AN EGOCENTRIC LOGIC OF KNOWING HOW TO TELL THEM APART

PAVEL NAUMOV AND JIA TAO

Abstract. Traditionally, the formulae in modal logic express properties of possible worlds. Prior introduced "egocentric" logics that capture properties of agents rather than of possible worlds. In such a setting, the article proposes the modality "know how to tell apart" and gives a complete logical system describing the interplay between this modality and the knowledge modality. An important contribution of this work is a new matrix-based technique for proving completeness theorems in an egocentric setting.

§1. Introduction. In this article, we propose a logical system for reasoning about an agent's abilities to tell apart groups of agents. Under the traditional Kripke approach to semantics of modal logics [12], modal formulae are interpreted as properties of possible worlds. For example, we write $w \Vdash$ "Earth is round" to express the fact that world w has the property of Earth being round. Prior [16] introduced the term "egocentric logic" for logical systems that capture properties of agents rather than possible worlds. Using his idea, we can write $a \Vdash$ "is sick" to express the fact that agent a is sick. We can use usual Boolean connectives to construct more complicated formulae. For example, the statement

$$a \Vdash \text{"is a doctor"} \wedge \neg \text{"is sick"}$$

means that agent a is a doctor and is not sick. Seligman et al. [18, 19] proposed a "for all friends" modality F for egocentric setting. In their notations, the statement $a \Vdash \mathsf{F}$"is sick" means that all friends of agent a are sick and the statement

$$a \Vdash \text{"is sick"} \wedge \neg\mathsf{F}\neg\text{"is a doctor"}$$

means that agent a is sick and one of their friends is a doctor. Modality F is also used in [2, 3]. Jiang and Naumov [11] introduced "likes those who" modality L. For example, by $a \Vdash \mathsf{L}$"is a doctor" they denoted the sentence "agent a likes those who are doctors." As usual, modalities can be nested. For example, statement $a \Vdash \mathsf{LL}$"is a doctor" means that agent a likes those who like doctors.

Grove and Halpern [8–10] suggested to consider the ternary satisfaction relation $(a, w) \Vdash \varphi$ that means that, in world w, agent a has property φ. In philosophy of language, this approach is called *2D semantics* [17]. In such a setting, one can define knowledge modality K. For example, the statement $(a, w) \Vdash \mathsf{K}$"is sick" means that in world w the agent a knows that agent a is sick. This modality can be combined

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with modality F to say that in world w agent a knows that one of their friends is sick: $(a, w) \Vdash \mathsf{K}\neg\mathsf{F}\neg$"is sick." Epstein and Naumov [5] introduced modality W for "know who." For example, the statement $(a, w) \Vdash \mathsf{W}$"is a doctor" means that in world w agent a knows at least one person who is a doctor. The statement $(a, w) \Vdash \mathsf{WW}$"is a doctor" means that agent a knows who knows who is a doctor.

Knowing (at least one person) who is a doctor is different from the ability to *tell apart* doctors from non-doctors. This ability of an agent to tell apart (distinguish) agents who have property φ from those who do not have such a property is the subject of the current work. We write

$$(a, w) \Vdash \mathsf{A}\text{"is sick"}$$

to express that in world w agent a can tell apart (distinguish, classify) agents who are sick from those who are not sick. To connect with intuition, let us use the term "doctor" for the agents who can tell apart sick people from those who are not sick. Then, the statement

$$(a, w) \Vdash \mathsf{KA}\text{"is sick"}$$

means that agent a knows that agent a themself is a doctor. The statement

$$(a, w) \Vdash \mathsf{AA}\text{"is sick"}$$

means that agent a can tell apart doctors from non-doctors. Finally, the statement

$$(a, w) \Vdash \mathsf{AK}\text{"is sick"}$$

means that agent a can tall apart agents who know about themselves that they are sick from those who don't know that they are sick (whether they are actually sick or not). Such an agent a is probably a psychologist.

The contribution of this work is twofold. First, we propose a formal egocentric semantics for modality A and a sound, complete, and decidable logical system that describes the interplay between modality A and the knowledge modality K. Second, we introduce and use a new, matrix-based, technique for proving completeness results for 2D semantics. Section 6 discusses this technique in detail.

The rest of this article is structured as follows. In the next section, we introduce the syntax and semantics of our formal system. Section 3 discusses related literature. In Section 4 we list the axioms of our system. The soundness of these axioms is shown in Section 5. Section 6 proves the completeness of our system using the newly proposed technique. Section 7 shows its decidability. We discuss possible extensions of our system in Section 8. Section 9 concludes.

§2. Syntax and semantics. In this section, we present the syntax and the formal semantics of our logical system. The semantics is using epistemic models that we define below. Note that the only difference from the traditional S5 models is that propositional variables are interpreted as properties of pairs (a, w), where a is an agent and w is a world. As a result, valuation function π maps each propositional variable to a set of such pairs. Informally, $\pi(p)$ is the set of pairs for which property p is true. Throughout the article, we assume a fixed countable set of propositional variables.

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	w_1	w_2	w_3	w_4	partition
a_1	p	p	p		$\{w_1, w_2\}, \{w_3, w_4\}$
a_2					$\{w_1, w_4\}, \{w_2, w_3\}$
a_3	p	p	p	p	$\{w_1, w_2\}, \{w_3, w_4\}$

FIGURE 1. An epistemic model. Informally, propositional variable p means "is sick."

