

## DEDUCTIVE CARDINALITY RESULTS AND NUISANCE-LIKE PRINCIPLES

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**Abstract.** The *injective* version of Cantor's theorem appears in full second-order logic as the inconsistency of the abstraction principle, Frege's Basic Law V (BLV), an inconsistency easily shown using Russell's paradox. This incompatibility is akin to others—most notably that of a (Dedekind) infinite universe with the Nuisance Principle (NP) discussed by neo-Fregean philosophers of mathematics. This paper uses the Burali–Forti paradox to demonstrate this incompatibility, and another closely related, without appeal to principles related to the axiom of choice—a result hitherto unestablished. It discusses both the general interest of this result, its interest to neo-Fregean philosophy of mathematics, and the potential significance of the Burali–Fortian method of proof.

**§1. Introduction.** The main formal result of this paper—given in Section 8—is a proof that the Nuisance Principle (NP) is jointly *inconsistent* with Hume's Principle (HP). The NP is a perennial thorn in the neo-Fregean's side, and HP is a principle whose reputation the neo-Fregean wants very much to protect. Previously it was known that the two principles are jointly unsatisfiable, but not whether the two principles are jointly inconsistent. This paper closes this gap in our knowledge, and shows that if NP causes the neo-Fregean problems, it does so in deductive settings as well as in (standard) semantic settings.

This will be of interest to neo-Fregeans and their critics, as I review in Section 2. Neo-Fregeanism is a form of *abstractionism*, in that it takes at least some abstraction principles to have a privileged status. HP is the paradigm case of a principle neo-Fregeans would like to privilege. Frege's inconsistent BLV is one that must go without this privilege. Principles like NP cause the most problems: unlike BLV, NP is consistent. The difficulties in combining HP with principles like NP shows that privileged abstraction principles need to be more than just consistent. Specifying what else they must be is the difficulty the neo-Fregean must face.

Should anybody else care about NP? The rest of this paper—that is, Sections 3–7—argues they should. The fate of NP in deductive settings should be of interest to anyone who finds Cantor's theorem on the size of the power-set strange, beautiful, or both. This is because abstraction principles afford a way to investigate results like Cantor's theorem in a comparatively austere setting. As such they can sharpen our focus on features salient to results like Cantor's.

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In particular, there is at present an interesting asymmetry between methods of proving cardinality results like Cantor’s in this austere setting. The inconsistency of NP and HP is what I shall call a *Cantorian result*. These results can be stated using just *cardinal notions* in addition to the basic concepts of set theory<sup>1</sup>: these cardinal notions are those of injections, surjections, and bijections. What is curious is that though this inconsistency is a Cantorian result, its only known proof (presented in this paper) requires more than cardinal notions. For the proof requires developing what might be called the *ordinal* notions of ordinal embedding and isomorphism. So it is odd that a more conceptually direct proof for NP is either non-existent or hard to find.

I discuss the further significance of this asymmetry in Section 7: the proof reveals connections between the existence of injections, and the definability of well-orderings in certain systems of second-order logic. Thus this paper is a friendly addition to [6, 22, 23]: the connections between well-orderings and the existence of injections, noted in these papers, are shown to persist in relatively weak second-order contexts, and are shown to generalize in interesting ways.

The proof of the main result is quite involved, which is why it is left for the last section. It requires several intricate steps before one can give the paradoxical construction which proves the result.

**§2. A nuisance to neo-Fregeans.** Fregean logicism, as I shall use the term, is distinguished by the interpretation of arithmetic in a second-order *theory of extensions*. Such a theory presumes second-order logic with full comprehension (“full second-order logic”), and issues the principle BLV as describing the behavior of extensions:

$$(\forall X)(\forall Y)(\check{\varepsilon}X = \check{\varepsilon}Y \leftrightarrow X = Y), \tag{1}$$

where “ $X = Y$ ” abbreviates “ $(\forall x)(Xx \leftrightarrow Yx)$ .” Frege’s theory of extensions was found to be inconsistent by Russell.<sup>2</sup>

But to interpret arithmetic, Frege does not need to use BLV once he has the principle HP:

$$(\forall X)(\forall Y)(\#X = \#Y \leftrightarrow (\exists g)(g : X \xrightarrow[\text{onto}]{1-1} Y)). \tag{2}$$

*Frege’s Theorem* is the result that HP suffices to interpret second-order Peano Arithmetic in second-order logic with full comprehension. So if HP is analytic—which for our purposes will just mean that it has some privileged status preserved by second-order deduction—then so is arithmetic.

*Neo-Fregeans* suggest that HP may be analytic because it is an *abstraction principle*: it determines the behavior of an abstraction operator—a functor taking concepts and returning object called *abstracts*—to respect a given equivalence relation. In Frege’s phrase, such principles “carve up the content” in a new way [16, Section 64]. The neo-Fregean’s hope is that if the original content is sufficiently logical, its re-carving will be analytic.

The complication for the neo-Fregean comes in the fact that being an abstraction principle on an equivalence relation definable in second-order logic is not sufficient

<sup>1</sup> The concepts of set and membership and the construction of new sets via Boolean operations.

<sup>2</sup> See [32, pp. 124–128], [15, 17, Appendix to Volume 2], [3, p. 253–7, 279–89], and also Russell’s [27, Chapter X].

for analyticity.<sup>3</sup> For BLV meets these conditions, and yet is inconsistent. So any argument for HP’s analyticity must distinguish HP, in a principled way, from the abstraction principles that are not analytic. Of course HP is distinguished from BLV by its consistency in full second-order logic. But what if there are abstraction principles that are consistent, like HP, but jointly inconsistent with HP? Then the neo-Fregean must give criteria for the acceptability of abstraction principles that sorts the sheep from the goats, and HP with the sheep.

Heck [20] shows that it is easy to construct such goats by abstracting on an equivalence relation which is a disjunction of “ $X = Y$ ” with any condition incompatible with HP. For example, the condition that the universe—which we will denote with “ $V$ ”—is Dedekind finite.<sup>4</sup> Such principles, however, have the air of contrivance<sup>5</sup>; their contrivance renders these principles dubious as challenges to the neo-Fregean project.

Are there any goats—principles inconsistent with HP—that are also mathematically interesting? There are several candidates, the simplest of which is the NP given by:

$$(\forall X)(\forall Y)(\S X = \S Y \leftrightarrow |X \Delta Y| < \omega). \tag{4}$$

We can characterize NP by: the *nuisance* of  $X$  is identical to the *nuisance* of  $Y$  just in case the symmetric difference  $X \Delta Y = (X - Y) \cup (Y - X)$  is Dedekind finite. Notice that NP’s equivalence relation does not contrive inconsistency with HP by disjoining an inconsistent “out clause.” It is also of some mathematical interest, as this equivalence relation is studied in point-set topology.<sup>6</sup>

<sup>3</sup> See [19] for just one part of the large literature on this question.

<sup>4</sup> This is the simplest example of this kind:

$$(\forall X)(\forall Y)(\partial X = \partial Y \leftrightarrow (|V| < \infty \vee X = Y)). \tag{3}$$

The above “Heck” principle acts like BLV on models of infinite size; otherwise it creates a trivial equivalence class, and so is satisfiable in finite models. Since HP (deductively) implies the universe is infinite, it triggers the BLV aspect of the “Heck” principle, and so the two are inconsistent.

<sup>5</sup> There are many complaints that principles like (3) are somehow fishy. How “contrived” is spelled out need not concern us here; suffice it to say that such principles have no obvious freestanding mathematical interest. Additionally, as Wright notes in the appendix to [37] in [19], interesting consequences of “Cambridge” principles, whose equivalence relations are of the form  $\Psi \vee X = Y$  with  $\Psi$  closed, come about in a suspicious way: by doing something other than putting requirements on the abstracts. This is not so, however, for the “distraction” principles discussed by Weir in [35]; those of the form

$$(\forall X)(\forall Y)(\delta X = \delta Y \leftrightarrow (\Phi(X) \wedge \Phi(Y)) \vee X = Y),$$

where  $\Phi(X)$  “is a second-order property of properties for which equinumerosity is a congruence (a cardinality property)” [35, p. 16]. Though such principles do have “escape clauses” like (3), those escape clauses are about  $X$  and  $Y$ , and (like (3)) can have consequences that arguably should not be derivable from an abstraction principle. For instance,  $\text{NewV}$  (from [4]), which sets  $\Psi$  as  $X \approx Y \approx V$ , implies that  $V$  can be well-ordered (see [31]). See [35] for further variations on this theme, as well as Wright’s discussion of the variety of oddly-generated abstraction principles in [36].

<sup>6</sup> For example, one can correlate automorphisms of the quotient of power-set of  $\mathbb{N}$  under the equivalence relation  $|X \Delta Y| < \omega$  with self-homeomorphisms of the *Stone-Ćech remainder*  $\beta\mathbb{N} - \mathbb{N}$ . One can use this correlation to prove the independence from ZFC of the claim that every self-homeomorphism of  $\beta\mathbb{N} \setminus \mathbb{N}$  generates from a bijection between co-finite subsets of  $\mathbb{N}$ . See [34, Chapter 15] as well as [33].

The existing literature does not show NP and HP are jointly inconsistent. It shows that they are jointly unsatisfiable: there are no models of both principles in which the second-order domain is the full power-set of the first-order domain. Proofs of this fact, however, are not obviously adaptable to the restricted deductive context of second-order logic.<sup>7</sup> Since the deductive system for second-order logic is not complete with respect to standard semantics, the existing results do not settle the question of the deductive compatibility of HP and NP.

The Heck-like examples, as well as BLV and NP, are considered by neo-Fregeans to be “Bad Company”: the most natural criteria by which HP is privileged let in at least some of these principles. The neo-Fregean can, of course, make the rather desperate argument that at least the most challenging bad companion makes a tolerable guest: NP, at least, is not known to be inconsistent with HP. The results of this paper show this to be a losing strategy.

Of course, neo-Fregeans have hardly been so unwise as to embrace such a strategy: they have attempted to find criteria according to which NP is defective, and HP not. Even so, the most prominent programs have been put in model-theoretic terms: HP, and not NP, is said to be conservative, unbounded, stable, and irenic.<sup>8</sup> All but the first of these criteria are described in terms of standard models. Much of the research has therefore been set in semantic terms. But this makes the philosophical sacrifice described by Linnebo:

[W]hen we chose to study model-theoretic rather than proof-theoretic conservativeness of higher-order theories, we already gave up on any requirement of a close link with actual (or even mildly idealized) human capacities [24, p. 333–4].<sup>9</sup>

Perhaps for this reason, Wright’s early call for deductive investigation of NP’s has persisted:

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Likewise Boolos’s Parity Principle, an earlier principle with similar properties to NP, is based on an equivalence relation also mathematically quite natural and interesting.

<sup>7</sup> For instance, the proof given in [1] (which is much the same as that of [36]) deploys both the fact that  $X \times X \approx X$  for all infinite  $X$ , and the fact that  $\mathcal{P}(X)$  is equinumerous with the number of equivalence classes for NP’s equivalence relation. The former requires the axiom of choice, while it is hard to see how to establish the latter in second-order logic, even with NP as an axiom.

<sup>8</sup> See [9].

<sup>9</sup> See, however, Shapiro’s remarks in [30] that deductive conservativity will not suit for justifying a neo-Fregean theory of the real numbers. By a result proved in that paper,

... if neo-logicism is to have any chance of success, then deductive conservativeness is the wrong requirement. Whether the [particular programs to generate reals from abstraction principles] succeeds or not, at some point the neo-logicist is going to (try to) introduce the real numbers from abstraction principles, and derive the axioms of second-order real analysis from those. The resulting theory will thus fail to be deductively conservative over Hume’s principle. In general, strong theories are not deductively conservative over weaker ones. If the neo-logicist wants to develop theories as strong as classical real analysis, she must eschew deductive conservativeness [30, p. 235].

These are matters for further study, but who is to say that the model-theoretic inconsistency between Hume's Principle and the Nuisance Principle ... does not ... prefigure a purely logical inconsistency between them? ... I know of no definite finding on this matter at present. [19, p. 295; and footnote 46 on that page]

As we will show below, deductive matters are as bad as feared for the neo-Fregean. But "as bad as feared" is not the same as "as bad as can be." In other words, that NP is inconsistent with HP raises the question of whether mathematically natural principles jointly inconsistent with HP are all of the same type—are they all, in some sense, like NP? It would be bad news for the neo-Fregean if the bad companions were multitude in kind as well as in number.<sup>10</sup> This question is answered in Theorem 2, a corollary of the main theorem of [13]: mathematically natural bad companions can be classified into just two types. Whether that is few enough for the neo-Fregean I will leave for another occasion.

**§3. Abstraction principles and Cantorian results.** The joint inconsistency of NP and HP is of interest to neo-Fregeans and their critics. But the result is interesting beyond these discussions in the philosophy of mathematics, for the reason that the result is of the same kind as Cantor's theorem in set theory. This point can be made with two observations: first, the traditional identification of BLV's inconsistency with Cantor's theorem,<sup>11</sup> and second, that NP and BLV are instances of the same parameterized abstraction principle.

