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Hodge Theory of Cyclic Covers Branched over a Union of Hyperplanes

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Abstract. Suppose that Y is a cyclic cover of projective space branched over a hyperplane arrangement D and that U is the complement of the ramification locus in Y. The first theorem in this paper implies that the Beilinson–Hodge conjecture holds for U if certain multiplicities of D are coprime to the degree of the cover. For instance, this applies when D is reduced with normal crossings. The second theorem shows that when D has normal crossings and the degree of the cover is a prime number, the generalized Hodge conjecture holds for any toroidal resolution of Y. The last section contains some partial extensions to more general nonabelian covers.

The principal goal of this paper is to verify some standard conjectures in Hodge theory for a natural class of examples. Fix integers $d > 1$, $m, n \geq 1$, and consider the cyclic cover $\pi\colon Y \to \mathbb{P}^n_{\mathbb{C}}$ defined by the weighted homogeneous equation $y^d = f(x_0, \dots, x_n)$, where f is a product of md distinct linear forms. Let D be the divisor defined by $f = 0$ and let $E = \pi^{-1}D$. Our first theorem is that the Beilinson–Hodge conjecture, as formulated in [AS], holds for $U = Y - E$ if, for instance, D has normal crossings. This means that all weight $2j$ Hodge cycles in $H^j(U, \mathbb{Q})$ lie in the image of the cycle map from motivic cohomology. The key point is to show that the weight $2j$ Hodge cycles on U come from $\mathbb{P}^n - D$. Then the theorem is almost immediate. We note that the theorem is valid even in some cases when D fails to be reduced or have normal crossings. The precise condition is that the multiplicities of the components of D, and their sums at points where D fails to have normal crossings, should be coprime to d.

For the second result, we assume that D is an arrangement of d hyperplanes with normal crossings (so that $m = 1$). Then Y is a singular toroidal variety, so we may choose a toroidal desingularization $X \to Y$ [M]. Our second theorem implies that the generalized Hodge conjecture [G] holds for X when d is prime. Another notable consequence of the theorem is that when n is odd and d prime, the maximal abelian subvariety of the intermediate Jacobian $J^n(X)$ is zero. The verification of the Hodge conjecture for X goes as follows. We check that all the Hodge cycles on X are either algebraic or are pullbacks of Hodge cycles from Y. To analyze the cycles in the second category, we employ a nice trick used by Shioda [S] in a similar context. By exploiting the action of $\mathbb{Z}/d\mathbb{Z}$ on cohomology, we obtain a very strong bound on the dimension of the space of Hodge cycles on Y. When d is prime, it will imply that there are no transcendental Hodge cycles.

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Let us now indicate the nature of the bounds used above. Given a smooth projective variety Y, or more generally an orbifold as above, the dimension of the space of Hodge cycles of type (p,p) on it is, of course, bounded by the Hodge number h^{pp}. This can be written as the $2p$-th Betti number minus $2\dim T$, where $T \subset H^{2p}(Y)$ is the space of (a,b) classes with $a < b$. If a possibly nonabelian finite group $G = \{1, g_1, \dots\}$ acts on Y, then $2\dim T$ can be replaced by the dimension of the smallest rationally defined G-module containing $T + \overline{T}$. In some cases this yields a huge improvement. For example, if G acts irreducibly on T with a nonreal character χ, then the factor in front of $\dim T$ jumps from 2 to the product of the degree of the number field $\mathbb{Q}(\chi(g_1), \dots)$ times the Schur index of χ. This follows from more general results given in the final section of the paper. The main result in this section is that under suitable conditions, there is an explicit bound for the dimension of the space of Hodge cycles on a branched G-cover in terms of branching data.

1 Preliminaries on Cyclic Covers

We start with a slightly more general setup than given in the introduction. Let Z be an n-dimensional smooth projective variety with a line bundle L. Let $D \subset Z$ be a not necessarily reduced effective divisor with simple normal crossings such that $\mathcal{O}_Z(D) \cong L^d$. Then we form the normal d-fold cyclic cover

$$Y = \text{Normalization of } \mathbf{Spec}\Big(\bigoplus_{i=0}^{d-1} L^{-i}\Big) \xrightarrow{\pi} Z$$

branched over D (*cf.* [EV2, §3]). Let $V = Z - D$, $E = \pi^{-1}D$, and $U = Y - E$. It is convenient to set $\Omega^k_Z(\log D) = \Omega^k_Z(\log D_{\mathrm{red}})$ below. The following is probably known to experts, but we do not know of a good reference.

Lemma 1.1 *Y is local analytically isomorphic to a toric variety with finite quotient singularities.*

Proof Local analytically, Y looks like the normalization of an affine variety of the form

$$y^d = x_1^{a_1} \cdots x_n^{a_n} \tag{1.1}$$

Let $R = R_{(a_1,\dots,a_n,d)}$ be the quotient of $\mathbb{C}[x_1,\dots,x_n,y]$ by the ideal generated by the difference of the two sides of this equation, and let $\widetilde{R}_{(a_1,\dots,a_n,d)}$ denote its normalization. The most important case for us is when all the exponents $a_i = 1$. In this case, the lemma is easy to see directly. The ring R is the ring of invariants of $\mathbb{C}[u_1,\dots,u_n,v]$ under $(\zeta_j) \in (\mu_d)^n$ acting by

$$u_j \longmapsto \zeta_j u_j;\ v \longmapsto \prod \zeta_j^{-1} v$$

Therefore, in this case, R is already normal. The fact that it is also toric is immediate from the shape of the equation (1.1), which is the equality of two monomials.

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For the general case, we will use toric methods more explicitly. But first, we make a series of reductions. Let $g = \gcd(a_1, \dots, a_n, d)$. If $g > 1$, then Spec R is reducible and the components are isomorphic to Spec $R_{(a_1/g,\dots,a_n/g,d/g)}$. Therefore, we may reduce the proof to the case that $g = 1$.

If some $a_i = 0$, then

$$R = R_{(a_1,\dots,\widehat{a_i},a_n,d)} \otimes \mathbb{C}[x_i].$$

Thus we may assume that all $a_i > 0$. Let $S \subset \mathbb{Z}^n$ be the subsemigroup generated by $v_1 = (d, 0, \dots 0), \dots, v_n = (0, \dots, d)$, $v_{n+1} = (a_1, \dots, a_n)$, and let $L \supset S$ denote the sublattice generated by the same vectors. The semigroup ring $\mathbb{C}[S]$ can be identified with the subring of $\mathbb{C}[u_1, \dots, u_n]$ generated by $u_1^d, \dots, u_n^d$ and $u_1^{a_1} \cdots u_n^{a_n}$. The homomorphism

$$\mathbb{C}[x_1, \dots, x_n, y] \longrightarrow \mathbb{C}[S]$$

defined by

$$x_i \longmapsto u_i^d, \quad y \longmapsto u_1^{a_1} \cdots u_n^{a_n}$$

gives an isomorphism $R \cong \mathbb{C}[S]$. The normalization $\widetilde{R}$ of R is given by the semigroup ring $\mathbb{C}[\widetilde{S}]$, where $S \subset \widetilde{S} \subset L$ is the saturation [CLS, Chap 1,§3], which is the intersection of L with the real cone

$$\left\{ \sum r_i v_i \in L \otimes \mathbb{R} \mid r_i \geq 0 \right\}.$$

Then $\widetilde{S}$ is simplicial, since it is generated by the real basis vectors $v_1, \dots, v_n$. Therefore Spec R has quotient singularities [CLS, Chap 3, §1]. ■

Thus Y is an orbifold, which for our purposes simply means that it has finite quotient singularities. Note also that the singularities lie over the singular locus $D_{\text{sing}} \subset D$. Bailey [Ba], and later Steenbrink [St], observed that most of the standard Hodge theoretic statements generalize from smooth varieties to orbifolds. We list the results that we need from the second reference.