DEFINITION 2.1. A tuple $(Ag, W, \{\sim_a\}_{a\in Ag}, \pi)$ is an epistemic model if:

1. Ag is a set of "agents,"
2. W is a set of all "worlds,"
3. $\sim_a$ is an "indistinguishability" equivalence relation on W,
4. $\pi(p) \subseteq Ag \times W$ for each propositional variable p.

Figure 1 depicts an epistemic model capturing our introductory example. This model has three agents, a_1, a_2, and a_3, as well as four worlds, w_1, w_2, w_3, and w_4. Each indistinguishability relation is specified by listing its equivalence classes in the partition column. For example, agent a_1 cannot distinguish world w_1 from world w_2 and they also cannot distinguish world w_3 from world w_4. In this example, we assume that our language has a single propositional variable p, which means "is sick." The set $\pi(p)$ consists of all pairs (a, w) such that agent a is sick in world w. We visualize set $\pi(p)$ on the left side of Figure 1. For example, propositional variable p in the first row, the first column of the table means that $(a_1, w_1) \in \pi(p)$.

The language Φ of our logical system is defined by the grammar:

$$\varphi := p \mid \neg\varphi \mid \varphi \to \varphi \mid \mathsf{K}\varphi \mid \mathsf{A}\varphi,$$

where p is a propositional variable. We read $\mathsf{K}\varphi$ as "knows about themself" and $\mathsf{A}\varphi$ as "knows how to tell apart those who." We assume that conjunction $\wedge$, disjunction $\vee$, biconditional $\leftrightarrow$, and false $\perp$ are defined in the standard way. The semantics of our logical system is given in the definition below.

DEFINITION 2.2. For any epistemic model $(Ag, W, \{\sim_a\}_{a\in Ag}, \pi)$, any agent $a \in Ag$, any world $w \in W$, and any formula $\varphi \in \Phi$, the satisfaction relation $(a, w) \Vdash \varphi$ is defined recursively as follows:

1. $(a, w) \Vdash p$ if $(a, w) \in \pi(p)$.
2. $(a, w) \Vdash \neg\varphi$ if $(a, w) \nVdash \varphi$.
3. $(a, w) \Vdash \varphi \to \psi$ if $(a, w) \nVdash \varphi$ or $(a, w) \Vdash \psi$.
4. $(a, w) \Vdash \mathsf{K}\varphi$ when for each world $u \in W$, if $w \sim_a u$, then $(a, u) \Vdash \varphi$.
5. $(a, w) \Vdash \mathsf{A}\varphi$ when for each agent $b \in Ag$ and any worlds $u, u' \in W$, if $w \sim_a u$, $w \sim_a u'$, and $(b, u) \Vdash \varphi$, then $(b, u') \Vdash \varphi$.

Note that the meaning of the statement $(a, w) \Vdash \mathsf{K}\varphi$, as defined in item 4 above, is "in world w, agent a knows φ about themself." It is the agent a who knows because the item is using the indistinguishability relation $\sim_a$. It is the agent a about whom the statement φ is known because, in that item, the statement $(a, u) \Vdash \varphi$ refers to agent a.

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	w_1	w_2	w_3	w_4
a_1	$p, \mathsf{K}p, \mathsf{A}p, \mathsf{AK}p$	$p, \mathsf{K}p, \mathsf{A}p, \mathsf{AK}p$	$p, \mathsf{AK}p$	$\mathsf{AK}p$
a_2		$\mathsf{A}p$	$\mathsf{A}p$	
a_3	$p, \mathsf{K}p, \mathsf{A}p, \mathsf{AK}p$	$p, \mathsf{K}p, \mathsf{A}p, \mathsf{AK}p$	$p, \mathsf{K}p, \mathsf{AK}p$	$p, \mathsf{K}p, \mathsf{AK}p$

FIGURE 2. Examples of formulae satisfied in the epistemic model shown in Figure 1.

The meaning of the statement $(a, w) \Vdash \mathsf{A}\varphi$ is "in world w, agent a knows how to tell apart those for whom φ is true." Indeed, item 5 above says that whether φ holds for an arbitrary agent b is consistent among all words indistinguishable by agent a. In our example, the statement $(a, w) \Vdash \mathsf{A}p$ means that any agent b is either (i) sick in all worlds indistinguishable by agent a from world w or (ii) not sick in all worlds indistinguishable to agent a from world w.

Figure 2 shows examples of formulae that are true about different agents in different worlds of the epistemic model depicted in Figure 1. For instance, observe in Figure 1 that agent a_1 cannot distinguish worlds w_1 and w_2. At the same time, agent a_1 is sick in both of these worlds. Thus, by item 4 of Definition 2.2, in both of these worlds, agent a_1 knows that agent a_1 is sick. We denote this by formula $\mathsf{K}p$ in cells (a_1, w_1) and (a_1, w_2) of Figure 2. Agent a_1 also cannot distinguish worlds w_3 and w_4. They are sick in world w_3, but not w_4. Thus, in both of these worlds, agent a_1 does not know that agent a_1 is sick.