**3.1. Cantor's theorem in weak systems.** "Cantor's theorem" means different things in different contexts. Its usual context is in the theory of sets, it can be stated in the following ways:

CANTOR'S THEOREM (INJECTIONS). *For any set  $X$  there is no injection from the power-set of  $X$  into  $X$ .*

CANTOR'S THEOREM (SURJECTIONS). *For any set  $X$ , there is no surjection from  $X$  to its power-set.*

But in this context (where "the theory of sets" means something like ZF or ZFC), there are two elements that might be regarded as extraneous. One is the mention of the power-set: it is the fact that the elements of the power-set are subsets of  $X$  that matters, not so much that they can themselves be gathered together into a set. The other is the meaning, in set theory, of "the existence of an injection (surjection)." In this context that means the existence of a specific kind of subset of a certain Cartesian product. But of course this can be removed as well.

In stripping away these specifically set-theoretic elements, one is left with a more general statement of "Cantor's theorem": there is no way to correlate all the sub-collections of a given collection with the objects of that collection so that certain

<sup>10</sup> Thanks to Sean Walsh for urging this question upon me.

<sup>11</sup> The connection between the results, of course, has been documented from the beginning; see Russell [27, Section 100]; also [18].

cardinality properties are preserved.<sup>12</sup> Since the “core” of Cantor’s theorem is something that does not need all of set theory, it makes sense to study the phenomenon in weaker theories. A relatively weak theory relating objects to their collections, of course, is second-order logic (with full comprehension, but without axioms of choice).<sup>13</sup>

While there is, in pure second-order logic, a way to express the surjective version of Cantor’s theorem, there is no easy method by which to express the specifically injective version, since a formula defining a function from the sub-concepts of  $X$  to its objects will automatically define a surjection from the latter to the former.<sup>14</sup> For this reason, something more is needed to study the injective version of Cantor’s theorem in second-order logic. Type-lowering operators offer a natural fix for this problem. With an operator  $\tilde{\epsilon}$  taking concepts to objects, the existence of an injection from second-level objects (concepts) to first-level objects (objects) would then be represented by

$$(\forall X)(\forall Y)(\tilde{\epsilon}X = \tilde{\epsilon}Y \rightarrow X = Y). \tag{7}$$

Concepts and sub-concepts, in the context of Cantor’s result, are extensional, and the natural way to represent this is that  $\tilde{\epsilon}$  is not extensionally ill-defined: there are not

<sup>12</sup> This is why there are both the injective and surjective versions of the theorem in set theory: both injective and surjective functions are suitable indications of relative cardinality. See [5, 22].

<sup>13</sup> A more detailed description of this logic is given in Section 8.1.

<sup>14</sup> Consider, for example, Shapiro’s treatment of the result:

In [pure second-order logic], Cantor’s theorem amounts to a statement that no binary relation can represent the collection of all subsets of its domain, or, in other words, for every relation there is a set that is not represented by it. [29, p. 103]

Shapiro gives the result as follows: one can represent a correspondence between concepts (second-order objects) and (first-order) objects using relations. Given a binary relation  $R$ , let  $R_x = \{y \mid Rxy\}$ . Then the correspondence effected by the second-order formula abbreviated by  $Y = R_x$  is a many-one correspondence between objects and concepts. Thus, using this machinery, one can express the *surjective* version of Cantor’s theorem by

$$\neg(\exists R)(\forall Y \subseteq X)(\exists x)(Xx \wedge Y = R_x). \tag{5}$$

But it is not so easy to represent the *injective* version of Cantor’s theorem in second-order logic. The injective version, in set theory, denies the existence of a kind of function defined on all sub-concepts of  $X$ , with range contained in  $X$ . How would one express the existence of such a function using the above method of binary relations? The natural way to express that  $R$  is functional in just such a way, is with

$$(\forall Y \subseteq X)(\exists x)(Xx \wedge Y = R_x). \tag{6}$$

But this is just an instance of the negation of (5). In trying to define something like a *function* from concepts to objects, one ends up with what is the correlate of a surjection from objects to concepts. Thus adding a condition of injectivity of  $R$  to (6) would be superfluous. Another consideration, pointed out to me by Graham Leach-Krouse, is that (in the meta-theory)  $f : X \rightarrow P(X)$  if and only if in a model  $\mathcal{M}$  containing as concepts  $X$  and  $f(x)$  for each  $x \in X$ ,  $\mathcal{M} \models (\exists R)(\forall x, y \in X)(R(x, y) \leftrightarrow y \in [f(x)])$ . But no analogous biconditional can be true in general for  $f : P(X) \rightarrow X$ . For if  $\mathcal{M}$  is a model with  $X$  a subset of its first-order domain, there are at most  $2^{|X^2|}$  relations  $R \subseteq X \times X$ . But there are  $2^{(2^{|X|})}$  functions from  $P(X)$  to  $X$ .

co-extensive  $X$  and  $Y$  that return different objects under  $\check{\epsilon}$ :

$$(\forall X)(\forall Y)(X = Y \rightarrow \check{\epsilon}X = \check{\epsilon}Y). \tag{8}$$

Putting together (7) and (8) yields Frege’s BLV (1). So a natural way to present the injective version of Cantor’s theorem in second-order logic is by means of BLV.

Cantor’s theorem is robust, then, in the sense that both its surjective and injective versions can be obtained in treatments of concepts and objects far weaker than ZF and ZFC. But Cantor’s theorem is just the most distinguished member of a family of results. For extensional equality between concepts is an equivalence relation between concepts—the finest such extensional relation. What the respective versions of Cantor’s theorem do is show that, given a notion of relative cardinality (specified by either something like injections or something like surjections), there are more equivalence classes of *this finest relation* than there are objects.

Putting it this way invites the question of whether the same is true for other equivalence relations. Here, the argument for the use of abstraction principles to represent injections translates neatly to the more general case. That is, suppose one has defined an equivalence relation  $E$  on concepts of a domain of objects, and one wants to know how many equivalence classes there are. In particular, one wants to know if there are more equivalence classes than there are objects. One can pursue these questions in second-order logic by adding an abstraction principle

$$(\forall X)(\forall Y)(\check{h}X = \check{h}Y \leftrightarrow E(X, Y)) \tag{9}$$

provided  $E$  is expressible in second-order logic. If, for example, the principle (9) only has models whose first-order domain is in a certain range of cardinalities, that means that on all other cardinalities, there is no injection from those  $E$ -equivalence classes to that many objects. In what follows, we will refer to results of the form “Abstraction principle  $A_E$  only has models of such and such cardinalities” as *Cantorian cardinality results* (“Cantorian results” for short).

**3.2. Uniform separations and complementations.** A second source of interest in this paper’s main result depends on that result being Cantorian. Clearly the inconsistency of BLV is, by the above reckoning, a Cantorian result. The purpose of this section is two-fold: first, to show that the joint inconsistency of HP and NP is, by similar reckoning, also a Cantorian result. In fact, NP and BLV are just instances of the same parameterized abstraction principle, and so are in the class of *uniform separations*. The second purpose of this section is to describe yet another parameterized abstraction principle, which determines the class of *uniform complementations*. As will be shown below, the results we prove for NP—and so those for uniform separations—have analogues for uniform complementations.

The first step, then, is to describe how BLV and NP are similar.

The equivalence relation of NP, recall, is  $|X \Delta Y| < \omega$ —that is, that the symmetric difference of  $X$  and  $Y$  is Dedekind finite. Notice now that  $X = Y$  is equivalent to  $|X \Delta Y| < 1$ . So both BLV and NP can be considered to be *uniform separations*; that is, abstraction principles  $US(\kappa)$  of the form

$$(\forall X)(\forall Y)(\sigma X = \sigma Y \leftrightarrow |X \Delta Y| < \kappa), \tag{10}$$



where “ $|A| < \kappa$ ” is definable in pure second-order logic for the parameter  $\kappa$ .<sup>15</sup> So, for example, “ $|A| < \omega$ ” will express that  $A$  is Dedekind finite, while “ $|A| < \omega_1$ ” will express that if a Dedekind infinite concept injects into  $A$ , then  $A$  injects into every Dedekind infinite concept. Thus the principle BLV is  $US(1)$  and NP is  $US(\omega)$ . As described in Section 3.1, the inconsistency of BLV can be regarded as a Cantorian result. So, however, can the joint inconsistency of NP and HP, since it is a corollary of the fact that NP deductively implies the universe is finite.

This again raises the question (of Section 3.1) of whether there are other kinds of abstraction principles jointly inconsistent with HP that are sufficiently unlike NP. There are: consider an abstraction principle that “halves” BLV, the Liberated Complementation Principle LCP of [13]:

$$(\forall X)(\forall Y)(\ell X = \ell Y \leftrightarrow X = Y \vee X = V - Y). \tag{11}$$

The equivalence relation of this principle sorts concepts with, and only with, their complements. Now, LCP is not inconsistent; it is easy to verify that it has standard models with first-order domain of sizes 1 and 2. However, one cannot go beyond that.

**PROPOSITION 1.** *LCP deductively implies  $|V| \leq 2$ .*

*Proof.* If  $\ell\emptyset = \ell\{\ell\emptyset\}$  then  $V = \{\ell\emptyset\}$ . So we may assume that  $\ell\emptyset \neq \ell\{\ell\emptyset\}$ .

Suppose  $V' = V - \{\ell\emptyset, \ell\{\ell\emptyset\}\} \neq \emptyset$ . For any  $X \subseteq V'$ , let  $X^* = X \cup \{\ell\{\ell\emptyset\}\}$ . Now if  $\ell X^* = \ell\emptyset$  or  $\ell X^* = \ell\{\ell\emptyset\}$ , then either  $X^* = \emptyset$ ,  $X^* = V$ ,  $X^* = \{\ell\emptyset\}$ , or  $X^* = V - \{\ell\emptyset\}$ . Each contradicts the fact that  $\emptyset \neq X^* \subseteq V - \{\ell\emptyset\}$ . Thus for each  $X \subseteq V'$ ,  $\ell X^*$  falls under  $V'$ .

But then consider the concept  $R$  defined by

$$Rx \leftrightarrow (\exists Y \subseteq V')(x = \ell Y^* \wedge \neg Yx);$$

we recreate the Russell paradox: if  $\ell R^*$  falls under  $R$ , then let  $Y \subseteq V'$  with  $\ell R^* = \ell Y^*$ . Since  $Y^* \cap R^* \neq \emptyset$ , it follows that  $Y = R$ ; and so we have that  $\neg R(\ell R^*)$ . So  $\ell R^*$  must not fall under  $R$ . But then by the definition of  $R$ , as  $R \subseteq V'$ , we have that  $R(\ell R^*)$ , a contradiction.  $\square$

Though LCP is not like BLV in one way (it is not a uniform separation), it is in another: a Russellian diagonal construction can be deployed to obtain the cardinal interval on which it has models (standard or not).

We generalized BLV by introducing uniform separations; we can do the same for LCP. Say that a *uniform complementation* is an abstraction principles  $UC(\kappa)$  of the form

$$(\forall X)(\forall Y)(\chi X = \chi Y \leftrightarrow |X \Delta Y| < \kappa \vee |X \Delta (V - Y)| < \kappa)$$

again for definable parameter  $\kappa$ . It is easily seen that LCP is just  $UC(1)$ .

Finally, I mentioned earlier (Section 3) that deductive bad companions—principles inconsistent with HP—can be classified into two kinds. This follows from the main theorem of [13],<sup>16</sup> which can be stated thus:

<sup>15</sup> It is worth noting that if  $\kappa \neq 1$  and is finite then  $E$  is not an equivalence relation; so  $US(\omega) = NP$  is very much the “next” separation after  $US(1) = BLV$ .

<sup>16</sup> This is in turn a refinement of [14, Theorem 4 (p. 142)].



EQUIVALENCE CLASSIFICATION THEOREM ECT. *Let  $E$  be an equivalence relation definable in pure second-order logic,<sup>17</sup> and let  $\mathcal{M}$  be a model with universe  $V$  infinite and well-ordered in  $\mathcal{M}$ . For  $X \in \mathcal{M}$  let*

$$[X]_{E(\boxminus)} = \{Y \subseteq V \mid E(X, Y) \wedge |X| = |Y| \wedge |V - X| = |V - Y|\}.$$

*Exactly one of the following holds for each  $X \in \mathcal{M}$ :*

$$[X]_{E(\boxminus)} = \{Y \subseteq V \mid |X| = |Y| = |V - X| = |V - Y|\}, \tag{12}$$

$$[X]_{E(\boxminus)} \subseteq \{Y \subseteq V \mid |X \Delta Y| < \omega \vee |V - (X \Delta Y)| < \omega\}. \tag{13}$$

*The latter divides into two cases, depending on whether*

$$[X]_{E(\boxminus)} \subseteq \{Y \subseteq V \mid |X \Delta Y| < \omega\} \tag{14}$$

*holds. Where  $\mathcal{M} \models |V| \geq \omega$  and (12) holds for  $X$  with  $|X| = |V - X|$  we say that  $E$  has a bicardinal top in  $\mathcal{M}$ . If both (13) and (14), we say that  $E$  has a separative top in  $\mathcal{M}$ . If (13) but not (14) hold we say that  $E$  has a complementative top in  $\mathcal{M}$ .*

It is plain that if  $E$  has a separative top in  $\mathcal{M}$ , then it behaves like  $\text{US}(|V|)$  on those concepts  $X$  with  $|X| = |V - X|$ , and if it has a complementative top, it behaves like  $\text{UC}(|V|)$  on those same concepts. The following result shows that any abstraction principles jointly inconsistent with HP can be characterized as “like” either  $\text{US}(\omega)$  or  $\text{UC}(\omega)$  based on how their equivalence relations behave in countable standard models.

