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Explicit Wiener–Hopf factorization and nonlinear Riemann–Hilbert problems

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A method for explicit Wiener–Hopf factorization of 2×2 matrix-valued functions is presented together with an abstract definition of a class of functions, $C(Q_1, Q_2)$, to which it applies. The method involves the reduction of the original factorization problem to certain nonlinear scalar Riemann–Hilbert problems, which are easier to solve. The class $C(Q_1, Q_2)$ contains a wide range of classes dealt with in the literature, including the well-known Daniele–Khrapkov class. The structure of the factors in the factorization of any element of the class $C(Q_1, Q_2)$ is studied and a relation between the two columns of the factors, which gives one of the columns in terms of the other through a linear transformation, is established. This greatly simplifies the complete determination of the factors and gives relevant information on the nature of the factorization. Two examples suggested by applications are completely worked out.

1. Introduction

The present paper deals with a general method for explicit Wiener–Hopf factorization of non-rational matrix-valued functions that appear in several areas of mathematics and its applications, such as linear operator theory, diffraction theory [8, 11, 13] and integrable systems [17].

Before we define in a more concrete way the main objective of the paper, let us give in very general terms what we mean by a Wiener–Hopf factorization of a bounded measurable matrix-valued function G invertible in $(L_\infty(\mathbb{R}))^{n\times n}$. This is a representation of G in the form

$$G = G_- D G_+,$$

where D is a diagonal rational matrix function ($D = \operatorname{diag}(r^{k_1}, r^{k_2}, \dots, r^{k_n})$, with $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_n$ integers, $r(\xi) = (\xi - \mathrm{i})/(\xi + \mathrm{i})$) and $G_\pm$ and their inverses belong to appropriate Hardy spaces of analytic functions in the upper (G_+) and lower (G_-) half-planes of the complex plane $\mathbb{C}$ (if we impose the condition that the factors be bounded, these spaces are $(H^\pm_\infty(\mathbb{R}))^{2\times 2}$ (cf. § 2 for a more precise definition)).

Perhaps the best-known class of non-rational matrix functions among those that have been studied in the specialized literature is the Daniele–Khrapkov class, here denoted C_{DK}, which is usually defined as

$$C_{\mathrm{DK}} = \{G \in L^{2\times 2}_\infty(\mathbb{R}) : G = \alpha I + \beta R,\ \alpha, \beta \in L_\infty(\mathbb{R})\}, \tag{1.1}$$

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where I is the identity in $\mathbb{C}^{2\times 2}$ and R is a bounded rational matrix function such that $R^2 = qI$ [4,5,8–10,16].

In this paper we start from a more abstract definition. We consider the class of matrix functions in $L_\infty^{2\times 2}(\mathbb{R})$ that satisfy the relation,

$$G^{\mathrm{T}} Q_1 G = h Q_2, \tag{1.2}$$

where 'T' denotes matrix transposition, $h \in L_\infty(\mathbb{R})$ and Q_1, Q_2 are rational matrix functions in $L_\infty^{2\times 2}(\mathbb{R})$. This class, here denoted by $C(Q_1, Q_2)$, includes, besides the Daniele–Khrapkov class, other classes that have some relevance in terms of applications and permits, in a natural way, a generalization to $n \times n$ matrix functions [6]. If $Q_2 = Q_1$, the relation (1.2) defines an infinite-dimensional Lie group, a fact that the authors believe may be of interest in the context of applications to integrable systems.

Some of the ideas behind the method to be expounded in the following sections have already been used by the first authors in concrete examples closely related to the Daniele–Khrapkov class. However, the main results presented here have a generality that goes well beyond those examples.

Next we describe briefly the main contents of the paper. In § 2 we give some preliminary results that are needed in the following sections, and in § 3 it is shown that the class $C(Q_1, Q_2)$ includes several different classes that have been studied in the specialized literature. A characterization of this class in a more concrete way is also given in § 3, making it easier to recognize whether a given matrix function belongs to $C(Q_1, Q_2)$. In § 4 our method for studying the factorization problem (existence and calculation of the factors) is expounded. As is known [7,12,14], investigating the existence of a factorization with $D = I$ (canonical factorization) involves studying the existence of non-trivial solutions of the Riemann–Hilbert homogeneous problem

$$G\phi^+ = \phi^- \tag{1.3}$$

(e.g. with $\phi^\pm \in (H_2^\pm(\mathbb{R})^2)$) and the calculation of the factors involves solving the problem

$$G\phi^+ = r\phi^-, \tag{1.4}$$

for example, in $(H_2^\pm(\mathbb{R}))^2$ (here, $r(\xi) = (\xi - \mathrm{i})/(\xi + \mathrm{i})$). It is shown in § 4 that the solution of problems (1.3) and (1.4) can be reduced to two scalar nonlinear Riemann–Hilbert equations, which we call the product and quotient equations. For (1.3), the product equation is simply obtained by multiplying on the left this equation firstly by Q_1 and afterwards by its transpose, leading to

$$(\phi^+)^{\mathrm{T}} G^{\mathrm{T}} Q_1 G \phi^+ = (\phi^-)^{\mathrm{T}} Q_1 \phi^-,$$

which becomes

$$h(\phi^+)^{\mathrm{T}} Q_2 \phi^+ = (\phi^-)^{\mathrm{T}} Q_1 \phi^- \tag{1.5}$$

after using definition (1.2). Equation (1.5) is relatively easy to solve because it is a scalar Riemann–Hilbert equation. In some cases, the existence problem also involves the quotient equation, which is derived in a less obvious manner (see theorems 4.1 and 4.2). The calculation of the factors involves analogous equations obtained from (1.4), which tend to be computationally more complicated, as explained in § 4. Although the product and quotient equations have appeared in other papers by the first two authors [1, 2], they appear here, for the first time, in full generality.

Section 5 contains, perhaps, the most important results of the paper. The method expounded in § 4 gives one column of the factors G_+^{-1} and G_-. The usual procedure to calculate the factors would involve repeating the calculation for the second columns and proving that the matrices formed by the columns are invertible in their appropriate Hardy spaces; this is what was done in [1, 2, 5]. In § 5 it is shown that the second column is easily obtained from the first by a rational transformation, which in some cases even belongs to $C(Q_1, Q_2)$. The main result is theorem 5.2, which we summarize as follows. If $f^{\pm}$ are the first columns of the factors G_+^{-1} and G^-, the second columns $s^{\pm}$ are obtained through the formulae

$$s^- = M_1 f^-, \qquad s^+ = M_2 f^+$$

with $M_1 = r_1^{-1}(\tilde{r}_1 I - JQ_1)$, $M_2 = r_1^{-1}(\tilde{r}_1 I - JQ_2)$, where J is a known constant matrix and r_1, $\tilde{r}_1$ are rational functions obtained from the calculation of $f^{\pm}$.

In § 6 the paper concludes with two examples that illustrate the general results of §§ 4 and 5. These examples were chosen to point out some of difficulties that may occur in the application of those results but, at the same time, with the preoccupation of reducing the computation complexity to a minimum in order to avoid obfuscating the main ideas by mere questions of algebraic calculations. It may be worth noting, however, that the first example corresponds to a problem from diffraction theory not dealt with in the literature.

2. Preliminaries

Let $L_2(\mathbb{R})$ denote the space of all complex-valued measurable square-integrable functions defined on $\mathbb{R}$ with the norm

$$\|f\|_2 = \left(\int_{\mathbb{R}} |f(t)|^2 \, \mathrm{d}t\right)^{1/2}.$$

As is known, the Cauchy's singular integral operator S, given by

$$(Sf)(t) = \frac{1}{\pi \mathrm{i}} \int_{\mathbb{R}} \frac{f(\tau)}{\tau - t} \, \mathrm{d}\tau, \quad t \in \mathbb{R}, \tag{2.1}$$

is bounded on $L_2(\mathbb{R})$ (in (2.1), the integral is understood in the sense of Cauchy's principal value). In $L_2(\mathbb{R})$, we can define two complementary projections, $P^{\pm} = \frac{1}{2}(I \pm S)$, where I is the identity operator. These projections lead to a direct sum decomposition of $L_2(\mathbb{R})$,

$$L_2(\mathbb{R}) = L_2^+(\mathbb{R}) \oplus L_2^-(\mathbb{R}),$$

where the subspaces $L_2^{\pm}(\mathbb{R}) = P^{\pm}(L_2(\mathbb{R}))$ can be identified with the corresponding Hardy spaces, $H_2^{\pm}$, of functions analytic on $\mathbb{C}^{\pm} = \{z \in \mathbb{C} : \pm \operatorname{Im} z > 0\}$.

By $L_\infty(\mathbb{R})$, sometimes simply denoted by L_∞, we denote the space of all essentially bounded measurable functions on $\mathbb{R}$. The subspaces of functions that have bounded analytic extensions to $\mathbb{C}^{\pm}$ will be denoted by $L_\infty^{\pm}$ (these spaces may be

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identified with the Hardy spaces $H_\infty(\mathbb{C}^\pm)$ of bounded analytic functions in $\mathbb{C}^\pm$). $\mathcal{R}(\mathbb{R})$ denotes the space of all rational functions in $L_\infty(\mathbb{R})$. In what follows, we also need the space $C_\mu(\dot{\mathbb{R}})$ of all continuous functions satisfying a Hölder condition of order μ on the one-point compactification of the real line.

Let $\mathcal{G}(A)$ be the group of invertible elements in an algebra A and let

$$r(\xi) = (\xi - \mathrm{i})/(\xi + \mathrm{i}), \quad r_\pm(\xi) = 1/(\xi \pm \mathrm{i}), \quad \xi \in \mathbb{R}.$$

Definition 2.1. By a *generalized factorization* of an invertible matrix-valued function $G \in \mathcal{G}(L_\infty^{n\times n}(\mathbb{R}))$ (relative to $L_2(\mathbb{R})$), we mean a factorization of the form [7, 12, 14]

$$G = G_- D G_+,$$

where $D = \mathrm{diag}(r^{k_1}, r^{k_2}, \dots, r^{k_n})$, $k_j \in \mathbb{Z}$, $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_n$ and G_-, G_+ satisfy the conditions

(i) $r_+ G_+^{\pm 1} \in (L_2^+(\mathbb{R}))^{n\times n}$;

(ii) $r_- G_-^{\pm 1} \in (L_2^-(\mathbb{R}))^{n\times n}$;

(iii) $G_+^{-1} P^+ G_-^{-1} I$ is an operator defined on a dense subset of $(L_2(\mathbb{R}))^n$, e.g. the rational functions in $(L_2(\mathbb{R}))^n$ possessing a bounded extension to $(L_2(\mathbb{R}))^n$.

(A geometrical approach to the factorization problem can be found in [15, ch. 8].)

The generalized factorization is said to be *canonical* if all the partial indices k_j ($j = 1, 2, \dots, n$) are equal to zero. The sum of all partial indices, $k = \sum_{j=1}^n k_j$, is the *total index* of G. The factorization is said to be a *bounded factorization* if the factors G_+, G_- and their inverses are bounded in $\mathbb{C}^+$, $\mathbb{C}^-$, respectively.

$G \in C_\mu^{n\times n}(\dot{\mathbb{R}})$ admits a generalized factorization in $C_\mu^{n\times n}(\dot{\mathbb{R}})$ (hence bounded) if and only if $\det G(\xi) \neq 0$ for all $\xi \in \dot{\mathbb{R}}$. In this case, the total index is $k = \mathrm{ind}\det G$, where ind denotes the usual winding number of non-vanishing continuous functions on $\dot{\mathbb{R}}$ relative to the origin. If $G \in C_\mu^{n\times n}(\dot{\mathbb{R}})$ has total index zero, then it possesses a canonical factorization if and only if the homogeneous Riemann–Hilbert problem

$$G\phi^+ = \phi^-, \quad \phi^\pm \in (L_2^\pm(\mathbb{R}))^n, \tag{2.2}$$

admits only the trivial solution $\phi^+ = \phi^- = 0$.

Although we assume that the matrix functions studied in this paper belong to $C_\mu^{n\times n}(\dot{\mathbb{R}})$, the results obtained here can easily be extended to more general settings.

In most of this paper, we shall be concerned with 2×2 matrix-valued functions, and thus the results that follow refer only to this case. The following result is taken from [5], where it is proven.