To illustrate modality A, observe again that agent a_1 cannot distinguish worlds w_1 and w_2. Note that each of the agents in the model (a_1, a_2, and a_3) is sick in world w_1 if and only if the same agent is sick in world w_2. Thus, in world w_1, agent a_1 knows which of the agents is sick and which is not. In other words, agent a_1 knows how to tell apart those who are sick from those who are not. Using the informal language from our introduction, agent a_1 is a "doctor" in world w_1. In the formal language of our logical system, $(a_1, w_1) \Vdash \mathsf{A}p$.

Consider now agent a_2 in the same world w_1. This agent cannot distinguish world w_1 from world w_4. Additionally, agent a_1 is sick in world w_1 and is not sick in world w_4. Thus, in world w_1, agent a_2 does not know if agent a_1 is sick or not. Hence, in world w_1 agent a_2 is not a "doctor": $(a_2, w_1) \nVdash \mathsf{A}p$.

Let us consider again world w_1 where agent a_1 cannot distinguish the current world from world w_2. As shown in Figure 2, for each agent in the model, the agent knows about themself that they are sick (formula $\mathsf{K}p$) in world w_1 iff they know the same in world w_2. Thus, in world w_1, agent a_1 knows how to tell apart the agents for whom $\mathsf{K}p$ is true. Using our informal language from the introduction, agent a_1 is a "psychologist" in world w_1. Formally, we write that $(a_1, w_1) \Vdash \mathsf{AK}p$.

Finally, observe that, in our model, each agent a in each world w can tell psychologists apart: $(a, w) \Vdash \mathsf{AAK}p$. At the same, in our model, each agent in each world cannot tell doctors apart: $(a, w) \nVdash \mathsf{AA}p$.

§3. Related work. In this section, we discuss how else the notion "knowing how to tell apart" could be formalized and also how our modality fits into the larger fields of logics of know-wh.

3.1. Another formalization. We are not aware of any other existing attempts to formalize the notion of "knowing how to tell apart" as a modality, but, as is often the case with modalities, this notion can be expressed using quantifiers. More specifically, our language Φ can be translated into the language of an epistemic modal logic with quantifiers over agents. An example of such a language is in our own work [15], but it is not sufficiently rich. Namely, in [15], we assume that propositional variables are statements about worlds. For the formalization to work, we need a language $\mathcal{Q}$ that includes "agent predicates" $P(x_1, \dots, x_n)$. The validity of such a predicate depends not only on the world we consider, but also on the values of agent variables $x_1, \dots, x_n$.

Next, we describe a translation τ_x from language Φ into language $\mathcal{Q}$ for each agent variable x. For any propositional variable p, the value $\tau_x(p)$ is an agent predicate $p(x)$ with a single agent variable x. In addition, let

$$\begin{aligned}
\tau_x(\varphi \to \psi) &= \tau_x(\varphi) \to \tau_x(\psi),\\
\tau_x(\neg\varphi) &= \neg\tau_x(\varphi),\\
\tau_x(\mathsf{K}\varphi) &= \mathsf{K}_x\tau_x(\varphi),\\
\tau_x(\mathsf{A}\varphi) &= \forall y\, (\mathsf{K}_x\tau_y(\varphi) \vee \mathsf{K}_x\neg\tau_y(\varphi)).
\end{aligned}$$

We believe that under any reasonable semantics $\Vdash_{\mathcal{Q}}$ of language $\mathcal{Q}$ defined using epistemic models from Definition 2.1, for any world w of such a model and any formula $\varphi \in \Phi$,

$$(\sigma(x), w) \Vdash \varphi \quad \text{iff} \quad w \Vdash_{\mathcal{Q}} \tau_x(\varphi)[\sigma],$$

where σ is any assignment of agents to agent variables.

3.2. Knowing-wh logics. Modality A expresses a particular type of knowledge that an agent might have—knowledge of how to classify people into those who have a given property and those who do not. As such, it belongs to the growing class of *know-wh* [20] modalities: know how [7, 14], know who [5], know why [21], and know value [1, 4]. Among such modalities, the closest one to ours is probably the "know whether" modality Kw proposed by Fan, Wang, and van Ditmarsch [6]. Note that in the traditional, non-egocentric setting, modality Kw is definable through individual knowledge modality: $\mathsf{Kw}_a\varphi \equiv \mathsf{K}_a\varphi \vee \mathsf{K}_a\neg\varphi$. In the egocentric setting, the formula $\mathsf{A}\varphi$ denotes the ability of an agent to tell apart those agents for whom φ is true from those for whom φ is false among *all agents*. At the same time, knowledge modality K expresses knowledge of an agent *about themself*. It is clear that A cannot be defined through K. However, if a property φ is true for an agent, then the agent's ability to decide φ about all agents implies their knowledge of the property about themself: $\varphi \to (\mathsf{A}\varphi \to \mathsf{K}\varphi)$. This formula is one of the axioms of our logical system that we introduce in the next section.

§4. Axioms. In addition to tautologies in language Φ, our logical system contains the following axioms:

1. Truth: $\mathsf{K}\varphi \to \varphi$.
2. Distributivity: $\mathsf{K}(\varphi \to \psi) \to (\mathsf{K}\varphi \to \mathsf{K}\psi)$.
3. Negative Introspection: $\neg\mathsf{K}\varphi \to \mathsf{K}\neg\mathsf{K}\varphi$.