**THEOREM 2.** *Let  $E$  be an equivalence relation definable in pure second-order logic. If  $E$  has a bicardinal top in a countable standard model, then there is a countable standard model of  $A_E$  and HP, and so  $A_E$  and HP are jointly consistent.*

*If  $A_E$  and HP are not jointly consistent, then  $E$  has a separative or complementative top on all countable standard models.*

*Proof.* Suppose for all countable standard models  $\mathcal{M}$ , (12) holds for  $X \in \mathcal{M}$  with  $|X| = |V - X|$ . Let  $\mathcal{M}$  be a standard model with first-order domain  $V = \omega$ , and let  $f$  map the finite and co-finite subsets of  $\omega$  to  $\omega - \{0\}$ . Set  $@_E X = 0$  if  $|X| = |V - X|$  and  $@_E X = f(X)$  otherwise; and  $\#X = 0$  if  $|X| = \omega$  and  $\#X = |X| + 1$  otherwise. It is plain that  $(\omega, P(\omega), \dots, \#, @_E)$  satisfies  $A_E \wedge \text{HP}$ . This establishes the first assertion.

The second assertion follows from ECT and the contrapositive of the first. □

Thus each abstraction principle inconsistent with HP is of one of two types; *a fortiori* this holds of all mathematically interesting bad companions.

**§4. Cantorian results, cardinal notions, and Russellian methods.** As described in Section 3, when described in set-theoretic terms, Cantorian cardinality results concern the existence or non-existence of certain kinds of functions. Translating this into the context of second-order logic, the injective versions of Cantor’s theorem were Cantorian in the sense that the abstraction operator  $\check{\epsilon}$ , governed by BLV, took the place of the injection asserted by the set-theoretic formulation of the theorem.

In both cases, however, to *state* the Cantorian result requires only that there be formal machinery answering to these set-theoretic notions: the cardinal notions of

<sup>17</sup> Meaning: suppose  $E(X, Y)$  is a formula of the language of pure second-order logic, and that  $E$  is provably an equivalence relation.

injections, surjections, and bijections. This is true of Cantorian results in general, especially in a second-order context. For example, the Cantorian result at the center of this paper can be given by “ $\models \text{NP} \rightarrow |V| < \omega$ ”: it is a theorem of second-order logic that NP implies the (Dedekind) finitude of the universal concept  $V$ . All that is used in the statement of this result are the cardinal notions in second-order logic sufficient to characterize infinity, and the added machinery needed for NP. But that machinery, as we argued above, is just the machinery asserting what (in ZF would be) the existence of an equivalence-respecting injection. So to state this result, we need only deploy these cardinal notions.

To the extent that Cantor’s theorem on power-sets typifies Cantorian results, the diagonal method of proving it typifies the proofs of such results. Which conceptual resources are used in the diagonal construction? In what we will call *Russellian methods*—those deployed in the classic derivation of Russell’s paradox from BLV—one diagonalizes using only cardinal notions.

Diagonalizing using such notions is a powerful tool: in semantic settings, one can prove a large class of Cantorian results by mimicking the derivation of Russell’s paradox.<sup>18</sup> Let  $\mathcal{M}$  be a standard model with  $\mathcal{M} \models \text{US}(|V|) \wedge |V| \geq \omega$ . Since  $\mathcal{M}$  is an infinite standard model, there is an injection  $\langle \cdot, \cdot \rangle : V \times V \rightarrow V$ . Letting

$$\langle X, V \rangle = \{ \langle x, v \rangle \mid x \in X, v \in V \}, \tag{15}$$

we can see that if  $X \neq Y$ , then  $|\langle X, V \rangle \Delta \langle Y, V \rangle| \not\leq |V|$ . Clearly if  $X = Y$  then  $|\langle X, V \rangle \Delta \langle Y, V \rangle| < |V|$ . So we have shown that

$$(\forall X)(\forall Y)(\sigma \langle X, V \rangle = \sigma \langle Y, V \rangle \leftrightarrow X = Y). \tag{16}$$

Notice the similarity of (16) to BLV: one can, as in [12], define the Russell set by

$$R = \{ \sigma \langle X, V \rangle \mid \neg X(\sigma \langle X, V \rangle) \}.$$

From here, one can use the argument used in getting Russell’s paradox out of BLV. As noted earlier, the mathematical machinery built in to standard models suffices to make the last step of the proof “Russellian.”

The same can be done for  $\text{UC}(|V|)$ : if  $X \neq Y$ , then by the above argument  $|\langle X, V \rangle \Delta \langle Y, V \rangle| \not\leq |V|$ . Likewise if  $X \neq V - Y$ , then  $|\langle X, V \rangle \Delta \langle V - Y, V \rangle| \not\leq |V|$ . So, as above, we obtain

$$(\forall X)(\forall Y)(\chi \langle X, V \rangle = \chi \langle Y, V \rangle \leftrightarrow X = Y \vee X = V - Y). \tag{17}$$

One can then replicate the Russellian argument of Proposition 1 to obtain a contradiction from (17). Again, then, exclusively cardinal notions suffice for the last step of establishing the semantic Cantorian result:

**PROPOSITION 3.** *For  $\kappa$  infinite, neither  $\text{US}(\kappa)$  nor  $\text{UC}(\kappa)$  has a standard model with first-order domain of size  $\kappa$ .*

But this only shows that, once one has imported the mathematical machinery of the semantic setting, only cardinal notions are required at the last step of the proof. So, these diagonal proofs used in semantic settings use extra-cardinal notions to prove Cantorian results. As such, to determine the conceptual resources needed to prove

<sup>18</sup> These results are implied by the results of [13, section 6]; I here present them in brief, with just enough detail for the reader to see how the arguments will go.

Cantorian results, it is best to proceed in a deductive, rather than a full semantic, setting.

This raises the question of whether all Cantorian results can be proved using just cardinal notions. Of course *some* can: the inconsistency of BLV is one; Proposition 1 is another. For such results as these, there is a kind of harmony between the resources needed to state the result, and those needed to prove it.<sup>19</sup>

It would be interesting if there is a Cantorian result that could not be proved using just cardinal notions. And it would be of further interest if that result could be obtained by generating, from a specified injection from concepts to objects, some of the mathematical machinery one finds in the standard semantics for second-order logic. A result that could not be proved using Russellian methods, but could be proved using additional notions, would be interesting for the lack of availability of the simpler methods.

**§5. On the difficulties of Russellian diagonalization.** The joint inconsistency of NP and HP is, as I have argued, a deductive Cantorian result. And while I cannot prove that Russellian methods are insufficient to prove the joint inconsistency of NP and HP, no such proof has thus far been found. This is evidence that if Russellian methods are available at all, they are not *easily* available. By examining the structure of Russellian methods, we can see why such proofs are hard to find, if they exist at all.

Russell himself gave a compelling general analysis of set-theoretic paradox in [26, p. 142–3].<sup>20</sup> Transferred into the second-order context, Russell provides the following theorem schema:

**THEOREM 4.** (Russell<sup>21</sup>). *Let  $f$  be a functor from concepts to objects, and  $\phi(x)$  be any formula with the first-order variable  $x$  free. Then second-order logic (in the language including  $f$ ) proves the following:*

$$(\forall U)[(\forall x)(Ux \rightarrow \phi(x)) \rightarrow \phi(fU) \wedge \neg U(fU)] \rightarrow \neg(\exists W)(\forall y)(Wy \leftrightarrow \phi(y)). \quad (18)$$

<sup>19</sup> See, for example, [2] for a general discussion of “purity of methods” in proofs.

A different version of this point is made in van Heijenoort’s discussion of Russell’s paradox in [32], though he overstates his case. What is interesting for van Heijenoort in Russell’s paradox is the fact that its derivation uses only the “bare notions of set and element.” These notions, of course, are not those of the second-order context we have been discussing, but they correlate sufficiently to make the needed point. Sets are extensional: they are equal if and only if they contain exactly the same members. Thus, any function from definable collections of objects, to the sets of those objects, is injective. Accordingly, Russell’s paradox shows that not every collection has a set.

Van Heijenoort concludes, however, that the exclusive use of these “bare notions” lands the paradox “squarely in the field of logic” [32, p. 124]. It is of course highly contentious whether such a notion of set that generates the paradox is “logical” in the sense that van Heijenoort desires.

<sup>20</sup> I am grateful to an anonymous reviewer for this journal, who prompted me to examine Russell’s treatment. See also [22, p. 295–6].

<sup>21</sup> It is worth noting the similarity of Theorem 4 to what Cook calls “Frege’s Little Theorem” in [8, p. 278]: “for any function  $f$  from concepts to objects one can prove:

$$(\exists X)(\exists Y)(f(X) = f(Y) \wedge X(f(X)) \wedge \neg Y(f(X))”$$

Cook attributes the theorem to Frege in the appendix to *Grundgesetze* volume II [3, p. 285–8].

By our stipulation “second-order logic” includes full comprehension, so (18) is equivalent to the negation of

$$(\forall U)((\forall x)(Ux \rightarrow \phi(x)) \wedge \phi(fU) \wedge \neg U(fU)). \tag{19}$$

*Proof.* Assume (19); let  $W$  witness  $(\exists W)(\forall y)(Wy \leftrightarrow \phi(y))$ . By the definition of  $W$ ,  $(\forall x)(Wx \rightarrow \phi(x))$  holds; so by (19),  $\phi(fW)$  and  $\neg W(fW)$ . However, the former and the definition of  $W$  combine to give  $W(fW)$ , a contradiction.  $\square$

As Russell shows, both the paradox bearing his name and the Burali–Forti paradox can be obtained by the “recipe” [26, p. 144] given in Theorem 4. For the former, let  $f$  be  $\check{\epsilon}$  and  $\phi(x)$  be  $(\forall X)(x = \check{\epsilon}X \rightarrow \neg Xx)$ . For the latter, one generates from any concept  $X$  using familiar methods (“ $y$  follows in the  $f$ -sequence that starts with  $fX$ ”) a sequence

$$\begin{aligned} f_0X &= fX \\ f_1X &= f(X \cup \{f_0X\}) \\ &\vdots \\ f_\omega X &= f(X \cup \{f_0X, f_1, \dots\}) \\ &\vdots \end{aligned}$$

One can then let  $\phi(x)$  be “ $x$  is in the  $f$ -sequence that starts with  $fX$ .” Provided the members of this sequence never repeat (as if  $f$  is  $\check{\epsilon}$  from BLV), this sequence is order-isomorphic to the  $f$ -sequence generated when  $X = \emptyset$ . These ordinal properties suffice to establish (19).<sup>22</sup>

One might conclude from Russell’s analysis that cardinal notions (as deployed in Russell’s paradox) and ordinal notions (as deployed in Burali–Forti’s) are symmetric in their ability to generate Cantorian results. But Russell’s analysis, as it stands, does not work for  $US(\omega)$ , either using cardinal or ordinal notions. The trouble is that the clauses  $\phi(fU)$  and  $\neg U(fU)$  in (19) discriminate more finely than the equivalence relation of  $US(\omega)$ . This can be seen from the following analysis.

First, observe that in the case of BLV, the Russell concept can be constructed like this:

$$Rx \leftrightarrow (\exists Y)(\check{\epsilon}Y = x \wedge \{x\} \not\subseteq Y). \tag{20}$$

Looked at this way, to construct the Russell concept, first recover a concept from an abstract (in the above,  $Y$ ), then associate the abstract with another concept ( $\{x\}$ ), and then assert that some defined relation ( $\subseteq$ ) between those two concepts fails to hold. So we can describe the general form of a Russell set by:

$$Rx \leftrightarrow (\exists Y)(@Y = x \wedge \neg\Phi_E(Y, F_E(x))). \tag{21}$$

Here  $F_E(x)$  is the concept associated with the object  $x$ , and  $\Phi_E$  is the defined relation said not to hold between the recovered concept  $Y$  and the determined concept  $F_E(x)$ . Both are indexed by the equivalence relation  $E$  since these constructions will vary depending on the equivalence relation in question,  $E$ .

<sup>22</sup> As the same reviewer remarked, the development of preordinals and Nuisance ordinals in Sections 8.3–8.7 of this paper is an adjustment and elaboration of this construction for the case of  $f = \check{\epsilon}$ .

One can discern the general problem by seeing first what goes *right* with the argument in the case of BLV. Notice that concepts (in this case  $R$ ) are defined in the finest terms: by specifying exactly the objects that fall under them. Thus, the left-hand side of (21) involves the application of a concept to an object. We can see that in the case of BLV,  $\Phi_{=} (Y, F_{=}(x))$  is calibrated exactly to this degree of specification: for BLV,  $\Phi_{=}$  is  $\subseteq$ , and  $F_E(x)$  is  $\{x\}$ , and so  $\Phi_{=}(Y, F_{=}(x))$  is implied by (and in fact is equivalent to) the predication of  $Y$  to  $x$ . That is why it is so easy to obtain a contradiction: both  $R$  and  $\Phi_{=}(Y, F_{=}(x))$  are specified with an equally fine grain. That is, in the case of BLV, we have:

$$(\forall Z)(Z = Y \rightarrow (F_{=}(x) \subseteq Z \rightarrow \Phi_{=}(Y, F_{=}(x)))) \tag{22}$$

But in general the relation described by  $\Phi_E(Y, F_E(x))$  is not calibrated to the fine grain of predication, as occurs in the left of definitions like (21). More specifically, if the equivalence relation  $E$  is coarser than equality, as it is for separations where  $\kappa \neq 1$ , then it is natural that  $\Phi_E(Y, F_E(x))$  would be coarser than is defined for BLV, and so that (22)’s correlate will fail.