Proposition 2.2. *Let G possess a canonical generalized factorization. Then this factorization can be obtained by determining two solutions, (ϕ^+, ϕ^-) and (ψ^+, ψ^-), to the equation*

$$G\phi^+ = r\phi^-, \quad \phi^\pm \in (L_2^\pm(\mathbb{R}))^2, \tag{2.3}$$

such that

$$\det[\phi^+ \psi^+](\xi) \neq 0 \quad \textit{for some } \xi \in \mathbb{C}^+. \tag{2.4}$$

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Remark 2.3. It can easily be checked that in (2.4) the given condition on the solution of the Riemann–Hilbert problem (2.3) may be a replaced by corresponding condition for $\xi \in \mathbb{C}^-$.

3. Classes of functions

To begin with, we define a family of classes of functions that will play a central role in all that follows. We shall see in the following sections that elements of this class can be explicitly factorized by the method of § 4.

Definition 3.1. Let $C(Q_1, Q_2)$ be the set of all bounded invertible measurable $n \times n$ matrix functions G on $\mathbb{R}$ satisfying the relation

$$G^{\mathrm{T}} Q_1 G = h Q_2, \tag{3.1}$$

where 'T' denotes matrix transposition, Q_1, Q_2 are given $n \times n$ rational matrix functions with poles off $\mathbb{R}$ and h is a scalar-valued function invertible in $L_\infty(\mathbb{R})$ and depending on G.

The following remarks help to understand some of the reasons behind definition 3.1.

Remark 3.2. The fact that matrix transposition appears in the first factor in relation (3.1) is going to be crucial in the theory of § 4. Matrix transposition is an anti-homomorphism in the algebra of matrix functions $L_\infty^{n\times n}(\mathbb{R})((G_1G_2)^{\mathrm{T}} = G_2^{\mathrm{T}} G_1^{\mathrm{T}})$ and this property ensures that the factors $G_{1,2}^{\pm}$ appear in the desired order when one tries to split the left- and right-hand sides of (3.1) in terms of their factors in a Wiener–Hopf factorization.

Remark 3.3. If $Q_1 = Q_2 = Q$, it is easy to see that the class given in definition 3.1 is a multiplicative group of $n \times n$ matrix functions. This fact gives the elements of the class some interesting algebraic properties that may be useful in a more abstract framework. If $Q_1 = Q_2 = Q$, we denote $C(Q, Q)$ simply by $C(Q)$.

Remark 3.4. It should be noted that definition 3.1 makes sense for $n \times n$ matrix functions, although in the following sections we concentrate on the factorization of 2×2 matrix functions, since the factorization method developed in § 4 requires considerable modifications to be applicable to $n \times n$ matrix functions (cf. [6] for a partial answer to this question).

Now we examine a few classes corresponding to special cases of definition 3.1. Most of theses classes are important from the point of view of applications.

Daniele–Khrapkov class [8, 9]

Let $R \in L_\infty^{2\times 2}(\mathbb{R})$ be a rational function with zero trace, which implies that $R^2 = qI$ for $q = -\det R$ (I is the identity matrix in $\mathbb{C}^{2\times 2}$). Let us assume, moreover, that $q \neq 0$. Then the Daniele–Khrapkov class associated with R is defined as

$$C_{\mathrm{DK}} = \{G \in L_\infty^{2\times 2}(\mathbb{R}) : G = \alpha I + \beta R,\ \alpha, \beta \in L_\infty(\mathbb{R})\}.$$

We show that C_{DK} is contained in $C(Q)$ for a certain Q related to R. In fact, the two classes coincide when Q is invertible, as we shall see later.

Given R with trace zero, define Q to be a symmetric matrix satisfying

$$QR + R^{\mathrm{T}}Q = 0. \tag{3.2}$$

It is clear that $Q = R^{\mathrm{T}}J$ satisfies this relation, for

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{3.3}$$

and Q is invertible (as a rational matrix function, meaning that $\det Q$ is not identically zero). Note that Q is unique up to normalization. Taking

$$R = \begin{bmatrix} 0 & 1 \\ q & 0 \end{bmatrix},$$

as is often chosen, we get $Q = \mathrm{diag}(-q, 1)$.

Now calculate

$$\begin{aligned} G^{\mathrm{T}}QG &= (\alpha I + \beta R^{\mathrm{T}})Q(\alpha I + \beta R) \\ &= \alpha^2 Q + \alpha\beta(R^{\mathrm{T}}Q + QR) + \beta^2 R^{\mathrm{T}}QR. \end{aligned}$$

The second term in the last expression vanishes by (3.2). As to $R^{\mathrm{T}}QR$, we have

$$R^{\mathrm{T}}QR = -QR^2 = -qQ.$$

Hence

$$G^{\mathrm{T}}QG = (\alpha^2 - q\beta^2)Q,$$

and thus $G \in C(Q)$ with $h = \alpha^2 - q\beta^2$.

Class $\mathcal{D} - \mathcal{N}$ [1]

As is clear from the definition given above, the Daniele–Khrapkov class consists of matrix functions that have two rationally independent elements. This is a consequence of the defining relation $G = \alpha I + \beta R$, where R is fixed and α, β are arbitrary L_∞ functions. It is possible to obtain from definition 3.1 a class of matrix functions with three rationally independent elements by appropriately choosing the matrix function Q. In order to avoid getting involved in rather cumbersome calculations, we give a direct definition of the class and show that it is included in $C(Q)$ for an appropriate Q. We take the set of all $G \in L_\infty^{2\times 2}(\mathbb{R})$ such that

$$G = \alpha I + \beta R + \gamma N, \tag{3.4}$$

where $R^2 = qI$ (for $q \neq 0$), $N^2 = 0$ and $RN + NR = 0$. If R is given by

$$R = \begin{bmatrix} 0 & 1 \\ q^{-2} & 0 \end{bmatrix}, \tag{3.5}$$

then, apart from a scalar multiplicative function,

$$N = \begin{bmatrix} 1 & q \\ -q^{-1} & -1 \end{bmatrix}. \tag{3.6}$$

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We assume that R and N are given by (3.5), (3.6), since any symbol $\tilde{G}$ in the class $\mathcal{D}-\mathcal{N}$ has the form $\tilde{G} = AGA^{-1}$, with G given by (3.4), (3.5), (3.6) and rational A.

Choosing Q to be the symmetric singular 2×2 rational matrix function given by $Q = JN$, it may easily be verified that

$$R^{\mathrm{T}}Q + QR = 2q^{-1}Q, \qquad N^{\mathrm{T}}Q = 0 = QN, \qquad R^{\mathrm{T}}QR = q^{-2}Q. \tag{3.7}$$

From the definition (3.4) of the class, together with (3.5) and (3.7), we obtain

$$\begin{aligned} G^{\mathrm{T}}QG &= (\alpha I + \beta R^{\mathrm{T}} + \gamma N^{\mathrm{T}})Q(\alpha I + \beta R + \gamma N) \\ &= \alpha^2 I + \alpha\beta(R^{\mathrm{T}}Q + QR) + \alpha\gamma(N^{\mathrm{T}}Q + QN) \\ &\qquad + \beta\gamma(R^{\mathrm{T}}QN + N^{\mathrm{T}}QR) + \beta^2 R^{\mathrm{T}}QR + \gamma^2 N^{\mathrm{T}}QN \\ &= (\alpha + \beta q^{-1})^2 Q, \end{aligned}$$

i.e. $G \in C(Q)$ for the singular Q given above.

Generalized Daniele–Khrapkov class

This is a subclass of $C(Q_1, Q_2)$ that includes an interesting example from diffraction theory [2, 11]. It consists of the set of all matrix functions of the form

$$G = \begin{bmatrix} \alpha & \rho_1\theta \\ q\theta & \alpha\rho_1 \end{bmatrix},$$

where $\alpha, \theta \in C_\mu(\bar{\mathbb{R}})$ and $q_1 = \rho_1^2$ and q are both quotients of two first-degree polynomials with zeros off $\mathbb{R}$. Such matrices satisfy (3.1) for

$$\begin{aligned} Q_1 &= \mathrm{diag}(1, -q^{-1}), \\ Q_2 &= \mathrm{diag}(1, -q_1 q^{-1}), \\ h &= \alpha^2 - q\theta^2. \end{aligned}$$

Rawlins–Williams class

This class is defined as the set of all 2×2 bounded matrix functions of the form

$$\begin{bmatrix} 1 & a \\ b & -ab \end{bmatrix},$$

where $a, b \in L_\infty(\mathbb{R})$ admit bounded factorizations and $a = a_- a_+$ with a_-^2 rational or $b = b_- b_+$ with b_+^2 of the same type [1]. It can also be defined by a slight generalization of relation (3.1) as the set of all 2×2 bounded matrix functions satisfying

$$G^{\mathrm{T}} Q_1^- G = h Q_2^+,$$

where $Q_1^- \in (\mathcal{R}(\mathbb{R}) + L_\infty^-(\mathbb{R}))^{2\times 2}$, $Q_2^+ \in (\mathcal{R}(\mathbb{R}) + L_\infty^+(\mathbb{R}))^{2\times 2}$. In particular, if a_-^2 is rational, we have

$$Q_1^- = b_-^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad Q_2^+ = 2b_+ \begin{bmatrix} 1 & 0 \\ 0 & -a^2 \end{bmatrix}.$$

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It will be clear in § 4 that the method proposed in the present paper still applies to this class.

Next *we characterize* $C(Q_1, Q_2)$ in a more concrete way.

THEOREM 3.5. *Let* Q_1*,* Q_2 *be invertible rational matrix functions in* $L_\infty^{2\times 2}(\mathbb{R})$*. Assume that there exists an invertible matrix function* H *in* $C(Q_1, Q_2)$*, i.e.*

$$H^{\mathrm{T}} Q_1 H = h_0 Q_2 \tag{3.8}$$

for some $h_0 \in L_\infty(\mathbb{R})$ *and let* G *be a matrix function in* $\mathcal{G}L_\infty^{2\times 2}(\mathbb{R})$*. The following statements are equivalent.*

(i) G *belongs to* $C(Q_1, Q_2)$*.*

(ii) $G = H\tilde{G}$ *for some* $\tilde{G} \in C(Q_2)$*.*

(iii) $G = \tilde{G}H$ *for some* $\tilde{G} \in C(Q_1)$*.*

Proof. (i) ⇒ (ii). Assume first that $G \in C(Q_1, Q_2)$. Since H is invertible, we can write

$$G = H\tilde{G},$$

and substituting this relation in (3.1) we obtain

$$\tilde{G}^{\mathrm{T}} H^{\mathrm{T}} Q_1 H \tilde{G} = h Q_2.$$

But, in view of (3.8), it follows that

$$\tilde{G}^{\mathrm{T}} Q_2 \tilde{G} = h_0^{-1} h Q_2,$$

which implies that $\tilde{G} \in C(Q_2)$, with $\tilde{h} = h_0^{-1} h$. For the converse (i.e. (ii) ⇒ (i)), assume that $G = H\tilde{G}$, with $\tilde{G}$ satisfying $\tilde{G}^{\mathrm{T}} Q_2 \tilde{G} = \tilde{h} Q_2$. Then

$$G^{\mathrm{T}} (H^{-1})^{\mathrm{T}} Q_2 H^{-1} G = \tilde{h} Q_2,$$

which, using (3.8), leads to

$$G^{\mathrm{T}} Q_1 G = h_0 \tilde{h} Q_2,$$

i.e. $G \in C(Q_1, Q_2)$, with $h = h_0 \tilde{h}$.

The proof that (i) ⇔ (iii) is entirely analogous. □

To come down to more concrete classes, consider the case where

$$Q_1 = \operatorname{diag}(1, q_1), \qquad Q_2 = \operatorname{diag}(1, q_2), \tag{3.9}$$

with $q_1, q_2 \in R(\mathbb{R})$, as in (iii). Then it is easy to check that

$$D = \operatorname{diag}(q_1^{1/2}, q_2^{1/2}) \tag{3.10}$$

belongs to $C(Q_1, Q_2)$. On the other hand, taking $R_2 = JQ_2$ for J defined by (3.3), any matrix function $\tilde{G} \in C(Q_2)$ can be put in the form

$$\tilde{G} = \tilde{\alpha} I + \tilde{\beta} R_2 (\tilde{\alpha}, \tilde{\beta} \in L_\infty(\mathbb{R})),$$

as we show later in § 4. Hence

$$G = \alpha D + \beta A,$$

where D is the diagonal matrix (3.10) and A is an anti-diagonal matrix

$$A = \begin{bmatrix} 0 & -\sqrt{q_1 q_2} \\ 1 & 0 \end{bmatrix}$$

is the general form of any element of $C(Q_1, Q_2)$ for Q_1, Q_2 as given above. We also see that, defining Q_1, Q_2 as in (3.9), the class $C(Q_1, Q_2)$ is not empty and an invertible matrix H (see theorem 3.5) can be immediately defined from Q_1 and Q_2. In fact, this is true for any Q_1, Q_2, as we will show later in the next section.