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4. Introspection of Knowing All: $\mathsf{A}\varphi \to \mathsf{KA}\varphi$.
5. Self-Knowledge: $\varphi \to (\mathsf{A}\varphi \to \mathsf{K}\varphi)$.
6. Negation: $\mathsf{A}\varphi \to \mathsf{A}\neg\varphi$.
7. Conjunction: $\mathsf{A}\varphi \wedge \mathsf{A}\psi \to \mathsf{A}(\varphi \wedge \psi)$.

We say that formula $\varphi \in \Phi$ is a *theorem* of our logical system and write $\vdash \varphi$ if φ is derivable from the above axioms using the Modus Ponens, two forms of the Necessitation, and the Substitution inference rules:

$$\frac{\varphi, \varphi \to \psi}{\psi} \qquad \frac{\varphi}{\mathsf{K}\varphi} \qquad \frac{\varphi}{\mathsf{A}\varphi} \qquad \frac{\varphi \leftrightarrow \psi}{\mathsf{A}\varphi \to \mathsf{A}\psi}.$$

We write $X \vdash \varphi$ if a formula $\varphi \in \Phi$ is derivable from the *theorems* of our logical system and an additional set of assumptions $X \subseteq \Phi$ using *only* the Modus Ponens inference rule. It is easy to see that $\varnothing \vdash \varphi$ iff $\vdash \varphi$. We say that set X is consistent if $X \nvdash \perp$.

The Truth, the Distributivity, and the Negative Introspection axioms are the standard axioms of epistemic logic S5. The Introspection of Knowing All axiom states that if an agent knows how to tell apart agents with a given property, then the agent knows that the agent has such an ability. The Self-Knowledge axiom states that if a statement φ is true about an agent and the agent knows how to tell apart the agents for whom φ is true, then the agent must know φ about themself. The Negation axiom states that if an agent knows how to tell apart those for whom φ is true, then it also knows how to tell apart those for whom φ is false. The Conjunction axiom states that if an agent knows separately how to tell apart agents who have each of the two properties, then the agent knows how to tell apart those who have both properties.

The next three lemmas capture well-known results in modal logic. To make the article self-contained, we reproduce their proofs in the Appendix.

LEMMA 4.1 (deduction). *If $X, \varphi \vdash \psi$, then $X \vdash \varphi \to \psi$.*

LEMMA 4.2. *If $\varphi_1, \dots, \varphi_n \vdash \psi$, then $\mathsf{K}\varphi_1, \dots, \mathsf{K}\varphi_n \vdash \mathsf{K}\psi$.*

LEMMA 4.3. $\vdash \mathsf{K}\varphi \to \mathsf{KK}\varphi$.

LEMMA 4.4. $\vdash \neg\mathsf{A}\varphi \to \mathsf{K}\neg\mathsf{A}\varphi$.

PROOF. By the Introspection of Knowing All axiom, $\vdash \mathsf{A}\varphi \to \mathsf{KA}\varphi$. Thus, $\vdash \neg\mathsf{KA}\varphi \to \neg\mathsf{A}\varphi$ by the contrapositive. Hence, $\vdash \mathsf{K}(\neg\mathsf{KA}\varphi \to \neg\mathsf{A}\varphi)$ by the Necessitation inference rule. Then, by the Distributivity axiom and the Modus Ponens inference rule $\vdash \mathsf{K}\neg\mathsf{KA}\varphi \to \mathsf{K}\neg\mathsf{A}\varphi$. Thus, by the Negative Introspection axiom and the laws of propositional reasoning, we have $\vdash \neg\mathsf{KA}\varphi \to \mathsf{K}\neg\mathsf{A}\varphi$. Note that $\neg\mathsf{A}\varphi \to \neg\mathsf{KA}\varphi$ is the contrapositive of the Truth axiom. Therefore, by the laws of propositional reasoning, $\vdash \neg\mathsf{A}\varphi \to \mathsf{K}\neg\mathsf{A}\varphi$. $\dashv$

LEMMA 4.5 (Lindenbaum). *Any consistent set of formulae can be extended to a maximal consistent set of formulae.*

PROOF. The standard proof of Lindenbaum's lemma [13, Proposition 2.14] applies. $\dashv$

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§5. Soundness. In this section, we prove the soundness of our logical system. The soundness of the Truth, the Distributivity, and the Negative Introspection axioms as well as of the Modus Ponens and the two forms of the Necessitation inference rules is straightforward. We prove the soundness of each of the remaining axioms and of the Substitution rule below.

LEMMA 5.1. *If* $(a, w) \Vdash \mathsf{A}\varphi$, *then* $(a, w) \Vdash \mathsf{KA}\varphi$.

PROOF. Consider any world $u \in W$ such that

$$w \sim_a u. \tag{1}$$

By item 4 of Definition 2.2, it suffices to show that $(a, u) \Vdash \mathsf{A}\varphi$.

Next, consider any agent $b \in Ag$ and any worlds $u', u'' \in W$, such that $u \sim_a u'$, $u \sim_a u''$, and $(b, u') \Vdash \varphi$. By item 5 of Definition 2.2, it suffices to prove that $(b, u'') \Vdash \varphi$. Indeed, statement (1) implies that $w \sim_a u'$, $w \sim_a u''$ because $\sim_a$ is an equivalence relation. Thus, $(b, u'') \Vdash \varphi$ by item 5 of Definition 2.2 and the assumption $(a, w) \Vdash \mathsf{A}\varphi$ of the lemma. ⊣

LEMMA 5.2. *If* $(a, w) \Vdash \varphi$ *and* $(a, w) \Vdash \mathsf{A}\varphi$, *then* $(a, w) \Vdash \mathsf{K}\varphi$.