(H1) The mixed Hodge structure on $H^i(Y)$ is pure of weight i. We can identify $Gr_F^k H^i(Y)$ with $H^{i-k}(Y, \widetilde{\Omega}_Y^k)$ where $\widetilde{\Omega}_Y^k := (\Omega_Y^k)^{**} = j_*\Omega_W^k$ and $j\colon W \to Y$ is the embedding of the smooth locus.

(H2) The hard Lefschetz theorem holds.

(H3) There is an (noncanonical) isomorphism

$$H^i(Y - D, \mathbb{C}) \cong \bigoplus_{a+b=i} H^a\big(Y, \widetilde{\Omega}_Y^b(\log D)\big),$$

where $\widetilde{\Omega}_Y^k(\log D) = j_*\Omega_W^k(\log D \cap W)$. (Although this is not explicitly stated there, it follows from the discussion in [St, 1.17–1.20] and the fact that the spectral sequence associated with $(\widetilde{\Omega}_Y^\bullet(\log D), \widetilde{\Omega}_Y^{\geq k}(\log D))$ degenerates at E_1, because it is part of a cohomological mixed Hodge complex [D2, 8.1.9].)

The group of d-th roots of unity $\mu_d \cong \mathbb{Z}/d$ acts on Y, and we will need to analyze the eigenspaces on cohomology. Let $\epsilon\colon \mu_d \to \mathbb{C}^*$ denote the standard character given by $\epsilon(\zeta) = \zeta$. Any $\mathbb{C}[\mu_d]$-module T can be decomposed as a sum $T = \bigoplus_{i=0}^{d-1} T_{\epsilon^i}$, where T_{ϵ^i} is the maximal submodule where $\zeta \in \mu_d$ acts by multiplication by $\epsilon^i(\zeta) = \zeta^i$. Define the nontrivial part of T by $T_{nt} = \bigoplus_{i=1}^{d-1} T_{\epsilon^i}$. Let

$$H^{jk}_{\epsilon^i}(Y) = H^k(Y, \widetilde{\Omega}^j_Y)_{\epsilon^i} \quad \text{and} \quad H^{jk}_{nt}(Y) = H^k(Y, \widetilde{\Omega}^j_Y)_{nt}.$$

Lemma 1.2 (Esnault–Viehweg) *Let $D = \sum a_j D_j$, where D_j are the irreducible components. Let $[rD] = \sum [a_j r] D_j$, $L^{(-i)} = L^{-i}([\frac{i}{d}D])$, and let $D^{(i)}$ be the sum of components D_j such that $d \not| \, ia_j$. Then*

$$H^j_{\epsilon^i}(Y, \widetilde{\Omega}^k_Y) = \begin{cases} H^j(\Omega^k_Z) & \text{if } i = 0, \\ H^j(\Omega^k_Z(\log D^{(i)}) \otimes L^{(-i)}) & \text{otherwise,} \end{cases}$$

$$H^j_{\epsilon^i}(Y, \widetilde{\Omega}^k_Y(\log E)) = \begin{cases} H^j(\Omega^k_Z(\log D)) & \text{if } i = 0, \\ H^j(\Omega^k_Z(\log D) \otimes L^{(-i)}) & \text{otherwise.} \end{cases}$$

Proof Since π is finite, we have $H^j(Y, \widetilde{\Omega}^k_Y) = H^j(Z, \pi_* \widetilde{\Omega}^k_Y)$ as $\mathbb{C}[\mu_d]$-modules. Let $W = Z - D_{\text{sing}}$. Esnault and Viehweg [EV2, lemma 3.16] have shown that

$$(\pi_* \widetilde{\Omega}^k_Y)_{\epsilon^i}|_W = \begin{cases} \Omega^k_W & \text{if } i = 0, \\ \Omega^k_W(\log D^{(i)}) \otimes L^{(-i)} & \text{otherwise.} \end{cases}$$

Equality extends to Z, because these sheaves are reflexive.

The second part also follows from [loc. cit] by the same argument. ∎

Corollary 1.3 *The invariant part $H^*_{\epsilon^0}(Y)$ is isomorphic to $H^*(Z)$.*

Remark 1.4 The above formulas simplify under the following assumptions

(i) If the coefficients a_j are coprime to d, then $D^{(i)} = D_{\text{red}}$ for all $1 \leq i \leq d-1$. This coprimality condition is equivalent to the map $Y \to Z$ being totally ramified along D.

(ii) If D is reduced, then additionally $L^{(-i)} = L^{-i}$

The following is a special case of much more general vanishing theorems due to Esnault and Viehweg [EV1].

Lemma 1.5 *Suppose that L is ample and that the coefficients of D are coprime to d. If $m + k \neq n$ and $1 \leq i \leq d - 1$, then $H^m(\Omega^k_Z(\log D) \otimes L^{(-i)}) = 0$.*

Proof We have

$$H^m\big(\Omega^k_Z(\log D) \otimes L^{(-i)}\big) \subset H^{m+k}(Y - E, \mathbb{C})$$

by Lemma 1.2 and item (H3). Since $Y - E$ is affine, the right side vanishes when $m + k > n$ [EV1, 1.5].

For the remaining case $m + k < n$, we use hard Lefschetz on Y with respect to the pullback of L. This is compatible with the μ_d-action and therefore induces

$$H^m\big(\Omega^k_Z(\log D) \otimes L^{(-i)}\big) \cong H^{m+j}\big(\Omega^{k+j}_Z(\log D) \otimes L^{(-i)}\big),$$

where $j = n - (m + k)$. ∎

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Corollary 1.6 *If in addition to the above assumptions $i \neq n$, then $H^i(Y,\mathbb{Q}) \cong H^i(Z,\mathbb{Q})$.*

2 Beilinson–Hodge

We say that the Beilinson–Hodge conjecture holds for a smooth variety V (in degree j) if the cycle map on the higher Chow group

$$CH^j(V, j) \otimes \mathbb{Q} \to \mathrm{Hom}_{\mathrm{MHS}}\big(\mathbb{Q}(-j), H^j(V,\mathbb{Q})\big)$$

is surjective (for the given j). See [AS, AK, B, J] for more background.

For the next lemma, we use the same notation as in Section 1, that Z is smooth projective and $Y \to Z$ is a d-sheeted normal cyclic cover branched over a normal crossing divisor D.