It may be worth noting that we no longer have the limitation of only two rationally independent matrix elements in G that characterize the Daniele–Khrapkov class $C(Q)$ for invertible Q.

4. Product and quotient equations

In this section we present our nonlinear approach for Wiener–Hopf factorization. This will be done in connection with (3.1), which defines a family of classes of symbols. We shall confine our treatment to the case $n = 2$ and Q_1, Q_2 rational, but with slight modifications the method applies to generalizations such as the class defined by (3.4) (cf. [1]).

The study of the factorization problem involves two parts: (i) investigation of the existence of canonical factorization; and (ii) calculation of the factors. Because of its computational simplicity, in this paper we only consider the case when the factorization is canonical.

Given a function $G \in \mathcal{G}(L_\infty^{2\times 2}(\mathbb{R}))$, we assume that the Toeplitz operator

$$T = P^+ G I_+$$

is Fredholm of index zero (here, P^+ is the orthogonal projection $P^+ : (L_2(\mathbb{R}))^2 \to (L_2^+(\mathbb{R}))^2$ defined in § 2 and I_+ is the identity on $(L_2^+(\mathbb{R}))^2$). If $\operatorname{ind} T = 0$, the factorization of G is canonical if and only if the Riemann–Hilbert problem

$$G\phi^+ = \phi^-, \quad \phi^\pm \in (L_2^\pm(\mathbb{R}))^2, \tag{4.1}$$

has only the trivial solution. As is known, if G is continuous on $\dot{\mathbb{R}}$, $\operatorname{ind} T = \operatorname{ind}\det G$, where $\operatorname{ind}\det G$ is the index of the function $\det G : \mathbb{R} \to \mathbb{C}$ (the winding number relative to the origin of the path in $\mathbb{C}$ defined by $\det G$).

Let $G \in C(Q_1, Q_2)$. Applying matrix transposition to (4.1) and multiplying in the appropriate order the transposed and the original equation, we obtain

$$(\phi^+)^{\mathrm{T}} G^{\mathrm{T}} Q_1 G \phi^+ = (\phi^-)^{\mathrm{T}} Q_1 \phi^-.$$

Using here the relation $G^{\mathrm{T}} Q_1 G = h Q_2$ leads to

$$h(\phi^+)^{\mathrm{T}} Q_2 \phi^+ = (\phi^-)^{\mathrm{T}} Q_1 \phi^-. \tag{4.2}$$

This is the first basic equation, which we call the *product equation*. Note that if we ignore the poles of Q_1, Q_2, equation (4.2) is essentially a scalar Riemann–Hilbert problem for the unknowns $(\phi^+)^{\mathrm{T}} Q_2 \phi^+$ and $(\phi^-)^{\mathrm{T}} Q_1 \phi^-$.

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In (4.2), we assume that h has a canonical bounded factorization $h = h_- h_+$. This is a natural assumption, since a factor of the form r^k in the generalized factorization of h can always be included in Q_1 or Q_2. Thus we obtain from (4.2)

$$h_+ (\phi^+)^{\mathrm{T}} Q_2 \phi^+ = h_-^{-1} (\phi^-)^{\mathrm{T}} Q_1 \phi^-, \tag{4.3}$$

and we see that both sides of (4.3) must be equal to a scalar rational function $r_1 \in L_1(\mathbb{R})$. In some examples, this immediately implies that $r_1 = 0$. However, this is generally not the case (if r_1 has two or more poles, $r_1 \in L_1(\mathbb{R})$ does not imply $r_1 = 0$). In this case, to determine the functions $\phi^\pm$, we can use another equation, which we proceed to derive.

Let J be defined by (3.3) and consider the alternate bilinear form in $\mathbb{C}^2$,

$$X^{\mathrm{T}} J Y, \quad X, Y \in \mathbb{C}^2.$$

For each $\xi \in \mathbb{R}$, we have

$$(\phi^-(\xi))^{\mathrm{T}}(\xi) J \phi^-(\xi) = 0,$$

and using this property in (4.1) leads to

$$(\phi^-)^{\mathrm{T}} J G \phi^+ = 0, \tag{4.4}$$

which we shall call the *quotient equation*, for reasons that will become apparent later. We state the above results in the following theorem.

THEOREM 4.1. *Let $\phi^\pm \in [L_2^\pm(\mathbb{R})]^2$ be solutions to the Riemann–Hilbert problem*

$$G\phi^+ = \phi^-,$$

where $G \in L_\infty^{2\times 2}(\mathbb{R})$ satisfies (3.1). Then $\phi^\pm$ satisfy the nonlinear equations

$$\begin{aligned} h(\phi^+)^{\mathrm{T}} Q_2 \phi^+ &= (\phi^-)^{\mathrm{T}} Q_1 \phi^-, \\ (\phi^-)^{\mathrm{T}} J G \phi^+ &= 0. \end{aligned}$$

Equations (4.3) and (4.4) are nonlinear Riemann–Hilbert problems that are, in general, difficult to solve by a direct attack. We are going to adopt, instead, an alternative approach based on transforming them into equivalent equations easier to solve.

Let Q be a general symmetric 2×2 rational matrix, which, apart from a rational scalar multiplicative function, can be put in the form

$$Q = \begin{bmatrix} q_o & q_1 \\ q_1 & q_2 \end{bmatrix}, \quad q_0 = 0 \quad \text{or} \quad q_0 = 1. \tag{4.5}$$

We then have

$$Q = \tfrac{1}{2} S^{\mathrm{T}} \tilde{J} S,$$

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with

$$\tilde{J} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{4.6}$$

$$S = \begin{bmatrix} 1 & q_1 + \rho \\ 1 & q_1 - \rho \end{bmatrix}, \quad \text{with } \rho^2 = -\det Q = q_1^2 - q_2 \quad \text{if } q_0 = 1, \tag{4.7}$$

$$S = \begin{bmatrix} 0 & 1 \\ 2q_1 & q_2 \end{bmatrix} \quad \text{if } q_0 = 0. \tag{4.8}$$

In the same way, we have (with S_1, S_2 analogous to S for $Q = Q_1$ and $Q = Q_2$, respectively),

$$Q_1 = \tfrac{1}{2} S_1^{\mathrm{T}} \tilde{J} S_1, \qquad Q_2 = \tfrac{1}{2} S_2^{\mathrm{T}} \tilde{J} S_2, \tag{4.9}$$

and using this in (4.3) we obtain

$$h_+ (S_2 \phi^+)^{\mathrm{T}} \tilde{J} (S_2 \phi^+) = h_-^{-1} (S_1 \phi^-)^{\mathrm{T}} \tilde{J} (S_1 \phi^-).$$

Let

$$S_2 \phi^+ = \begin{bmatrix} \psi_s^+ \\ \psi_d^+ \end{bmatrix}, \qquad S_1 \phi^- = \begin{bmatrix} \psi_s^- \\ \psi_d^- \end{bmatrix}; \tag{4.10}$$

then we see that the product equation (4.3) is equivalent to

$$h_+ \psi_s^+ \psi_d^+ = h_-^{-1} \psi_s^- \psi_d^-. \tag{4.11}$$

It should be noted here that in (4.11) we deviated from standard notation in that $\psi_d^\pm$, $\psi_s^\pm$ are in general not analytic in $\mathbb{C}^\pm$, due to the presence of $\rho_1 = (-\det Q_1)^{1/2}$ in $S_1 \phi^-$ and $\rho_2 = (-\det Q_2)^{1/2}$ in $S_2 \phi^+$.

We also remark that if Q_1 is not invertible ($\det Q_1 = 0$), then Q_2 is also non-invertible ($\det Q_2 = 0$), admitting that $G \in \mathcal{G}L_\infty^{2\times 2}$ and $h \in \mathcal{G}L_\infty$. In this case, $\psi_s^\pm = \psi_d^\pm$ and the product equation (4.11) can be reformulated as a linear Riemann–Hilbert problem.

From now on we concentrate on the more interesting case where Q_1, Q_2 are invertible matrices whose determinants are not the square of a rational function. In this case, it is clear that S_1, S_2 are also invertible. Let

$$D = S_1 G S_2^{-1}. \tag{4.12}$$

From (3.1) and (4.9), it follows that

$$D^{\mathrm{T}} \tilde{J} D = h \tilde{J},$$

or, equivalently,

$$\tilde{J} D = h (D^{\mathrm{T}})^{-1} \tilde{J}. \tag{4.13}$$

On the other hand, from

$$h^2 = \det G^2 \cdot \det Q_1 \cdot \det Q_2^{-1} = \det G^2 \cdot \det S_1^2 \cdot \det S_2^{-2},$$

we see that

$$h = \pm \det S_1 G S_2^{-1} = \pm \det D. \tag{4.14}$$

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It follows from this and from (4.13) that D is diagonal (if $h = \det D$) or anti-diagonal (if $h = -\det D$).

Considering now equation (4.4), we have

$$(\phi^-)^{\mathrm{T}} J S_1^{-1} D S_2 \phi^+ = 0, \tag{4.15}$$

but, since

$$J S_1^{-1} = \det S_1^{-1} S_1^{\mathrm{T}} J,$$

equation (4.15) is equivalent to

$$(S_1 \phi^-)^{\mathrm{T}} J D (S_2 \phi^-) = 0. \tag{4.16}$$

If, for instance, $D = \mathrm{diag}(d_1, d_2)$, using the notation defined in (4.10), we see that (4.4) becomes

$$\frac{d_1}{d_2} \frac{\psi_s^+}{\psi_d^+} = \frac{\psi_s^-}{\psi_d^-}, \tag{4.17}$$

and an analogous expression can be obtained if D is anti-diagonal. The form of (4.17) justifies the terminology 'quotient equation'.

For the sake of simplicity, we assume in what follows that h in (3.1) is equal to $\det D$ (so that D is diagonal).

We can now state the result that gives us the basic tool to solve the Riemann–Hilbert problem (4.1).

THEOREM 4.2. *Let $G \in \mathcal{G}(L_\infty^{2\times 2}(\mathbb{R}))$ satisfy (3.1), where Q_1, Q_2 are symmetric invertible rational matrix function. If (ϕ^+, ϕ^-), $\phi^\pm \in (L_2^\pm(\mathbb{R}))^2$, is a solution to the Riemann–Hilbert problem*

$$G\phi^+ = \phi^-,$$

then the functions $\phi^\pm = (\phi_1^\pm v \phi_2^\pm)$ satisfy the nonlinear equations

$$h \psi_s^+ \psi_d^+ = \psi_s^- \psi_d^-, \tag{4.18}$$

$$\frac{d_1}{d_2} \frac{\psi_s^+}{\psi_d^+} = \frac{\psi_s^-}{\psi_d^-}, \tag{4.19}$$

where

$$\psi_s^+ = (\phi_1^+ + \tilde{q}_1 \phi_2^+) + \rho_2 \phi_2^+, \qquad \psi_d^+ = (\phi_1^+ + \tilde{q}_1 \phi_2^+) - \rho_2 \phi_2^+,$$
$$\psi_s^- = (\phi_1^- + q_1 \phi_2^-) + \rho_1 \phi_2^-, \qquad \psi_d^- = (\phi_1^- + q_1 \phi_2^-) - \rho_1 \phi_2^-$$

for q_1, $\tilde{q}_1$, ρ_1, $\tilde{\rho}_1$ such that

$$Q_1 = \begin{bmatrix} 1 & q_1 \\ q_1 & q_2 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} 1 & \tilde{q}_1 \\ \tilde{q}_1 & \tilde{q}_2 \end{bmatrix},$$
$$\rho_1^2 = -\det Q_1, \qquad \rho_2^2 = -\det Q_2$$

and d_1, d_2 are the diagonal elements in D defined by (4.12).

In the above theorem, the solution of (4.19) can be obtained by applying logarithms and using the product equation (4.18), if d_1/d_2 admits a generalized factorization. Some non-trivial difficulties occur, however, if ρ_1, ρ_2 involve more than

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two distinct branch points (cf. [4,5]). This problem will be addressed in the second example chosen for § 6.