It is then hard to obtain the paradoxical construction as in the case of BLV. For if  $F_E(x)$  is not just  $\{x\}$ , then it is hard to find an appropriate  $\Phi_E$  that will force  $@R$  outside of  $R$  in the event that  $\neg\Phi_E(Y, F_E(x))$ . For if  $F_E(x)$  is not a singleton, then even if  $\Phi_E$  is as fine-grained as  $\subseteq$ , then failure of  $\Phi_E$  can occur when any of the many members of  $F_E(x)$  falls outside of  $Y$ . As such, even if  $@R$  is in  $F_E(@R)$ ,  $F_E(@R) \not\subseteq R$  does not imply that  $\neg R(@R)$ .

That is if one can determine  $Y$ . Of course, where  $E$  is coarser than extensional equality, there will be more than one concept represented by a given abstract. So here again, even if  $\Phi_E$  is as fine-grained as subset,

$$(\forall Z)(E(Y, Z) \rightarrow (F_E(x) \subseteq Z \rightarrow \Phi_E(Y, F_E(x)))) \tag{23}$$

might fail. This is what happens with a straightforward attempt to use Russellian diagonalization on NP. From

$$Rx \leftrightarrow (\exists Y)(x = \S Y \wedge \{x\} \not\subseteq Y) \tag{24}$$

one can easily obtain that  $R(\S R)$ , since  $\S R = \S(R - \{\S R\})$  whether  $R$  is finite or not. No contradiction beckons—not obviously anyway—since there seems not to be a way to obtain something like (23).

None of this is to say that Russellian methods cannot be used to show NP deductively implies a Dedekind finite universe—someone may prove that tomorrow, for all I know. But proofs of this kind have been elusive, and I think this explains why: if one uses only notions like “set and element” or “object and predicate,” it is hard to hit upon just the right  $F_E$  and  $\Phi_E$  that will ensure that  $@R$  falls outside of  $R$  in case  $\neg\Phi_E(R, F_E(@R))$ .

This point is interesting in light of the proof given below that NP implies a Dedekind finite universe. That proof uses ordinal notions to generate a contradiction. I have just said that Russellian proofs are hard to find for uniform separations and complementations with  $\kappa \neq 1$ , because when  $\kappa > 1$ , our diagonal constructions lose the degree of control needed for easy exit from the diagonal concept. It is easier to recover at least some of that control if the locus of the contradiction we seek is in ordinal notions, rather than in the logical notion of predication.

This can be seen by imposing the structure of (21) on the Burali–Forti paradox for those ordinal notions. As a first pass consider how to generate the Burali–Forti

paradox in second-order logic plus BLV: one would first develop a membership relation  $\eta$  between extensions (that is, BLV-abstracts), and then characterize certain abstracts as satisfying some condition  $\Psi$  that qualifies them as BLV-ordinals.

$$Ox \leftrightarrow (\exists Y)(x = \check{\epsilon}Y \wedge \Psi(Y)). \tag{25}$$

The usual construction will characterize ordinals as the extensions of concepts that are  $\eta$ -transitive and well-ordered by  $\eta$ .

This does not look immediately like the kind of construction in (21), but a little massaging shows that it is. For what differentiates  $\Psi$  from something like  $\Phi_E(Y, F_E(x))$  is the absence of two things: invocation of the equivalence relation  $E$ , and a concept determined by the abstract  $x$ . However, in the case of (26), this is supplied by theorems in second-order logic about well-orderings. For where  $\Psi(Y)$  implies that  $(\eta, Y)$  is a well-ordering, it also implies that  $\check{\epsilon}Y$  does not fall under  $Y$ —for otherwise  $(\eta, \{y \in Y \mid y\eta(\check{\epsilon}Y)\})$  would be an isomorphic proper initial segment of  $(\eta, Y)$ . As such, we have (25) is equivalent to

$$Ox \leftrightarrow (\exists Y)(x = \check{\epsilon}Y \wedge \Psi(Y) \wedge \{\check{\epsilon}Y\} \not\subseteq Y). \tag{26}$$

One can thus reproduce the reasoning of the Russell paradox on an ordinal construction by using the fact that  $\neg Y(\check{\epsilon}Y)$  if  $(\eta, Y)$  is a well-ordering, for clearly, aligning with (22), we have

$$(\forall Z)(Y = Z \rightarrow (\{\check{\epsilon}Y\} \subseteq Z \rightarrow \neg(\Psi(Y) \wedge \{\check{\epsilon}Y\} \not\subseteq Y))). \tag{27}$$

Shoehorning the Burali–Forti paradox into Russell’s diagonal form reveals the contrast between the bare Russellian methods and those which use ordinal constructions. The difficulty in using Russellian methods in obtaining Cantorian results from separations with  $\kappa > 1$  comes from finding a formula  $\Phi_E(Y, F_E(x))$  that satisfies (22). In the case of ordinal constructions, it is not easy, but it is comparatively easier: one needs only to use the at-hand theorems of second-order logic that prohibit well-orderings from being isomorphic to their proper initial segments.

The suggestion of constructing ordinals in turn suggests a format for the proof. In [22] (see also [23]), Kanamori examines Zermelo’s proof of the Well-ordering Theorem in [38]. Kanamori’s analysis separates it into two parts:

**THEOREM 5 (Zermelo, Kanamori).** *Given any function  $F : P(X) \rightarrow X$ , there is an unique (strict) well-ordering  $(W, R)$  with  $W \subseteq X$  such that (i)  $F(\{y \in W \mid yRx\}) = x$  for all  $x \in W$ , and (ii)  $F(W) \in W$ .*

**COROLLARY 6 (Zermelo’s Well-ordering Theorem).** *If  $P(X)$  has a choice function, it is well-orderable.*

But Theorem 5 has another curious corollary: a strong version of Cantor’s theorem.<sup>23</sup>

**COROLLARY 7 (Cantor’s Theorem).** *For any  $F : P(X) \rightarrow X$  there are distinct sets  $Y$  and  $W$ , both definable from  $F$ , such that  $F(Y) = F(W)$ . Thus,  $F$  cannot be injective.*

*Proof.* By (ii) of Theorem 5,  $F(W) \in W$ , so by (i) of the same,  $F(\{y \in W \mid yR(F(W))\}) = F(W)$ . But  $Y = \{y \in W \mid yR(F(W))\} \neq W$  since  $F(W) \notin Y$ .  $\square$

<sup>23</sup> See also [5].

This proof of Cantor's Theorem uses the fact that a well-ordered set can be constructed from any function from a power-set to its base set. It is not hard to see that the proof given in [22] of Theorem 5 can be adapted to a second-order setting, where abstraction operators take the place of the function  $F : P(X) \rightarrow X$ .

However, adaptation of Corollary 7 is more difficult. The problem is that one needs more information about  $W$  than is given in the proof of Theorem 5: in particular, that the nature of  $W$  depends partly on  $F$  (or its correlate, the abstraction operator  $\S$ ), and partly on the size of the universe. This is essentially what we do in proving our main result: construct from  $\S$  and an infinite universe a well-ordered concept. Our task for the next section, then, is to sketch this proof.

**§6. An outline of the proof.** Given what we have said so far, we can state the main result of this paper as follows:

**CARDINAL LIMITATION THEOREM CLT.** *In second-order logic, both  $US(\omega)$  and  $UC(\omega)$  deductively imply that the universe is Dedekind finite.*

We will refine the statement of the CLT before proving it; it needs, for example, a more precise statement of the background logic. In any case, it is a direct corollary of CLT that HP, conjoined with either of  $US(\omega)$  or  $UC(\omega)$ , is inconsistent, since HP deductively implies the universe is Dedekind infinite. (In Section 7 I will discuss the reasons why the proof of this theorem does not easily generalize to  $\kappa > \omega$ .)

The case of  $UC(\omega)$  reduces to that of  $US(\omega)$  (Proposition 10), so all the work of the proof concerns the latter principle.

The first thing to do is to define a relation between nuisances:  $\S A \triangleleft \S B$  if and only if  $B - A$  is finite but  $A - B$  is not. Next, if a concept  $P$  is infinite and well-ordered by  $\triangleleft$ , we will say that the nuisance of that concept is a *preordinal*. Now, in second-order logic, a Dedekind infinite universe allows for the definition of a natural number structure  $N$ . The key, then is to show that from this structure we can define nested sequences of concepts with infinite differences. In particular, we show that there is an infinite preordinal  $\S\Omega$ , and that given a preordinal  $p$  and natural numbers  $m$  and  $k$  we can define further preordinals  $p + n\S\Omega^k$ . These defined preordinals behave as one would expect:  $p + n\S\Omega^j \triangleleft p + m\S\Omega^k$  for nonzero natural numbers  $n \leq m$  and  $j \leq k$ , with at least one of these inequalities strict (Preordinal Arithmetic 26).

Preordinals are not well-behaved: there are too many distinct preordinals of similar order-type to be well-ordered. However, because of the properties of well-orderings in second-order logic, we can define a relation  $\ll$  between preordinals. Modulo a defined equivalence ( $\asymp$  of Definition 16), preordinals satisfy trichotomy with respect to  $\ll$ . This then enables us to collect preordinals into concepts containing all and only the  $\ll$ -predecessors of a given preordinal  $p$ . We designate the nuisance of said concept  $O_p$ . The  $O_p$ s are, unlike the preordinals themselves, quite well-behaved: they are well-ordered by  $\triangleleft$ . We therefore may designate these nuisances to be our *nuisance ordinals*. The construction of the nuisance ordinals gives us a large degree of control over their behavior given the preordinals we use to determine them; in particular, we show in the Ordering Lemma 33 that for preordinals  $p$  and  $q$ ,  $p \ll q$  if and only if  $O_p \triangleleft O_q$ .

These facts enable us to prove the Translation Lemma 37, according to which two truncations of the nuisance ordinal sequence are isomorphic:

$$(\triangleleft, Ord_{O_p} + \Omega^n) \cong (\triangleleft, Ord_{O_{p+\S\Omega^{n+1}}}).$$



But this in turn enables the definition of an infinite decreasing subsequence of nuisance ordinals: where  $Ord$  is the concept containing all and only nuisance ordinals, we have

$$\dots \triangleleft O_{\S Ord} O_{\S Ord} O_{\S Ord} \triangleleft O_{\S Ord} O_{\S Ord} \triangleleft O_{\S Ord}.$$

But this, of course, contradicts the fact that  $Ord$  is well-ordered by  $\triangleleft$ .

**§7. On the strength of various proof methods.** I have endeavored to argue in this paper that it is worth examining the methods of proof used for Cantorian results in the relatively modest setting of second-order logic. With the exception of the full proof (given in Section 8), all that remains to do is to draw some tentative conclusions.

As shown in Section 3.2, Russellian methods work for a large class of semantic Cantorian results. We might say, in light of this, that Russellian methods for *semantic* Cantorian cardinality results generalize both *vertically* and *horizontally*. They generalize vertically in the sense that, the known result that  $US(\kappa)$  has no standard model of size at least  $\kappa$  remains true as no matter how one varies the parameter  $\kappa$ . They generalize horizontally in that the arguments establishing these results for uniform *separations* easily adapt to establish correlated results for uniform *complementations*.

Things are less settled when it comes to *deductive* Cantorian results in second-order logic. Russellian methods certainly work for  $BLV = US(1)$  and  $LCP = UC(1)$ , so on the lowest level, they generalize horizontally. But do the Russellian methods apply deductively for anything other than  $\kappa = 1$ ? This is anybody’s guess, but I have given reasons in Section 5 to think that such proofs are not among the low-hanging fruit.

On the other hand, what I have called Burali–Fortian methods—those that construct well-orderings—generalize both horizontally and, to a degree, vertically. With the usual construction described in Section 5, one can prove  $BLV$  inconsistent using the regular Burali–Forti paradox. This method generalizes horizontally, since one can adapt this proof as we adapted the Russellian method in Proposition 1: copy that proof until construction of the diagonal concept  $R$ , and instead define the relation  $\eta^*$  so that for abstracts  $a$  and  $b$  in  $V'$ ,  $a\eta^*b$  holds just if there is a concept  $B \subseteq V'$  with  $Ba$ ; the ordinals are then constructed as usual with the relation  $\eta^*$ . The proof of  $CLT$  shows that in fact the Burali–Fortian method generalizes vertically from  $US(1)$  to  $US(\omega)$ , and horizontally from  $US(\omega)$  to  $UC(\omega)$ .

I have said that these results may reveal connections between Cantorian cardinality results and the availability of well-orderings. It is tempting in such a context—especially since a Russellian proof of these results is presently unavailable—that the core explanation of the Cantorian phenomena is something to do with well-orderings. But such an assertion would be rash. Even if Russellian proofs are permanently unavailable, such a claim would be unwise in the absence of a more considered account of what makes a mathematical explanation the right one. Having not examined these, I refrain from the more robust conclusion.

