \in -P$. In other words, $A_f \in Y_u$ if and only if $f \in \mathcal{F}_i$.

Conversely, suppose $A \in Y_u$. By Lemma 4.4, there exists a unique continuous function $f_i^A : Q_i \to \mathbb{R}$ such that

$$A = A_{f_i^A} = \left\{ \sum_{j=1}^{d} x_j v_j \in \mathbb{R}^d : x_i \leq f_i^A \left(\sum_{k \neq i} x_k v_k \right) \right\}.$$

Since $-P + A \subset A$, $f_i^A \in \mathcal{F}_i$. Hence, the map

$$Y_u \in A \to f_i^A \in \mathcal{F}_i$$

is a bijection. Clearly,

$$X_u = \{A \in Y_u : f_i^A(0) \geq 0\}.$$

Proposition 4.6 *Suppose $(A_n)_{n\in\mathbb{N}}$ is a sequence in Y_u such that $A_n \to A$ in Y_u. Then, the sequence $\{f_i^{A_n}\}_{n\in\mathbb{N}}$ converges pointwise to f_i^A for each $i \in \{1, 2, \ldots, d\}$.*

Proof Fix $i \in \{1, 2, \ldots, d\}$. Suppose (A_n) is a sequence in Y_u converging to $A \in Y_u$. For simplicity, we denote $f_i^{A_n}$ and f_i^A by f_n and f respectively. From the proof of [14, Proposition II.13], $1_{A_n}(a) \to 1_A(a)$ pointwise for every $a \in \mathbb{R}^d \backslash \partial A$, where ∂A is the boundary of A. Recall that

$$A = \left\{ \sum_{i=1}^{d} x_j v_j \in \mathbb{R}^d : x_i \leq f \left(\sum_{k \neq i} x_k v_k \right) \right\}.$$

Let $x \in Q_i$ and $t > 0$. Let $y = x + (f(x) - t)v_i$. Since $y \in Int(A)$, $y \in A_n$ eventually, i.e., for $n \in \mathbb{N}$ sufficiently large, there exists $s_n \geq 0$ and $x_n \in Q_i$ such that $y = x_n + (f_n(x) - s_n)v_i$. Now,

$$f(x) - t = f_n(x) - s_n.$$

for sufficiently large $n \in \mathbb{N}$. In particular,

$$f_n(x) - f(x) \geq -t$$

for large $n \in \mathbb{N}$.

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Let $z = x + (f(x) + t)v_i$. Then, $z \in \mathbb{R}^d \backslash A$. Therefore, $z \in \mathbb{R}^d \backslash A_n$ eventually, i.e., for sufficiently large $n \in \mathbb{N}$, there exists $t_n > 0$ and $z_n \in Q_i$ such that $z = z_n + (f_n(x) + t_n)v_i$. Now,

$$f(x) + t = f_n(x) + t_n$$

for large $n \in \mathbb{N}$. Therefore, for large n,

$$f_n(x) - f(x) \le t. \tag{4.1}$$

Since $t > 0$ is arbitrary, we conclude that $f_n(x) \to f(x)$. This completes the proof. ■

For $i \in \{1, 2, \ldots, d\}$, let $Y_u^i := \{A \in Y_u : f_i^A(0) = 0\}$. Then, Y_u^i is a closed subset of Y_u by Proposition 4.6. Define $\Psi_i : Y_u^i \times \mathbb{R} \to Y_u$ by

$$\Psi_i(A, t) = A + tv_i.$$

The map Ψ_i is a homeomorphism with the inverse given by $\Psi_i^{-1}(A) = (A - f_i^A(0)v_i, f_i^A(0))$. Observe that

$$\Psi_i(A, s + t) = \Psi_i(A, s) + tv_i \tag{4.2}$$

for $A \in Y_u$, $s, t \in \mathbb{R}$.

Remark 4.7 Let $i \in \{1, 2, \ldots, d\}$, and let $X_{u+}^{(i)} := \{A \in X_u : f_i^A(0) > 0\}$. Then, $X_{u+}^{(i)}$ is open in Y_u whose closure is X_u. This is because Ψ_i is a homeomorphism, $X_u = \Psi_i(Y_u^{(i)} \times [0, \infty))$ and $X_{u+}^{(i)} = \Psi_i(Y_u^{(i)} \times (0, \infty))$.

Let μ be an invariant, non-zero Radon measure on Y_u which is fixed until further mention. The measure that we consider on $Y_u \times Y_u$ is the product measure $\mu \times \mu$.

Lemma 4.8 *With the forgoing notation,*

(i) *for each $i \in \{1, 2, \ldots, d\}$, $Y_u^i \subset Y_u$ has measure zero, and*
(ii) *the set $\mathcal{N} = \{(A, B) : f_i^A(0) = f_i^B(0)\} \subset Y_u \times Y_u$ has measure zero.*

Proof Fix $i \in \{1, 2, \ldots, d\}$. Let $\nu = (\Psi_i^{-1})_* \mu$. Since μ is invariant, it follows from (4.2) that ν is invariant under the action of $\mathbb{R}$ on $Y_u^i \times \mathbb{R}$ given by $s.(A, t) = (A, s + t)$. Hence, ν is a product measure of the form $\nu_0 \times m$ where m is the Lebesgue measure on $\mathbb{R}$. Now, $\mu(Y_u^i) = \nu(Y_u^i \times \{0\}) = 0$. This proves (i).

Fix $B \in Y_u$, and let $t := f_i^B(0)$. Then, $1_{\mathcal{N}}(A, B) = 1$ if and only if $f_i^A(0) = f_i^B(0) = t$. But $\{A \in Y_u : f_i^A(0) = t\} = Y_u^i + tv_i$ is a null set. Now (ii) follows from Fubini's theorem. ■

For $n \in \mathbb{N} \cup \{0\}$, we denote the projection from $\Gamma_a(L^2(X_u))$ onto its n-particle space by P_n. For $A, B \in Y_u$, define

$$\varepsilon_i(A, B) := \begin{cases} 1 & \text{if } f_i^A(0) > f_i^B(0), \\ -1 & \text{if } f_i^A(0) < f_i^B(0), \\ 0 & \text{if } f_i^A(0) = f_i^B(0). \end{cases}$$

Consider the additive cocycle $\eta = \{\eta_a\}_{a \in P}$ for V^μ, where $\eta_a = 1_{X_u \backslash X_u + a}$ for $a \in P$. Let Exp_i denote the exponential map of the one parameter product system $\{E^\mu(tv_i)\}_{t \ge 0}$.

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Lemma 4.9 *For $i \in \{1,2,\dots,d\}$ and $t > 0$,*

$$P_2\Big(Exp_i(\eta_{tv_i})\Big)(A,B) = \frac{1}{\sqrt{2}}\varepsilon_i(A,B)\eta_{tv_i}(A)\eta_{tv_i}(B)$$

for almost every $(A,B) \in X_u \times X_u$.

Proof The computation is similar to the computation done in [1, Proposition 5]. Fix $t > 0$ and $i \in \{1,2,\dots,d\}$. For simplicity, write $\eta_{sv_i} = \eta_s$ for $s \geq 0$. Let $x_s^{(1)} = \eta_s$.

By definition,

$$P_2\Big(Exp_i(\eta_{tv_i})\Big) = \int_0^t x_s^{(1)}\, d\eta_s \tag{4.3}$$

$$= \lim_{n\to\infty}\sum_{j=0}^{n-1} \eta_{\frac{jt}{n}} . \eta_{\frac{t}{n}}$$

$$= \lim_{n\to\infty}\sum_{j=0}^{n-1} V_{\frac{jtv_i}{n}}\eta_{\frac{t}{n}} \wedge \eta_{\frac{jt}{n}}$$

$$= \lim_{n\to\infty}\sum_{j=0}^{n-1} (\eta_{\frac{(j+1)t}{n}} - \eta_{\frac{jt}{n}}) \wedge \eta_{\frac{jt}{n}}$$

$$= \lim_{n\to\infty}\sum_{j=0}^{n-1} \eta_{\frac{(j+1)t}{n}} \wedge \eta_{\frac{jt}{n}}. \tag{4.4}$$

Let $s_n = \sum_{j=0}^{n-1} \eta_{\frac{(j+1)t}{n}} \wedge \eta_{\frac{jt}{n}}$ for $n \in \mathbb{N}$. The proof will be over if we show that

$$\lim_{n\to\infty} s_n(A,B) = \frac{1}{\sqrt{2}}\varepsilon_i(A,B)\eta_{tv_i}(A)\eta_{tv_i}(B)$$

for almost every $(A,B) \in X_u\backslash(X_u + tv_i) \times X_u\backslash(X_u + tv_i)$.