For the class $C(Q)$ ($Q_1 = Q_2 = Q$), the results of the last theorem become simpler and equations (4.18) and (4.19) can be obtained directly from a diagonalization of G. This is related to the following proposition, which characterizes that class completely.

COROLLARY 4.3. *Let $G \in C(Q)$, where Q is an invertible symmetric rational matrix. Then $G = \alpha I + \beta R$, where $\alpha, \beta \in L_\infty(\mathbb{R})$, I is the identity 2×2 matrix and R is a rational matrix such that* $\operatorname{tr} R = 0$.

Proof. In this case, equation (4.12) becomes

$$D = SGS^{-1},$$

with $D = \operatorname{diag}(d_1, d_2)$ (we assume that $h = \det D$ in (3.1)), so that G is diagonalizable. Let

$$\alpha = \frac{d_1 + d_2}{2}, \qquad \beta = \frac{d_1 - d_2}{2\rho}$$

for $\rho = (\det Q)^{1/2}$, Q given by (4.5). Then

$$G = S^{-1} D S = \alpha I + \beta R$$

for

$$R = \rho S^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} S.$$

Thus we see that $R^2 = \rho^2 I$ and $\operatorname{tr} R = 0$. It only remains to show that R is rational.

We have

$$\begin{aligned} Q &= \tfrac{1}{2} S^{\mathrm{T}} \tilde{J} S \\ &= \tfrac{1}{2} S^{\mathrm{T}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} J S \\ &= \tfrac{1}{2} S^{\mathrm{T}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (S^{\mathrm{T}})^{-1} (S^{\mathrm{T}} J S) \\ &= \tfrac{1}{2} \rho^{-1} R^{\mathrm{T}} \cdot (\det S \cdot J) \\ &= \tfrac{1}{2} \rho^{-1} \det S (R^{\mathrm{T}} J). \end{aligned}$$

Since $\det S = -2\rho$, we see that

$$Q = -R^{\mathrm{T}} J,$$

so that $R = -JQ^{\mathrm{T}}$ is rational. □

The last result shows that the class $C(Q)$, with Q invertible and taking $h = \det G$ in (3.1), coincides with the Daniele–Khrapkov class, as was said before. If we take a matrix function G in this class and diagonalize it, $G = S^{-1} D S$, we get from $G\phi^+ = \phi^-$,

$$DS^{-1}\phi^+ = S^{-1}\phi^-,$$

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and calculating the product and quotient of the corresponding scalar equations we obtain (4.18) and (4.19) directly (cf. [4, 5]).

Remark 4.4. It is worth noting also that from (4.9) we have

$$(S_1^{\mathrm{T}})^{-1}Q_1S_1^{-1} = (S_2^{\mathrm{T}})^{-1}Q_2S_2^{-1},$$

so that

$$(S_1^{-1}S_2)^{\mathrm{T}}Q_1(S_1^{-1}S_2) = Q_2,$$

which means that $S_1^{-1}S_2 \in C(Q_1, Q_2)$. This shows, in particular, that $C(Q_1, Q_2)$ is not empty, for any invertible Q_1, Q_2 (as we mentioned in § 4), and taking theorem 3.5 and corollary 4.3 into account, we see that the general form for any $G \in C(Q_1, Q_2)$ is

$$S_1^{-1}S_2(\alpha I + \beta R_2),$$

with $R_2 = -JQ_2$.

To end this section we note that to calculate the factors of the factorization $G = G_-G_+$, once we know that the factorization is canonical, we have to solve the homogeneous Riemann–Hilbert problem in $[L_2^{\pm}(\mathbb{R})]^2$ (cf. § 2),

$$G\phi^+ = r\phi^-, \quad \phi^{\pm} \in [L_2^{\pm}(\mathbb{R})]^2 \tag{4.20}$$

for $r(\xi) = (\xi - \mathrm{i})/(\xi + \mathrm{i})$, with some normalization condition, e.g. $\phi^+(\mathrm{i}) = (1, 0)$. Apart from additional computational difficulties, the development of the calculations follow the same lines as for the equation $G\phi^+ = \phi^-$ (see § 6 for examples).

Of course, solving (4.20) gives us just one column of each factor G_+^{-1}, G_-. At first sight, according to proposition 2.2, we would have to solve a similar problem with a different normalization condition in order to obtain another linearly independent column. However, this can be avoided if we know the structure of the factors in such a way that the second column can be obtained from the first. This is what is studied next.

5. Structure of the factors

Here we show that for the class $C(Q_1, Q_2)$ the second column in G_+^{-1} and G_- can be obtained from the first one by a rational transformation. This constitutes the main result of this section.

Let $G \in C(Q_1, Q_2)$, i.e.

$$G^{\mathrm{T}}Q_1G = gQ_2, \tag{5.1}$$

where we assume that $g = \det G$ (i.e. $\det Q_1 = \det Q_2$) and g admits a canonical generalized factorization $g = g_-g_+$. Moreover, let $G = G_-G_+$ be a canonical generalized factorization for G.

Remark 5.1. Assuming that $g = \det G$, in (5.1), implies no loss of generality, as far as all the essential results are concerned. In fact, if we just impose that $g \in \mathcal{G}L_\infty(\mathbb{R})$, $g = g_-g_+$ and $\det G = d = d_-d_+$, we have

$$G^{\mathrm{T}}\tilde{Q}_1G = d\tilde{Q}_2 \tag{5.2}$$

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where $\tilde{Q}_1 = d_- g_-^{-1} Q_1$, $\tilde{Q}_2 = d_+^{-1} g_+ Q_2$. Thus

$$\tilde{Q}_1 \in (\mathcal{R}(\mathbb{R}) + L_\infty^-(\mathbb{R}))^{2\times 2}, \qquad \tilde{Q}_2 \in (\mathcal{R}(\mathbb{R}) + L_\infty^+(\mathbb{R}))^{2\times 2},$$

and it is easy to see that all the results stand true, except that Q_1 and Q_2 are replaced by $\tilde{Q}_1$ and $\tilde{Q}_2$, which are no longer rational.

Denoting by f^+ and s^+, respectively, the first and second columns in G_+^{-1}, and by f^- and s^- the first and second columns in G_-, we have

$$Gf^+ = f^-, \qquad Gs^+ = s^-. \tag{5.3}$$

From the product equation (4.3), we see then that

$$g_+(f^+)^{\mathrm{T}} Q_2 f^+ = g_-^{-1}(f^-)^{\mathrm{T}} Q_1 f^- = r_1, \tag{5.4}$$

with $r_1 \in \mathcal{R}(\mathbb{R})$. On the other hand, applying matrix transposition to the second equality in (5.3) and multiplying in the appropriate way by the first equality in (5.3), we obtain

$$(s^+)^{\mathrm{T}} G^{\mathrm{T}} Q_1 G f^+ = (s^-)^{\mathrm{T}} Q_1 f^-,$$

which, together with (5.1), yields the following cross-product equation:

$$g_+(s^+)^{\mathrm{T}} Q_2 f^+ = g_-^{-1}(s^-)^{\mathrm{T}} Q_1 f^-. \tag{5.5}$$

Since the left-hand side is meromorphic in $\mathbb{C}^+$ and the right-hand side is meromorphic in $\mathbb{C}^-$, and taking into account definition 2.1, we conclude that both sides of (5.5) must represent a rational function, so that

$$g_+(s^+)^{\mathrm{T}} Q_2 f^+ = g_-^{-1}(s^-)^{\mathrm{T}} Q_1 f^- = \tilde{r}_1, \tag{5.6}$$

with $\tilde{r}_1 \in \mathcal{R}(\mathbb{R})$.

From (5.4) and (5.6), we now obtain the structure of the factors G_+ and G_-, as stated in the following theorem.

THEOREM 5.2. *Let $G \in C(Q_1, Q_2)$ possess a canonical generalized factorization $G = G_- G_+$ and let f^+ and s^+ be the first and second columns in G_+^{-1}, respectively, and f^- and s^- be the first and second columns in G_-. Then there exists $\tilde{r}_1 \in \mathcal{R}(\mathbb{R})$ such that*

$$s^- = M_1 f^-, \qquad s^+ = M_2 f^+, \tag{5.7}$$

where

$$M_1 = r_1^{-1}(\tilde{r}_1 I - JQ_1), \qquad M_2 = r_1^{-1}(\tilde{r}_1 I - JQ_2), \tag{5.8}$$

with r_1 given by (5.4).

Proof. We have, from (5.4) and (5.6),

$$(f^+)^{\mathrm{T}} Q_2 f^+ = g_+^{-1} r_1, \tag{5.9}$$

$$(s^+)^{\mathrm{T}} Q_2 f^+ = g_+^{-1} \tilde{r}_1. \tag{5.10}$$

Multiplying both sides of (5.10) by f^+, we get

$$f^+(s^+)^{\mathrm{T}} Q_2 f^+ = g_+^{-1} \tilde{r}_1 f^+, \tag{5.11}$$

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and since

$$f^+(s^+)^{\mathrm{T}} = s^+(f^+)^{\mathrm{T}} + g_+^{-1}J,$$

we have, from (5.11),

$$s^+(f^+)^{\mathrm{T}}Q_2f^+ = g_+^{-1}\tilde{r}_1f^+ - g_+^{-1}JQ_2f^+.$$

Taking (5.9) into account, it follows that

$$s^+ = r_1^{-1}(\tilde{r}_1I - JQ_2)f^+. \tag{5.12}$$

Analogously, we obtain

$$s^- = r_1^{-1}(\tilde{r}_1I - JQ_1)f^-. \tag{5.13}$$

□

Corollary 5.3. *With the same assumptions as in theorem 5.2, if $Q_1 = Q_2 = Q$ (i.e. $G \in C(Q)$), then $M_1 = M_2 \in C(Q)$.*

Proof. It is enough to see that $M = \tilde{r}_1I - JQ$ satisfies the relation

$$M^{\mathrm{T}}QM = (\tilde{r}_1^2 + \det Q)Q.$$

□

Remark 5.4. As is clear from the proof of theorem 5.2, r_1 is explicitly obtained from the product equation (5.4) and depends only on the first columns f^+, f^-.

We deal with the determination of $\tilde{r}_1$ in the results that follow.

Theorem 5.5. *Let G, f^+ and f^- satisfy the assumptions of theorem 5.2 and let r_1 be defined by (5.4). Then, for any $\tilde{r}_1 \in \mathcal{R}(\mathbb{R})$, the functions s^+, s^- defined by (5.12) and (5.13) satisfy the Riemann–Hilbert problem $Gs^+ = s^-$, as well as the relations*

$$(f^+)^{\mathrm{T}}Js^+ = g_+^{-1}, \qquad (f^-)^{\mathrm{T}}Js^- = g_-. \tag{5.14}$$

Proof. We start by showing that, for G satisfying (5.1), the following relation holds:

$$GJQ_2 = JQ_1G. \tag{5.15}$$

In fact,

$$Q_1G = g(G^{-1})^{\mathrm{T}}Q_2. \tag{5.16}$$

Since

$$GJG^{\mathrm{T}} = \det(G)J = gJ,$$

we have $G^{-1} = -JG^{\mathrm{T}}J/\det G$ and therefore, from (5.16),

$$JQ_1G = gJ(-g^{-1}JGJ)Q_2 = GJQ_2.$$

Now,

$$\begin{aligned} Gs^+ &= G[r_1^{-1}(\tilde{r}_1I - JQ_2)]f^+ \\ &= r_1^{-1}(\tilde{r}_1Gf^+ - GJQ_2f^+) \end{aligned}$$

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but, taking the equality $Gf^+ = f^-$ and (5.15) into account, we see that this is equivalent to

$$Gs^+ = r_1^{-1}(\tilde{r}_1 I - JQ_1)f^- = s^-.$$

As for the first equality in (5.14), we have

$$\begin{aligned}(f^+)^{\mathrm{T}} Js^+ &= (f^+)^{\mathrm{T}} Jr_1^{-1}(\tilde{r}_1 I - JQ_2)f^+ \\ &= r_1^{-1}(\tilde{r}_1 (f^+)^{\mathrm{T}} Jf^+ + (f^+)^{\mathrm{T}} Q_2 f^+) \\ &= g_+^{-1},\end{aligned}$$

where we took into account that $(f^+)^{\mathrm{T}} Jf^+ = 0$ and $(f^+)^{\mathrm{T}} Q_2 f^+ = r_1 g_+^{-1}$, according to (5.9). The second equality in (5.14) is proved analogously. □

As an immediate consequence of the last theorem, we give in corollary 5.6 below a criterion to determine $\tilde{r}_1$ such that s^+, s^-, defined by the expressions (5.12) and (5.13), respectively, can be taken as the second columns in G_+^{-1} and G_-. In fact, it shows that s_+, s_-, defined in that way, satisfy the equality $Gs_+ = s_-$ and are linearly independent from f_+, f_- (respectively) in the corresponding half-planes, independently of the choice of $\tilde{r}_1$. This leads to the conclusion that the only condition to impose on $\tilde{r}_1$ is that it must be such that $s^\pm$ are analytic in $\mathbb{C}^\pm$ and $r_\pm s_\pm \in (L_2^\pm(\mathbb{R}))^2$.