These considerations, however, raise further questions about a general Cantorian result. For what we will have shown with  $CLT$  is in effect the result that for all  $\kappa \leq \omega$ , if  $|X \triangle Y| < \kappa$  is an equivalence relation, then:

$$US(\kappa) \vdash_{SOL} |V| < \kappa \qquad UC(\kappa) \vdash_{SOL} |V| < \kappa. \tag{28}$$

The natural generalization is that (28) holds for all the various  $\kappa$  that can be characterized in second-order logic. But this question is open. In the proof of CLT we used the existence of Dedekind injections (Section 8.2.1)—essentially the fact that any concept  $X$  of size  $\geq \omega$  is the union of two concepts, one the same size as  $X$  and the other of size  $\omega$ . But the decomposition of sets of higher cardinalities is not guaranteed in the absence of choice-like principles (such as the existence of a pairing injection). As is shown in [10], ZF admits models with *quasi-amorphous* sets: uncountable sets that are not the union of two disjoint uncountable sets, and such that every uncountable subset contains a countably infinite subset. So for uncountable  $\kappa$ , one could not help oneself to the analogue of Dedekind injections, and so cannot follow the same method to construct  $US(\kappa)$ -ordinals as we construct the nuisance ordinals in Sections 8.3–8.7.

**§8. Proving the CLT.**

**8.1. Second-order logic.** To state CLT more clearly, we need to be more specific about what is meant by “full second-order logic.” I will assume general familiarity with second-order languages and second-order logic; the real question is in specifying what is meant by “full.” As for expressions of the language, I am going to abuse notation and use both  $Xt$  and  $t \in X$  to mean the predication of a monadic second-order term  $X$  of a first-order term  $t$ .<sup>24</sup> The language of *pure* second-order logic contains first- and second-order variables, the latter taking all finite arities, as well as the usual logical vocabulary “(,” “),” “ $\forall$ ,” “ $\neg$ ,” “ $\vee$ ,” and “=,” where “=” means identity of objects. However, I will abuse notation to use “=” also to mean coextensiveness of concepts so that “ $X = Y$ ” will mean  $(\forall z)(Xz \leftrightarrow Yz)$  as in (1).

A concept (or relation)  $X$  is said to be Dedekind infinite if there is an injection  $f : X \rightarrow X$  such that  $f(X) \neq X$ . A concept is (Dedekind) finite just if there is no such injection. Though it is an abuse of notation, I will use  $|X| < \omega$  to mean  $X$  is Dedekind finite, and  $|X| \not< \omega$  to mean  $X$  is Dedekind infinite. In what follows, “infinite” just means “Dedekind infinite.”

One can find in [29, Section 3.2, p. 108] a deductive system  $D2^*$  for second-order languages that is complete for the Henkin semantics; it is worth noting that this does not include any “axioms of choice” as part of our background logic. But in addition to the usual axioms governing quantifiers, identity, and logical connectives, we do include the axioms that make second-order logic “full” for a language. These are the comprehension axioms for every formula  $\Phi(\bar{X}, \bar{x})$  of that language; namely axioms of the form:

$$(\forall \bar{X})(\exists Z)(\forall \bar{x})(Z\bar{x} \leftrightarrow \Phi(\bar{X}, \bar{x})). \tag{29}$$

It is therefore somewhat misleading to state CLT as we have above: it is not that we are “adding” abstraction principles to pure, full, second-order logic; rather, we are adding the abstraction principles to a second-order theory that includes comprehension axioms for all formulae of the language of that theory—including those formulae in which the abstraction operator appears.

Let us then give names to these second-order theories. Abusing our notation somewhat, let us call  $US(\omega)$  the second-order theory which includes  $US(\omega)$  (that is, NP) as an axiom in addition to the usual axioms governing quantifiers, identity,

<sup>24</sup> The reader in need of review can consult [29].

and logical connectives, plus comprehension axioms for all formulas including those in which the abstraction operator  $\S$  appears. (Likewise for  $\text{UC}(\omega)$ .)

We aim to show that  $\text{US}(\omega)$  (respectively,  $\text{UC}(\omega)$ ) deductively implies  $|V| < \omega$  (that the universal concept is Dedekind finite); to do this it will suffice to show that the theory  $\text{US}(\omega) + |V| \not< \omega$  (respectively for  $\text{UC}(\omega)$ ) is deductively inconsistent. We will abbreviate these theories  $\text{US}(\omega)^\omega$  and  $\text{UC}(\omega)^\omega$ .

A more precise version of the **CLT**, can now be stated by:

**THEOREM 8.** *The theories  $\text{US}(\omega)^\omega$  and  $\text{UC}(\omega)^\omega$  are inconsistent.*

**8.2. Cardinality and arithmetic in second-order logic with infinity.** I will now review some results we need; these are all provable in what we have called “second-order logic.”

**SCHRÖDER-BERNSTEIN THEOREM SBT.** *If there are injections  $f : A \rightarrow B$  and  $g : B \rightarrow A$ , then there is a bijection  $h : A \rightarrow B$ . That is, if  $|A| \leq |B|$  and  $|B| \leq |A|$  then  $|A| = |B|$ . (The interested reader can find a proof in [29, Theorem 5.2, p. 102].)*

Injections and bijections are measures of the relative cardinality of different concepts and relations. The addition of an axiom of infinity provides more resources for dealing with concepts of cardinality corresponding to the natural numbers. In particular, as Dedekind shows,<sup>25</sup>

**INTERPRETATION OF PEANO ARITHMETIC IPA.** *If there is an infinite concept, then there are a concept  $N$ , an object  $0$ , a function  $s : N \rightarrow N$ , and a relation  $<$ , such that:*

- $(N, 0, s)$  interpret the five Dedekind-Peano axioms,
- $N$  is infinite and injects into every infinite concept,
- $(<, N)$  is a well-ordering, and
- One can define relations  $\text{sum}(x, y, z)$  and  $\text{mult}(x, y, z)$  in  $(N, 0, s)$  that are provably functional and satisfy the recursion axioms, respectively, for addition and multiplication (relativized to  $N$ ), which we state in functional notation:

$$(\forall x \in N)(x + 0 = x) \quad (\forall x, y \in N)(x + s(y) = s(x + y)), \quad (30)$$

$$(\forall x \in N)(x \cdot 0 = 0) \quad (\forall x, y \in N)(x \cdot s(y) = (x \cdot y) + y). \quad (31)$$

We will be working in a theory that assumes the existence of an infinite concept, so throughout the remainder of this proof we will use “ $N$ ,” “ $0$ ,” “ $s$ ,” and “ $<$ ” to be arbitrarily selected so as to satisfy **IPA**. We will also continue to use functional notation for addition and multiplication. Moreover, since  $N$  is well-ordered and satisfies induction, every function definable by recursion exists in  $\text{US}(\omega)$ ; we will therefore also help ourselves to exponentiation, again to be expressed in the usual functional notation (by  $x^y$ ).

One consequence of **IPA** we will use below is worth stating by itself, as we will use it in Section 8.4:

**PROPOSITION 9.** *There is a bijection  $\pi : N \rightarrow N \times N$ , and functions  $\pi_1, \pi_2 : N \rightarrow N$  such that  $\pi(n) = (\pi_1(n), \pi_2(n))$ .*

<sup>25</sup> See [11]. On the relationship to Frege’s Theorem, see [25, Chapter 3]. As Potter notes, HP serves the philosophical purpose of determining what are arguably *the* cardinal numbers (see also [20]); but for the purposes of this paper, HP serves really as an axiom of infinity.

8.2.1. *Dedekind injections.* For  $X$  an infinite concept, say that  $d_X : X \rightarrow X$  is a *Dedekind injection* for  $X$  just if  $d_X$  is an injection and  $|X - d_X(X)| \geq \omega$ .

EXISTENCE OF DEDEKIND INJECTIONS EDI. *If  $X$  is infinite, then there is a Dedekind injection for  $X$ . That is, if  $X$  is an infinite concept, there is a  $d : X \rightarrow X$  such that  $X - d(X)$  is Dedekind infinite, or equivalently,  $|X - d(X)| \geq |N|$ .*

We will really need just one Dedekind injection, so we fix one here for the rest of this section, and designate it “ $d$ .” For simplicity we will also assume that  $N \subseteq V - d(V)$ .

The existence of Dedekind injections will be crucial to the proof of the main result of this paper, but it has an immediate consequence that vastly simplifies our work. The proof given below to establish Theorem 8 shows the inconsistency of  $US(\omega)^\omega$ . The following proposition shows that this is sufficient to prove Theorem 8:

PROPOSITION 10. *If  $UC(\omega)^\omega$  is consistent then so is  $US(\omega)^\omega$ .*<sup>26</sup>

*Proof.* Let  $\mathcal{M} \models UC(\omega)^\omega$ , with  $\chi$  the abstraction operator. Let  $f : V \rightarrow V$  be a Dedekind injection in  $\mathcal{M}$ .

If  $|X \Delta Y| < \omega$ , then as  $f$  is injective  $|f(X) \Delta f(Y)| < \omega$ , and thus  $\chi f(X) = \chi f(Y)$  by  $UC(\omega)$ .

Conversely, assume  $|X \Delta Y| \not< \omega$ . As  $f$  is injective,  $|f(X) \Delta f(Y)| \not< \omega$ . Now notice first that  $V - f(V) \subseteq V - f(Y)$ , second that  $|V - f(V)| \not< \omega$ , and third that  $(V - f(V)) \cap f(X) = \emptyset$ . It follows that  $|f(X) \Delta (V - f(Y))| \not< \omega$ . By  $UC(\omega)$  it follows that  $\chi f(X) \neq \chi f(Y)$ .

So we have shown that

$$(\forall X)(\forall Y)(\chi f(X) = \chi f(Y) \leftrightarrow |X \Delta Y| < \omega). \tag{32}$$

Set  $\sigma X = \chi f(X)$  for each concept  $X$  in  $\mathcal{M}$ ; and let model  $\mathcal{M}'$  be the model extending  $\mathcal{M}$  with the abstraction operator  $\sigma$ . As  $\sigma$  is definable in  $\mathcal{M}$ , the concepts and relations of  $\mathcal{M}'$  are exactly those of  $\mathcal{M}$ ; namely  $f$  is in  $\mathcal{M}'$ . Thus  $\mathcal{M}'$  satisfies  $US(\omega)^\omega$ .<sup>27</sup>  $\square$

It therefore suffices in the below to show  $US(\omega)^\omega$  inconsistent.

8.2.2. *Well-orderings.* We also have at our disposal several results concerning well-orderings. A binary relation  $R$  on  $X$  is said to be a well-ordering of  $X$  (abbreviated

<sup>26</sup> Notice this Proposition generalizes: the same proof works for the theories  $US(\kappa)^\kappa$  and  $UC(\kappa)^\kappa$ , where the former is the theory  $US(\kappa)$  augmented by the assertion that there is a Dedekind injection  $d$  on the universe such that  $|V - d(V)| \geq \kappa$  (and likewise for  $UC(\kappa)^\kappa$ ).

<sup>27</sup> This proof adapts a method shown to me by Richard Kimberly Heck in private correspondence. A result closer to the one Heck proved in their correspondence is this:

PROPOSITION. *Let  $A$  be an abstraction principle that (deductively) implies*

$$(\forall X)(\forall Y)(|X|, |Y|, |V - X|, |V - Y| \not< \omega \rightarrow @X = @Y \leftrightarrow X = Y).$$

*Then  $A$  (deductively) implies that  $|V| < \omega$ .*

*Proof.* If  $|V| \not< \omega$  then let  $d : V \rightarrow V$  be a Dedekind injection, so  $V - d(V)$  and  $d(V - d(V))$  are both infinite. For any  $X$ , then,  $dd(X) \cap ((V - d(V)) \cup (d(V - d(V)))) = \emptyset$ , so  $dd(X) \cup d(V - d(V))$  is infinite and co-infinite. Set  $U = d(V - d(V))$  and we have:

$$(\forall X)(\forall Y)(@(dd(X) \cup U) = @(dd(Y) \cup Y) \leftrightarrow X = Y)$$

and so can reproduce the Russell paradox.  $\square$

$WO(R, X)$ ) just if  $R$  linearly orders  $X$  and for every non-empty  $Y \subseteq X$ , there is an  $R$ -least object falling under  $Y$ . Given  $R, S, X$ , and  $Y$  with  $WO(R, X)$  and  $WO(S, Y)$ , we write  $(R, X) \leq (S, Y)$  to mean that there is an order-isomorphism  $i : X \rightarrow Y$  where  $i(X)$  is an initial segment of  $Y$ . (Context will make clear when “ $\leq$ ” and its derivatives indicate relative cardinality, the well-ordering of  $N$ , or isomorphism to a proper initial segment of a well-ordering.) If  $i(X) = Y$  we write  $(R, X) \cong (S, Y)$ ; if  $i(X)$  is a proper initial segment of  $Y$ , we will write  $(R, X) < (S, Y)$ ; and when needed we will indicate the witnessing isomorphism by superscripting, as in “ $(R, X) \overset{i}{<} (S, Y)$ .”

All of this is expressible in second-order logic, and the following is a theorem of second-order logic.

**ORDER ISOMORPHISM THEOREM OIT.** *For any well-orderings  $(R, X)$  and  $(S, Y)$ , exactly one of*

$$(R, X) < (S, Y) \quad (R, X) \cong (S, Y) \quad (R, X) > (S, Y) \quad (33)$$

*obtains. Moreover, the witnessing isomorphisms are unique.*

*Further, no well-ordering is isomorphic to a proper initial segment of itself:  $(R, X) \not\prec (R, X)$ .*

The theorem OIT has the nice consequence that well-orderings are themselves “well-ordered” in the following sense:

**WELL-ORDERINGS OF WELL-ORDERINGS WOWO.** *Let  $\varphi(R, X)$  be a formula. The following is a theorem of second-order logic:*

$$\begin{aligned} (\exists R, X)(WO(R, X) \wedge \varphi(R, X) \rightarrow \\ (\exists S, Y)(WO(S, Y) \wedge \varphi(S, Y) \wedge \\ (\forall R, X)(WO(R, X) \wedge \varphi(R, X) \rightarrow (S, Y) \leq (R, X))). \end{aligned} \quad (34)$$

*In words: If there is a well ordering  $R$  of  $X$  such that  $\varphi$  holds of  $R$  and  $X$ , then there is a  $<$ -least well-ordering of some  $S$  and  $Y$  such that  $\varphi$  holds of  $S$  and  $Y$ . The proof is left as an exercise to the reader.*

We will also use the fact that in second-order logic, one can define functions and relations by recursion on well-ordered concepts. In particular, we will use recursion on the well-ordering  $(<, N)$ .