For $n \in \mathbb{N}$, suppose $A,B \in X_u$ are such that $f_i^A(0) < \frac{jt}{n}$, $\frac{jt}{n} \leq f_i^B(0) < \frac{(j+1)t}{m}$ for some $j \in \{1,2,\dots,n-1\}$. Then,

$$s_n(A,B) = 1_{X_u\backslash x_u + (\frac{(j+1)tv_i}{n})} \wedge 1_{X_u\backslash x_u+(\frac{jtv_i}{n})}(A,B).$$

Since $A \in X_u\backslash(X_u + \frac{jtv_i}{n})$ and $B \in (X_u + \frac{jtv_i}{n})\backslash(X_u + \frac{(j+1)tv_i}{n})$,

$$s_n(A,B) = 1_{X_u\backslash x_u + (\frac{(j+1)tv_i}{n})} \wedge 1_{X_u\backslash x_u+(\frac{jtv_i}{n})}(A,B) = -\frac{1}{\sqrt{2}}. \tag{4.5}$$

Let $A,B \in X_u\backslash X_u + tv_i$. Suppose $f_i^A(0) < f_i^B(0)$. For sufficiently large $n \in \mathbb{N}$, there exists a unique $j_n \in \{1,2,\dots,n-1\}$ such that $f_i^A(0) < \frac{j_n t}{n}$, $\frac{j_n t}{n} \leq f_i^B(0) < \frac{(j_n+1)t}{n}$. By (4.5), for sufficiently large n,

$$s_n(A,B) = -\frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\Big(\varepsilon_i(A,B)\eta_{tv_i}(A)\eta_{tv_i}(B)\Big).$$

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Similarly, it may be proved that, when $f_i^A(0) > f_i^B(0)$, for n sufficiently large,

$$s_n(A,B) = \frac{1}{\sqrt{2}}\varepsilon_i(A,B)\eta_{tv_i}(A)\eta_{tv_i}(B).$$

The result follows from Lemma 4.8. The proof is complete. ∎

Lemma 4.10 *Suppose $\{Exp_a(1_{X_u\backslash X_u+a})\}_{a\in P}$ is a unit for E^μ. Then, for every $i, j \in \{1, 2, \ldots, d\}$, $\varepsilon_i(A,B) = \varepsilon_j(A,B)$ for almost every $(A,B) \in X_u \times X_u$.*

Proof Write $V^\mu = V$, and let $\eta_a = 1_{X_u\backslash(X_u+a)}$ for $a \in P$. Let $i, j \in \{1, 2, \ldots, d\}$ be given. Since $Exp(\eta)$ is a unit,

$$Exp_i(\eta_{sv_i}).Exp_j(\eta_{tv_j}) = Exp_j(\eta_{tv_j}).Exp_i(\eta_{sv_i}) \tag{4.6}$$

for $i, j \in \{1, 2, \ldots, d\}$ and $s, t > 0$.

Hence,

$$P_2\Big(Exp_i(\eta_{sv_i}).Exp_j(\eta_{tv_j})\Big) = P_2\Big(Exp_j(\eta_{tv_j}).Exp_i(\eta_{sv_i})\Big) \tag{4.7}$$

for $i, j \in \{1, 2, \ldots, d\}$ and $s, t > 0$.

Thanks to Lemma 4.9,

$$P_2\Big(Exp_i(\eta_{sv_i}).Exp_j(\eta_{tv_j})\Big)(A,B) = \varepsilon_i(A,B)\eta_{sv_i}(A)\eta_{sv_i}(B) + V_{sv_i}\eta_{tv_j} \wedge \eta_{sv_i}(A,B) + \varepsilon_j(A - sv_i, B - sv_i)V_{sv_i}\eta_{tv_j}(A)V_{sv_i}\eta_{tv_j}(B)$$

for almost every $(A,B) \in X_u \times X_u$. Similarly,

$$P_2\Big(Exp_j(\eta_{tv_j}).Exp_i(\eta_{sv_i})\Big)(A,B) = \varepsilon_j(A,B)\eta_{tv_j}(A)\eta_{tv_j}(B) + V_{tv_j}\eta_{sv_i} \wedge \eta_{tv_j}(A,B) + \varepsilon_i(A - tv_j, B - tv_j)V_{tv_j}\eta_{sv_i}(A)V_{tv_j}\eta_{sv_i}(B)$$

for almost every $(A,B) \in X_u \times X_u$.

For $s > 0$ and $i \in \{1, 2, \ldots, d\}$, let $L_{i,s} := (1_{X_u}\backslash X_u + sv_i) \times (1_{X_u}\backslash X_u + sv_i)$. Let $s, t > 0$. For almost all $(A,B) \in L_{i,s} \cap L_{j,t}$,

$$P_2\Big(Exp_i(\eta_{sv_i}).Exp_j(\eta_{tv_j})\Big)(A,B) = \varepsilon_i(A,B)$$

and

$$P_2\Big(Exp_j(\eta_{tv_j}).Exp_i(\eta_{sv_i})\Big)(A,B) = \varepsilon_j(A,B).$$

Therefore, (4.7) implies,

$$\varepsilon_i(A,B) = \varepsilon_j(A,B)$$

for almost every $(A,B) \in L_{i,s} \cap L_{j,t}$. Now observe that $X_u \times X_u = \bigcup_{m,n\in\mathbb{N}} L_{i,m} \cap L_{j,n}$. Hence the proof follows. ∎

Notation: *Let $X := supp(\mu) \cap X_u$. Recall that $Y_u\backslash supp(\mu)$ is the largest open set that has μ measure zero. Since μ is invariant, $supp(\mu)$ is $\mathbb{R}^d$-invariant. It follows from Remark 4.7 that $\mu|_X$ has full support.*

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Proposition 4.11 *Suppose $\{Exp_a(1_{X_u\setminus X_u+a})\}_{a\in P}$ is a unit for E^μ. Then, for $A, B \in supp(\mu)$, $A \subset B$ or $B \subset A$.*

Proof Write $\eta_a = 1_{X_u\setminus X_u+a}$ for $a \in P$. Assume that $Exp(\eta)$ is a unit for E^μ. For $A, B \in Y_u$ and $i \in \{1, 2, \ldots, d\}$, note that $A \subset B$ if and only if $f_i^A(x) \leq f_i^B(x)$ for each $x \in Q_i$.

Let $A, B \in X$, and $i \in \{1, 2, \ldots, d\}$. Suppose $f_i^A(0) \leq f_i^B(0)$. Then, we claim the following.

(i) $f_j^A(0) \leq f_j^B(0)$ for each $j \in \{1, 2, \ldots, d\}$, and
(ii) if $t_j > 0$ for $j \in \{1, 2, \ldots, d-1\}$, then

$$f_d^A\left(-\sum_{j=1}^{d-1} t_j v_j\right) \leq f_d^B\left(-\sum_{j=1}^{d-1} t_j v_j\right).$$

Observe that X is a measurable subset of X_u such that $X + P \subset X$. Consider the set

$$N_{i,j} = \{(A', B') \in X \times X : f_i^{A'}(0) < f_i^{B'}(0), f_j^{A'}(0) > f_j^{B'}(0)\}$$

for $i, j \in \{1, 2, \ldots, d\}$. By Proposition 4.6, $N_{i,j}$ is an open set, and by Lemma 4.10, $N_{i,j}$ is a null set. Therefore, $N_{i,j}$ is empty. Similarly, consider the set

$$M_{i,j} = \{(A', B') \in X \times X : f_i^{A'}(0) = f_i^{B'}(0), f_j^{A'}(0) > f_j^{B'}(0)\}$$

for $i, j \in \{1, 2, \ldots, d\}$. Fix $i, j \in \{1, 2, \ldots, d\}$ with $i \neq j$. Suppose $(A', B') \in M_{i,j}$. Let $s, t > 0$ be sufficiently small such that $s < t$ and $f_j^{A'}(0) > f_j^{B'}(-tv_i)$. Then,

$$f_i^{A'+sv_i} = f_i^{A'}(0) + s < f_i^{B'}(0) + t = f_i^{B'+sv_i}.$$

Since $f_j^{A'} \in \mathcal{F}_j$,

$$f_j^{A'+sv_i}(0) = f_j^{A'}(-sv_i) \geq f_j^{A'}(0) > f_j^{B'}(-tv_i) = f_j^{B'+tv_i}(0).$$

This implies $(A' + sv_i, B' + tv_i) \in N_{i,j}$, which is a contradiction. Hence, $M_{i,j} = \emptyset$. Therefore, for $A', B' \in X$ and $i, j \in \{1, 2, \ldots d\}$, $f_i^{A'}(0) \leq f_i^{B'}(0)$ if and only if $f_j^{A'}(0) \leq f_j^{B'}(0)$, which proves (i).