Corollary 5.6. *If the assumptions of theorem 5.5 are satisfied, a canonical generalized factorization for* G *is* $G = G_- G_+$, *where* $G_- = [f^- s^-]$, $G_+^{-1} = [f^+ s^+]$, *with* s^+, s^- *given by (5.12) and (5.13) for any rational function* $\tilde{r}_1$ *such that* $r_\pm s^\pm \in (L_2^\pm(\mathbb{R}))^2$.

The explicit calculation of $\tilde{r}_1$ is addressed in the following two theorems, concerning matrix functions $G \in C(Q)$. In both cases, we assume that the assumptions of theorem 5.5 are satisfied and $\tilde{r}_1$ has the same meaning as in corollary 5.6.

Theorem 5.7. *Let* $G \in C(Q)$ *admit a canonical generalized factorization, with* $Q = \operatorname{diag}(1, -q)$, $q \in \mathcal{R}(\mathbb{R})$. *If* $q = p_1/p_2$, *where* p_1 *and* p_2 *are polynomials with no common zeros, such that* $\deg(p_1) \leqslant 1$, $\deg(p_2) = 1$, *we have*

$$r_1 = K_1 \in \mathbb{C} \setminus \{0\}, \qquad \tilde{r}_1 = K_2 \in \mathbb{C}$$

for appropriate normalization conditions on $f^\pm = (f_1^\pm, f_2^\pm)$.

Proof. Assume that p_2 has a zero $a_- \in \mathbb{C}^-$ and choose for (f^+, f^-) the normalization condition

$$f_2^-(a_-) = 0, \qquad f_1^-(a_-) = K_0 \neq 0. \tag{5.17}$$

The product equation (4.3) now takes the form

$$g_+[(f_1^+)^2 - q(f_2^+)^2] = g_-^{-1}[(f_1^-)^2 - q(f_2^-)^2] \tag{5.18}$$

and the normalization condition (5.17) implies that both sides of (5.18) represent analytic functions in the corresponding half-plane. Moreover, $r_\pm f^\pm \in (L_2^\pm(\mathbb{R}))^2$, so that the left-hand side represents a function F such that $r_+^2 F$ is in $L_1^+(\mathbb{R})$, and analogously for the right-hand side. Therefore, both sides are equal to a constant and thus $r_1 = K_1 \in \mathbb{C}$. It is clear, on the other hand, that K_1 must be different from zero, otherwise we would have $f^+ = f^- = 0$.

As for $\tilde{r}_1$, according to corollary 5.6, it must be such that

$$[r_\pm K_1^{-1}(\tilde{r}_1 I - JQ)f^\pm] \in (L_2^\pm(\mathbb{R}))^2 \tag{5.19}$$

and it is obvious, taking (5.17) into account, that (5.19) is satisfied for $\tilde{r}_1 = K_2 \in \mathbb{C}$.

We would proceed analogously if p_2 had a zero $a_+ \in \mathbb{C}^+$, choosing the normalization condition $f_1^+(a_+) = K_0 \neq 0$. □

Corollary 5.8. *With the same assumptions as in theorem 5.7, the factors G_-, G_+ in the canonical factorization of G belong to $C(Q)$ for appropriate normalization conditions on $f^\pm$.*

Proof. This is as immediate consequence of the previous result, since we can choose $\tilde{r}_1 = 0$. □

In the last two results, a particular form for Q was considered, which corresponds to a case that has drawn considerable attention in the literature (see [4, 5, 8, 10]). This corresponds, apart from a scalar factor, to taking $q_0 = 1$, $q_1 = 0$ in

$$Q = \begin{bmatrix} q_0 & q_1 \\ q_1 & q_2 \end{bmatrix} \tag{5.20}$$

(see § 4). A case that corresponds to $q_1 \neq 0$ is considered in our second example in § 6.

Next we consider the case where q is a quotient of two second-degree polynomials.

Theorem 5.9. *Let G satisfy the same assumptions as in theorem 5.7. If $q = p_1/p_2$, where p_1 and p_2 are polynomials without common zeros, such that $\deg(p_1) \leqslant 2$, $p_2(\xi) = (\xi - \alpha_1)(\xi - \alpha_2)$, with $\alpha_1 \neq \alpha_2$, $\alpha_1, \alpha_2 \notin \mathbb{R}$, r_1 takes the form*

$$r_1(\xi) = \frac{\alpha\xi + \beta}{\xi - \alpha_2}, \quad \alpha, \beta \in \mathbb{C},$$

for appropriate conditions on $f^\pm$ and $\tilde{r}_1$ is a constant.

Proof. Let us assume, for simplicity, that $\alpha_1, \alpha_2 \in \mathbb{C}^-$, i.e. $p_2(\xi) = (\xi - a_-)(\xi - b_-)$, with $a_-, b_- \in \mathbb{C}^-$ and $a_- \neq b_-$. Let us choose the following normalization condition:

$$f_2^-(a_-) = 0, \qquad f_1^-(a_-) = K_0 \neq 0. \tag{5.21}$$

The product equation (4.3) takes the form

$$g_+[(f_1^+)^2 - q(f_2^+)^2] = g_-^{-1}[(f_1^-)^2 - q(f_2^-)^2], \tag{5.22}$$

and the normalization condition (5.21) implies that, while the left-hand side of this equality represents a function analytic in $\mathbb{C}^+$, the right-hand side may have a pole for $\xi = b_-$. This is a situation different from the one we had in the proof of theorem 5.7 (see (5.18)) but following the same reasoning we conclude that both sides of (5.22) must represent a rational function r_1 of the form

$$r_1(\xi) = \frac{\alpha\xi + \beta}{\xi - b_-} = \frac{\alpha\xi + \beta}{\xi - b_-}, \quad \alpha, \beta \in \mathbb{C}. \tag{5.23}$$

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Let $\alpha\xi + \beta = \alpha(\xi - z_0)$, with $\alpha \neq 0$. We can assume that $\alpha = 1$. We then have

$$g_+[(f_1^+)^2 - q(f_2^+)^2] = g_-^{-1}[(f_1^-)^2 - q(f_2^-)^2] = \frac{(\xi - z_0)}{\xi - b_-}. \tag{5.24}$$

It is clear from this relation that $z_0 = b_-$ if and only if $f_2^-(b_-) = 0$.

Considering now $\tilde{r}_1$, we have

$$s_\pm = \frac{\xi - b_-}{\xi - z_0}\begin{bmatrix} \tilde{r}_1 f_1^\pm & +qf_2^\pm \\ f_1^\pm + \tilde{r}_1 f_2^\pm & \end{bmatrix}, \tag{5.25}$$

and thus $x_\pm$ is analytic in $\mathbb{C}^\pm$ if $\tilde{r}_1$ is such that the pole for $\xi = z_0$ is compensated and no other singularity is introduced.

If $z_0 = b_-$, in which case $f_2^-(z_0) = f_2^-(b_-) = 0$, it is clear that $s^\pm$ is analytic in $\mathbb{C}^\pm$ for any constant $\tilde{r}_1$. In particular, we can choose $\tilde{r}_1 = 0$ and obtain

$$s_\pm = \begin{bmatrix} qf_2^\pm \\ f_1^\pm \end{bmatrix}, \tag{5.26}$$

which means that $G_\pm \in C(Q)$.

If $z_0 \neq b_-$, $z_0 \in \mathbb{C}^+$, for instance, $\tilde{r}_1$ must be such that

$$\left.\begin{aligned} (\tilde{r}_1 f_1^+ + qf_2^+)(z_0) = 0, \\ (f_1^+ + \tilde{r}_1 f_2^+)(z_0) = 0. \end{aligned}\right\} \tag{5.27}$$

We remark here that (5.24) implies that

$$\begin{vmatrix} f_1^+ & qf_2^+ \\ f_2^+ & f_1^+ \end{vmatrix}(z_0) = 0, \tag{5.28}$$

and therefore $(f_1^+, f_2^+) = \lambda_0(qf_2^+, f_1^+)$ for some constant $\lambda_0 \in \mathbb{C} \setminus \{0\}$. So (5.27) is equivalent to

$$f_1^+(z_0) + \tilde{r}_1(z_0) f_2^+(z_0) = 0$$

and we can take

$$\tilde{r}_1 = \tilde{r}_1(z_0) = -\frac{f_1^+(z_0)}{f_2^+(z_0)}. \tag{5.29}$$

This is well defined, since $f_2^+(z_0) \neq 0$, as we mentioned concerning (5.24) (otherwise (5.28) would imply that $f_1^+(z_0)$ was also equal to zero, which is impossible since f^+ is the first column in G_+^{-1}).

We would have, analogously, $\tilde{r}_1 = -f_1^-(z_0)/f_2^-(z_0)$ if $z_0 \in \mathbb{C}^-$, $z_0 \neq b_-$.

The same reasoning applies if $z_0 \in \mathbb{R}$, and in this case we have

$$\tilde{r}_1 = -\frac{f_1^+(z_0)}{f_2^+(z_0)} = -\frac{f_1^-(z_0)}{f_2^-(z_0)}.$$

As for this last equality, it is clear from (5.24) that

$$\left(\frac{f_1^+(z_0)}{f_2^+(z_0)}\right)^2 = \left(\frac{f_1^-(z_0)}{f_2^-(z_0)}\right)^2 = q(z_0).$$

On the other hand, since $Gf^+ = f^-$, we have

$$\left.\begin{aligned} d_{1+}(f_1^+ + \rho f_2^+) &= d_{1-}^{-1}(f_1^- + \rho f_2^-), \\ d_{1+}(f_1^+ - \rho f_2^+) &= d_{2-}^{-1}(f_1^- - \rho f_2^-), \end{aligned}\right\} \tag{5.30}$$

with $\rho = q^{1/2}$. If $f_1^+(z_0)/f_2^+(z_0) = \rho(z_0)$, it follows from the second equation in (5.30) that we also have $f_1^-(z_0) = \rho(z_0)$ and, analogously, if $f_1^+(z_0)/f_2^+(z_0) = -\rho(z_0)$, it follows from the first equation in (5.30) that $f_1(z_0)/f_2^-(z_0) = -\rho(z_0)$, so that, in fact, $f_1^+(z_0)/f_2^+(z_0) = f_1^-(z_0)/f_2^-(z_0)$.