**DEFINITION BY RECURSION DBR.** *Let  $g, h : N^k \rightarrow N$  be functions,  $S(\bar{x})$  be a  $k$ -ary relation, and  $\Phi(R, S, y, \bar{x})$  be a formula with  $k + 1$ -ary relation variable  $R$  and  $k$ -ary relation variable  $S$  free, and  $y, \bar{x}$  be  $k + 1$  free variables. Then the equations in (35) suffice to define a function  $f : N^{k+1} \rightarrow N$ , and the equations in (36) suffice to define a  $k + 1$ -ary relation  $R$ .*

$$f(0, \bar{n}) = g(\bar{n}) \quad f(m + 1, \bar{n}) = h(f(m, \bar{n}), g(\bar{n})), \quad (35)$$

$$R(0, \bar{n}) \Leftrightarrow S(\bar{n}) \quad R(m + 1, \bar{n}) \Leftrightarrow \Phi(R(m, \bar{n}), S(\bar{n})). \quad (36)$$

Finally, we make the following observations:

**PROPOSITION 11.** *Let  $(R, X)$  be a well-ordering and let  $X$  be finite. Then  $(R, X) < (<, N)$ .*

*Proof.* If  $(R, X) \geq (<, N)$  then  $X$  is infinite, so the result follows by OIT. □

**8.3. Preordinals.** Our construction thus begins by defining a privileged relation  $\triangleleft$ , which I will call “subvention.” In what follows, we will take “ $\S$ ” as the abstraction operator for  $US(\omega)$ , as in (4).

DEFINITION 12. For all  $\S X, \S Y \in \text{rng}(\S)$ :

$$\S X \triangleleft \S Y \Leftrightarrow (\forall X', Y')(\S X = \S X' \wedge \S Y = \S Y' \rightarrow |X' - Y'| < \omega \wedge |Y' - X'| \geq \omega). \tag{37}$$

The subvention relation  $\triangleleft$  has certain desirable properties:

PROPOSITION 13. First, if  $\S X = \S Y$ , then

$$(\forall Z)(\S X \triangleleft \S Z \Leftrightarrow \S Y \triangleleft \S Z), \tag{38}$$

$$(\forall Z)(\S Z \triangleleft \S X \Leftrightarrow \S Z \triangleleft \S Y). \tag{39}$$

Thus  $\triangleleft$  holds independently of the concept  $X$  determining the abstract  $\S X$ .

PROPOSITION 14. If  $f : X \cup Y \rightarrow V$  is an injection and  $Z \cap f(X) = \emptyset$ , then:

$$\S X \triangleleft \S Y \Leftrightarrow \S(Z \cup f(X)) \triangleleft \S(Z \cup f(Y)). \tag{40}$$

PROPOSITION 15. The relation  $\triangleleft$  is irreflexive and antisymmetric.

DEFINITION 16. An abstract  $\S X$  is a preordinal ( $\S X \in POrd$ ) if and only if there is an infinite  $Y \subseteq \text{rng}(\S)$  with  $\S Y = \S X$  such that  $WO(\triangleleft, Y)$ .

If  $p, q$  are preordinals, we will say that  $p \asymp q$  just if  $\S P = p, \S Q = q$  such that  $(\triangleleft, P)$  and  $(\triangleleft, Q)$  are well-orderings and  $(\triangleleft, P) \cong (\triangleleft, Q)$ .

**8.4. Countable preordinals in  $US(\omega)^\omega$ .** The collection of preordinals is interesting in  $US(\omega)^\omega$  only if there is an infinite preordinal. Think for a moment how one would show, in ZF, that there is an infinite collection of well-ordered nested subsets, each differing infinitely from its predecessors. Two sequences can be generated in ZF that satisfy these desiderata: the sequences  $\omega, 2\omega, 3\omega, \dots$  and  $\omega, \omega^2, \omega^3, \dots$ . Both because they provide infinite preordinals, and because we will need them later to control certain constructions, we will show in this section how to get preordinals of these order-types.

Our first task will be to define preordinals  $\Omega^k$ , for each  $k$  in  $N$ . These are the preordinal surrogates for the ordinals  $\omega^k$  appearing in ZF.

To begin, we can define a relation on  $N$  recursively as follows:

$$R(0, n, m) \Leftrightarrow n < m, \tag{41}$$

$$R(k + 1, n, m) \Leftrightarrow R(k, \pi_1(n), \pi_1(m)) \vee (\pi_1(n) = \pi_1(m) \wedge \pi_2(n) < \pi_2(m)). \tag{42}$$

The relation  $R$  can be considered as the dictionary ordering on its second and third parameters  $n$  and  $m$ , while the first parameter  $k$  determines that they will be treated as  $k + 1$ -tuples. Since  $R(k, \cdot, \cdot)$  is the dictionary-order for  $k + 1$ -tuples, we will henceforth write  $R(k, n, m)$  by  $n <_{k+1} m$ . So, for example,  $n <_3 m$  holds just if  $n$  is less than  $m$  in the dictionary order when  $n$  and  $m$  are decoded as triples.

We now define another sequence of relations. Informally, start with  $N$  under the well-ordering  $<_k$ . Now notice that for  $m_1 < m_2$ , the concepts

$$\pi^{-1}(m_1, N) = \{\pi^{-1}(m_1, n) \mid n \in N\} \tag{43}$$

$$\pi^{-1}(m_2, N) = \{\pi^{-1}(m_2, n) \mid n \in N\} \tag{44}$$

are disjoint and infinite. Thus, if  $n_1 < n_2$ , we have that

$$\bigcup_{m <_k n_1} \pi^{-1}(m, N) \subset \bigcup_{m <_k n_2} \pi^{-1}(m, N) \tag{45}$$

with an infinite difference. We can collect the nuisances of the unions listed in (45) into a single concept, designated  $\Omega^k$ . More formally:

DEFINITION 17. Define  $\Omega^k(x)$  by

$$\Omega^k(x) \Leftrightarrow (\exists Y)(x = \S Y \wedge (\forall y)(Yy \leftrightarrow (\exists n, m \in N)(m <_k n \wedge \pi_1(y) = m))). \tag{46}$$

Equivalently,

$$\Omega^k = \left\{ \S \left( \bigcup_{m <_k n} \pi^{-1}(m, N) \right) \mid n \in N \right\}, \tag{47}$$

where  $(m, N) = \{(m, z) \in N \times N \mid z \in N\}$ . We stipulate that  $\Omega^0 = \emptyset$ .

The concept  $\Omega^k$  is constructed to have the same structure under  $\triangleleft$  as  $N$  under the ordering  $<_k$ :

$\Omega^k$ -WELL-ORDERING LEMMA 18. For each  $k > 0$ ,  $(\triangleleft, \Omega^k) \cong (<_k, N)$ . Thus each  $(\triangleleft, \Omega^k)$  is a well-ordering.

*Proof.* We proceed by induction; let  $k = 1$ . Let  $F^1 : N \rightarrow \Omega$  be defined by

$$F^1(n) = \S \left( \bigcup_{j < n} \pi^{-1}(j, N) \right). \tag{48}$$

Clearly  $F^1$  is a bijection; to see it is an isomorphism it is enough to show

$$m < n \rightarrow F^1(m) \triangleleft F^1(n). \tag{49}$$

For (49) implies its converse, since if  $m \not< n$  then either  $n = m$  or  $n < m$ . If the former,  $F^1(m) \not\triangleleft F^1(n)$  by the irreflexivity of  $\triangleleft$ , and if the latter,  $F^1(m) \not\triangleleft F^1(n)$  by (49) and the antisymmetry of  $\triangleleft$ .

Towards (49), observe that if  $m < n$ , then

$$\left( \bigcup_{j < m} \pi^{-1}(j, N) \right) \subset \left( \bigcup_{j < n} \pi^{-1}(j, N) \right) \tag{50}$$

with  $\pi^{-1}(m, N)$  an infinite subset of the difference. Thus  $F^1(m) \triangleleft F^1(n)$ .

Assume now that  $(\triangleleft, \Omega^k) \stackrel{F^k}{\cong} (<_k, N)$ , and consider  $(\triangleleft, \Omega^{k+1})$ . Define  $\triangleleft$  to be a relation on  $N \times N$  by

$$(a, m) \triangleleft (b, n) \Leftrightarrow a <_k b \vee (a = b \wedge m < n). \tag{51}$$



Clearly  $(\prec, N \times N) \cong^i (\prec_{k+1}, N)$ , so it will suffice to show that  $(\prec, N \times N) \cong (\prec, \Omega^{k+1})$ . Define  $F^{k+1} : N \times N \rightarrow \Omega^{k+1}$  by

$$F^{k+1}(b, n) = \S \left( \bigcup_{(a,m) \prec (b,n)} \pi^{-1}(i((a, m), N)) \right), \tag{52}$$

where  $(i(a, m), N) = \{(i(a, m), x) \mid x \in N\}$

As in the case of (49), it is enough to show

$$(a, m) \prec (b, n) \rightarrow F^{k+1}(a, m) \triangleleft F^{k+1}(b, n) \tag{53}$$

since, like (49), (53) proves its converse.

But clearly if  $(a, m) \prec (b, n)$  then

$$\left( \bigcup_{(c,j) \prec (a,m)} \pi^{-1}(i(c, j), N) \right) \subset \left( \bigcup_{(c,j) \prec (b,n)} \pi^{-1}(i(c, j), N) \right). \tag{54}$$

Moreover,  $\pi^{-1}(i(a, m), N)$  is an infinite subset of the difference. So again,  $F^{k+1}(a, m) \triangleleft F^{k+1}(b, n)$ , as required.  $\square$

In addition to the above exponential expansion of the preordinal  $\S\Omega$ , we will need to be able to multiply these preordinals by natural numbers—allowing us to obtain preordinals that behave like the set-theoretic ordinals  $m\omega$  and  $m\omega^k$ . These ordinals, of course, behave like  $N^m = \underbrace{N \times \dots \times N}_m$  and  $(N^k)^m = \underbrace{N^k \times \dots \times N^k}_m$  under their respective dictionary orderings. If we were working in a normal set-theoretic environment, we would consider  $N^m$  and  $(N^k)^m$  with the sets

$$\{(n, j) \in N \times N \mid j \leq m\} \subseteq N \times N$$

and

$$\{(n, j) \in N^k \times N \mid j \leq m\} \subseteq N^{k+1},$$

respectively. Under their respective dictionary orderings, these are order-isomorphic to  $m\omega$  and  $m\omega^k$ .

The next construction gives us these preordinal surrogates, but using the ordering  $\prec_{k+1}$  of  $N$  so that  $N$  behaves like  $N^{k+1}$ . It is then easily obtainable by the  $\Omega^k$ -Well-Ordering Lemma 18. Let

$$N_m = \{n \in N \mid \pi_2(n) \leq m\} \tag{55}$$

and consider  $N_m$  under the ordering  $\prec_{k+1}$ . Clearly  $(\prec_{k+1}, N_m) \prec (\prec_{k+1}, N)$  for each  $m$ . The embedding  $e_m^{k+1}$  witnessing this is unique, as is the isomorphism  $i^{k+1}$  witnessing  $(\prec_{k+1}, N) \cong (\prec, \Omega^{k+1})$ . Thus we may give the following:

DEFINITION 19. For  $k, m > 0$ , let

$$(m + 1)\Omega^k = \{i^{k+1}(e_m^{k+1}(n)) \mid n \in N_m\}.$$

We stipulate that  $0\Omega^k = \emptyset$ .

Set  $m\S\Omega^k = \S(m\Omega^k)$ .

It follows now by the construction of  $m\S\Omega^k$ :

**COROLLARY 20.** *For each  $k, m \in N$  with  $k, m > 0$ ,  $m\Omega^k$  is a preordinal. Clearly, as well, for each  $w \in m\Omega^k$ , there is a  $W \subseteq N$  with  $w = \S W$ .*

*Further, if  $(R, X)$  is a well-ordering and  $(\triangleleft, m\Omega^k) < (R, X)$  for all  $m$ , then  $(\triangleleft, \Omega^{k+1}) \leq (R, X)$ .*

**8.5. Preordinal addition and the “move over” construction.** The next task is to define functions on preordinals, which we will call “Addition by  $\S m\Omega^k$ .” Again, recall from Section 8.2.1 that  $d$  is a Dedekind injection, and we assume  $N \subseteq V - d(V)$ .

We need to show that for any  $k, m \in N$  and for any preordinal  $p$ , the expression  $p + \S m\Omega^k$  determines a unique preordinal. Because (as we shall see in Proposition 27) distinct preordinals may be the nuisances of  $\triangleleft$ -isomorphic concepts, we will also need to show that if  $\S P$  and  $\S Q$  are distinct preordinals with  $p \asymp q$ , then  $p + m\Omega^k \asymp q + m\Omega^k$ .