PROOF. Consider any world $u \in W$ such that

$$w \sim_a u. \tag{2}$$

By item 4 of Definition 2.2, it suffices to show that $(a, u) \Vdash \varphi$.

Note $w \sim_a w$ because $\sim_a$ is an equivalence relation. Thus, the assumptions $(a, w) \Vdash \mathsf{A}\varphi$ and $(a, w) \Vdash \varphi$ of the lemma and statement (2) imply $(a, u) \Vdash \varphi$ by item 5 of Definition 2.2. ⊣

LEMMA 5.3. *If* $(a, w) \Vdash \mathsf{A}\varphi$, *then* $(a, w) \Vdash \mathsf{A}\neg\varphi$.

PROOF. Consider any agent $b \in Ag$ and any worlds $u, u' \in W$ such that

$$w \sim_a u \quad \text{and} \quad w \sim_a u'. \tag{3}$$

By item 5 of Definition 2.2, it suffices to prove that if $(b, u) \Vdash \neg\varphi$, then $(b, u') \Vdash \neg\varphi$. Then, by item 2 of Definition 2.2 and the law of contraposition, it suffices to show that if $(b, u') \Vdash \varphi$, then $(b, u) \Vdash \varphi$. The last statement is true by the assumption $(a, w) \Vdash \mathsf{A}\varphi$ of the lemma, statements (3), and item 5 of Definition 2.2. ⊣

LEMMA 5.4. *If* $(a, w) \Vdash \mathsf{A}\varphi$ *and* $(a, w) \Vdash \mathsf{A}\psi$, *then* $(a, w) \Vdash \mathsf{A}(\varphi \wedge \psi)$.

PROOF. Consider any agent $b \in Ag$ and any worlds $u, u' \in W$ such that

$$w \sim_a u \quad \text{and} \quad w \sim_a u' \tag{4}$$

and

$$(b, u) \Vdash \varphi \wedge \psi. \tag{5}$$

By item 5 of Definition 2.2, it suffices to prove that if $(b, u') \Vdash \varphi \wedge \psi$.

The assumption (5) implies $(b, u) \Vdash \varphi$ and $(b, u) \Vdash \psi$. Thus, $(b, u') \Vdash \varphi$ and $(b, u') \Vdash \psi$ by the assumptions $(a, w) \Vdash \mathsf{A}\varphi$ and $(a, w) \Vdash \mathsf{A}\psi$ of the lemma, statement (5), and item 5 of Definition 2.2. Then, $(b, u') \Vdash \varphi \wedge \psi$. ⊣

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LEMMA 5.5. *If* $(a, w) \Vdash \varphi \leftrightarrow \psi$ *for each agent a and each world w of each epistemic model, then* $(a, w) \Vdash \mathsf{A}\varphi \to \mathsf{A}\psi$ *for each agent a and each world w of each epistemic model.*

PROOF. Consider any agent a and any world w of an epistemic model such that $(a, w) \Vdash \mathsf{A}\varphi$. By item 3 of Definition 2.2, it suffices to prove that $(a, w) \Vdash \mathsf{A}\psi$.

By item 5 of Definition 2.2, the statement $(a, w) \Vdash \mathsf{A}\varphi$ implies that for each agent $b \in Ag$ and any worlds $u, u' \in W$, if $w \sim_a u$, $w \sim_a u'$, and $(b, u) \Vdash \varphi$, then $(b, u') \Vdash \varphi$. Then, by the assumption of the lemma, for each agent $b \in Ag$ and any worlds $u, u' \in W$, if $w \sim_a u$, $w \sim_a u'$, and $(b, u) \Vdash \psi$, then $(b, u') \Vdash \psi$. Therefore, $(a, w) \Vdash \mathsf{A}\psi$ again by item 5 of Definition 2.2. ⊣

§6. Completeness. In this section, we prove the completeness of our logical system using a canonical model construction. Usually, in modal logic, the canonical model construction defines possible worlds as maximal consistent sets. In such proofs, the truth lemma states that a formula belongs to a maximal consistent set if and only if it is satisfied at the world defined by this set. In our egocentric semantics, a formula is satisfied not at a world, but at an agent–world pair. Thus, to use the traditional approach, we must learn how to “split” a maximal consistent set into an agent and a world. How easy such a split is depends on if we require each agent to be present in each world.

In [5], the split is done by an introduction of a tree-like structure on maximal consistent sets. Then, two equivalence relations are defined on the nodes of the tree: “agent”-equivalent and “world”-equivalent. The equivalence classes of these two relations are defined to be the agents and the worlds of the model, respectively. It is then shown that each world and each agent might share at most one node of the tree. If a world and an agent are disjoint, then the agent is assumed not to be present in the world. Otherwise, the formulae in the shared node (maximal consistent set) are the formulae true about the given agent in the given world. This construction relies on the fact that not all agents are required to be present in all worlds. To the best of our knowledge, the only other work that does such a split is [10], in which the authors go even further by stipulating that “each agent exists in just one world” [10, Appendix D].

We think that the semantics of modality A is the most natural in the setting where all agents are present in all worlds. After all, what does it mean to decide if φ is true about an agent which might not even exist? To guarantee that all agents are present in all worlds, in this article, we devise a *matrix-based technique* for constructing a canonical model.