For each $j \in \{1, 2, \ldots, d-1\}$, let $t_j > 0$. Let $A, B \in X$ be such that $f_i^A(0) \leq f_i^B(0)$ for some i. Since $f_i^A(0) \leq f_i^B(0)$, by (i), $f_1^A(0) \leq f_1^B(0)$. Since $f_1^{A+t_1v_1}(0) \leq f_1^{B+t_1v_1}(0)$, by (i), $f_2^{A+t_1v_1}(0) \leq f_2^{B+t_1v_1}(0)$, i.e.,

$$f_2^A(-t_1v_1) \leq f_2^B(-t_1v_1).$$

Since $f_2^{A+t_1v_1}(0) + t_2 \leq f_2^{B+t_1v_1}(0) + t_2$,

$$f_2^{A+t_1v_1+t_2v_2}(0) \leq f_2^{B+t_1v_1+t_2v_2}(0).$$

Once again by (i),

$$f_3^A(-t_1v_1 - t_2v_2) = f_3^{A+t_1v_1+t_2v_2}(0) \leq f_3^{B+t_1v_1+t_2v_2}(0) = f_3^B(-t_1v_1 - t_2v_2).$$

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Inductively,

$$f_d^A\left(-\sum_{j=1}^{d-1} t_j v_j\right) \leq f_d^B\left(-\sum_{j=1}^{d-1} t_j v_j\right). \tag{4.8}$$

We have proved (ii).

It follows from (i) that for $i \in \{1,2,\dots,d\}$ and $A, B \in X$, $f_i^A(0) > f_i^B(0)$ if and only if $f_j^A(0) > f_j^B(0)$ for every $j \in \{1,2,\dots,d\}$. Suppose $f_i^A(0) > f_i^B(0)$ for $A, B \in X$ and $i \in \{1,2,\dots,d\}$. Arguing as before, we see that

$$f_d^A\left(-\sum_{j=1}^{d-1} t_j v_j\right) > f_d^B\left(-\sum_{j=1}^{d-1} t_j v_j\right) \tag{4.9}$$

whenever $t_j > 0$ for $j \in \{1,2,\dots,d-1\}$.

Let $A, B \in X$. Without loss of generality, we can assume that $f_d^A(0) \leq f_d^B(0)$. Then,

$$f_d^A\left(-\sum_{j=1}^{d-1} t_j v_j\right) \leq f_d^B\left(-\sum_{j=1}^{d-1} t_j v_j\right) \tag{4.10}$$

whenever $t_j > 0$ for $j \in \{1,2,\dots,d-1\}$.

Let $x \in Q_d$. Choose $t > 0$ such that $A - x + tv_d \in X$ and $B - x + tv_d \in X$, i.e.,

$$f_d^{A-x+tv_d}(0) = f_d^A(x) + t \geq 0$$

and

$$f_d^{B-x+tv_d}(0) = f_d^B(x) + t \geq 0.$$

Let $\widetilde{A} = A - x + tv_d$ and $\widetilde{B} = B - x + tv_d$. Suppose $f_d^A(x) > f_d^B(x)$. Then,

$$f_d^{\widetilde{A}}(0) = f_d^A(x) + t > f_d^B(x) + t = f_d^{\widetilde{B}}(0).$$

By (4.9),

$$f_d^{\widetilde{A}}\left(-\sum_{j=1}^{d-1} r_j v_j\right) > f_d^{\widetilde{B}}\left(-\sum_{j=1}^{d-1} r_j v_j\right)$$

whenever $r_j > 0$ for $j = 1,2,\dots,d-1$. This implies

$$f_d^A\left(\sum_{j=1}^{d-1} (x_j - r_j) v_j\right) > f_d^B\left(\sum_{j=1}^{d-1} (x_j - r_j) v_j\right)$$

whenever $r_j > 0$ for $j = 1,2,\dots,d-1$. This contradicts (4.10). Hence, $f_d^A(x) \leq f_d^B(x)$ for every $x \in Q_d$, i.e., $A \subset B$.

Now suppose $A, B \in supp(\mu)$. Then, there exists $t > 0$ such that $A + tv_1, B + tv_1 \in X$. Without loss of generality, assume that $A + tv_1 \subset B + tv_1$. Then, $A \subset B$. Therefore, if $A, B \in supp(\mu)$, $A \subset B$ or $B \subset A$. The proof is complete. ■

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Theorem 4.12 *The following statements are equivalent.*

(i) $\{Exp(1_{X_u \backslash X_u + a})\}_{a \in P}$ *is a unit for* E^μ.
(ii) *There exists* $\lambda \in S(P^*)$ *such that* $supp(\mu) = Y_u^\lambda$.

Proof By Lemma 4.2, if $supp(\mu) = Y_u^\lambda$ for some $\lambda \in S(P^*)$, then $\{Exp(1_{X_u \backslash X_u + a})\}_{a \in P}$ is a unit for E^μ.

To prove the converse, let $A \in supp(\mu)$. Since μ is invariant, $supp(\mu)$ is $\mathbb{R}^d$-invariant. By Lemma 4.11, for $x \in \mathbb{R}^d$, either $A + x \subset A$ or $A \subset A + x$. Define $Q_A := \{x \in \mathbb{R}^d : A + x \subset A\}$. Then, $Q_A \cup -Q_A = \mathbb{R}^d$. Note that $-P \subset Q_A$ and $-P + Q_A \subset Q_A$. Let $Q := Q_A \cap -Q_A$. We claim the following:

(i) $\partial Q_A = Q$, where ∂Q_A is the boundary of Q_A, and
(ii) Q is a vector subspace of $\mathbb{R}^d$.

We identify $span\{v_1, v_2, \ldots v_{d-1}\}$ with $\mathbb{R}^{d-1}$. Let $f : \mathbb{R}^{d-1} \to \mathbb{R}$ be a continuous function such that $A = \{(x, t) : t \leq f(x)\}$. Note that

$$Q_A = \{(y, t) \in \mathbb{R}^{d-1} \times \mathbb{R} : f(x) \leq f(x + y) - t \text{ for all } x \in \mathbb{R}^{d-1}\},$$

and

$$-Q_A = \{(y, t) \in \mathbb{R}^{d-1} \times \mathbb{R} : f(x) \geq f(x + y) - t \text{ for all } x \in \mathbb{R}^{d-1}\}.$$

It is not difficult to deduce using the fact that $Q_A \cup -Q_A = \mathbb{R}^d$ that

$$\partial Q_A \subset \{(y, t) \in \mathbb{R}^{d-1} \times \mathbb{R} : f(x) = f(x + y) - t \text{ for all } x \in \mathbb{R}^{d-1}\} = Q.$$

Suppose $(y, t) \in Q$. Set $t_n := t - \frac{1}{n}$ and $s_n := t + \frac{1}{n}$. Then, $(y, t_n) \in Q_A$ and $(y, t_n) \to (y, t)$. Also, $(y, s_n) \notin Q_A$ and $(y, s_n) \to (y, t)$. Therefore, $(y, t) \in \partial Q_A$. Hence, $\partial Q_A = Q$.

Let $g : \mathbb{R}^{d-1} \to \mathbb{R}$ be a continuous function such that $Q_A = \{(x, t) : t \leq g(x)\}$. Clearly, $Q = \{x \in \mathbb{R}^d : A + x = A\}$ is a closed subgroup of $\mathbb{R}^d$. Since $Q = \partial Q_A$ is the graph of the continuous function g, it is connected. Therefore, Q is a vector space and consequently, g is linear. Hence, there exists $\lambda \in \mathbb{R}^d$, $||\lambda|| = 1$ such that

$$Q_A = \{y \in \mathbb{R}^d : \langle \lambda | y \rangle \leq 0\}.$$

Since $-P \subset Q_A$, $\lambda \in S(P^*)$.