The method of defining addition by  $\S \Omega^k$  could be called the “move-over” construction: the selected Dedekind injection  $d$  “moves over” all of  $V$  to make room for countable concepts.

**DEFINITION 21.** *We will say that a concept  $Q$  is numerically based if for each  $w \in Q$ , there is a  $W \subseteq N$  with  $w = \S W$ . Further, if  $q$  is a preordinal, we will say that it is numerically based just if there is a numerically based  $Q$  such that  $(\triangleleft, Q)$  is a well-ordering and  $q = \S Q$ .*

**DEFINITION 22** (The “move over” construction). *Let  $(\triangleleft, P)$  be a well-ordering with  $P \subseteq \text{rng}(\S)$ , and  $(\triangleleft, U)$  a well-ordering with  $U$  numerically based. Then define*

$$P + U = \{\S d(Q) \mid \S Q \in P\} \cup \{\S(d(V) \cup W) \mid \S W \in U \text{ and } W \subseteq N\}. \tag{56}$$

The Dedekind injection “moves over” the concepts  $Q$  and  $V$  so that each  $d(Q)$ , and  $d(V)$ , are disjoint from each infinite  $W \subseteq N$ :

$$d(Q) \cap W = \emptyset \qquad d(V) \cap W = \emptyset. \tag{57}$$

This ensures that  $\S d(Q) \triangleleft \S(d(V) \cup W)$  for each such  $W$ . Moreover, if  $\S W \triangleleft \S W'$  with  $\S W, \S W' \in U$ , then (57) requires that  $\S(d(V) \cup W) \triangleleft \S(d(V) \cup W')$ .

Finally, it is easy to see that  $P + U$  does not vary according to the choices of  $Q$  and  $W$  for  $\S Q \in P$  and  $\S W \in U$ . For if  $\S Q' = \S Q$ , clearly then  $\S d(Q) = \S d(Q')$ . If  $\S W' = \S W$ , then as  $W \triangle W'$  is finite,  $(d(V) \cup W) \triangle (d(V) \cup W')$  will also be finite (again by (57)).

**PROPOSITION 23.** *Let  $(\triangleleft, P), (\triangleleft, Q)$  be well-orderings with  $\S P = \S Q$ . Then for each  $k, m \in N$ ,  $\S(P + m\Omega^k) = \S(Q + m\Omega^k)$ .*

*Proof.* Clearly  $(\triangleleft, P \cap Q)$  is a well-ordering and  $\S(P \cap Q) = \S P = \S Q$ . By the construction of  $P + m\Omega^k$  and  $Q + m\Omega^k$ ,  $\S(P + m\Omega^k) = \S((P \cap Q) + m\Omega^k) = \S(Q + m\Omega^k)$ . □

**DEFINITION 24.** *Given Proposition 23, we may define: for a given preordinal  $p$ , let  $p + m\Omega^k = \S(P + m\Omega^k)$  for each  $k, m \in N$ .*

**PROPOSITION 25.** *Let  $p, q$  be preordinals with  $p \asymp q$ . Then for each  $k, m \in N$ ,  $p \triangleleft p + \S m\Omega^k \asymp q + m\Omega^k$ .*

*Proof.* That  $p \triangleleft p + m\Omega^k$  follows from the construction of  $p + m\Omega^k$ .

As  $p \asymp q$ , let  $p = \S P, q = \S Q$  and  $(\triangleleft, P) \cong (\triangleleft, Q)$  be well-orderings. It is easy to see then that  $(\triangleleft, P + m\Omega^k) \cong (\triangleleft, Q + m\Omega^k)$ , and thus that  $p + m\Omega^k = q + m\Omega^k$ . □

At last we obtain the distilled essentials we will need from preordinal addition:

PREORDINAL ARITHMETIC 26. *Let  $p$  be any preordinal and  $j, k, m, n \in N$  with  $0 < j < k$  and  $0 < n < m$ . Then:*

$$\S\Omega^j \triangleleft n\S\Omega^j \triangleleft m\S\Omega^j \triangleleft \S\Omega^k, \tag{58}$$

$$p + \S\Omega^j \triangleleft p + n\S\Omega^j \triangleleft p + m\S\Omega^j \triangleleft p + \S\Omega^k. \tag{59}$$

*Proof.* The first, (58), is an immediate consequence of the  $\Omega^k$ -Well-Ordering Lemma 18: it follows from the fact that

$$\langle \langle_j, N^j \rangle \rangle < \langle \langle_{j+1}, N_n^{j+1} \rangle \rangle < \langle \langle_{j+1}, N_m^{j+1} \rangle \rangle < \langle \langle_k, N^k \rangle \rangle.$$

The next, (59), then follows from (58) and the construction of  $p + q$ , where  $q$  is numerically based. Note that  $m\S\Omega^j$ , etc., are by construction numerically based.  $\square$

We will conclude with another result of the “move over” construction, which will be important for the transition from preordinals to full-blown nuisance ordinals.

PROPOSITION 27. *Let  $p$  be a preordinal. There are infinitely many preordinals  $q$  such that  $p \asymp q$ .*

*Proof.* Let  $(\triangleleft, P)$  be a well-ordering and  $\S P = p$ ; again let  $d$  be our designated Dedekind injection. Now define, for each  $k \in N$ :

$$2^k N = \{2^k n \mid n \in N\}. \tag{60}$$

Observe that for distinct  $j < k \in N$ ,

$$2^j N \subset 2^k N \qquad |2^k N - 2^j N| \not\prec \omega. \tag{61}$$

Thus, set

$$P^k = \{\S(d(Q) \cup 2^k N) \mid \S Q \in P\}. \tag{62}$$

The arguments given following Definition 22 and in proving Proposition 23 show that each  $\S P^k$  is uniquely determined and that  $p \asymp \S P^k$  for each  $k$ . Further, by (61),  $\S P^j \triangleleft \S P^k$  for  $j < k$ , so  $\S P^j \neq \S P^k$  by the asymmetry of  $\triangleleft$ .  $\square$

**8.6. The relation  $\ll$ .** The astute reader may have complained up until now, in the back of her mind, that “preordinals” are rather ill-behaved for our purposes: for in ZF, a set which represents one order type is distinct from the set which represents any other. This is because sets are distinguished extensionally, and so if, e.g.,  $\omega$  differs from  $\omega + 1$  (as it does) by just one element, then  $\omega \neq \omega + 1$ .

But this is not the case for preordinals as we have defined them, since preordinals are nuisances, and nuisances are differentiated by infinite difference between the concepts they represent. As such, consider that  $(\triangleleft, \Omega)$  is (in ZF) of order-type  $\omega$ , while  $(\triangleleft, \Omega \cup \{\S N\})$  is of order-type  $\omega + 1$ . However, since  $\Omega \triangle (\Omega \cup \{\S N\}) = \{\S N\}$ , the nuisances of these two concepts are identical:  $\S\Omega = \S(\Omega \cup \{\S N\})$ .

But we need not give up hope. Our goal is to generate the Burali–Forti paradox, and it will be enough if we can generate a theory of nuisance ordinals based on *infinite differences* between order-types. For this reason, define the following relation:

DEFINITION 28. *Let  $p, q$  be preordinals. Then we say that  $p \ll q$  if and only if there are  $P, Q$  such that  $p = \S P$  and  $q = \S Q$  with  $(\triangleleft, P) \overset{i}{\prec} (\triangleleft, Q)$  and  $|Q - i(P)| \not\prec \omega$ .*

Notice that the relation  $\ll$  is well-defined:

**PROPOSITION 29.** *If  $p = \S P = \S P'$  and  $q = \S Q = \S Q'$ , then the following two either both obtain or both fail:*

$$(\triangleleft, P) \overset{i}{<} (\triangleleft, Q), \quad |Q - i(P)| \not\prec \omega, \tag{63}$$

$$(\triangleleft, P') \overset{i'}{<} (\triangleleft, Q'), \quad |Q' - i'(P')| \not\prec \omega. \tag{64}$$

*Proof.* We will just prove one case to give a flavor for how to proceed: As  $P, P'$  and  $Q, Q'$  are all well-ordered by  $\triangleleft$ , suppose

$$(\triangleleft, P) \overset{j}{<} (\triangleleft, P'), \tag{65}$$

$$(\triangleleft, Q) \overset{j'}{<} (\triangleleft, Q'). \tag{66}$$

Since  $\S P = \S P'$ ,  $P' - j(P)$  is finite. Since it is moreover well-ordered, by Proposition 11,  $(\triangleleft, P' - j(P)) < (\triangleleft, \Omega) \leq (\triangleleft, Q - i(P))$  since  $(\triangleleft, N) \cong (\triangleleft, \Omega)$ . So assuming (63), let  $k$  witness  $(\triangleleft, P' - j(P)) < (\triangleleft, Q - i(P))$ .

Then the homomorphism  $i'$  in (64) is witnessed by

$$i'(x) = \begin{cases} i(x) & x \in P, \\ j'(k(x)) & x \in P' - j(P). \end{cases} \tag{67}$$

□

**PROPOSITION 30.** *For  $\S P, \S Q$  preordinals with  $(\triangleleft, P)$  and  $(\triangleleft, Q)$  well-orderings, if  $\S P \not\ll \S Q$  and  $\S Q \ll \S P$  then  $\S P \asymp \S Q$ .*

*Proof.* Suppose by WOWO that  $(\triangleleft, P) \overset{i}{<} (\triangleleft, Q)$ . As  $\S P \not\ll \S Q$ ,  $Q - i(P)$  is finite. Thus  $\S Q = \S i(P)$ . But clearly  $(\triangleleft, P) \cong (\triangleleft, i(P))$ . If  $(\triangleleft, P) > (\triangleleft, Q)$  the argument is the same. □

Adding the relation  $\ll$  allows us to state more facts about preordinal addition:

**PREORDINAL ARITHMETIC 31.** *Let  $p$  be any preordinal and  $j, k, m, n \in N$  with  $0 < j < k$  and  $0 < n < m$ . Then:*

$$n\S\Omega^j \ll m\S\Omega^j \ll \S\Omega^k, \tag{68}$$

$$p + n\S\Omega^j \ll p + m\S\Omega^j \ll p + \S\Omega^k. \tag{69}$$

*Proof.* Like (58), (68) follows from the constructions of  $m\Omega^j$  and  $\Omega^k$ ; for by the  $\Omega^k$ -Well-Ordering Lemma 18,  $n\Omega^j$ ,  $m\Omega^j$ , and  $\Omega^k$  are respectively isomorphic to the structures in the sequence:

$$(\triangleleft_{j+1}, N_n^j) < (\triangleleft_{j+1}, N_m^j) < (\triangleleft_k, N^k) \tag{70}$$

with an infinite remainder in each case—in fact, remainder on the left is isomorphic to  $(\triangleleft, N)$ , and on the right isomorphic to  $(\triangleleft_{k-j}, N^{k-j})$ .

The equation (69) follows from (68) and the construction of  $p + m\S\Omega^j$ , etc. □

**8.7. Nuisance ordinals.** As mentioned above, we will define nuisance ordinals as concepts of preordinals.

**DEFINITION 32.** *Say that  $x$  is a nuisance ordinal (which we will write with “ $x \in Ord$ ”) just if there is a preordinal  $p$  such that  $x = \S\{q \mid q \ll p\}$ . The nuisance ordinal determined by the preordinal  $p$  we will designate  $O_p$ . We will let:*

$$POrd = \{p \mid p \text{ is a preordinal}\} \qquad Ord = \{O_p \mid p \in POrd\}. \tag{71}$$

It is clear from the definition that if  $p \asymp q$ , then  $O_p = O_q$ . From this it follows that the nuisance ordinals behave like the limit ordinals in ZF:

**ORDERING LEMMA 33.** *For any well-orderings  $(\triangleleft, P)$  and  $(\triangleleft, Q)$ :*

$$\S P \ll \S Q \leftrightarrow O_{\S P} \triangleleft O_{\S Q}. \tag{72}$$

*Proof.* ( $\rightarrow$ ): Suppose first that  $\S P \ll \S Q$ . Clearly  $O_{\S P} \subseteq O_{\S Q}$  with  $\S P \in O_{\S Q}$ . By Proposition 27, there are infinitely many  $\S P'$  with  $\S P' \asymp \S P$ , and so each such  $\S P' \ll \S Q$  as well; thus each such  $\S P' \in O_{\S Q}$  but not in  $O_{\S P}$ . Hence  $O_{\S P} \triangleleft O_{\S Q}$ .

( $\leftarrow$ ): By  $\rightarrow$  and the asymmetry of  $\triangleleft$ , if  $O_{\S P} \triangleleft O_{\S Q}$  then  $\S Q \not\ll \S P$ . Likewise, if  $\S P \asymp \S Q$ , then  $O_{\S P} = O_{\S Q}$ , contradicting  $O_{\S P} \triangleleft O_{\S Q}$ . Thus by Proposition 30,  $\S P \ll \S Q$ . □

**REMARK 34.** We have required that each preordinal be the nuisance of an infinite concept. Notice that the Ordering Lemma 33 fails if we allow preordinals  $\S P$  for  $P$  finite. For if  $P$  is finite and  $Q$  is  $\Omega$ , then we have:

$$\begin{aligned} \S P &= \S \emptyset & O_{\S P} &= \S\{r \mid r \ll \S P\} = \S \emptyset \\ \S Q &= \S \Omega & O_{\S \Omega} &= \S\{r \mid r \ll \S \Omega\} \\ & & &= \S\{\S X \mid |X| < \omega\} \\ & & &= \S\{\S \emptyset\} = \S \emptyset. \end{aligned}$$

However, we do have as a corollary to Lemma 33:

**WELL-ORDERING OF  $Ord$  35.**  $(\triangleleft, Ord)$  is a well-ordering.

*Proof.* By Proposition 30, for any  $p, q \in POrd$ , exactly one of the following holds:

$$p \ll q \qquad q \ll p \qquad O_p = O_q. \tag{73}$$

And so by Lemma 33, we have exactly one of

$$O_p \triangleleft O_q \qquad O_q \triangleleft O_p \qquad O_p = O_q. \tag{74}$$

So  $(\triangleleft, Ord)$  is a linear ordering.