Lemma 2.1 *Suppose that the coefficients of D are coprime to d, that the Beilinson–Hodge conjecture holds for $V = Z - D$ in degree j, and that $W_j H^j(V,\mathbb{Q}) = 0$. Then the Beilinson–Hodge conjecture holds for $U = Y - E$ in degree j.*

Proof We can assume that $j > 0$, since otherwise the statement is vacuous. Since $H^j(Y)$ is pure of weight j, $\mathrm{Hom}(\mathbb{Q}(-j), H^j(Y)) = 0$. Therefore we have an injection

$$\mathrm{Hom}_{\mathrm{MHS}}\big(\mathbb{Q}(-j), H^j(U,\mathbb{Q})\big) \hookrightarrow \mathrm{Hom}_{\mathrm{MHS}}\big(\mathbb{Q}(-j), H^j(U,\mathbb{Q})/\operatorname{im} H^j(Y,\mathbb{Q})\big).$$

By Lemma 1.2 and (H3), there is an isomorphism

$$H^j(U,\mathbb{C}) = \bigoplus_k H^{j-k}\big(\Omega_Z^k(\log D)\big) \oplus \bigoplus_k \bigoplus_{i=1}^{d-1} H^{j-k}\big(\Omega_Z^k(\log D) \otimes L^{(-i)}\big),$$

$$H^j(Y,\mathbb{C}) = \bigoplus_k H^{j-k}(\Omega_Z^k) \oplus \bigoplus_k \bigoplus_{i=1}^{d-1} H^{j-k}\big(\Omega_Z^k(\log D) \otimes L^{(-i)}\big).$$

Therefore

$$H^j(U,\mathbb{Q})/\operatorname{im} H^j(Y,\mathbb{Q}) \cong H^j(V,\mathbb{Q})/\operatorname{im} H^j(Z,\mathbb{Q}) \cong H^j(V,\mathbb{Q}), \tag{2.1}$$

where the last isomorphism follows from our assumption that

$$W_j H^j(Z) = \operatorname{im} H^j(Z) = 0.$$

Thus we have a commutative diagram

$$\begin{array}{ccc}
CH^j(U, j) \otimes \mathbb{Q} & \longleftarrow & CH^j(V, j) \otimes \mathbb{Q} \\
\downarrow r & & \downarrow \\
\mathrm{Hom}(\mathbb{Q}(-j), H^j(U)) & & \mathrm{Hom}(\mathbb{Q}(-j), H^j(V)) \\
\downarrow & \swarrow \sim & \\
\mathrm{Hom}(\mathbb{Q}(-j), H^j(U)/\operatorname{im} H^j(Y)), & &
\end{array}$$

which implies that the map r is necessarily surjective. ■

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We sketch an alternate proof of Lemma 2.1 that is a bit more conceptual.

Outline of second proof Consider the diagram

$$\begin{array}{ccccccc} H^j(Y) & \longrightarrow & H^j(U) & \longrightarrow & H^{j+1}_E(Y) & \longrightarrow & H^{j+1}(Y) \\ \uparrow & & \uparrow & & \uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\beta} \\ H^j(Z) & \longrightarrow & H^j(V) & \longrightarrow & H^{j+1}_D(Z) & \longrightarrow & H^{j+1}(Z) \end{array}$$

where the coefficients are $\mathbb{Q}$. Since $Y \to Z$ is totally ramified at D, we can see that D and E have homotopy equivalent tubular neighbourhoods. Therefore α is an isomorphism. If $X \to Y$ is a resolution of singularities, then the map $H^{j+1}(Z) \to H^{j+1}(X)$ is injective, because the normalized pushforward gives a left inverse. Since the map factors through β, β must also be injective. This implies (2.1) by a diagram chase. The rest of the proof is identical to the one above. ∎

We can extend this lemma to more general branch divisors, but we have to worry about the effect of the singularities. Given a divisor $D \subset Z$, a *log resolution* of (Z, D) is a resolution of singularities $p \colon Z' \to Z$ such that p^*D has simple normal crossings.

Corollary 2.2 *Assume all of the conditions of Lemma 2.1 with one exception, that D is only effective. Then the Beilinson–Hodge conjecture holds in degree j for U if in addition there exists a log resolution $p \colon Z' \to Z$ such that p^*D has all coefficients prime to d.*

We will say that $D = \sum a_i D_i$ is of *arrangement type* if the components are all smooth and D is local analytically isomorphic to a hyperplane arrangement in affine space. This, of course, includes the case where $D \subset \mathbb{P}^n$ is itself a hyperplane arrangement. If $p \in D$, let us say that the *incidence number* at p is the sum of all coefficients a_j for components D_j containing p. In particular, for a reduced divisor, this is precisely the number of components containing p. The set of *essential incidence numbers* of D is the set of incidence numbers of those $p \in D$ at which D fails to have normal crossing singularities.

Lemma 2.3 *Suppose that $Y \to Z$ is a d-sheeted cyclic cover branched over an effective divisor $D = \sum a_i D_i \subset Z$ of arrangement type. Suppose that the coefficients of D and the essential incidence numbers are coprime to d, the Beilinson–Hodge conjecture holds for $Z - D$ in degree j, and that $W_j H^j(Z - D) = 0$. Then the Beilinson–Hodge conjecture holds for the preimage of $Z - D$ in the same degree.*

Proof The key point is that we can resolve the singularities of D in an explicit fashion and keep track of the multiplicities. For hyperplane arrangements this resolution goes back to De Concini and Procesi [DP], although we will follow the simplified presentation of [BS, §2.1]. Since their procedure is canonical, it applies to our more general case as well.

Let $D^{\mathrm{nnc}} \subset D$ be the nonnormal crossing locus. This is the largest closed subset for which $D \cap (Z - D^{\mathrm{nnc}})$ has normal crossings. We form the set of centres

$$S_i = \Big\{ D_J = \bigcap_{j \in J} D_j \mid D_J \subseteq D^{\mathrm{nnc}}, \dim D_J = i \Big\}$$

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for our subsequent blow ups. We define a sequence of smooth varieties as follows. Take $Z_0 = Z$ and let $Z_1 \to Z_0$ be the blow up of Z_0 at the union of centres in S_0. Let Z_2 be the blow up of Z_1 at the union (which is a disjoint union) of the strict transforms varieties in S_1 and so on. Finally set $Z' = Z_{n-1}$. Then it follows from [BS, §2.1] that the pullback D' of D to Z' is a divisor with normal crossings. We claim moreover that the coefficients of D' are coprime to d. This can be checked by induction. Let $D_{(i+1)}$ be the pullback of $D = D_{(0)}$ to Z_{i+1}. This is the sum of the strict transform of $D_{(i)}$ with a sum of exceptional divisors $\sum m_{ij}F_{ij}$. The coefficients m_{ij} are the multiplicities of $D_{(i)}$ along the centres of the blow ups, which are precisely the essential incidence numbers.

The result now follows from Lemma 2.1 ∎

Theorem 2.4 *The Beilinson–Hodge conjecture holds for U in the following cases:*

(i) *U is the complement of $f(x_0,\dots,x_n) = 0$ in the variety defined by $y^d = f(x_0,\dots,x_n)$, where f is a product of linear forms such that the divisor in $\mathbb{P}^n$ defined by f satisfies the conditions of Lemma 2.3.*

(ii) *U is the complement of $f(x_0,x_1,x_2) = 0$ in the variety defined by $y^d = f(x_0,x_1,x_2)$, such that f is divisible by a linear form and its divisor in $\mathbb{P}^2$ satisfies the conditions of Corollary 2.2.*

Proof In case (i), let $f = \prod h_i^{a_i}$ be the factorization as a product of linear forms. Define $V = \mathbb{P}^n - \{f = 0\}$ as usual. Then the classes $1/(2\pi\sqrt{-1})d\log h_i$ lie in the image of the cycle map from $CH^1(V,1)$ essentially by definition. Since the cycle map is multiplicative, the Beilinson–Hodge conjecture holds for V, because its cohomology is generated as an algebra by the forms $1/(2\pi\sqrt{-1})d\log h_i$ by Brieskorn [Br, lemma 5]. This also implies that $H^j(V)$ has weight $2j$, so that $W_jH^j(V) = 0$.