Since $Q_A + A \subset A$, there exists $y \in \mathbb{R}^d$ such that $A = y + Q_A$, i.e., $A \in Y_u^\lambda$. Since $supp(\mu) \subset Y_u$ is invariant, $x + A = x + y + Q_A \in supp(\mu)$ for each $x \in \mathbb{R}^d$, i.e., $Y_u^\lambda \subset supp(\mu)$. To stress the dependence of λ on A, we write $\lambda = \lambda_A$. We have proved that

$$supp(\mu) = \bigcup_{A \in supp(\mu)} Y_u^{\lambda_A}.$$

Suppose $A, B \in supp(\mu)$. Note that $Q_A \in Y_u^{\lambda_A} \subset supp(\mu)$ and $Q_B \in Y_u^{\lambda_B} \subset supp(\mu)$. Then, by Proposition 4.11, either $Q_A \subset Q_B$ or $Q_B \subset Q_A$. But this can happen only if $\lambda_A = \lambda_B$. In that case, $Y_u^{\lambda_A} = Y_u^{\lambda_B}$. Therefore, there exists $\lambda \in S(P^*)$ such that $supp(\mu) = Y_u^\lambda$. Now the proof is complete. ∎

Notation and Convention: *Let* V *be an isometric representation of* P *on a Hilbert space* $\mathcal{H}$. *Let* $\xi = \{\xi_a\}_{a \in P} \in \mathcal{A}(V)$. *We define* $\mathcal{H}^\xi$ *to be the smallest, closed, reducing subspace of* $\mathcal{H}$ *containing* $\{\xi_a : a \in P\}$. *(Hereafter, all reducing subspaces will be assumed to be*

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closed.) If W is a direct summand of V, we view the product system E^W as a subsystem of E^V. Similarly, we view $\mathcal{A}(W)$ as a subspace of $\mathcal{A}(V)$.

Proposition 4.13 *Let V be a pure isometric representation of P on a Hilbert space $\mathcal{H}$ with commuting range projections. Let $\xi = \{\xi_a\}_{a \in P} \in \mathcal{A}(V)$ be non-zero. Then,*

1. *there exists an $\mathbb{R}^d$-invariant, non-zero Radon measure μ on Y_u such that $V|_{\mathcal{H}^\xi}$ is unitarily equivalent to V^μ, and*
2. *if $Exp(\xi)$ is a unit for E^V, then there exists $\lambda \in S(P^*)$ such that $V|_{\mathcal{H}^\xi}$ is unitarily equivalent to S^λ.*

Proof For the proof of (1), we refer the reader to [34, Theorem 3.16 and Remark 3.17]. From (1), $V^\xi := V|_{\mathcal{H}^\xi}$ is unitarily equivalent to V^μ. We can see from the proof of [34, Theorem 3.16] that the unitary U intertwining V^ξ and V^μ can be chosen such that $U\xi_a = 1_{X_u \backslash X_u + a}$ for $a \in P$. Suppose $Exp(\xi)$ is a unit for E^V. Note that the subsystem E^{V^ξ} contains $Exp(\xi)$. Hence, $\{Exp(1_{X_u \backslash X_u + a})\}_{a \in P}$ is a unit for E^{V^μ}. It follows from Theorem 4.12 and Lemma 4.2 that V^μ is unitarily equivalent to S^λ for some $\lambda \in S(P^*)$. This proves (2). ∎

Before proceeding further, let us recall the notation introduced in the introduction. For $\lambda \in S(P^*)$ and $k \in \mathbb{N}_\infty$, let $S^{(\lambda,k)}$ denote the isometric representation $\{S_{\langle \lambda | a \rangle} \otimes 1\}_{a \in P}$ of P acting on the Hilbert space $L^2[0,\infty) \otimes \mathcal{K}$ where $\mathcal{K}$ is a Hilbert space of dimension k. For $k = 1$, we denote $S^{(\lambda,1)}$ by S^λ. For a non-empty countable set I, an injective map $\lambda : I \to S(P^*)$ and a function $k : I \to \mathbb{N}_\infty$, set

$$S^{(\lambda,k)} := \bigoplus_{i \in I} S^{(\lambda_i,k_i)}.$$

Let $E^{(\lambda,k)}$ be the product system of the CAR flow associated with $S^{(\lambda,k)}$.

Remark 4.14 A few properties concerning the representation $S^{(\lambda,k)}$ are summarized below.

1. For $\lambda_1, \lambda_2 \in S(P^*)$ and for $k_1, k_2 \in \mathbb{N}_\infty$, $S^{(\lambda_1,k_1)}$ is unitarily equivalent to $S^{(\lambda_2,k_2)}$ if and only if $\lambda_1 = \lambda_2$ and $k_1 = k_2$. Also, S^λ is irreducible for every $\lambda \in S(P^*)$, i.e., it has no non-zero non-trivial reducing subspace. Thus, if $\lambda_1 \neq \lambda_2$, the representations $S^{(\lambda_1,k_1)}$ and $S^{(\lambda_2,k_2)}$ are disjoint. Recall that two isometric representations V and W, acting on $\mathcal{H}$ and $\mathcal{K}$, respectively, are said to be disjoint if
$$\{T \in B(\mathcal{H},\mathcal{K}) : TV_a = W_a T, TV_a^* = W_a^* T \text{ for all } a \in P\} = 0.$$
2. Consider a countable (non-empty) indexing set I. Let $\lambda : I \to S(P^*)$ be injective, and let $k : I \to \mathbb{N}_\infty$ be a map. Consider $V = S^{(\lambda,k)} = \bigoplus_{i \in I} S^{(\lambda_i,k_i)}$ acting on the Hilbert space $\mathcal{H} = \bigoplus_{i \in I} L^2[0,\infty) \otimes \mathcal{K}_i$, where $\mathcal{K}_i$ is of dimension k_i. Write $V^{(i)} = S^{(\lambda_i,k_i)}$. Then, $V^{(i)}$ acts on $\mathcal{H}_i := L^2[0,\infty) \otimes \mathcal{K}_i$. Since $\lambda_i \neq \lambda_j$ whenever $i \neq j$, the isometric representations $V^{(i)}$ and $V^{(j)}$ are disjoint wh enever $i \neq j$.

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Thus, a bounded operator $T \in \{V_a, V_a^* : a \in P\}'$ if and only if there exists $T_i \in \{V_a^{(i)}, V_a^{(i)*} : a \in P\}'$ such that $T|_{\mathcal{H}_i} = T_i$ for each $i \in I$. Moreover, $\{V_a^{(i)}, V_a^{(i)*} : a \in P\}' = \{1 \otimes R : R \in B(\mathcal{K}_i)\}$. Hence,

$$\{V_a, V_a^* : a \in P\}' = \bigoplus_{i \in I} B(\mathcal{K}_i).$$

(3) It follows from (2) that the reducing subspaces of V are of the form $\bigoplus_{j \in J} L^2[0,\infty) \otimes \mathcal{W}_j$ for some non-empty subset J of I, and where, for j, $\mathcal{W}_j$ is a subspace of $\mathcal{K}_j$. Suppose $\mathcal{W}$ is a non-zero reducing subspace for V such that $V|_{\mathcal{W}}$ is irreducible. Then, there exists $i \in I$ and a one dimensional subspace $\mathcal{W}_i$ of $\mathcal{K}_i$ such that $\mathcal{W} \subset \mathcal{H}_i = L^2[0,\infty) \otimes \mathcal{K}_i$ and $\mathcal{W} = L^2[0,\infty) \otimes \mathcal{W}_i$. Moreover, $V|_{\mathcal{W}}$ is unitarily equivalent to S^{λ_i}.

The next proposition is the "only if part" of Theorem 1.2. The "uniqueness" part of Theorem 1.2 follows from the fact that if an isometric representation admits a direct sum decomposition of irreducible representations, then the decomposition is "unique".

Proposition 4.15 *Let V be a pure isometric representation of P with commuting range projections on a Hilbert space $\mathcal{H}$. Suppose the product system E^V of the CAR flow associated with V is type I. Then, there exists a non-empty, countable set I, a map $\lambda : I \to S(P^*)$ which is injective, and a map $k : I \to \mathbb{N}_\infty$ such that V is unitarily equivalent to $S^{(\lambda,k)}$. Equivalently, E^V is isomorphic to $E^{(\lambda,k)}$.*

Proof A family $\mathscr{S}$ of closed subspaces of $\mathcal{H}$ is said to be "shift reducing" if

(1) each $\mathcal{K} \in \mathscr{S}$ is a non-zero reducing subspace for V,
(2) for each $\mathcal{K} \in \mathscr{S}$, there exists $\lambda \in S(P^*)$ such that $V|_{\mathcal{K}}$ is unitarily equivalent to S^λ, and
(3) if $\mathcal{K}, \mathcal{L} \in \mathscr{S}$ and $\mathcal{K} \neq \mathcal{L}$, then $\mathcal{K} \perp \mathcal{L}$.