Finally, if $\alpha = 0$, we have $r_1^{-1} = \beta^{-1}(\xi - b_-)$ and we see that $\tilde{r}_1$ must be defined, taking into account the behaviour of $s^\pm$ at ∞. Thus we have

$$\tilde{r}_1 = -\frac{f_1^+(\infty)}{f_2^+(\infty)} = -\frac{f_1^-(\infty)}{f_2^-(\infty)}.$$

□

REMARK 5.10. We see from the proof of theorem 5.9 that $\tilde{r}_1 = K_0 \in \mathbb{C}$, where

$$K_0 = \begin{cases} 0 & \text{if } \alpha\xi + \beta = \alpha(\xi - \alpha_2), \quad \text{with } \alpha \neq 0 \\ & \quad \text{(in which case we have} \\ & \quad\quad \text{a factorization in } C(Q)), \\ -\dfrac{f_1^\pm(z_0)}{f_2^\pm(z_0)} & \text{if } \alpha\xi + \beta = \alpha(\xi - z_0), \\ & \quad \text{with } z_0 \in \mathbb{C}^\pm \setminus \{\alpha_2\}, \quad \alpha \neq 0, \\ -\dfrac{f_1^+(z_0)}{f_2^+(z_0)} = -\dfrac{f_1^-(z_0)}{f_2^-(z_0)} & \text{if } \alpha\xi + \beta = \alpha(\xi - z_0), \\ & \quad \text{with } z_0 \in \mathbb{R}, \quad \alpha \neq 0, \\ -\dfrac{f_1^+(\infty)}{f_2^+(\infty)} = -\dfrac{f_1^-(\infty)}{f_2^-(\infty)} & \text{if } \alpha = 0. \end{cases}$$

REMARK 5.11. Comparing the results of theorems 5.7 and 5.9, we see that in the first case we can always obtain a canonical generalized factorization within the same class $C(Q)$, while this does not happen in general in the second case. However, we can always obtain a meromorphic factorization for G (see [3]) within the same class. In fact, $[f^\pm s^\pm]$ can be written in the form

$$[f^\pm s^\pm] = \begin{bmatrix} f_1^\pm & r_1^{-1}(\tilde{r}_1 f_1^\pm + q f_2^\pm) \\ f_2^\pm & r_1^{-1}(f_1^\pm + \tilde{r}_1 f_2^\pm) \end{bmatrix} = \begin{bmatrix} f_1^\pm & q f_2^\pm \\ f_2^\pm & f_1^\pm \end{bmatrix} \begin{bmatrix} 1 & r_1^{-1}\tilde{r}_1 \\ 0 & r_1^{-1} \end{bmatrix},$$

and taking $M^\pm$ equal to the first factor on the right-hand side of this equality, we see that $G = M_- M_+^{-1}$ is a meromorphic factorization with factors belonging to $C(Q)$.

In the case considered in corollary 5.3, the following question naturally arises: when can we obtain a canonical factorization within the same group, i.e. $G = G_- G_+$ with $G_\pm \in C(Q)$?

Let $G_- = [f^- s^-]$, with s^- related to f^- by

$$s^- = r_1^{-1}[\tilde{r}_1 I - JQ]f^-. \tag{5.31}$$

We have

$$G_-^{\mathrm{T}} Q G_- = \begin{bmatrix} (f^-)^{\mathrm{T}} \\ (s^-)^{\mathrm{T}} \end{bmatrix} Q[f^- s^-] = \begin{bmatrix} (f^-)^{\mathrm{T}} Q f^- & (f^-)^{\mathrm{T}} Q s^- \\ (s^-)^{\mathrm{T}} Q f^- & (s^-)^{\mathrm{T}} Q s^- \end{bmatrix}. \tag{5.32}$$

Since

$$(f^-)^{\mathrm{T}} Q f^- = g_- r_1,$$
$$(f^-)^{\mathrm{T}} Q s^- = g_- \tilde{r}_1 = (s^-)^{\mathrm{T}} Q f^-$$

and, taking (5.31) into account,

$$\begin{aligned} (s^-)^{\mathrm{T}} Q s^- &= r_1^{-2}[(\tilde{r}_1)^2 (f^-)^{\mathrm{T}} Q f^- - (f^-)^{\mathrm{T}} QJQJQ f^-] \\ &= r_1^{-2}[(\tilde{r}_1)^2 g_- r_1 + \det Q (f^-)^{\mathrm{T}} Q f^-] \\ &= r_1^{-1} g_- [(\tilde{r}_1)^2 + \det Q], \end{aligned}$$

we see from (5.32) that $G_-^{\mathrm{T}} Q G_- = \gamma Q$, with $\gamma \in L_\infty(\mathbb{R})$, if and only if

$$g_- r_1 \begin{bmatrix} 1 & r_1^{-1}\tilde{r}_1 \\ r_1^{-1}\tilde{r}_1 & r_1^{-2}(\tilde{r}_1^2 + \det Q) \end{bmatrix} = \gamma Q = \gamma \begin{bmatrix} 1 & q_1 \\ q_1 & q_1^2 + \det Q \end{bmatrix}$$

(see (4.5)). It is clear that we must have $r_1^{-2} = 1$, which means that in the product equation (4.3) both sides are equal to a constant. It is also easy to see that the converse is also true. We thus have the following result.

Corollary 5.12. *With the same assumptions as in corollary 5.3, G admits a canonical factorization with $G_\pm \in C(Q)$ if and only if both sides of the product equation (4.3) are equal to a constant.*

6. Examples

In this section we consider two examples of matrix symbols of the class $C(Q)$ defined in § 3. These examples were chosen having in mind illustrating the main difficulties in the application of the method presented in § 4 and showing how to overcome such difficulties, but limiting the computation complexity to a minimum. Some motivation from applications is, however, behind our choice of examples: the first one corresponds to a subclass of the Daniele–Khrapkov class, which, to the authors' knowledge, is not dealt with in the literature, and the second one illustrates some of the features of the case where the rational function q in (3.2) is a quotient of second-degree polynomials without, however, getting into the computational difficulties that occur in the treatment of the general case [4, 5].

Let us now consider our first example. This corresponds to a symbol $G \in C(Q)$ of the form

$$G = \begin{bmatrix} 1 & g \\ qg & 1 \end{bmatrix}, \tag{6.1}$$

where $g \in C_\mu(\dot{\mathbb{R}})$ and $q(\xi) = 1/(\xi + \mathrm{i})$. In this case, $Q = \mathrm{diag}(-q, 1)$. We wish to determine conditions (if possible, necessary and sufficient) under which G possesses a canonical factorization. When this factorizations exists, we calculate the corresponding factors. The main results are stated in the following theorem.

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THEOREM 6.1. *Let G be given by (6.1), with $g \in C_\mu(\dot{\mathbb{R}})$, $\det G \neq 0$ in $\dot{\mathbb{R}}$, $\operatorname{ind}\det G = 0$ and assume that $d_1 = 1+\rho g$, $d_2 = 1-\rho g$, with $\rho(\xi) = (\xi+\mathrm{i})^{-1/2}$. Then G possesses a canonical factorization with factors in the same class $C(Q)$, $G = G_- G_+$, where the factors are given by*

$$G_+^{-1} = \begin{bmatrix} g_2^+ & g_1^+ \\ qg_1^+ & g_2^+ \end{bmatrix}, \qquad G_- = \begin{bmatrix} g_2^- & g_1^- \\ qg_1^- & g_2^- \end{bmatrix}, \tag{6.2}$$

with

$$\left.\begin{aligned} g_1^+ &= d_+^{-1/2}\rho^{-1}\operatorname{sh}(\tfrac{1}{2}\rho^{-1}F^+), \\ g_2^+ &= d_+^{-1/2}\operatorname{ch}(\tfrac{1}{2}\rho^{-1}F^+), \\ g_1^- &= d_-^{1/2}\operatorname{sh}(\tfrac{1}{2}\rho^{-1}F^-), \\ g_2^- &= d_-^{1/2}\operatorname{ch}(\tfrac{1}{2}\rho^{-1}F^-), \end{aligned}\right\} \tag{6.3}$$

where $F^\pm \in C_\mu^\pm(\dot{\mathbb{R}})$ are such that, for $F = \rho\log(d_1/d_2)$, we have $F = F_- + F_+$.

Proof. Firstly, we prove that the factorization is canonical. Since G is continuous on $\dot{\mathbb{R}}$ and $\det G = 1 - qg^2$ possesses a canonical factorization, the operator

$$P^+GI_+ : [L_2^+(\mathbb{R})]^2 \to [L_2^+(\mathbb{R})]^2$$

is Fredholm of index zero. Hence it is invertible if and only if it is injective, i.e. G possesses a canonical factorization if and only if the Riemann–Hilbert problem in $[L_2^\pm(\mathbb{R})]^2$,

$$G\phi^+ = \phi^-, \tag{6.4}$$

has only the trivial solution. To answer this question, we begin with the product equation (4.2),

$$(1 - qg^2)[-q(\phi_1^+)^2 + (\phi_2^+)^2] = -q(\phi_1^-)^2 + (\phi_2^-)^2, \tag{6.5}$$

where $(\phi_1^\pm, \phi_2^\pm) = \phi^\pm$. Since $1 - qg^2$ possesses a canonical factorization, we write $1 - qg^2 = d_- d_+$. Hence, from (6.5), we get

$$d_+[-q(\phi_1^+)^2 + (\phi_2^+)^2] = d_-^{-1}[-q(\phi_1^-)^2 + (\phi_2^-)^2]. \tag{6.6}$$

In (6.6), the left-hand side is analytic in $\mathbb{C}^+$ and the right-hand side is analytic in $\mathbb{C}^-$, except for a pole at $\xi = -\mathrm{i}$. On the other hand, $\phi_1^\pm, \phi_2^\pm \in L_2(\mathbb{R})$, which implies that the two sides in (6.6) are equal to an L_1 rational function $\gamma q (\gamma \in \mathbb{C})$. However, this does not belong to $L_1(\mathbb{R})$ unless $\gamma = 0$. Thus

$$-q(\phi_1^-)^2 + (\phi_2^-)^2 = 0 \quad \Rightarrow \quad \phi_2^- = \pm\rho\phi_1^-$$

for $\rho = q^{1/2}$, and the last equality implies $\phi_1^- = \phi_2^- = 0$. Hence the factorization is canonical.

We now proceed to calculate the factors G_+^{-1}, G_-. As pointed out in § 4 (cf. (4.20)), the first columns $f^\pm$ in these factors can be determined from a solution to the Riemann–Hilbert problem,

$$G\phi^+ = r\phi^-, \tag{6.7}$$

where $r(\xi) = (\xi - \mathrm{i})/(\xi + \mathrm{i})$, with a normalization condition on ϕ^+ or ϕ^-. The product equation corresponding to (6.7) now takes the form

$$d_+[-q(\phi_1^+)^2 + (\phi_2^+)^2] = r^2 d_-^{-1}[-q(\phi_1^-)^2 + (\phi_2^-)^2]. \tag{6.8}$$

Let us look for a solution to (6.7) satisfying the normalization condition $\phi^-(-\mathrm{i}) = (0, K_0)$, $K_0 \neq 0$. In this case, a similar argument to the one used above tells us that both sides of (6.8) are equal to a rational function (say, q_1), now with a second-order pole at $\xi = -\mathrm{i}$. The L_1 argument gives

$$q_1(\xi) = \frac{\gamma}{(\xi + \mathrm{i})^2}, \quad \gamma \in \mathbb{C}.$$

To determine the functions $\phi_1^\pm$, $\phi_2^\pm$, we have to resort to the quotient equation (4.19). This leads to

$$\frac{d_1}{d_2}\frac{\phi_2^+ + \rho\phi_1^+}{\phi_2^+ - \rho\phi_1^+} = \frac{\phi_2^- + \rho\phi_1^-}{\phi_2^- - \rho\phi_1^-}, \tag{6.9}$$

where $d_1 = 1 - \rho g$, $d_2 = 1 + \rho g$. Note that ρ is a function analytic in $\mathbb{C}^+$, but not in $\mathbb{C}^-$. To solve the scalar Riemann–Hilbert problem (6.9), we apply the logarithm function to both sides of (6.9) and multiply by ρ, obtaining

$$\rho \log\frac{d_1}{d_2} + \rho\log\biggl(\frac{\phi_2^+ + \rho\phi_1^+}{\phi_2^+ - \rho\phi_1^+}\biggr) = \rho\log\biggl(\frac{\phi_2^- + \rho\phi_1^-}{\phi_2^- - \rho\phi_1^-}\biggr). \tag{6.10}$$

In (6.10), we can see that the second term in the left-hand side and the right-hand side are analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively.

Note that, since both sides in (6.8) equal q_1, we have

$$-q(\phi_1^+)^2 + (\phi_2^+)^2 = \frac{\gamma d_+^{-1}}{(\xi + \mathrm{i})^2}, \qquad -q(\phi_1^-)^2 + (\phi_2^-)^2 = \frac{\gamma d_-}{(\xi - \mathrm{i})^2}, \tag{6.11}$$

i.e. the functions in the arguments of the logarithm in (6.10) have neither zeros nor poles in their half-planes of analyticity. On the other hand, a series expansion argument easily shows that the function on the right-hand side of (6.9) is analytic at $\xi = -\mathrm{i}$.