For (ii), we use [C] and the fact that $W_2H^2(V) \subseteq \operatorname{im} H^2(\mathbb{C}^2) = 0$ ∎

Corollary 2.5 *In case* (i)*, it suffices that the branch divisor is reduced with normal crossings.*

It is worth remarking that Beilinson [B] made a stronger conjecture that amounts to the surjectivity of the cycle map

$$CH^i(U,j)\otimes\mathbb{Q} \longrightarrow \operatorname{Hom}_{\mathrm{MHS}}\big(\mathbb{Q}(-i), H^{2i-j}(U,\mathbb{Q})\big)$$

for all i, j. It is now known to be overly optimistic in general [J]. However, in the main case of Theorem 2.4(i), it is vacuously true for $0 < j < i$, because the above arguments show that

$$H^i(U)/\operatorname{im} H^i(Y) = H^i(\mathbb{P}^n - (f))/\operatorname{im} H^i(\mathbb{P}^n)$$

is a sum of Tate structures $\mathbb{Q}(-i)$. The case of $j = 0$ is more subtle. It is essentially the ordinary Hodge conjecture for Y, and this will be studied in the next section.

3 Cohomology of Toroidal Resolutions

Let Y be defined by $y^d = f$, where f is a product of d distinct linear forms h_i as in the introduction. We now assume that the divisor $D = \sum D_i \subset \mathbb{P}^n$ defined by $f = 0$ is a divisor with normal crossings. Let $E = \pi^{-1}D$ and $U = Y - E$. Then (Y, U) is a toroidal embedding in the sense of [M]. This means that about each point of $y \in Y$, there is a neighbourhood Y_y that is isomorphic to an étale open subset of a toric variety in such a way that $U \cap Y_y$ maps to the torus. To see this, we can assume that after a linear change of coordinates, y lies over $[1, 0, \dots, 0] \in \mathbb{P}^n$. Write $f(1, x_1 \cdots x_n) = x_1 \cdots x_k g(x_1, \dots, x_n)$, where $g(0) \neq 0$. Then

$$Y \supset Y_y = \operatorname{Spec} \mathbb{C}[x_1, \dots, x_n, y]/(y^d - f(1, x_1, \dots x_n)) \xrightarrow{F_y} M_k,$$

where the map F_y, given by projection, is an open immersion into the toric variety

$$M_k = \operatorname{Spec} \mathbb{C}[x_1, \dots, x_n, y]/(y^d - x_1 \cdots x_k).$$

Later on, we will need to consider the more general case where D is a normal crossing divisor in a smooth variety; then (Y, U) is still toroidal, but the local toric models $M_{(a)} = \operatorname{Spec} \widetilde{R}_{(a_1, \dots, a_n, d)}$ need to be chosen as in the proof of Lemma 1.1, and the corresponding map F_y is only étale. By [M, p. 94], there exists a toroidal resolution of singularities $\rho \colon X \to Y$. In other words, for each y, there is a commutative diagram

$$\begin{array}{ccccc} X & \longleftarrow & \rho^{-1}(Y_y) & \xrightarrow{\widetilde{F}_y} & \widetilde{M}_{(a)} \\ \downarrow & & \downarrow & & \downarrow \tau \\ Y & \longleftarrow & Y_y & \xrightarrow{F_y} & M_{(a)} \end{array}$$

where the map τ is toric, F_y is étale and the right-hand square is cartesian.

Remark 3.1 J. Włodarczyk pointed out to us that such resolutions are very natural in the sense that any canonical resolution algorithm, such as Hironaka's, will yield a toroidal resolution of Y.

As a prelude to the next theorem, we recall that Grothendieck's amended form of the generalized Hodge conjecture [G] says that sub-Hodge structures of cohomology are induced from subvarieties of expected codimension. More precisely, recall that the level of a Hodge structure $H \otimes \mathbb{C} = \bigoplus H^{pq}$ is the maximum of $|p - q|$ over the nonzero H^{pq}. The conjecture states that given an irreducible sub-Hodge structure $H \subset H^i(X, \mathbb{Q})$ of level $\leq i - 2k$, there exists a subvariety $\iota : T \subset X$ of codimension at least k and a desingularization $\kappa \colon \widetilde{T} \to T$ such that $H \subset (\iota \circ \kappa)_* H^{i-2k}(\widetilde{T}, \mathbb{Q})$. This includes the usual Hodge conjecture, which corresponds to the case where $i = 2k$. Our goal is to prove the following theorem.

Theorem 3.2 *When d is prime, any irreducible sub-Hodge structure of $H^r(X,\mathbb{Q})$ of level at most $r-2$ is spanned by an algebraic cycle. In particular, the generalized Hodge conjecture holds for X.*

Remark 3.3 The level and weight have the same parity. So the statement can be "strengthened" by replacing $r-2$ with $r-1$.

This will be deduced from another more general theorem. Before stating it, it is convenient to recall the notion of motivic dimension introduced in [A]. Given a smooth projective variety Z, $\mu(Z)\in\mathbb{N}$ roughly measures how much transcendental cohomology Z has. So $\mu(Z)=0$ holds precisely when all the cohomology is generated by algebraic cycles. In general, $\mu(Z)$ is the smallest nonnegative integer such that $H^*(Z)$ is generated by Gysin images of classes of degree at most $\mu(Z)$. The definition can be extended to arbitrary varieties. The basic facts we need are these:

- $\mu(Z)\le\mu(Z')$, when $Z'\to Z$ is proper and surjective [A, prop 1.1];
- $\mu(Z)\le\max(\mu(Z'),\mu(Z-Z'))$, when $Z'\subset Z$ is closed [A, prop 1.1];
- $\mu(Z)\le\mu(Z_s)+\mu(S)$, when $Z\to S$ is a topologically trivial smooth projective map [A, cor 2.7].

Theorem 3.4 *Suppose that Z is a smooth projective variety and that $D\subset Z$ is an effective divisor with simple normal crossings such that $\mathcal{O}_Z(D)=L^d$. Let X be a toroidal resolution of the cyclic d-fold cover Y determined by the data (D,L). We will assume that*

(i) *the motivic dimensions $\mu(Z)=0$ and $\mu(D_{i_1}\cap\dots D_{i_k})=0$ for all $\{i_1,\dots,i_k\}$;*

(ii) *the inequality*

$$\phi(d)h^r(Z,L^{(-d+1)})\ge\dim H^r_{nt}(Y)=\sum_{k=0}^{r}\sum_{i=1}^{d-1}h^{r-k}\big(\Omega^k_Z(\log D^{(i)})\otimes L^{(-i)}\big)$$

holds, where ϕ is the Euler function.

Then any irreducible sub-Hodge structure of $H^r(X,\mathbb{Q})$ of level at most $r-2$ is spanned by an algebraic cycle.

Remark 3.5 The proof will show that inequality (ii) is necessarily an equality.

For the ensuing discussion, let us assume that we are in the more general situation of Theorem 3.4. Since Y is an orbifold, it is a rational homology manifold. Therefore the natural map $\pi^*\colon H^i(Y,\mathbb{Q})\to H^i(X,\mathbb{Q})$ is injective, since $\frac{1}{d}\pi_*$ gives a left inverse. Also, by Poincaré duality we get a cycle map on the Chow group of codimension k cycles $CH^k(Y)\otimes\mathbb{Q}\to H^{2k}(Y,\mathbb{Q})$.