Let $\mathcal{W} := \{\mathscr{S} : \mathscr{S}$ is shift reducing$\}$. Note that $\mathcal{W}$ is partially ordered where the partial order on $\mathcal{W}$ is given by inclusion.

Since E^V is type I, there exists a non-zero additive cocycle $\xi = \{\xi_a\}_{a \in P}$ such that $Exp(\xi)$ is a unit for E^V. Hence, $\{\mathcal{H}^\xi\}$ is in $\mathcal{W}$ by Proposition 4.13. Therefore, $\mathcal{W}$ is non-empty. A routine application of Zorn's lemma allows us to get a maximal shift reducing family $\mathscr{K}$ of closed subspaces.

Define

$$\mathcal{K} := \bigoplus_{W \in \mathscr{K}} W.$$

Note that for each $W \in \mathcal{K}$, there exists $\lambda_W \in S(P^*)$ such that $V|_W$ is unitarily equivalent to S^{λ_W}. Hence, $V|_{\mathcal{K}} = \bigoplus_{W \in \mathcal{K}} S^{\lambda_W}$. Let $\mathcal{L} := \mathcal{K}^\perp$. Note that $\mathcal{K}$ and $\mathcal{L}$ are reducing subspaces for V. Denote the restriction of V to $\mathcal{K}$ and $\mathcal{L}$ by $V^{(1)}$ and $V^{(2)}$, respectively. Also, denote the projection of $\mathcal{H}$ onto $\mathcal{L}$ by Q.

It suffices to prove that $V = V^{(1)}$. Denote by $E^{V^{(1)}}$ the product system of the CAR flow associated with $V^{(1)}$. We consider $E^{V^{(1)}}$ as a subsystem of E^V. We claim that

$E^{V^{(1)}} = E^V$. Suppose $u = \{u_a\}_{a\in P}$ is a unit for E^V. We claim that $E^{V^{(1)}}$ contains u. Without loss of generality, we can assume u is exponential. Then, there exists $\xi = \{\xi_a\}_{a\in P} \in \mathcal{A}(V)$ such that $u_a = Exp_a(\xi_a)$ for $a \in P$. For $a \in P$, let $\xi_a^{(1)} = (1-Q)\xi_a$ and $\xi_a^{(2)} = Q\xi_a$. Then, $\xi^{(i)} = \{\xi_a^{(i)}\}_{a\in P} \in \mathcal{A}(V^{(i)})$ for $i = 1, 2$.

Note that $\Omega = \{\Omega_a\}_{a\in P} : E^V \to E^V$ given by

$$E^V(a) \ni u \mapsto \Gamma(Q)u \in E^V(a)$$

is multiplicative, where $\Gamma(Q)$ is the second quantisation of Q. Also, $\Gamma(Q)u_a = Exp_a(\xi_a^{(2)})$. Therefore, $\{Exp_a(\xi_a^{(2)})\}_{a\in P}$ is a unit for E^V.

Assume that $\xi^{(2)}$ is non-zero. Consider the reducing subspace $\mathcal{H}^{\xi^{(2)}}$ of V. Then, by Proposition 4.13 there exists $\lambda \in S(P^*)$ such that $V|_{\mathcal{H}^{\xi^{(2)}}}$ is unitarily equivalent to S^λ. Now, $\mathscr{K} \cup \{\mathcal{H}^{\xi^{(2)}}\} \in \mathcal{W}$ and contains $\mathscr{K}$ as a proper subset. This contradicts the maximality of $\mathscr{K}$. Therefore, $\xi^{(2)} = 0$. Thus, $E^{V^{(1)}}$ contains u. Hence, every unit of E^V is contained in $E^{V^{(1)}}$. Since E^V is type I, $E^V = E^{V^{(1)}}$.

Since

$$E^{V^{(1)}}(a) = \Gamma_a(Ker(V_a^{(1)*})) = \Gamma_a(Ker(V_a^*)) = E^V(a)$$

for $a \in P$, $Ker(V_a^{(1)*}) = Ker(V_a^*)$ for $a \in P$. Since $\bigcup_{a\in P} Ker(V_a^*)$ is dense in $\mathcal{H}$, $\mathcal{H} = \mathcal{K}$. Therefore, $V = V^{(1)}$. The proof is complete. ∎

Fix a countable (non-empty) indexing set I. Let $\lambda : I \to S(P^*)$ be injective, and let $k : I \to \mathbb{N}_\infty$ be a map. Let $V = S^{(\lambda,k)}$. Denote the space $\bigoplus_{i\in I} L^2[0,\infty) \otimes \mathcal{K}_i$ (on which V acts) by $\mathcal{H}$, where $\mathcal{K}_i$ is a Hilbert space of dimension $k_i \in \mathbb{N}_\infty$. For each $i \in I$, let $V^{(i)} = S^{(\lambda_i,k_i)}$, and write $\mathcal{H}_i := L^2[0,\infty) \otimes \mathcal{K}_i$ (the Hilbert space on which $V^{(i)}$ acts). Note that for $i \in I$, we may view $\mathcal{A}(S^{(\lambda_i,k_i)})$ as a subspace of $\mathcal{A}(V)$ under the natural inclusion.

Proposition 4.16 *Let $\xi \in \mathcal{A}(V)$. Then, $\{Exp_a(\xi_a)\}_{a\in P}$ is a unit of E^V if and only if there exists $i \in I$ such that $\xi \in \mathcal{A}(S^{(\lambda_i,k_i)})$.*

Proof Suppose $\xi = \{\xi_a\}_{a\in P} \in \mathcal{A}(V)$ is non-zero and is such that $Exp(\xi)$ is a unit of E^V. Consider the reducing subspace $\mathcal{H}^\xi$ of V. By Proposition 4.13, $V|_{\mathcal{H}^\xi}$ is unitarily equivalent to S^λ for some $\lambda \in S(P^*)$. Since S^λ is irreducible, by Remark 4.14, $\mathcal{H}^\xi$ is a subspace of $\mathcal{H}_i$ and $\lambda = \lambda_i$ for some $i \in I$. In particular, $\xi \in \mathcal{A}(S^{\lambda_i,k_i})$.

Conversely, suppose $\xi = \{\xi_a\}_{a\in P} \in \mathcal{A}(S^{(\lambda_i,k_i)})$ for some $i \in I$. Then, there exists an additive cocycle $\eta = \{\eta_a\} \in \mathcal{A}(S^{\lambda_i})$ and $\gamma \in \mathcal{K}_i$ such that $\xi_a = \eta_a \otimes \gamma$ for $a \in P$. In turn, there is an additive cocycle $\widetilde{\eta} = \{\widetilde{\eta}_t\}_{t\geq 0}$ of $\{S_t\}_{t\geq 0}$ such that $\eta_a = \widetilde{\eta}_{\langle \lambda_i|a\rangle}$ for $a \in P$. Denote the exponential map of the product system of the 1-parameter CAR flow associated with $\{S_t \otimes 1\}_{t\geq 0}$ acting on $\mathcal{H}_i$ by Exp. By Remark 2.2,

$$Exp(\widetilde{\eta}_s \otimes \gamma).Exp(\widetilde{\eta}_t \otimes \gamma) = Exp(\widetilde{\eta}_{s+t} \otimes \gamma)$$

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for $s, t \geq 0$. Then, for $a, b \in P$,

$$\begin{aligned} Exp_a(\xi_a).Exp_b(\xi_b) &= Exp_a(\eta_a \otimes \gamma).Exp_b(\eta_b \otimes \gamma) \\ &= Exp(\widetilde{\eta}_{\langle \lambda_i | a \rangle} \otimes \gamma).Exp(\widetilde{\eta}_{\langle \lambda_i | b \rangle} \otimes \gamma) \\ &= Exp(\widetilde{\eta}_{\langle \lambda_i | a+b \rangle} \otimes \gamma) \\ &= Exp_{a+b}(\xi_{a+b}) \end{aligned}$$

for $a, b \in P$, i.e., $Exp(\xi)$ is a unit. This completes the proof. ∎

The following is the "if part" of Theorem 1.2.