Let  $X \subseteq Ord$  be non-empty; we show  $X$  has a  $\triangleleft$ -least element. Let  $Y = \{p \in POrd \mid O_p \in X\}$ ; note that  $Y \neq \emptyset$ . Now let  $\varphi(R, Z)$  be the following formula

$$(\forall p)(\forall q)(Rpq \leftrightarrow p \triangleleft q) \wedge (\exists P)(\S P = \S Z \in Y \wedge WO(R, P)). \tag{75}$$

It follows from the fact that there is a  $\S P \in Y$  that  $\varphi(R, Z)$  for some  $R$  and  $Z$ . By WOWO, there is then a relation  $R$  and a concept  $Z$  such that  $WO(R, Z)$ ,  $\varphi(R, Z)$ , and for any  $(R', Z')$  with  $WO(R', Z')$  and  $\varphi(R', Z')$ ,  $(R, Z) \leq (R', Z')$ . Each such  $R'$  must be  $\triangleleft$ , so there is a  $P$  such that  $WO(\triangleleft, P)$  with  $\S P \in Y$  and  $(\triangleleft, P) \leq (\triangleleft, P')$  for every  $\S P' \in Y$ . By the definition of  $Y$ ,  $O_{\S P} \in X$ .

It remains to show that  $O_{\S P} \triangleleft O_q$  for all remaining ordinals in  $X$ . If not, then by the linear ordering of the  $O_q \triangleleft O_{\S P}$ , the Ordering Lemma 33 yields  $q \ll \S P$ . But then there is a  $Q$  with  $WO(\triangleleft, Q)$  and  $\S Q \in Y$  such that  $(\triangleleft, Q) < (\triangleleft, P)$ , contradicting the choice of  $P$ . So  $O_{\S P}$  is the least element of  $X$ , and so  $(\triangleleft, Ord)$  is a well-ordering.  $\square$

**8.8. Contradiction via the Translation Lemma.** Observe that we are quite close to the Burali–Forti paradox. For any  $O_p \in Ord$ , let

$$Ord_{O_p} = \{O_q \in Ord \mid O_q \triangleleft O_p\}. \tag{76}$$

Clearly, for any  $O_p \in Ord$ ,  $(\triangleleft, Ord_{O_p})$  is an initial segment of  $(\triangleleft, Ord)$ . Now, since  $(\triangleleft, Ord)$  is a well-ordering, it follows that  $\S Ord \in POrd$ . And thus, that there is a nuisance ordinal  $O_{\S Ord}$ . By what was just said, it follows that  $(\triangleleft, Ord_{O_{\S Ord}})$  is an initial segment of  $(\triangleleft, Ord)$ .

If we could show that  $(\triangleleft, Ord_{O_{\S Ord}}) \cong (\triangleleft, Ord)$ , we would have the Burali–Forti contradiction. Doubtless this can be done, but I think what is given below is a more direct route to a contradiction. The Burali–Forti paradox finds a well-ordering that is isomorphic to a proper initial segment of itself, and that is a problem because such an isomorphism generates a non-empty set of ordinals with no least element. In what follows below, we will skip the isomorphism and go right to the construction of a subconcept of nuisance ordinals without a  $\triangleleft$ -least object.

Our goal will be to define a function  $B : N \rightarrow Ord$  such that

$$B(0) \triangleright B(1) \triangleright B(2) \triangleright \dots B(n) \triangleright B(n + 1) \dots \tag{77}$$

We will define this function  $B$ , and an auxiliary function  $A$ , by recursion (see DBR):

$$A(0) = \S Ord \qquad B(0) = O_{\S Ord} = O_{A(0)}, \tag{78}$$

$$A(n + 1) = \S Ord_{O_{A(n)}} \qquad B(n + 1) = O_{\S Ord_{B(n)}} = O_{A(n)}. \tag{79}$$

Notice now that if we can prove the following lemma, then we have a contradiction:

LEMMA 36. For all  $n, m \in N$ ,

$$n < m \leftrightarrow B(m) \triangleleft B(n). \tag{80}$$

For if Lemma 36 holds, then  $B(N) \subseteq Ord$  is non-empty with no  $\triangleleft$ -least object falling under it.

But to prove Lemma 36, we need to peer deep beneath the surface so that we can focus on how nuisance ordinals determined by a sum  $p + m\S\Omega^k$  relate to preordinal sums, and so to the entire structure  $(\triangleleft, Ord)$ . This is what we show in the following lemma.

TRANSLATION LEMMA 37. For any preordinal  $p$  and any  $n \in N$  with  $n > 0$ ,

$$(\triangleleft, Ord_{O_p + \Omega^n}) \cong (\triangleleft, Ord_{O_{p+\S\Omega^{n+1}}}). \tag{81}$$

*Proof.* Note first that

$$(\triangleleft, (Ord_{O_p + \Omega^n} - Ord_{O_p})) \cong (\triangleleft, \Omega^n), \tag{82}$$

$$(\triangleleft, Ord_{O_{p+\S\Omega^{n+1}}} - Ord_{O_p}) \cong (\triangleleft, Ord_{O_{\S\Omega^{n+1}}}). \tag{83}$$

So it will suffice to show that  $(\triangleleft, \Omega^n) \cong (\triangleleft, Ord_{O_{\S\Omega^{n+1}}})$ .

Next observe that  $(\triangleleft, \Omega^{n+1}) \cong (\triangleleft, \Omega^n \times N)$ , where  $\triangleleft$  is as in (51); let  $g$  witness this order isomorphism. Now define  $G : \Omega^n \rightarrow \Omega^{n+1}$  by

$$G(a) = g(a, 0). \tag{84}$$

To see what is going on here, recall that the objects falling under  $\Omega^n$  are ordered by  $\triangleleft$  like  $n$ -tuples are ordered by the dictionary ordering. So  $G$  starts with such an object, first sends it to the pair  $(a, 0)$  in  $\Omega^n \times N$ , and then sends it, via  $g$ , to the object in  $\Omega^{n+1}$  that has the same place in the structure  $(\triangleleft, \Omega^{n+1})$  as does the object  $(a, 0)$  in the structure  $(\triangleleft, \Omega^n \times N)$ .

Now, let  $a \in \Omega^n$  and  $b$  the  $\triangleleft$ -least object in  $\Omega^n$  such that  $a \triangleleft b$ . If  $x \in \Omega^{n+1}$  with  $G(a) \triangleleft x \triangleleft G(b)$ , then  $a \asymp x$ : for then  $g^{-1}(x) = (a, k)$  for some  $k$ , and thus while  $(\triangleleft, (\Omega^{n+1})_{G(a)}) < (\triangleleft, (\Omega^{n+1})_x)$ , the difference is finite. Consequently,  $\S(\Omega^{n+1})_{G(a)} \asymp \S(\Omega^{n+1})_x \ll \S(\Omega^{n+1})_{G(b)}$ . It follows from the Ordering Lemma 33 that

$$O_{\S(\Omega^{n+1})_{G(a)}} = O_{\S(\Omega^{n+1})_x} \triangleleft O_{\S(\Omega^{n+1})_{G(b)}} \triangleleft O_{\S\Omega^{n+1}}. \tag{85}$$

Now consider the function  $H : \Omega^n \rightarrow Ord_{\S\Omega^{n+1}}$  defined by:

$$H(a) = O_{\S(\Omega^{n+1})_{G(a)}}. \tag{86}$$

By (85),  $\text{rng}(H) \subseteq Ord_{\S\Omega^{n+1}}$ , and  $H$  is injective and  $\triangleleft$ -preserving. For surjectivity, suppose that  $O_p \in Ord_{\S\Omega^{n+1}}$ , and let  $a \in \Omega^n$  be the  $\triangleleft$ -least element of  $\Omega^n$  such that for all  $b \in \Omega^n$ , if  $a \triangleleft b$  then  $O_p \triangleleft H(b)$ . It follows then that  $H(a) \triangleleft O_p \triangleleft H(a')$  if  $H(a) \neq O_p$ , where  $a'$  is the successor of  $a$  in  $(\triangleleft, \Omega^n)$ . But if  $H(a) \neq O_p$ , then by the Ordering Lemma 33,

$$\S(\Omega^{n+1})_{G(a)} \ll p \ll \S(\Omega^{n+1})_{G(a')}. \tag{87}$$

Thus there is an  $x \in \Omega^{n+1}$  and a  $P$  with  $p = \S P$  such that  $(\triangleleft, (\Omega^{n+1})_x) \cong (\triangleleft, P)$ . As above, then,

$$(\triangleleft, (\Omega^n \times N)_{(a,0)}) \cong (\triangleleft, (\Omega^{n+1})_a) \tag{88}$$

$$< (\triangleleft, (\Omega^{n+1})_x) \tag{89}$$

$$\cong (\triangleleft, (\Omega^n \times N)_{(a,k)}) \tag{90}$$

$$< (\triangleleft, (\Omega^n \times N)_{(a',0)}) \tag{91}$$

$$\cong (\triangleleft, (\Omega^{n+1})_{G(a')}) \tag{92}$$

for some  $k \in N - \{0\}$ . But then  $\S(\Omega^{n+1})_{G(a)} \ll \S(\Omega^{n+1})_x$ , whence also  $\S(\Omega^{n+1})_{G(a)} \ll p$  and so  $\S(\Omega^{n+1})_{G(a)} \asymp p$  (by Proposition 30). Thus

$$H(a) = O_{\S(\Omega^{n+1})_{G(a)}} = \S O_p. \tag{93}$$

□

COROLLARY 38. For any  $n \in N - \{0\}$ ,

$$\S Ord_{O_p} + \S \Omega^n = \S Ord_{O_p + \S \Omega^{n+1}}. \tag{94}$$



As a consequence we have:

**BOUNDEDNESS LEMMA 39.** *Let  $p$  be a preordinal and  $n \in N$ . Notice that  $\S Ord_{O_p}$  is then a preordinal. Then*

$$\S Ord_{O_p} + \S \Omega^n \ll \S Ord. \tag{95}$$

*Proof.* By (69) of Preordinal Arithmetic 31 and Corollary 38:

$$\S Ord_{O_p} + \S \Omega^n \ll \S Ord_{O_p} + \S \Omega^{n+1} = \S Ord_{O_{p+\S \Omega^{n+1}}}. \tag{96}$$

The result follows from the fact that  $(\triangleleft, Ord_{O_{p+\S \Omega^{n+1}}}) < (\triangleleft, Ord)$ . □

The Translation Lemma 37, its Corollary 38, and the Boundedness Lemma 39 were all in the service of proving Lemma 36, which finally we can prove. We proceed by induction on  $n$  to show that for each  $n, k \in N$ ,

$$A(n + 1) + \Omega^k \ll A(n) \qquad B(n + 1) \triangleleft B(n) \tag{97}$$

(see (78) and (79)).

For the base case, first notice by the Boundedness Lemma 39 that  $A(1) + \S \Omega^k = \S Ord_{O_{\S Ord}} + \S \Omega^k \ll \S Ord = A(0)$ . It follows then by the Ordering Lemma 33 that  $B(1) = O_{\S Ord_{\S Ord}} \triangleleft O_{\S Ord} = B(0)$ .

Now suppose that  $A(n + 1) + \S \Omega^k \ll A(n)$  for all  $k \in N$ , and  $B(n + 1) \triangleleft B(n)$ . We obtain, starting from the former, the sequence:

$$A(n + 1) + \S \Omega^{k+2} \ll A(n), \tag{98}$$

$$O_{A(n+1)+\S \Omega^{k+2}} \triangleleft O_{A(n)}, \tag{99}$$

$$(\triangleleft, Ord_{O_{A(n+1)+\Omega^{k+2}}}) < (\triangleleft, Ord_{O_{A(n)}}), \tag{100}$$

$$(\triangleleft, Ord_{O_{A(n+1)}} + \Omega^k) < (\triangleleft, Ord_{O_{A(n+1)}} + \Omega^{k+1}) < (\triangleleft, Ord_{O_{A(n)}}). \tag{101}$$

Line (99) follows from the Ordering Lemma 33. Line (100) follows from the preceding line since both terms are ordinals. The right-hand relation of (101) is then a result of the Translation Lemma 37; the left is by the definition of  $\Omega^k$ . Notice that the difference on the left of (101) is infinite. It follows then that:

$$A(n + 2) + \S \Omega^k = \S Ord_{O_{A(n+1)}} + \S \Omega^k \ll \S Ord_{O_{A(n)}} = A(n + 1), \tag{102}$$

$$B(n + 2) = O_{\S Ord_{O_{A(n+1)}}} \triangleleft O_{\S Ord_{O_{A(n)}}} = B(n + 1). \tag{103}$$

Line (103) follows from (102) by the Ordering Lemma 33.

The induction is complete, and thus so is the proof of Lemma 36. And with it, then, the proof of the Cardinal Limitation Theorem CLT.

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