Proposition 3.6 *$H^i(X,\mathbb{Q})=H^i(Y,\mathbb{Q})\oplus A^i$, where A^i is generated by algebraic cycles.*

Proof It is more convenient to work in homology. Let $F\subset X$ denote the reduced preimage of E. We will show that $\mu(F)=0$ (note that motivic dimension is defined for singular varieties as well). We have a stratification of D by

$$D_I=D_{i_1}\cap\dots\cap D_{i_\ell},\quad D'_I=D_I-\bigcup_{j\notin I}D_j,\tag{3.1}$$

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where $I = \{i_1, \dots, i_\ell\}$. Let $F'_I \subseteq F$ be the preimage of D'_F and let $a(I) = (a_{i_1}, \dots, a_{i_\ell})$. Over a neighbourhood of $p \in D'_I$, the map $X \to Y$ is locally isomorphic to the model $\psi_{a(I)} : \widetilde{M}_{(a(I))} \times \mathbb{C}^{n-\ell} \to M_{(a(I))} \times \mathbb{C}^{n-\ell}$. Consequently, $F'_I \to D'_I$ is a Zariski locally trivial fibration with fibres isomorphic to $\psi^{-1}_{a(I)}(0)$. Let Ψ_I be a toric resolution of $\psi^{-1}_{a(I)}(0)$. Then, using the previously stated inequalities, we obtain

$$\mu(F) \le \max_I \mu(F'_I) \le \max_I \big(\mu(D'_I) + \mu(\Psi_I)\big) = 0$$

Suppose that $\alpha \in H_i(X)$ lies in the kernel of π_*. From the diagram

$$\begin{array}{ccccc} H_i(F) & \longrightarrow & H_i(X) & \longrightarrow & H_i(X,F) \\ \downarrow & & \downarrow \pi_* & & \downarrow \cong \\ H_i(E) & \longrightarrow & H_i(Y) & \longrightarrow & H_i(Y,E) \end{array}$$

we see that α is the image of a class in $H_i(F)$ that is algebraic because $\mu(F) = 0$. Dualizing shows that $H^i(X)$ is generated by $H^i(Y)$ and algebraic cycles. ■

Proof of Theorem 3.4 Let $H \subset H^r(X, \mathbb{Q})$ be an irreducible sub-Hodge structure of level at most $r-2$. By Proposition 3.6 and Corollary 1.3, we can decompose $H^r(X,\mathbb{Q}) = H^r_{nt}(Y,\mathbb{Q}) \oplus A$ where A is spanned by algebraic cycles. So H is either spanned by an algebraic cycle or it lies in $H^r_{nt}(Y)$. Assuming the latter, we will show that $H = 0$. Let $H' = \sum \zeta^i H$, where $\zeta \in \mu_d$ is a generator. This is a μ_d-invariant Hodge structure containing H and with the same level as H. Thus $H' \otimes \mathbb{C} \subset H^{1,r-1}_{nt}(Y) \oplus \cdots \oplus H^{r-1,1}_{nt}(Y)$ so that $H^r(Y, \mathcal{O}_Y) \cap H' \otimes \mathbb{C} = 0$. Let $H^* = H^r_{nt}(Y)/H'$. Then $H^r(Y,\mathcal{O}_Y)$ injects into $H^* \otimes \mathbb{C}$. Let $N = \mathbb{Q}[t]/(P(t))$, where

$$P(t) = \prod_{\gcd(i,d)=1} (t - \zeta^i)$$

is the cyclotomic polynomial. This is the unique irreducible $\mathbb{Q}[\mu_d]$-module for which $N \otimes \mathbb{C} \supset \mathbb{C}_{\epsilon^{d-1}}$. It follows that H^* must contain the m-fold sum N^m, where $m = \dim(H^* \otimes \mathbb{C})_{\epsilon^{d-1}}$. Since

$$(H^* \otimes \mathbb{C})_{\epsilon^{d-1}} \supseteq H^r_{\epsilon^{d-1}}(Y, \mathcal{O}_Y) = H^r(L^{-d+1}),$$

we must have $m \ge h^r(L^{-d+1})$. Therefore

$$\dim H^* \ge \phi(d) h^r(L^{-d+1}) \ge \dim H^r_{nt}(Y)$$

by condition (ii). Therefore $H' = 0$ as claimed, and the proof is complete. ■

We turn to the proof of Theorem 3.2. Our main task is to compute the Hodge numbers.

Lemma 3.7 (Hirzebruch)

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\chi\big(\Omega^k_{\mathbb{P}^n}(i)\big)\,y^k z^n = \frac{(1+yz)^{i-1}}{(1-z)^{i+1}}.$$

Proof This is a special case of the formula from [H, p. 160]. ∎

Lemma 3.8 *Let Y be as in Theorem 3.2. Then for each k,*

$$h^{k,n-k}_{nt}(Y) = \binom{d-1}{n+1} = \frac{d-1}{n+1}\binom{d-2}{n}.$$

In particular, this vanishes for $d-1 < n+1$.

Proof From the residue isomorphism [D2]

$$Gr^W_r\Omega^k_{\mathbb{P}^n}(\log D) \cong \bigoplus \Omega^{k-r}_{D_{i_1}\cap\cdots\cap D_{i_r}}.$$

Thus we deduce

$$\chi(\Omega^k_{\mathbb{P}^n}(\log D)(-i)) = \sum_{r=0}^{d}\binom{d}{r}\chi\big(\Omega^{k-r}_{\mathbb{P}^{n-r}}(-i)\big).$$

Therefore

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\chi\big(\Omega^k_{\mathbb{P}^n}(\log D)(-i)\big)\,y^k z^n = (1+yz)^d\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\chi\big(\Omega^k_{\mathbb{P}^n}(-i)\big)\,y^k z^n.$$

Combining this with Lemma 3.7 yields

$$\begin{aligned}\sum_{i=1}^{d-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\chi(\Omega^k_{\mathbb{P}^n}(\log D)(-i))y^k z^n &= \sum_{i=1}^{d-1}(1-z)^{i-1}(1+yz)^{d-i-1}\\ &= \frac{(1+yz)^{d-1}-(1-z)^{d-1}}{(1+yz)-(1-z)} = \frac{(1+yz)^{d-1}-(1-z)^{d-1}}{z(1+y)}\\ &= \frac{1}{z[y-(-1)]}\sum_{n=0}^{d-2}\binom{d-1}{n+1}[y^{n+1}-(-1)^{n+1}]z^{n+1}\\ &= \sum_{n=0}^{d-2}\binom{d-1}{n+1}[y^n - y^{n-2}+\cdots\pm 1]z^n.\end{aligned}$$

By Lemmas 1.2 and 1.5,

$$h^{k,n-k}_{nt}(Y) = (-1)^{n-k-1}\sum_{i=1}^{d-1}\chi\big(\Omega^k_{\mathbb{P}^n}(\log D)(-i)\big).$$

This, together with the previous formula, implies the lemma. ∎

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Proof of Theorem 3.2 When $r \neq n$, by Proposition 3.6 and Corollary 1.6, $H^r(X,\mathbb{Q})$ is spanned by algebraic cycles. So we only need consider $r = n$. In this case, we apply Theorem 3.4. Condition (i) of this theorem is clear. For (ii), we observe that by the previous lemma, we have

$$(d-1)h^n\big(\mathcal{O}_{\mathbb{P}^n}(-d+1)\big) = (d-1)\binom{d-2}{n} = \sum_k h^{k,n-k}_{nt}(X). \qquad \blacksquare$$

We can handle some related examples in a similar way.

Corollary 3.9 *Let d be prime. The generalized Hodge conjecture holds for a toroidal resolution of the cyclic branched cover of $(\mathbb{P}^1)^n$ given by*

$$y^d = \prod_{i=1}^{n}\prod_{j=1}^{d}(x_i - a_{ij}),$$

where $a_{i1}, a_{i2}, \dots, a_{i,p}$ are distinct for each i.