Proposition 4.17 *Keeping the forgoing notation, the CAR flow associated with V is type I.*

Proof Let F be a subsystem of E^V containing all the units of E^V. For $a \in P$, let Ψ_a be the projection of $E^V(a)$ onto $F(a)$. Then, $\Psi = \{\Psi_a\}_{a \in P}$ is multiplicative, i.e.,

$$\Psi_a.\Psi_b = \Psi_{a+b}$$

for $a, b \in P$. Let $a \in P$. Consider the one parameter product system $E_a = \{E(ta)\}_{t \geq 0}$ and the multiplicative section of maps $\{\Psi_{ta}\}_{t \geq 0}$. Let $\widetilde{E}_a$ be the product system of the 1-parameter CCR flow associated with $\{V_{ta}\}_{t \geq 0}$. Suppose $\widetilde{\Psi} = \{\mathcal{Q}_t\}_{t \geq 0} : \widetilde{E}_a \to \widetilde{E}_a$ is such that

(i) $\mathcal{Q}_t$ is a projection for $t \geq 0$, and
(ii) $\mathcal{Q}_s.\mathcal{Q}_t = \mathcal{Q}_{s+t}$ for $s, t \geq 0$.

Then, by [10, Theorem 7.6] (see also [36, Proposition 6.12]), there exists an additive cocycle $\{\xi_t^a\}_{t \geq 0}$ of $\{V_{ta}\}_{t \geq 0}$, a projection $Q^a \in \{V_{ta}, V_{ta}^* : t \geq 0\}'$ such that $(1 - Q^a)\xi_t^a = \xi_t^a$ for $t \geq 0$, and $\mu_a \in \mathbb{R}$ such that

$$\widetilde{\Psi}_t e(\eta) = e^{\mu_a t} e^{\langle \eta | \xi_t^a \rangle} e(Q^a \eta + \xi_t^a)$$

for $\eta \in Ker(V_{ta}^*)$ and $t \geq 0$.

By Lemma 3.1, the map $\widetilde{E}_a(t) \ni exp(\xi) \to Exp_a(\xi) \in E_a(t)$ extends to an isomorphism from $\widetilde{E}_a$ onto E_a. Therefore, there exists an additive cocycle $\{\xi_t^a\}_{t \geq 0}$ of $\{V_{ta}\}_{t \geq 0}$, a projection $Q^a \in \{V_{ta}, V_{ta}^* : t \geq 0\}'$ with $(1 - Q^a)\xi_t^a = \xi_t^a$ for $t \geq 0$ and $\mu_a \in \mathbb{R}$, such that

$$\Psi_{ta} Exp_a(\eta) = e^{\mu_a t} e^{\langle \eta | \xi_t^a \rangle} Exp_a(Q^a \eta + \xi_t^a)$$

for $\eta \in Ker(V_{ta}^*)$.

For $a \in P$, let $\xi_a := \xi_1^a$. Proceeding as in the proof of Theorem 3.3, we can prove that

(i) $\{\xi_a\}_{a \in P}$ is an additive cocycle for V,
(ii) there exists a projection $Q \in \{V_a, V_a^* : a \in P\}'$ such that $Q|_{Ker(V_a^*)} = Q_a$
(iii) the map $P \ni a \mapsto \mu_a$ is a continuous homomorphism. Hence, there exists $\mu \in \mathbb{R}^d$ such that $\mu_a = \langle \mu | a \rangle$ for $a \in P$.

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Therefore, there exists an additive cocycle $\xi = \{\xi_a\}_{a\in P}$, a projection $Q \in \{V_a, V_a^* : a \in P\}'$ and a vector $\mu \in \mathbb{R}^d$ such that

$$\Psi_a Exp_a(\eta) = e^{\langle\mu|a\rangle} e^{\langle\eta|\xi_a\rangle} Exp_a(Q\eta + \xi_a) \tag{4.11}$$

for $a \in P, \eta \in Ker(V_a^*)$.

Suppose $u = \{u_a\}_{a\in P}$ is an exponential unit for E^V. Let $\eta = \{\eta_a\}_{a\in P} \in \mathcal{A}(V)$ be such that $u_a = Exp_a(\eta_a)$ for $a \in P$. Since F contains all the units, it follows that $\Psi_a Exp_a(\eta_a) = Exp_a(\eta_a)$ for $a \in P$. In particular,

$$\Psi_a Exp_a(0) = Exp_a(0)$$

for $a \in P$. By (4.11),

$$e^{\langle\mu|a\rangle} Exp_a(\xi_a) = Exp_a(0)$$

for $a \in P$. This implies, $\xi_a = 0$ for $a \in P$ and $\mu = 0$.

Note that since $Q \in \{V_a, V_a^* : a \in P\}'$, by Remark 4.14, Q is a diagonal operator, i.e., for $i \in I$, there exists a projection $Q^{(i)} \in \{V_a^{(i)}, V_a^{(i)*} : a \in P\}'$ such that $Q|_{\mathcal{H}_i} = Q^{(i)}$. Fix $i \in I$. By Proposition 4.16, for any $\eta = \{\eta_a\}_{a\in P} \in \mathcal{A}(V^{(i)})$, $Exp(\eta)$ is a unit for E^V. Hence

$$\Psi_a Exp_a(\eta_a) = Exp_a(\eta_a)$$

for $\eta = \{\eta_a\}_{a\in P} \in \mathcal{A}(V^{(i)})$. This implies that

$$Q\eta_a = Q^{(i)}\eta_a = \eta_a \tag{4.12}$$

for $a \in P$ and $\eta = \{\eta_a\}_{a\in P} \in \mathcal{A}(V^{(i)})$. Note that for $\gamma \in \mathcal{K}_i$, $\{1_{(0,\langle\lambda_i|a\rangle)} \otimes \gamma\}_{a\in P}$ is an additive cocycle for $V^{(i)}$. Therefore, the set $\{\eta_a : \{\eta_a\}_{a\in P} \in \mathcal{A}(V^{(i)}), a \in P\}$ is total in $\mathcal{H}_i$. Now, (4.12) implies that

$$Q^{(i)}\eta = \eta$$

for $\eta \in \mathcal{H}_i$ for $i \in I$, i.e., Q is the identity operator. Hence,

$$\Psi_a Exp_a(\xi) = Exp_a(\xi)$$

for $\xi \in Ker(V_a^*)$ and $a \in P$. Since $\{Exp_a(\xi) : \xi \in Ker(V_a^*)\}$ is total in $E^V(a)$, it follows that Ψ_a is the identity operator for each $a \in P$. Therefore, $F = E^V$. Hence the proof. ■

5 Computation of index and gauge Group

In this section, we compute the index and the gauge group of the product system $E^{(\lambda,k)}$. Arveson's definition of index for a 1-parameter product system was extended to the multiparameter case in [24] which we first recall. Let E be a product system over P. Denote the set of units of E by $\mathcal{U}_E$. Assume that $\mathcal{U}_E \neq \emptyset$. Fix $a \in Int(P)$. For $u, v \in \mathcal{U}_E$, let $c_a(u,v) \in \mathbb{C}$ be such that

$$\langle u_{ta}|v_{ta}\rangle = e^{tc_a(u,v)}$$

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for $t \geq 0$. The function $\mathcal{U}_E \times \mathcal{U}_E \ni (u,v) \to c_a(u,v) \in \mathbb{C}$ is conditionally positive definite and is called the *covariance function* of E with respect to a.

Let $\mathcal{H}(\mathcal{U}_E)$ be the Hilbert space obtained from the covariance function using the GNS construction. For the sake of completeness and also to fix notation, we brief the construction of $\mathcal{H}(\mathcal{U}_E)$. Let $\mathbb{C}_c(\mathcal{U}_E)$ denote the vector space of finitely supported complex valued functions on $\mathcal{U}_E$. Set

$$\mathbb{C}_0(\mathcal{U}_E) := \{f \in \mathbb{C}_c(\mathcal{U}_E) : \sum_{u \in \mathcal{U}_E} f(u) = 0\}.$$

Define a semi-definite inner product on $\mathbb{C}_0(\mathcal{U}_E)$ by

$$\langle f|g\rangle = \sum_{u,v \in \mathcal{U}} c_a(u,v) f(u)\overline{g(v)}.$$

Let $\mathcal{H}(\mathcal{U}_E)$ be the Hilbert space obtained by completing the semi-definite inner product space $\mathbb{C}_0(\mathcal{U}_E)$.