Keeping this in mind, we may decompose $F = \rho\log(d_1/d_2)$ in (6.10) as $F = F^+ + F^-$, with $F^\pm \in P^\pm[C_{\mu,0}(\dot{\mathbb{R}})]$ (here, $C_{\mu,0}(\dot{\mathbb{R}})$ denotes the subspace of $C_\mu(\dot{\mathbb{R}})$ whose elements vanish at ∞) and get

$$\rho\log\frac{\phi_2^\pm + \rho\phi_1^\pm}{\phi_2^\pm - \rho\phi_1^\pm} = \mp F^\pm$$

(for the details, see [2]). From this result and (6.11), we obtain

$$f_2^+ = r_+^{-1}\phi_2^+ = -\gamma^{1/2} d_+^{-1/2}\,\mathrm{ch}(\tfrac{1}{2}\rho^{-1}F^+), \tag{6.12}$$

$$f_1^+ = r_+^{-1}\phi_1^+ = \gamma^{1/2} d_+^{-1/2}\rho^{-1}\,\mathrm{sh}(\tfrac{1}{2}\rho^{-1}F^+), \tag{6.13}$$

$$f_2^- = r_-^{-1}\phi_2^- = \gamma^{1/2} d_-^{1/2}\,\mathrm{ch}(\tfrac{1}{2}\rho^{-1}F^-), \tag{6.14}$$

$$f_1^- = r_-^{-1}\phi_1^- = \gamma^{1/2} d_-^{1/2}\rho^{-1}\,\mathrm{sh}(\tfrac{1}{2}\rho^{-1}F^-). \tag{6.15}$$

These results correspond to one of the columns in G_+^{-1} and G_-, which we choose to be the second one. The first column can be obtained from theorem 5.7 (using $Q = \mathrm{diag}(-q, 1)$ instead of $Q = \mathrm{diag}(1, -q)$). So we have a factorization $G = G_-G_+$, where G_-, G_+ are given by (6.2), (6.3) and belong to the same class as G. □

We now consider our second example. Let $G \in C(Q)$ be given by

$$G = \begin{bmatrix} 1 & \rho g \\ \rho^{-1} g & 1 \end{bmatrix}, \tag{6.16}$$

where $g \in C_\mu(\dot{\mathbb{R}})$ and

$$\rho(\xi) = \frac{(\xi - \mathrm{i})}{[(\xi + \mathrm{i})(\xi + 2\mathrm{i})]^{1/2}}. \tag{6.17}$$

This is what we call a degenerate second-degree case, since $Q = \mathrm{diag}(1, -q)$, where $q = \rho^2$ is the quotient of second-degree polynomials, but ρ has only two branch points, whereas there are four in the general case. The study of this class carries some of the features of the second-degree case treated in [4, 5], but without many of its computational difficulties.

Let $d_1 = 1 - g$ and $d_2 = 1 + g$ admit canonical generalized factorizations $d_1 = d_{1-}d_{1+}$, $d_2 = d_{2-}d_{2+}$ and let $d = \det G = d_1 d_2$. The results concerning existence of canonical factorization are stated in the next theorem.

THEOREM 6.2. *Let G be given by (6.16), with $g \in C_\mu(\dot{\mathbb{R}})$ and ρ given by (6.17). Then if* $\mathrm{ind}(1 - g^2) = 0$, *$G$ possesses a canonical generalized factorization if and only if*

$$\frac{1}{\pi \mathrm{i}} \int_{\mathbb{R}} \frac{\rho_+(t)}{t - \mathrm{i}} \log\left(-\frac{1 - g(t)}{1 + g(t)}\right) \mathrm{d}t \neq 0,$$

where $\rho_+(t) = [(t + \mathrm{i})(t + 2\mathrm{i})]^{-1/2}$.

Proof. We know that G possesses a canonical factorization if and only if the Riemann–Hilbert problem in $[L_2^\pm(\mathbb{R})]^2$,

$$G\phi^+ = \phi^-, \tag{6.18}$$

has only the trivial solution. As in the proof of theorem 6.1, we start by writing the product equation,

$$(1 - g^2)[(\phi_1^+)^2 - \rho^2(\phi_2^+)^2] = (\phi_1^-)^2 - \rho^2(\phi_2^-)^2. \tag{6.19}$$

Since $\mathrm{ind}(1 - g^2) = 0$, we write $d = 1 - g^2 = d_- d_+$ and using this in (6.19), we have

$$d_+[(\phi_1^+)^2 - \rho^2(\phi_2^+)^2] = d_-^{-1}[(\phi_1^-)^2 - \rho^2(\phi_2^-)^2]. \tag{6.20}$$

By the same reasoning of the previous proposition, both sides of (6.20) are equal to a rational function, say q_1, with two poles at $\xi = -\mathrm{i}$ and $\xi = -2\mathrm{i}$ and, bearing in mind that $q_1 \in L_1(\mathbb{R})$, we have

$$q_1(\xi) = \frac{\gamma}{(\xi + \mathrm{i})(\xi + 2\mathrm{i})}, \quad \gamma \in \mathbb{C}. \tag{6.21}$$

Contrary to what happened in the previous example regarding the Riemann–Hilbert problem (6.18), we cannot conclude from (6.21) that the factorization is canonical, since γ may be different from zero. Thus we have to pass on to the quotient equation. Let us assume that $\gamma \neq 0$. Then we see from (6.20), (6.21) that

$$(\phi_1^+)^2 - \rho^2(\phi_2^+)^2 = \frac{\gamma d_+^{-1}}{(\xi + \mathrm{i})(\xi + 2\mathrm{i})}, \tag{6.22}$$

$$(\phi_2^-)^2 - \rho^{-2}(\phi_1^-)^2 = \frac{\gamma d_-}{(\xi - \mathrm{i})^2}, \tag{6.23}$$

so that the left-hand sides of (6.22) and (6.23) do not vanish in $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively, and

$$\phi_1^+(\mathrm{i}) \neq 0, \qquad \phi_2^-(-\mathrm{i}) \neq 0, \qquad \phi_2^-(-2\mathrm{i}) \neq 0.$$

Using again the procedure of § 4, we arrive at the equation

$$-\frac{1-g}{1+g} \cdot \frac{\phi_1^+ - \rho\phi_2^+}{\phi_1^+ + \rho\phi_2^+} = \frac{\phi_2^- - \rho^{-1}\phi_1^-}{\phi_2^- + \rho^{-1}\phi_1^-}. \tag{6.24}$$

Similarly to the example of theorem 6.1, the functions

$$\psi^+ = \rho_+ \log\frac{\phi_1^+ - \rho\phi_2^+}{\phi_1^+ + \rho\phi_2^+}, \qquad \psi^- = \rho_+ \log\frac{\phi_2^- - \rho^{-1}\phi_1^-}{\phi_2^- + \rho^{-1}\phi_1^-}, \tag{6.25}$$

with $\rho_+(\xi) = [(\xi+\mathrm{i})(\xi+2\mathrm{i})]^{-1/2}$, are analytic in the half-planes of analyticity of $\phi_{1,2}^+$, $\phi_{1,2}^-$, respectively. In fact, $\psi^\pm \in L_2^\pm(\mathbb{R})$ (cf. [4]). Hence we can obtain from (6.24) a scalar Riemann–Hilbert problem, which we proceed to solve.

We have

$$\rho_+ \log\left(-\frac{1-g}{1+g}\right) + \psi^+ = \psi^-, \tag{6.26}$$

and therefore

$$\psi^+ = -F^+, \quad \psi^- = F^-, \quad \text{with } F^\pm = P^\pm\left(\rho_+ \log\left(-\frac{1-g}{1+g}\right)\right). \tag{6.27}$$

It is clear from (6.25) that $\psi^+(i) = 0$, which implies that

$$F^+(\mathrm{i}) = 0. \tag{6.28}$$

This is therefore a necessary condition for having $\gamma \neq 0$ in (6.21). Otherwise, both sides of (6.20) are equal to zero and it follows that $G\phi^+ = \phi^-$ admits only the trivial solution $\phi^+ = \phi^- = 0$.

To prove that (6.28) is also a sufficient condition for existence of non-trivial solutions to (6.18), we now calculate these solutions. From (6.25) and (6.27), we get

$$\frac{\phi_1^+ - \rho\phi_2^+}{\phi_1^+ + \rho\phi_2^+} = \exp(-\rho_+^{-1}F^+), \qquad \frac{\phi_2^- - \rho^{-1}\phi_1^-}{\phi_2^- + \rho^{-1}\phi_1^-} = \exp(\rho_+^{-1}F^-),$$

which, together with the product equation (6.20) and taking (6.21) into account, yields, apart from a multiplicative constant,

$$\phi_1^+ = d_+^{-1/2}\rho_+ \operatorname{ch}(\tfrac{1}{2}\rho_+^{-1}F^+), \tag{6.29}$$

$$\phi_2^+ = d_+^{-1/2}r_- \operatorname{sh}(\tfrac{1}{2}\rho_+^{-1}F^+), \tag{6.30}$$

$$\phi_2^- = d_-^{1/2}r_- \operatorname{ch}(\tfrac{1}{2}\rho_+^{-1}F^-), \tag{6.31}$$

$$\phi_1^- = -d_-^{1/2}r_-\rho^{-1} \operatorname{sh}(\tfrac{1}{2}\rho_+^{-1}F^-). \tag{6.32}$$

We remark that the right behaviour for ϕ_2^+ in the neighbourhood of $\xi = \mathrm{i}$ is guaranteed by (6.28). □

Corollary 6.3. *Let G satisfy the assumptions of theorem 6.2 and, moreover, let*

$$\frac{1}{\pi\mathrm{i}}\int_{\mathbb{R}} \frac{\rho_+(t)}{t-\mathrm{i}} \log\left(-\frac{1-g(t)}{1+g(t)}\right) \mathrm{d}t = 0. \tag{6.33}$$

Then, apart from a multiplicative constant, the solutions to $G\phi^+ = \phi^-$ are given by (6.29)–(6.32).

Let us assume now that the condition for existence of a canonical factorization for G is satisfied. To complete the study of the factorization problem, it remains to calculate the factors, which we propose to do next.

Following the technique used in the proof of theorem 6.1, we calculate a column of the factors G_+^{-1} and G_- by solving the vector Riemann–Hilbert problem (6.7). For this problem, which can be put in the form

$$DS^{-1}\phi^+ = rS^{-1}\phi^-, \tag{6.34}$$

since $G = SDS^{-1}$ as in the proof of corollary 4.3, the product equation is

$$(1-g^2)[(\phi_1^+)^2 - \rho^2(\phi_2^+)^2] = r^2[(\phi_1^-)^2 - \rho^2(\phi_2^-)^2],$$

and using the factorization $1 - g^2 = d_-d_+$, we obtain

$$d_+[(\phi_1^+)^2 - \rho^2(\phi_2^+)^2] = r^2 d_-^{-1}[(\phi_1^-)^2 - \rho^2(\phi_2^-)^2] = q_1, \tag{6.35}$$

where q_1 is a rational function. To determine q_1, we choose the normalization conditions (see 5.21),

$$\phi_1^-(-\mathrm{i}) = K_1 \neq 0, \qquad \phi_2^-(-\mathrm{i}) = 0. \tag{6.36}$$

It is now clear from these conditions, and the fact that both sides of (6.34) belong to $L_1(\mathbb{R})$, that q_1 must have the form

$$q_1(\xi) = \frac{\alpha\xi + \beta}{(\xi+\mathrm{i})^2(\xi+2\mathrm{i})}, \tag{6.37}$$

where $\alpha, \beta \in \mathbb{C}$ are constants to be determined later.

We now consider the quotient equation. This equation coincides, actually, with (6.24), because r cancels out when we take the quotient of the two scalar equations corresponding to (6.7). The difference to the corresponding equation encountered in theorem 6.2 is that, because q_1 in (6.37) may have a zero at $\xi = -\beta/\alpha$ (if $\alpha \neq 0$),

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we cannot guarantee the analyticity of the functions given by expressions (6.25). However, this difficulty can be overcome. Consider the two scalar equations corresponding to the Riemann–Hilbert problem (6.7), written in the form

$$\left.\begin{aligned}(1-g)(\phi_1^+ - \rho\phi_2^+) &= r(\phi_1^- - \rho\phi_2^-),\\ (1+g)(\phi_1^+ + \rho\phi_2^+) &= r(\phi_1^- + \rho\phi_2^-).\end{aligned}\right\} \tag{6.38}$$

The product equation obtained from these is (6.35). Now, we can write the factor $(\alpha\xi+\beta)/(\xi+2\mathrm{i})$, which appears on the right-hand side of (6.37), as a product of the form

$$\frac{\alpha\xi+\beta}{\xi+2\mathrm{i}} = K(1+\mu\tilde{\rho}_+)(1-\mu\tilde{\rho}_+), \tag{6.39}$$

where

$$\tilde{\rho}_+^2(\xi) = \frac{\xi+\mathrm{i}}{\xi+2\mathrm{i}} = (r^{-2}\rho^2)(\xi) \tag{6.40}$$

for conveniently chosen values of K and μ ($\mu^2 = 1$, $K = -\mathrm{i}\beta$ for $\alpha = 0$; $\mu^2 = (\beta - 2\mathrm{i}\alpha)/(\beta - \mathrm{i}\alpha)$, $K = \alpha/(1-\mu^2)$ for $\alpha \neq 0$), except in the case where $\beta = \mathrm{i}\alpha$, corresponding to a zero of $\alpha\xi+\beta$ for $\xi = -\mathrm{i}$. However, this does not happen due to the normalization condition (6.35).