Proof Let $D \subset (\mathbb{P}^1)^n$ be the divisor given by the union of $x_i - a_{ij} = 0$, and let $L = \mathcal{O}(1) \boxtimes \cdots \boxtimes \mathcal{O}(1)$. We have only to check Theorem 3.4(ii) for $r = n$. We can compute $h^n(L^{-d+1}) = (d-2)^n$ immediately. For the other side, we define the generating function

$$\chi_{n,i}(y) = \sum_k \chi\big(\Omega^k_{(\mathbb{P}^1)^n}(\log D)\otimes L^{-i}\big)\, y^k.$$

Then by Künneth's formula, we obtain

$$\chi_{n,i}(y) = \chi_{1,i}(y)^n = \big(1 - i + (d-i-1)y\big)^n.$$

We have

$$\sum_{i=1}^{d-1}\sum_k h^{r-k}\big(\Omega^k(\log D)\otimes L^{-i}\big) = (-1)^n\sum_{i=1}^{d-1}\chi_{n,i}(-1) = (d-1)(d-2)^n,$$

which implies (ii). $\blacksquare$

Suppose that $\dim X = n = 2m-1$ is odd. Then we have the Abel–Jacobi map

$$\alpha\colon CH^m(X)_{\text{hom}} \to J^n(X) = \frac{H^n(X,\mathbb{C})}{F^m + H^n(X,\mathbb{Z})}$$

from the homologically trivial part of the Chow group to the intermediate Jacobian. Nori [N] has constructed a filtration

$$A_0CH^m(X) \subseteq \cdots \subseteq A_{n-m}CH^m(X) = CH^m(X)_{\text{hom}}$$

where A_0 is the subgroup of cycles algebraically equivalent to 0. In general, a cycle lies in A_r if it is induced via a correspondence from a homologically trivial r-cycle on another variety.

Corollary 3.10 *Suppose that X is either a variety of the type given in Theorem 3.2 or Corollary 3.9 with n odd. Then $\alpha(A_{n-m-1}CH^m(X)) = 0$. In particular, $\alpha(\eta) = 0$ for any cycle η algebraically equivalent to* 0.

Proof The image $\alpha(A_{n-m-1}CH^m(X))$ lies in the subtorus determined by the maximal integral Hodge structure contained in $F^1H^n(X)$ [N]. The theorem implies that this Hodge structure is zero. ∎

This argument also shows that $J^n(X)_{alg} = 0$, where $J^n(X)_{alg} \subset J^n(X)$ is the maximal abelian subvariety [V, §8.2.1].

4 Nonabelian Covers

Our goal is to extend the previous estimates to situations where a possibly nonabelian finite group G acts on a variety. This will apply in particular to G-covers. Let $\widehat{G}$ be the set of characters of irreducible $\mathbb{C}[G]$-modules, and $1 \in \widehat{G}$ the character of the trivial module. Given a character χ of an irreducible $\mathbb{C}[G]$-module, let

$$e_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \overline{\chi}(g) g$$

denote the corresponding central idempotent [CR, thm 33.8]. This determines the χ-isotypic submodule of a $\mathbb{C}[G]$-module M by $M_\chi = e_\chi M$. Let $M_{nt} = \sum M_\chi$, as χ ranges over the nontrivial characters.

We introduce an invariant that will measure the difference between the complex and rational representation theory. Given a finite dimensional $\mathbb{C}[G]$-module M, we define the *rational span* as the minimal (with respect to dimension) $\mathbb{Q}[G]$-module M' such that $M' \otimes \mathbb{C} \supseteq M$. Of course, the span is only an isomorphism class, but its character is well defined, as are the numbers

$$\sigma(M) = \dim_{\mathbb{Q}} M', \quad \Phi(M) = \frac{\sigma(M)}{\dim_{\mathbb{C}} M}.$$

A character will be called rational if the associated $\mathbb{C}[G]$-module is realizable over $\mathbb{Q}$. Given a character $\sum_{\chi \in \widehat{G}} n_\chi \chi$, the character of its rational span can be characterized as the rational character $\sum r_\chi \chi$ with $r_\chi \geq n_\chi$ such that $\sum r_\chi$ is minimal. We let $\mathbb{Q}(\chi)$ denote the extension of $\mathbb{Q}$ obtained by adjoining the values $\chi(g)$. The Schur index $m(\chi)$ is the degree of the smallest extension of $\mathbb{Q}(\chi)$ over that M can be realized; *cf.* [CR, 41.4]. The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\widehat{G}$; $\mathrm{orb}(\widehat{G})$ will denote the set of orbits. The orbit $\mathrm{orb}(\chi)$ of a given χ is the set of Galois conjugates $\gamma\chi$ with $\gamma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$, and these are all distinct.

Lemma 4.1

(i) *If $\chi \in \widehat{G}$, then $\Phi(\chi)$ is the product of $m(\chi)$ and the degree of $\mathbb{Q}(\chi)$ over $\mathbb{Q}$. In particular, it is an integer.*

(ii) *For a non-irreducible character* $\xi = \sum_{\chi\in\widehat{G}} n_\chi \chi$,

$$\sigma(\xi) = \sum_{\Gamma\in \mathrm{orb}(\widehat{G})} \max_{\chi\in\Gamma} \left\lceil \frac{n_\chi}{m(\chi)} \right\rceil \sigma(\chi),$$

where $\lceil\cdot\rceil$ *is the round up or ceiling function.*

Proof (i) is an immediate consequence of [CR, thm 70.15], which implies that the character of the span of $\chi \in \widehat{G}$ is $\sum_{\chi'\in\mathrm{orb}(\chi)} m(\chi)\chi'$. This also implies (ii) by the above remarks. ∎

Remark 4.2 When $\chi \in \widehat{G}$, $m(\chi)$ and $\sigma(\chi)$ are Galois invariant, so we can write these as functions of the orbit. So the formula (ii) can be simplified ever so slightly to

$$\sigma(\xi) = \sum_{\Gamma\in \mathrm{orb}(\widehat{G})} \sigma(\Gamma) \max_{\chi\in\Gamma} \left\lceil \frac{n_\chi}{m(\Gamma)} \right\rceil.$$

Armed with this formula, and standard facts from [CR, §28, §70] we can compute a number of examples:

(a) If $G = \mathbb{Z}/d\mathbb{Z}$, then $\Phi(\epsilon^i) = \phi(d/\gcd(i,d))$.
(b) If $G = S_N$ is the symmetric group, $\Phi(\chi) = 1$ for all χ.
(c) If $G = \{\pm 1, \pm i, \pm j, \pm k\}$ is the quaternion group, and χ the character of the unique 2 dimensional irreducible complex representation, then $\Phi(\chi) = m(\chi) = 2$.

We come to the key estimate. We first note that if M is $\mathbb{Q}[G]$-module, then so is $M_{nt} \cong M/M_1$.

Proposition 4.3 *Suppose that G is a finite group of automorphisms of a rational pure effective Hodge structure H of weight i. The dimension of any sub-Hodge structure* $H' \subset H_{nt}$ *of level less than* $i-2k$ *is bounded above by the difference*

$$\dim H_{nt} - \sigma\left(H_{nt}^{0i} \oplus \cdots \oplus H_{nt}^{k,i-k} \oplus \overline{H_{nt}^{0i} \oplus \cdots \oplus H_{nt}^{k,i-k}}\right).$$

Proof Let $T = H_{nt}^{0i} \oplus \cdots \oplus H_{nt}^{k,i-k} \oplus \overline{H_{nt}^{0i} \oplus \cdots \oplus H_{nt}^{k,i-k}}$. Given a sub-Hodge structure H' of level $< i-2k$, by replacing it with $\sum gH'$, we can assume without loss of generality that it is G-invariant. By the level assumption, $T \cap (H' \otimes \mathbb{C}) = 0$. Thus H_{nt}/H' is a $\mathbb{Q}[G]$-module containing T after extending scalars. Therefore H/H' contains the rational span of T. ∎

Putting everything together yields the following corollary.