For $u \in \mathcal{U}_E$, let $\delta_u : \mathcal{U}_E \to \mathbb{C}$ be the indicator function $1_{\{u\}}$. Clearly, $\{\delta_u - \delta_v : u,v \in \mathcal{U}_E\}$ is total in $\mathcal{H}(\mathcal{U}_E)$. The dimension of the space $\mathcal{H}(\mathcal{U}_E)$ is independent of the choice of $a \in Int(P)$ and is called the *index* of E denoted $Ind(E)$. The reader is referred to [24, Proposition 2.4] for a proof of this statement.

Consider the isometric representation $V = S^{(\lambda,k)}$ for a non-empty countable indexing set I, an injective map $\lambda : I \to S(P^*)$ and a map $k : I \to \mathbb{N}_\infty$. The isometric representation V shall remain fixed for the rest of this section. Let $\mathcal{H} := \bigoplus_{i \in I} L^2[0,\infty) \otimes \mathcal{K}_i$ be the Hilbert space on which V acts, where $\mathcal{K}_i$ is a Hilbert space of dimension $k_i \in \mathbb{N}_\infty$ for $i \in I$. For $i \in I$, let $V^{(i)} = S^{(\lambda_i,k_i)}$, which acts on $\mathcal{H}_i = L^2[0,\infty) \otimes \mathcal{K}_i$. Denote the set of units of E^V by $\mathcal{U}$. Similarly, denote the set of units of $E^{V^{(i)}}$ by $\mathcal{U}_i$ for each $i \in I$. Then, $\mathcal{U}_i \subset \mathcal{U}$ for $i \in I$. Let $c(.,.)$ be the covariance function of E^V with respect to a fixed $a \in Int(P)$. Let $\mathcal{U}^\Omega$, $\mathcal{U}_i^\Omega$ respectively denote the set of exponential units of E^V and $E^{V^{(i)}}$. By Proposition 4.16, $\mathcal{U}^\Omega = \bigcup_{i \in I} \mathcal{U}_i^\Omega$.

Remark 5.1 Let $u \in \mathcal{U}$. Let $\widetilde{u} \in \mathcal{U}^\Omega$ be given by $\widetilde{u}_b = \frac{u_b}{\langle u_b|\Omega_b\rangle}$ for $b \in P$, where Ω_b is the vacuum vector in $E^V(b)$. Fix $a \in Int(P)$. Now, for $v \in \mathcal{U}$,

$$\langle u_{ta}|v_{ta}\rangle = \langle u_{ta}|\Omega_{ta}\rangle e^{tc(\widetilde{u}_{ta},v_{ta})}.$$

Since the map $[0,\infty) \ni t \to \langle u_{ta}|\Omega_{ta}\rangle \in \mathbb{C}^*$ is multiplicative, there exists $z_u \in \mathbb{C}$ such that $\langle u_{ta}|\Omega_{ta}\rangle = e^{tz_u}$ for $t \geq 0$. Hence,

$$c(u,v) = z_u + c(\widetilde{u},v).$$

It is now routine to verify that if $w,z \in \mathcal{U}$, then

$$\langle \delta_u - \delta_v|\delta_w - \delta_z\rangle = \langle \delta_{\widetilde{u}} - \delta_{\widetilde{v}}|\delta_w - \delta_z\rangle$$

for every $u,v \in \mathcal{U}$. Thus, for $u,v \in \mathcal{U}$,

$$\delta_u - \delta_v = \delta_{\widetilde{u}} - \delta_{\widetilde{v}}$$

in $\mathcal{H}(\mathcal{U})$. Hence, the set $\{\delta_u - \delta_v : u,v \in \mathcal{U}^\Omega\}$ is total in $\mathcal{H}(\mathcal{U})$. Similarly, $\{\delta_u - \delta_v : u,v \in \mathcal{U}_i^\Omega\}$ is total in $\mathcal{H}(\mathcal{U}_i)$ for $i \in I$.

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Lemma 5.2 *If $u \in \mathcal{U}_i^\Omega$ and $v \in \mathcal{U}_j^\Omega$ for $i \neq j$, then $c(u,v) = 0$.*

Proof Let $\xi = \{\xi_b\}_{b\in P} \in \mathcal{A}(V^{(i)})$ and $\eta = \{\eta_b\}_{b\in P} \in \mathcal{A}(V^{(j)})$ be such that $u_b = Exp_b(\xi_b)$ and $v_b = Exp_b(\eta_b)$ for $b \in P$. Then, for $t \geq 0$,

$$\langle u_{ta}|v_{ta}\rangle = e^{\langle \xi_{ta}|\eta_{ta}\rangle} \text{ (by Proposition 2.1)}$$
$$= 1 \text{ (since} \langle \xi_{ta}|\eta_{ta}\rangle = 0).$$

Therefore, $c(u,v) = 0$. Hence the proof. ∎

Let $a \in Int(P)$ and $i \in I$. Denote the covariance function of E^V, $E^{V^{(i)}}$ (with respect to a) by c, c_i, respectively. Note that if $u,v \in \mathcal{U}_i$, $c_i(u,v) = c(u,v)$. For each $i \in I$, the map $\mathcal{H}(\mathcal{U}_i) \ni \delta_u - \delta_v \to \delta_u - \delta_v \in \mathcal{H}(\mathcal{U})$ extends to an isometry from $\mathcal{H}(\mathcal{U}_i)$ into $\mathcal{H}(\mathcal{U})$. Hence, we may consider $\mathcal{H}(\mathcal{U}_i)$ as a subspace of $\mathcal{H}(\mathcal{U})$.

Proposition 5.3 *With the foregoing notation, we have the following.*

(i) $\mathcal{H}(\mathcal{U}_i) \perp \mathcal{H}(\mathcal{U}_j)$ *if* $i \neq j$,
(ii) $\mathcal{H}(\mathcal{U}) = \bigoplus_{i\in I} \mathcal{H}(\mathcal{U}_i)$, *and*
(iii) $Ind(E^V) = \sum_{i\in I} Ind(E^{V^{(i)}})$.

Proof Clearly, it suffices to prove (i) and (ii). For $i \in I$, let $\mathcal{W}_i := \{\delta_u - \delta_v : u,v \in \mathcal{U}_i^\Omega\}$. As observed in Remark 5.1, $\mathcal{W}_i$ is total in $\mathcal{H}(\mathcal{U}_i)$ for $i \in I$. Using Lemma 5.2, it is routine to verify that

$$\langle \delta_{u_1} - \delta_{v_1}|\delta_{u_2} - \delta_{v_2}\rangle = 0$$

whenever $u_1, v_1 \in \mathcal{U}_i^\Omega$ and $u_2, v_2 \in \mathcal{U}_j^\Omega$ and $i \neq j$. Now (i) follows.

Consider $\mathcal{W} = \bigcup_{i\in I} \mathcal{W}_i \subset \bigoplus_{i\in I} \mathcal{H}(\mathcal{U}_i)$. Note that if $u \in \mathcal{U}_i^\Omega$, then $\delta_u - \delta_\Omega \in \mathcal{W}$, where Ω is the vacuum unit. Let $u,v \in \mathcal{U}^\Omega$. Then, there exist $i,j \in I$ such that $u \in \mathcal{U}_i^\Omega$ and $v \in \mathcal{U}_j^\Omega$. Then, $\delta_u - \delta_v = (\delta_u - \delta_\Omega) + (\delta_\Omega - \delta_v)$. Consequently, $\delta_u - \delta_v \in span(\mathcal{W})$. Therefore, $\mathcal{W}$ is total in $\mathcal{H}(\mathcal{U})$, and hence $\mathcal{H}(\mathcal{U}) = \bigoplus_{i\in I} \mathcal{H}(\mathcal{U}_i)$. This proves (ii). ∎

Let G denote the group of automorphisms of E^V, also called the gauge group of E^V. Denote the set of normalized units of E^V by $\mathcal{U}^n$. It follows from Proposition 4.16 that the normalized units of E^V are of the form $\left\{e^{i\langle\lambda|a\rangle} e^{\frac{-\|\xi_a\|^2}{2}} Exp_a(\xi_a)\right\}_{a\in P}$ for some $\xi = \{\xi_a\} \in \mathcal{A}(V^{(i)})$ and $\lambda \in \mathbb{R}^d$. For each $i \in I$, we denote the unitary group of $\mathcal{K}_i$ by $U(k_i)$.