The matrix technique is inspired by Figure 2. Note that the rows of this table represent agents and the columns represent possible worlds. The cells of the table contain sets of formulae. In our construction, we build matrices (tables) whose cells are *maximal consistent* sets of formulae. We interpret the rows of the matrix as agents and columns as worlds.

Instead of defining a matrix at once, we use a recursive procedure that starts with a single 1×1 matrix and adds to it either a single column or a row and a column at a time. In most cases, the final matrix is an infinite-sized matrix obtained in the

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limit of this recursive construction. We are not aware of a similar technique used in any other proof of completeness.

The rest of this section is structured as follows. In Section 6.1, we establish various properties of maximal consistent sets. These results are used in Section 6.2 to recursively construct a sequence of matrices. To finish the proof of completeness, in Section 6.3, we show how a matrix can be converted to a model of our logical system as specified in Definition 2.1.

6.1. Maximal consistent sets. In the standard proof of completeness for epistemic modal logic, worlds are maximal consistent sets of formulae. In such a construction, for each maximal consistent set (possible world) w and each formula φ such that $\mathsf{K}\varphi \notin w$, the proof of the completeness builds another maximal consistent set (possible word) u such that $\neg\varphi \in u$ and worlds w and u are indistinguishable. In our work, we also have such a lemma, it's Lemma 6.10. In conjunction with item 4 of Definition 2.2, this lemma is used in the proof of the "truth lemma," Lemma 6.19. In addition to this, we also need another lemma that would work in a similar way for modality A. Because item 5 of Definition 2.2 refers to two additional worlds, u and u' (not mentioning the agent b), the equivalent of Lemma 6.10 needs to construct two maximal consistent sets instead of one. We state this new lemma as Lemma 6.13 at the end of this section. Because the proof of Lemma 6.13 is long and non-trivial, we split it into a sequence of smaller lemmas that we present in this subsection.

LEMMA 6.1. *The set* $\{\neg\varphi\} \cup \{\psi \mid \mathsf{K}\psi \in X\} \cup \{\chi \mid \chi, \mathsf{A}\chi \in X\}$ *is consistent for any consistent set* $X \subseteq \Phi$ *and any formula* $\varphi \in \Phi$ *such that* $X \nvdash \mathsf{K}\varphi$.

PROOF. Suppose the opposite. Then, there are formulae

$$\mathsf{K}\psi_1, \dots, \mathsf{K}\psi_k \in X, \chi_1, \mathsf{A}\chi_1, \dots, \chi_\ell, \mathsf{A}\chi_\ell \in X \tag{6}$$

such that

$$\psi_1, \dots, \psi_k, \chi_1, \dots, \chi_\ell \vdash \varphi.$$

Thus, by Lemma 4.2,

$$\mathsf{K}\psi_1, \dots, \mathsf{K}\psi_k, \mathsf{K}\chi_1, \dots, \mathsf{K}\chi_\ell \vdash \mathsf{K}\varphi.$$

Hence, by the Self-Knowledge axiom applied ℓ times,

$$\mathsf{K}\psi_1, \dots, \mathsf{K}\psi_k, \chi_1, \mathsf{A}\chi_1, \dots, \chi_\ell, \mathsf{A}\chi_\ell \vdash \mathsf{K}\varphi.$$

Therefore, $X \vdash \mathsf{K}\varphi$ by statement (6), which contradicts the assumption $X \vdash \mathsf{K}\varphi$ of the lemma. ⊣

DEFINITION 6.2. For any set of formulae $X \subseteq \Phi$ and any formula $\varphi \in \Phi$, a (finite or infinite) sequence $\psi_1, \psi_2, \dots$ is (X, φ)-regular if for each k,

$$X \nvdash \mathsf{A}(\varphi \wedge \psi_1 \wedge \cdots \wedge \psi_k).$$

LEMMA 6.3. *For any set of formulae* $X \subseteq \Phi$, *any formula* $\varphi \in \Phi$ *such that* $X \nvdash \mathsf{A}\varphi$, *and any (finite or infinite) sequence* $\mathsf{A}\psi_1, \mathsf{A}\psi_2, \dots$ *of formulae from set* X, *there is an* (X, φ)*-regular sequence* $\psi'_1, \psi'_2, \dots$ *such that, for each* $i \geq 1$, *either* $\psi'_i \equiv \psi_i$ *or* $\psi'_i \equiv \neg\psi_i$.

PROOF. We prove the existence of such sequence $\psi'_1, \psi'_2, \dots$ by constructing it recursively and simultaneously proving that statement

$$X \nvdash \mathsf{A}(\varphi \wedge \psi'_1 \wedge \cdots \wedge \psi'_k) \tag{7}$$

holds after k-th step of the construction.

Base: $k = 0$. Then, it suffices to prove that $X \nvdash \mathsf{A}\varphi$, which is true by an assumption of the lemma.