Corollary 4.4 *If G is a finite group acting on a projective orbifold Y, the dimension of a sub-Hodge structure of* $H^i_{nt}(Y)$ *of level less than* $i-2k$ *is bounded above by the difference*

$$\dim H^i_{nt}(Y) - \sum_{\Gamma\in\mathrm{orb}(\widehat{G}-1)} \sum_{\substack{p+q=i \\ p-q\geq k}} |\Gamma| m(\Gamma) \max_{\chi\in\Gamma} \left\lceil \frac{h^{p,q}_\chi(Y)}{m(\Gamma)\chi(1)} \right\rceil. \tag{4.1}$$

In particular when i is even, we get a bound on the dimension the space of Hodge cycles by applying this with $k = i/2 - 1$.

The main remaining issue is whether we can actually compute this bound. We will work this out for covers. We will fix the following notation for the remainder of this section. Let Z be an n-dimensional smooth projective variety. Let $D \subset Z$ be a reduced effective divisor with simple normal crossings and let $V = Z - D$. Suppose that $\rho\colon \pi_1(V) \to G$ is a surjective homomorphism onto a finite group. We can construct the associated étale cover $U \to V$. Let $\pi\colon Y \to Z$ be the normalization of Z in the function field of $\mathbb{C}(U)$, and let $E = \pi^{-1}D$. We refer to the triple (Z, D, ρ) as the *branching data* for Y.

We first analyze the local picture. We can cover Z by coordinate polydisks Δ_i so that D is given by $x_1 \cdots x_{k_i} = 0$. Let us fix one of these, and suppress the subscript i below. Then the fundamental group $\pi_1(\Delta - D) = \mathbb{Z}^k$ with generators corresponding to loops around the coordinate hyperplanes. Thus the preimage of $\Delta - D$ in Y is given by a disjoint union of connected abelian covers of $\Delta - D$. We can describe these components explicitly.

Lemma 4.5 *A normal connected abelian cover $\widetilde{\Delta} \to \Delta$ is an open set (in the classical topology) of a normal affine toric variety with finite quotient singularities. The projection to Δ is flat.*

Proof The cover Δ is determined by a subgroup $\Gamma \subset \pi_1(\Delta - D) = \mathbb{Z}^k$ of finite index. By elementary divisor theory, a basis for Γ is given the columns of a diagonal matrix with positive entries d_i. Thus Δ is equivalent to a neighbourhood of the normalization $\operatorname{Spec} \widetilde{R}$ of the variety $\operatorname{Spec} R$ defined

$$y_i^{d_i} = \prod_{j \in J_i} x_j^{a_{ij}}, \quad i = 1 \dots k,$$

where the sets $J_i \subset \{1, \dots, k\}$ are disjoint. These are tensor products of the rings considered earlier in the proof of Lemma 1.1. The results proved there show that this is a toric variety with quotient singularities. Also the projection is flat, because $\widetilde{R}$ is a free module over $\mathbb{C}[x_1, \dots, x_n]$; *cf.* [EV2, §3]. ■

Corollary 4.6 *Y is a toroidal orbifold. Moreover, the map $Y \to Z$ is flat.*

It follows that we can construct a toroidal resolution X of Y as before. Fix one such. We will say that the *weak Lefschetz property* holds for π if the map $H^i(Z, \mathbb{Q}) \to H^i(Y, \mathbb{Q})$ is an isomorphism for $i \neq n$. For example, we saw that cyclic covers totally ramified along D have this property. We now come to the main result.

Theorem 4.7 *With the above notations:*

(i) *The dimension of any sub-Hodge structure of $H^i_{nt}(Y, \mathbb{Q})$ of level at most $i - 2k - 2$ is bounded above by the expression given in* (4.1) *of Corollary* 4.4*.*

(ii) *If this bound is zero and if the assumptions of Theorem* 3.4(i) *hold, then the generalized Hodge conjecture is valid for X.*

(iii) *If the weak Lefschetz property holds, then the bound (4.1) can be computed by an explicit formula involving only the branching data.*

Most of this follows from we have said previously. The only thing that we need to explain is statement (iii). This will require a bit of preparation.

The sheaf $\mathcal{V} = \pi_* \mathcal{O}_Y$ is a vector bundle with a G-action. Set $\mathcal{V}_\chi = e_\chi \mathcal{V}$ as usual. Then we can decompose

$$\mathcal{V} = \bigoplus_{\chi \in \widehat{G}} \mathcal{V}_\chi = \mathcal{O}_Z \oplus \mathcal{V}_{nt}.$$

We see that

$$H^i_\chi(Y, \mathcal{O}_Y) = e_\chi H^i(Z, \mathcal{V}) = H^i(Z, \mathcal{V}_\chi).$$

Now $\mathcal{V}$ carries an integrable logarithmic connection

$$\nabla \colon \mathcal{V} \longrightarrow \mathcal{V} \otimes \Omega^1_Z(\log D),$$

which is none other than the canonical extension of the Gauss–Manin connection given by the direct image of $d \colon \mathcal{O}_Y \to \widetilde{\Omega}^1_Y(\log E)$ [D1]. The restriction of ∇ to U can also be characterized by the fact that the underlying local system $\ker \nabla|_U$ is given by $\pi_* \mathbb{C}|_U$. All of this is compatible with the G-action. The restriction of ∇ to $\mathcal{V}_\chi$ is just $e_\chi \nabla$.

The fact that ∇ is a canonical extension, with finite monodromy, means that over a polydisk, we can choose local coordinates and a local frame $\{e_j\}$ for $\mathcal{V}_\chi$ so that

$$\nabla = d + \sum R_i \frac{dx_i}{x_i} \quad \text{with } R_i = \begin{pmatrix} r_{i1} & 0 & \cdots & 0 \\ 0 & r_{i2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \end{pmatrix},$$

where $r_{ij} \in [0,1) \cap \mathbb{Q}$. The above matrices are determined from monodromy by taking normalized logarithms [D1, p. 54]. More precisely, we can choose a matrix representation T isomorphic to the regular representation $\mathbb{C}[G]$, such that

$$T\big(\rho(\gamma_i)\big) = \exp(-2\pi\sqrt{-1} R_i).$$

We can compute the Hodge numbers in terms of logarithmic differentials. Following Timmerscheidt [T], we have a subbundle

$$W_0\big(\Omega^k_Z(\log D) \otimes \mathcal{V}\big) \subseteq \Omega^k_Z(\log D) \otimes \mathcal{V}$$

locally spanned by wedge products of

$$\begin{cases} e_j \otimes dz_i & \text{if } r_{ij} = 0, \\ e_j \otimes \frac{dz_i}{z_i} & \text{otherwise,} \end{cases}$$

for a frame chosen with diagonal connection matrix as above. The utility of this construction for us stems from the following lemma.