Proof of Theorem 1.3:

(1) Suppose that $I = \{i\}$ is singleton. Denote $\mathcal{K}_i$ by $\mathcal{K}$, λ_i by λ and k_i by k. Let $E^{(\lambda,k)}$ be the product system of the CAR flow associated with $V = S^{(\lambda,k)}$. Since V is a pull back of the one parameter shift semigroup $\{S_t \otimes 1\}$ on $L^2[0,\infty) \otimes \mathcal{K}$ by the homomorphism $P \ni a \to \langle\lambda|a\rangle \in \mathbb{C}$, and since the product systems of CAR and CCR flows associated with the one-parameter shift semigroup $\{S_t \otimes 1\}_{t\geq 0}$

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are isomorphic, $E^{(\lambda,k)}$ is isomorphic to the product system $F^{(\lambda,k)}$ of the CCR flow associated with V. In fact, the map $T = \{T_a\}_{a\in P} : E^{(\lambda,k)} \to F^{(\lambda,k)}$ defined by

$$T_a(Exp_a(\xi)) = e(\xi),$$

for $\xi \in Ker(V_a^*)$ and $a \in P$ is an isomorphism. Hence, G is isomorphic to the automorphism group of $F^{(\lambda,k)}$. Since the automorphism group of a CCR flow acts transitively on the set of normalized units, the conclusion follows. The fact that the gauge group of a CCR flow acts transitively on the set of normalized units can been seen from the explicit description of units and the gauge group obtained in [2, Theorems 5.1 and 7.3].

Since $E^{(\lambda,k)}$ and $F^{(\lambda,k)}$ are isomorphic, they have the same index. But, by [24, Proposition 2.7],

$$Ind(F^{(\lambda,k)}) = \dim \mathcal{A}(S^{(\lambda,k)}).$$

Note that for every $\eta \in \mathcal{K}$, $\{1_{(0,\langle\lambda|a\rangle)} \otimes \eta\}_{a\in P}$ is an additive cocycle for $S^{(\lambda,k)}$, and it is not difficult to see that every additive cocycle is of this form. Thus,

$$Ind(E^{(\lambda,k)}) = Ind(F^{(\lambda,k)}) = \dim \mathcal{K} = k.$$

(2) Next assume that I has at least two elements. It is clear from Proposition 5.3 and (1) that

$$Ind(E^V) = \sum_{i\in I} k_i.$$

Let $\Psi = \{\Psi_a\}_{a\in P} \in G$ be given. For $a \in P$, consider the one-parameter product system $E_a = \{E(ta)\}_{t\geq 0}$. By Lemma 3.2, there exists a unitary $U_a \in \{V_{ta}, V_{ta}^* : t \geq 0\}'$, an additive cocycle $\xi^a = \{\xi_t^a\}_{t\geq 0}$ and $\mu_a \in \mathbb{R}^d$ such that

$$\Psi_{ta}Exp_a(\eta) = e^{i\mu_a t}e^{-\frac{\|\xi_t^a\|^2}{2} - \langle U_a\eta|\xi_t^a\rangle}Exp_a(U_a\eta + \xi_t^a)$$

for $\eta \in Ker(V_{ta}^*)$. Proceeding as in the proof of Theorem 3.3, we see that there exists $\xi = \{\xi_a\}_{a\in P} \in \mathcal{A}(V)$, a unitary $U \in \{V_a, V_a^* : a \in P\}'$ and $\mu \in \mathbb{R}^d$ such that

$$\Psi_a Exp_a(\eta) = e^{i\langle\mu|a\rangle}e^{-\frac{\|\xi_a\|^2}{2} - \langle U\eta|\xi_a\rangle}Exp_a(U\eta + \xi_a) \tag{5.1}$$

for $\eta \in Ker(V_a^*)$ and $a \in P$.

Assume that the additive cocycle $\{\xi_a\}_{a\in P}$ is non-zero. Note that $\{\Psi_a Exp_a(0)\}_{a\in P}$ is a unit of E^V. Hence, $\{Exp_a(\xi_a)\}_{a\in P}$ is a unit for E^V (the map $P \ni a \to \langle\xi_a|\eta_a\rangle \in \mathbb{C}$ is additive if $\xi, \eta \in \mathcal{A}(V)$). By Proposition 4.16, there exists $i \in I$ such that $\{\xi_a\}_{a\in P} \in \mathcal{A}(V^{(i)})$. Let $j \in I$, $j \neq i$. Let $\{\eta_a\}_{a\in P} \in \mathcal{A}(V^{(j)})$ be non-zero. By Proposition 4.16, $\{Exp_a(\eta_a)\}_{a\in P}$ is a unit for E^V. Therefore, $\{\Psi_a Exp_a(\eta_a)\}_{a\in P}$ is a unit for E^V. (5.1) implies that $\{Exp_a(U\eta_a + \xi_a)\}_{a\in P}$ is a unit for E^V. Note that since $U \in \{V_a, V_a^* : a \in P\}'$, U is a diagonal operator, i.e., there exists a unitary operator $U_i \in \{V_{ta}^{(i)}, V_{ta}^{(i)*} : t \geq 0\}'$ such that $U|_{\mathcal{H}_i} = U_i$ for $i \in I$. Therefore, $U\eta_a \in Ker(V_a^{(j)*})$ for $a \in P$. Hence, $\{U\eta_a + \xi_a\}_{a\in P} \notin \mathcal{A}(V^{(\ell)})$

for any $\ell \in I$, contradicting Proposition4.16. As a consequence, $\xi_a = 0$ for $a \in P$. Therefore,

$$\Psi_a Exp_a(\xi) = e^{i\langle\mu|a\rangle} Exp_a(U\xi)$$

for $\xi \in Ker(V_a^*)$ and $a \in P$.

Suppose $\mu \in \mathbb{R}^d$ and $U \in \{V_a, V_a^* : a \in P\}'$. Then, there exists $\Psi^{(\mu,U)} \in G$ such that

$$\Psi_a^{(\mu,U)} Exp_a(\xi) = e^{i\langle\mu|a\rangle} Exp_a(U\xi)$$

for $\xi \in Ker(V_a^*)$ and $a \in P$. Note that $\Psi \in G$ because $\Psi_a = e^{i\langle\mu|a\rangle}\Gamma(U)$ for $a \in P$, where $\Gamma(U)$ is the second quantisation map.

Let $M := \{V_a, V_a^* : a \in P\}'$ and denote the unitary group of M by $\mathcal{U}(M)$. By Remark 4.14, we have $\mathcal{U}(M) = \prod_{i\in I} U(k_i)$. We have shown that the map

$$\mathbb{R}^d \times \mathcal{U}(M) \ni (\mu, U) \to \Psi^{(\mu,U)} \in G$$

is an isomorphism of groups. Hence, if Ψ is an automorphism of E^V, then for $a \in P$,

$$\Psi_a\Omega_a = \Psi_a Exp_a(0) = e^{i\langle\mu|a\rangle} Exp_a(0) = e^{i\langle\mu|a\rangle}\Omega_a \tag{5.2}$$

for some $\mu \in \mathbb{R}^d$. Let $\eta = \{\eta_a\}_{a\in P}$ be a non-zero additive cocycle such that $Exp(\eta)$ is a unit. Then, $u = \{u_a\}_{a\in P}$ given by $u_a = e^{-\frac{\|\eta_a\|^2}{2}} Exp(\eta_a)$ is a normalized unit, but, by (5.2), $\Psi.\Omega \neq u$ for every $\Psi \in G$. Hence, the action of G on $\mathcal{U}^n$ is not transitive.

■

Remark 5.4 Suppose $d \geq 2$. Then, $S(P^*)$ is uncountable. This is because, as P is pointed, P^* spans $\mathbb{R}^d$. It follows from Theorems 1.3 and 1.2 that there are uncountably many CAR flows that are type I and for which the action of the gauge group on the set of normalized units is not transitive.

Remark 5.5 It is immediate from Theorems 1.2 and 1.3 that if V is an isometric representation with commuting range projections such that E^V is type I and has index one, then V is unitarily equivalent to S^λ for some $\lambda \in S(P^*)$, i.e., E^V is "a pullback" of a one parameter CAR flow. The analogous statement for F^V, the product system of the CCR flow associated with V, is not true (see [24]).

Question *Is it possible to construct an isometric representation V of P such that E^V is type I, has index one, but V is not unitarily equivalent to S^λ for any $\lambda \in S(P^*)$?*

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