Step: $k > 0$. Suppose that statement (7) holds and, at the same time,

$$\begin{aligned} &X \vdash \mathsf{A}(\varphi \wedge \psi'_1 \wedge \cdots \wedge \psi'_k \wedge \psi_{k+1}) \quad \text{and} \\ &X \vdash \mathsf{A}(\varphi \wedge \psi'_1 \wedge \cdots \wedge \psi'_k \wedge \neg\psi_{k+1}). \end{aligned}$$

Thus, by the Negation axiom and the Modus Ponens inference rule,

$$\begin{aligned} &X \vdash \mathsf{A}\neg(\varphi \wedge \psi'_1 \wedge \cdots \wedge \psi'_k \wedge \psi_{k+1}) \quad \text{and} \\ &X \vdash \mathsf{A}\neg(\varphi \wedge \psi'_1 \wedge \cdots \wedge \psi'_k \wedge \neg\psi_{k+1}). \end{aligned}$$

Hence, by the Conjunction axiom and propositional reasoning,

$$X \vdash \mathsf{A}(\neg(\varphi \wedge \psi'_1 \wedge \cdots \wedge \psi'_k \wedge \psi_{k+1}) \wedge \neg(\varphi \wedge \psi'_1 \wedge \cdots \wedge \psi'_k \wedge \neg\psi_{k+1})).$$

Then, by the Negation axiom and the Modus Ponens inference rule,

$$X \vdash \mathsf{A}\neg(\neg(\varphi \wedge \psi'_1 \wedge \cdots \wedge \psi'_k \wedge \psi_{k+1}) \wedge \neg(\varphi \wedge \psi'_1 \wedge \cdots \wedge \psi'_k \wedge \neg\psi_{k+1})). \tag{8}$$

At the same time, note that formula

$$\begin{aligned} \neg(\neg(\varphi \wedge \psi'_1 \wedge \cdots \wedge \psi'_k \wedge \psi_{k+1}) \wedge \neg(\varphi \wedge \psi'_1 \wedge \cdots \wedge \psi'_k \wedge \neg\psi_{k+1})) \\ \leftrightarrow \varphi \wedge \psi'_1 \wedge \cdots \wedge \psi'_k \end{aligned}$$

is a propositional tautology. Thus, by the Substitution inference rule,

$$\begin{aligned} X \vdash \mathsf{A}\neg(\neg(\varphi \wedge \psi'_1 \wedge \cdots \wedge \psi'_k \wedge \psi_{k+1}) \wedge \neg(\varphi \wedge \psi'_1 \wedge \cdots \wedge \psi'_k \wedge \neg\psi_{k+1})) \\ \to \mathsf{A}(\varphi \wedge \psi'_1 \wedge \cdots \wedge \psi'_k). \end{aligned}$$

Hence, by statement (8) and the Modus Ponens inference rule,

$$X \vdash \mathsf{A}(\varphi \wedge \psi'_1 \wedge \cdots \wedge \psi'_k),$$

which contradicts our assumption that statement (7) holds. ⊣

LEMMA 6.4. *For any set of formulae $X \subseteq \Phi$, any formula $\varphi \in \Phi$, and any (X, φ)-regular sequence $\psi_1, \psi_2, \dots$, the set $\{\varphi, \psi_1, \psi_2, \dots\}$ is consistent.*

PROOF. Suppose that the set $\{\varphi, \psi_1, \psi_2, \dots\}$ is inconsistent. Thus, there is $n \geq 0$ such that $\vdash \neg(\varphi \wedge \psi_1 \wedge \cdots \wedge \psi_n)$. Then, $\vdash \mathsf{A}\neg(\varphi \wedge \psi_1 \wedge \cdots \wedge \psi_n)$ by the Necessitation inference rule. Hence, by the Negation axiom and the Modus Ponens inference rule,

$$\vdash \mathsf{A}\neg\neg(\varphi \wedge \psi_1 \wedge \cdots \wedge \psi_n). \tag{9}$$

Note that $\neg\neg(\varphi \wedge \psi_1 \wedge \cdots \wedge \psi_n) \leftrightarrow (\varphi \wedge \psi_1 \wedge \cdots \wedge \psi_n)$ is a propositional tautology. Thus, $\vdash \mathsf{A}(\varphi \wedge \psi_1 \wedge \cdots \wedge \psi_n)$ by the Substitution inference rule, statement (9), and the Modus Ponens inference rule. Therefore, by Definition 6.2, the sequence $\psi_1, \psi_2, \dots$ is not (X, φ)-regular, which contradicts an assumption of the lemma. ⊣

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LEMMA 6.5. *For any set of formulae $X \subseteq \Phi$, any formula $\varphi \in \Phi$, and any (X, φ)-regular sequence $\psi_1, \psi_2, \dots$, if $\mathsf{A}\psi_1, \dots, \mathsf{A}\psi_2, \dots \in X$, then the set $\{\neg\varphi, \psi_1, \psi_2, \dots\}$ is consistent.*

PROOF. Assume that the set $\{\neg\varphi, \psi_1, \psi_2, \dots\}$ is inconsistent. Thus, there is $n \geq 0$ such that $\vdash \neg(\neg\varphi \wedge \psi_1 \wedge \cdots \wedge \psi_n)$. Hence, by the Necessitation inference rule, $\vdash \mathsf{A}\neg(\neg\varphi \wedge \psi_1 \wedge \cdots \wedge \psi_n)$. Then,

$$X \vdash \mathsf{A}(\psi_1 \wedge \cdots \wedge \psi_n \wedge \neg(\neg\varphi \wedge \psi_1 \wedge \cdots \wedge \psi_n)) \tag{10}$$

by the assumption $\mathsf{A}\psi_1, \dots, \mathsf{A}\psi_2, \dots \in X$ of the lemma, the Conjunction axiom, and propositional reasoning. Next, note that the formula

$$(\psi_1 \wedge \cdots \wedge \psi_n \wedge \neg(\neg\varphi \wedge \psi_1 \wedge \cdots \wedge \psi_n)) \leftrightarrow \varphi \wedge \psi_1 \wedge \cdots \wedge \psi_n$$

is a tautology. Thus, by the Substitution inference rule, statement (10), and the Modus Ponens inference rule, $X \vdash \mathsf{A}(\varphi \wedge \psi_1 \wedge \cdots \wedge \psi_n)$. Therefore, by Definition 6.2, the sequence $\psi_1, \psi_2, \dots$ is not (X, φ)-regular, which contradicts an assumption of the lemma. $\dashv$