It is easy to see that if $K = 0$, $G\phi^+ = r\phi^-$ admits only the trivial solution $\phi^+ = \phi^- = 0$. So we have $K \neq 0$.

Substitution of (6.34) into (6.37) leads to a new product equation

$$d_+\frac{(\phi_1^+)^2 - \rho^2(\phi_2^+)^2}{1-\mu^2\tilde{\rho}_+^2} = r^2 d_-^{-1}\frac{(\phi_1^-)^2 - \rho^2(\phi_2^-)^2}{1-\mu^2\tilde{\rho}_+^2} = \tilde{q}_1 = \frac{K}{(\xi+\mathrm{i})^2}, \tag{6.41}$$

and thus

$$\frac{(\phi_1^+)^2 - \rho^2(\phi_2^+)^2}{1-\mu^2\tilde{\rho}_+^2} = \frac{d_+^{-1}K}{(\xi+i)^2}, \qquad \frac{(\phi_1^-)^2 - \rho^2(\phi_2^-)^2}{1-\mu^2\tilde{\rho}_+^2} = \frac{Kd_-}{(\xi-\mathrm{i})^2}, \tag{6.42}$$

which shows that the expressions on the left-hand side of these equalities are analytic and do not vanish in the corresponding half-planes ($\mathbb{C}^+$ and $\mathbb{C}^-$, respectively).

These expressions correspond precisely to the product equation, which is obtained from (6.38) when we divide both sides of its equations by $1 \pm \mu\tilde{\rho}_+$,

$$\left.\begin{aligned}d_{1+}\frac{\phi_1^+ - \rho\phi_2^+}{1-\mu\tilde{\rho}_+} &= rd_{1-}^{-1}\frac{\phi_1^- - \rho\phi_2^-}{1-\mu\tilde{\rho}_+},\\ d_{2+}\frac{\phi_1^+ + \rho\phi_2^+}{1+\mu\tilde{\rho}_+} &= rd_{2-}^{-1}\frac{\phi_1^- + \rho\phi_2^-}{1+\mu\tilde{\rho}_+}.\end{aligned}\right\} \tag{6.43}$$

So, if $\alpha\xi+\beta = \alpha(\xi - z_0)$, it is clear from the preceding discussion concerning (6.39) that we can find $\mu \in \mathbb{C}$ such that $z_0 = -\mathrm{i}((2-\mu^2)/(1-\mu^2))$, and this zero is compensated in (6.43). Now, the function represented by each of the left-hand sides of (6.43) is bounded and does not vanish in $\mathbb{C}^+$ (and, analogously, in $\mathbb{C}^-$, for the right-hand sides of (6.43)).

The quotient equation corresponding to this modified system of equations is

$$\frac{d_1}{d_2}\cdot\frac{\phi_1^+ - \rho\phi_2^+}{\phi_1^+ + \rho\phi_2^+}\cdot\frac{1+\mu\tilde{\rho}_+}{1-\mu\tilde{\rho}_+} = \frac{\phi_1^- - \rho\phi_2^-}{\phi_1^- + \rho\phi_2^-}\cdot\frac{1+\mu\tilde{\rho}_+}{1-\mu\tilde{\rho}_+}, \tag{6.44}$$

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which is equivalent to

$$-\frac{d_1}{d_2}\frac{\psi_1^+ - \tilde{\rho}_+\psi_2^+}{\psi_1^+ + \tilde{\rho}_+\psi_2^+} = \frac{\psi_1^- - \rho^{-1}\psi_2^-}{\psi_1^- + \rho^{-1}\psi_2^-} \tag{6.45}$$

for

$$\psi_1^+ = \phi_1^+ - \mu r\tilde{\rho}_+^2\phi_2^+, \tag{6.46}$$
$$\psi_2^+ = r\phi_2^+ - \mu\phi_1^+, \tag{6.47}$$
$$\psi_1^- = r^{-1}\tilde{\rho}_+^{-2}\phi_2^- - \mu\rho^{-2}\phi_1^-, \tag{6.48}$$
$$\psi_2^- = r^{-1}\tilde{\rho}_+^{-2}\phi_1^- - \mu\phi_2^-. \tag{6.49}$$

Equation (6.45) can be handled as (6.24), and we obtain

$$\rho_+\log\left(-\frac{d_1}{d_2}\right) + \rho_+\log\frac{\psi_1^+ - \tilde{\rho}_+\psi_2^+}{\psi_1^+ + \tilde{\rho}_+\psi_2^+} = \rho_+\log\frac{\psi_1^- - \rho^{-1}\psi_2^-}{\psi_1^- + \rho^{-1}\psi_2^-}. \tag{6.50}$$

Defining $F^{\pm}$ as in (6.27), we get

$$\frac{\psi_1^+ - \tilde{\rho}_+\psi_2^+}{\psi_1^+ + \tilde{\rho}_+\psi_2^+} = \exp(-\rho_+^{-1}F^+),$$
$$\frac{\psi_1^- - \rho^{-1}\psi_2^-}{\psi_1^- + \rho^{-1}\psi_2^-} = \exp(\rho_+^{-1}F^-).$$

Since we also have, from (6.42),

$$(\psi_1^+ - \tilde{\rho}_+\psi_2^+)(\psi_1^+ + \tilde{\rho}_+\psi_2^+) = (1-\mu^2\tilde{\rho}_+^2)^2\frac{Kd_+^{-1}}{(\xi+\mathrm{i})^2},$$
$$(\psi_1^- - \rho^{-1}\psi_2^-)(\psi_1^- + \rho^{-1}\psi_2^-) = -(1-\mu^2\tilde{\rho}_+^2)^2\rho^{-2}\left(\frac{\xi+2\mathrm{i}}{\xi-\mathrm{i}}\right)^2\frac{Kd_-}{(\xi-\mathrm{i})^2},$$

we obtain the expressions for $\psi_1^{\pm}$, $\psi_2^{\pm}$ and, taking (6.46)–(6.49) into account, we finally determine

$$\phi_1^+ = K^{1/2}d_+^{-1/2}r_+[\mathrm{sh}(\tfrac{1}{2}\rho_+^{-1}F^+) + \mu\tilde{\rho}_+\,\mathrm{ch}(\tfrac{1}{2}\rho_+^{-1}F^+)], \tag{6.51}$$
$$\phi_2^+ = K^{1/2}d_+^{-1/2}r_-[\mu\,\mathrm{sh}(\tfrac{1}{2}\rho_+^{-1}F^+) + \tilde{\rho}_+^{-1}\,\mathrm{ch}(\tfrac{1}{2}\rho_+^{-1}F^+)], \tag{6.52}$$
$$\phi_1^- = \mathrm{i}d_-^{1/2}K^{1/2}r_-[\mathrm{ch}(\tfrac{1}{2}\rho_+^{-1}F^-) - \mu\tilde{\rho}_+\,\mathrm{sh}(\rho_+^{-1}F^-)], \tag{6.53}$$
$$\phi_2^- = \mathrm{i}d_-^{1/2}K^{1/2}r_-[-\rho^{-1}\,\mathrm{sh}(\tfrac{1}{2}\rho_+^{-1}F^-) + \mu r^{-1}\,\mathrm{ch}(\tfrac{1}{2}\rho_+^{-1}F^-)]. \tag{6.54}$$

Due to the presence of the factor r_- in the expression defining ϕ_2^+, we see that μ must be such that

$$[\mu\,\mathrm{sh}(\tfrac{1}{2}\rho_+^{-1}F^+) + \tilde{\rho}_+^{-1}\,\mathrm{ch}(\tfrac{1}{2}\rho_+^{-1}F^+)]_{\xi=\mathrm{i}} = 0, \tag{6.55}$$

from which it follows that

$$\mu = -\sqrt{\tfrac{3}{2}}\coth(\tfrac{1}{2}\sqrt{6}\mathrm{i}F^+(\mathrm{i})). \tag{6.56}$$

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We remark that the value of μ is well defined by (6.55), since we must have $[\frac{1}{2}\,\mathrm{sh}(\rho_+^{-1}F^+)]_{(\mathrm{i})} \neq 0$. In fact, if this does not happen, it is clear from (6.29)–(6.32) that these expressions give a non-trivial solution to the homogeneous equation $G\phi^+ = \phi^-$. Therefore, G does not admit a canonical factorization.

On the other hand, it can be verified directly that for μ defined by (6.56), the expressions (6.51)–(6.54) give a solution to $G\phi^+ = r\phi^-$ satisfying $\phi_2^-(-\mathrm{i}) = 0$, $\phi_1^-(-\mathrm{i}) \neq 0$.

Taking $r_+^{-1}\phi^+$, $r_-^{-1}\phi^-(\phi^\pm = (\phi_1^\pm, \phi_2^\pm))$, given by (6.51)–(6.54), as one of the columns in G_+^{-1}, G_-, respectively, we still have to determine the other column. This can, of course, be done by solving the equation $G\phi^+ = r\phi^-$, subject to a convenient normalization condition such as $\phi_1^+(\mathrm{i}) = 0$, $\phi_2^+(\mathrm{i}) = K_1 \neq 0$. In fact, the second column is of the form $r_+^{-1}\tilde{\phi}^+$, $r_-^{-1}\tilde{\phi}^-$ for G_+^{-1} and G_-, respectively, where $(\tilde{\phi}^+, \tilde{\phi}^-)$ is a solution to (6.7), such that $\det[\phi^+\tilde{\phi}^+]_{(z_+)} \neq 0$ for some point $z_+ \in \mathbb{C}^+$ (or $\det[\phi^-\tilde{\phi}^-]_{(z_-)} \neq 0$, $z_- \in \mathbb{C}^-$ (see § 2)). However, we can avoid this procedure by once again using the results of § 5, namely theorem 5.9.

We state our conclusions in the following theorem.

THEOREM 6.4. *Let the assumptions of theorem 6.2 be satisfied, as well as the condition for existence of a canonical factorization. Then one such factorization is* $G = G_-G_+$, *where the first column in* G_- *(respectively,* G_+^{-1}*) is* $[f_1^-\,f_2^-]^{\mathrm{T}}$ *(respectively,* $[f_1^+\,f_2^+]^{\mathrm{T}}$*), where* $f_1^\pm = r_\pm^{-1}\phi_1^\pm$, $f_2^\pm = r_\pm^{-1}\phi_2^\pm$, *with* $\phi_1^\pm$, $\phi_2^\pm$ *defined by (6.51)–(6.54) for* μ *given by (6.56).*

In this case,

(i) *for* $\mu \neq 0$,

$$G_+^{-1} = \begin{bmatrix} f_1^+ & af_1^+ + qbf_2^+ \\ f_2^+ & bf_1^+ + af_2^+ \end{bmatrix}, \qquad G_- = \begin{bmatrix} f_1^- & af_1^- + qbf_2^- \\ f_2^- & bf_1^- + af_2^- \end{bmatrix},$$

with a *and* b *given by*

$$b = \frac{1}{1 - \mu^2\tilde{\rho}_+^2}, \qquad a = K_0 b,$$

where K_0 *is defined in theorem 5.9 and* $\tilde{\rho}_+^2$ *is given by (6.40);*

(ii) *for* $\mu = 0$, G *admits a canonical factorization within the same Daniele–Khrapkov class and we have*

$$G_+^{-1} = \begin{bmatrix} f_1^+ & qf_2^+ \\ f_2^+ & f_1^+ \end{bmatrix}, \qquad G_- = \begin{bmatrix} f_1^- & qf_2^- \\ f_2^- & f_1^- \end{bmatrix}.$$

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