Lemma 4.8 *We have*

$$\pi_*\widetilde{\Omega}_Y^k(\log E) = \Omega_Z^k(\log D)\otimes \mathcal{V},$$
$$\pi_*\widetilde{\Omega}_Y^k = W_0(\Omega_Z^k(\log D)\otimes \mathcal{V}).$$

Proof Since it suffices to check this locally away from the codimension two set D_{sing}, we can reduce almost immediately to a polydisk centered around a component of D, whence to the cyclic case. In this case, the arguments can be found in [EV2, §3]. To elaborate a bit more, the first equality follows from the projection formula and the fact that $\pi^*\Omega_Z^k(\log D) = \widetilde{\Omega}_Y^k(\log E)$. For the second equality, we will be content to work out the basic example of $y^d = x_1$ with the local frame $1, y, \dots, y^{d-1}$. We have

$$dy^i = \frac{(i-1)}{d} y^i \frac{dx_1}{x_1}.$$

Therefore, we see that both sides are spanned by products of

$$dx_1, dx_2, \dots\ ;\ y\frac{dx_1}{x_1}, ydx_2, \dots\ ;\quad \cdots\quad ;\ y^{d-1}\frac{dx_1}{x_1}, y^{d-1}dx_2, \dots. \qquad \blacksquare$$

Corollary 4.9

$$H_\chi^{p,q}(Y) \cong H^q\big(W_0(\Omega_Z^p(\log D)\otimes \mathcal{V}_{\overline{\chi}})\big).$$

The forms lying in W_0 should be thought of as nonsingular, in analogy with the usual case. The submodule $W_\ell(\Omega_Z^k(\log D)\otimes \mathcal{V}) \subseteq \Omega_Z^k(\log D)\otimes \mathcal{V}$ is defined by locally allowing sums of wedge products of at most ℓ forms singular forms. There is a Poincaré residue isomorphism [T],

$$Gr_\ell^W\big(\Omega_Z^k(\log D)\otimes \mathcal{V}\big) \cong \bigoplus_{|I|=\ell} W_0\big(\Omega_{D_I}^{k-\ell}(\log D_I'')\otimes \mathcal{V}_I\big), \tag{4.2}$$

where D_I, D_I' are as in (3.1), $D_I'' = D_I - D_I'$, and $\mathcal{V}_I \subset \mathcal{V}|_{D_I}$ is the subbundle corresponding to the local system $j_*(\ker \nabla|_U)|_{D_I'}$. In other words, the extension of $\mathcal{V}_I$ to a tubular neighbourhood corresponds to the maximal sublocal system of $\ker\nabla|_U$ with trivial monodromy around components of $D_i, i\in I$.

Lemma 4.10 *The class of $W_0(\Omega_Z^k(\log D)\otimes \mathcal{V})$ in the Grothendieck group $K_0(Z)$ is*

$$\sum_I (-1)^{|I|}\Omega_{D_I}^{k-|I|}(\log D_I'')\otimes \mathcal{V}_I.$$

A similar formula holds for each $\mathcal{V}_\chi$.

Proof By (4.2), we have

$$\Omega_Z^k(\log D)\otimes \mathcal{V} = \sum_I W_0\big(\Omega_{D_I}^{k-|I|}(\log D_I'')\otimes \mathcal{V}_I\big).$$

Then the lemma follows by the Möbius inversion formula [R, Prop 2]. Or, more directly, we can solve

$$W_0\big(\Omega^k_Z(\log D)\otimes \mathcal{V}\big) = \Omega^k_Z(\log D)\otimes \mathcal{V} - \sum_{I\neq\varnothing} W_0\big(\Omega^{k-|I|}_{D_I}(\log D_I'')\otimes \mathcal{V}_I\big).$$

We can assume that the lemma holds for each proper D_I by induction. Substituting the resulting expressions into the one above and simplifying, yields the lemma. ■

We are now ready to finish the proof of the main theorem.

Proof of Theorem 4.7(iii) It will be convenient to fix a choice of loops $\gamma_j \in \pi_1(V)$ around each component of D_j. It will be clear that the formulas will ultimately depend only on their conjugacy classes, which are completely determined by the branching data. Since G acts on the sheaf $\pi_*\mathbb{C}_Y$, we can decompose it as $\mathbb{C}_Z \oplus (\pi_*\mathbb{C}_Y)_{nt}$. So we have $\dim H^i_{nt}(Y) = \dim H^i(Y) - \dim H^i(Z)$, which is zero unless $i = n$. Thus we conclude that (4.1) is trivial for $i \neq n$ and that

$$(-1)^n \dim H^n_{nt}(Y) = e(Y) - e(Z),$$

where e denotes the topological Euler characteristic. The right-hand side is easily computed as

$$d(e(Z-D)) + \sum_{\substack{|J|>0,\\ D_J\neq\varnothing}} d_J e(D_J') - e(Z) = |G|(e(Z-D)) + \sum_{\substack{|J|>0,\\ D_J\neq\varnothing}} \frac{|G|}{|G(J)|} e(D_J') - e(Z),$$

where we write d_J for the number of sheets over D_J', and $|G(J)|$ for the order of the stabilizer of a component of $\pi^{-1}D_J'$. Then we see that $|G(J)|$ is the dimension of the intersection of kernels of the action of $\rho(\gamma_j)$, $j \in J$ on the regular representation $\mathbb{C}[G]$. Thus we have our desired formula for the first part $\dim H^n_{nt}(Y)$ in terms of branching data.

To finish the proof, it suffices to give formulas for the Hodge numbers $h^{p,i-p}_\chi(Y)$ with $\chi \neq 1$. The group

$$H^{p,i-p}_\chi(Y) = H^{i-p}\big(W_0(\Omega^p_Z(\log D)\otimes V_\chi)\big)$$

is a summand of $H^i_{nt}(Y)$, so it is zero when $i \neq n$. Therefore $h^{p,i-p}_\chi(Y)$ is a holomorphic Euler characteristic up to sign. When combined with Hirzebruch–Riemann–Roch [H], we obtain

$$(-1)^{n-p} h^{p,n-p}_\chi = \int_Z ch\big(W_0\big(\Omega^p_Z(\log D)\otimes V_\chi\big)\big)\, td(Z).$$

So the only thing remaining is to evaluate the Chern character in terms of the branching data.

By applying a result of Esnault and Verdier [EV1, appendix B], we obtain

(4.3)

$$\begin{aligned}\mathrm{ch}(V_\chi) &= \sum_p \sum_{m_1+m_2\cdots=p} \frac{(-1)^p}{p!}\binom{p}{m_1, m_2, \dots} \mathrm{tr}\big((e_\chi R_1)^{m_1}(e_\chi R_2)^{m_2}\cdots\big)[D_1]^{m_1}[D_2]^{m_2}\cdots \\ &= \sum_p \sum_{m_1+m_2\cdots=p} \frac{(-1)^p}{p!}\binom{p}{m_1, m_2, \dots} \mathrm{tr}\big(e_\chi R_1^{m_1} R_2^{m_2}\cdots\big)[D_1]^{m_1}[D_2]^{m_2}\cdots.\end{aligned}$$

Since this involves only the branching data, we have our desired formula for $h_\chi^{0,n}$. For the other p's there is one extra step. By Lemma 4.10 and the fact that ch is ring homomorphism, we have

$$\mathrm{ch}\Big(W_0\big(\Omega_Z^k(\log D)\otimes \mathcal{V}_\chi\big)\Big) = \sum(-1)^{|I|}\,\mathrm{ch}\big(\Omega_{D_I}^{k-|I|}(\log D_I'')\big)\,\mathrm{ch}(\mathcal{V}_I). \tag{4.4}$$

We have a formula for $\mathrm{ch}(V_{I,\chi})$ similar to (4.3) involving restrictions of the residues to D_I do that (4.4) can be expanded to a formula of the desired type. ∎

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