LEMMA 6.6. *For any maximal consistent set X and any formula $\mathsf{A}\varphi \notin X$, there are maximal consistent sets Y and Z such that*:

1. *$\varphi \in Y$, $\neg\varphi \in Z$.*
2. *$\psi \in Y$ iff $\psi \in Z$ for any formula $\mathsf{A}\psi \in X$.*

PROOF. The assumption of the article that the set of propositional variables is countable implies that set X is also countable. Let $\mathsf{A}\psi_1, \mathsf{A}\psi_2, \dots$ be an enumeration of all formulae of the form $\mathsf{A}\psi$ in set X. Then, $X \vdash \mathsf{A}\neg\psi_i$ for each $i \geq 1$ by the Negation axiom and the Modus Ponens inference rule. Hence, because X is a maximal consistent set, $\mathsf{A}\neg\psi_i \in X$ for each $i \geq 1$. Also, recall that $\mathsf{A}\psi_1, \mathsf{A}\psi_2, \dots$ is an enumeration of formulae from set X. Hence,

$$\mathsf{A}\psi_i, \mathsf{A}\neg\psi_i \in X \quad \text{for each } i \geq 1. \tag{11}$$

The assumption $\mathsf{A}\varphi \notin X$ of the lemma implies that $X \nvdash \mathsf{A}\varphi$ because X is a maximal consistent set. Thus, by Lemma 6.3, there is an (X, φ)-regular sequence $\psi'_1, \psi'_2, \dots$ such that, for each $i \geq 1$,

$$\psi'_i = \psi_i \quad \text{or} \quad \psi'_i = \neg\psi_i. \tag{12}$$

Then, by statement (11),

$$\mathsf{A}\psi'_i \in X \quad \text{for each } i \geq 1. \tag{13}$$

Consider the sets of formulae

$$Y_0 = \{\varphi, \psi'_1, \psi'_2, \dots\}, \tag{14}$$

$$Z_0 = \{\neg\varphi, \psi'_1, \psi'_2, \dots\}. \tag{15}$$

Note that set Y_0 is consistent by Lemma 6.4 because $\psi'_1, \psi'_2, \dots$ is an (X, φ)-regular sequence. Set Z_0 is consistent by Lemma 6.5, statement (13), and also because $\psi'_1, \psi'_2, \dots$ is an (X, φ)-regular sequence. By Lemma 4.5, sets Y_0 and Z_0 can be

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extended to maximal consistent sets Y and Z. Note that $\varphi \in Y_0 \subseteq Y$ and $\neg\varphi \in Z_0 \subseteq Z$ by statements (14) and (15), respectively.

To finish the proof of the lemma, it suffices to show that $\psi \in Y$ iff $\psi \in Z$ for any formula $\mathsf{A}\psi \in X$. Consider an arbitrary formula $\mathsf{A}\psi \in X$. Because $\mathsf{A}\psi_1, \mathsf{A}\psi_2, \dots$ is an enumeration of all formulae of the form $\mathsf{A}\psi$ in set X, there must exist $i_0 \geq 1$ such that $\psi = \psi_{i_0}$. By statement (12), either $\psi'_{i_0} = \psi_{i_0}$ or $\psi'_{i_0} = \neg\psi_{i_0}$. In the first case, $\psi = \psi_{i_0} = \psi'_{i_0} \in Y_0 \subseteq Y$ and $\psi = \psi_{i_0} = \psi'_{i_0} \in Z$ by statements (14) and (15), respectively. In the second case, $\neg\psi = \neg\psi_{i_0} = \psi'_{i_0} \in Y_0 \subseteq Y$ and $\neg\psi = \neg\psi_{i_0} = \psi'_{i_0} \in Z_0 \subseteq Z$ by statements (14) and (15), respectively. Hence, $\psi \notin Y$ and $\psi \notin Z$ because sets Y and Z are consistent. ⊣

6.2. Pseudo models. As discussed in the preamble to Section 6, we construct the canonical model by building a sequence of matrices of maximal consistent sets. Informally, the rows of the matrix represent the agents and the columns represent the worlds.

We assume that a matrix can have either finite or ω-many rows and columns. We also assume that matrix rows and columns are numbered starting with 0. Thus, for example, if a matrix has three rows and ω columns, then its rows are numbered 0, 1, and 2 and its columns are numbered by 0, 1, 2, ... (not including ω). In this case, we will also say that the matrix *size* is $3 \times \omega$. *Formally, a matrix of size $m \times n$ is an arbitrary function defined on the Cartesian product of ordinals m and n*. We say that a matrix of size $m \times n$ is *finite* if ordinals m and n are finite.

Technically, the canonical model is constructed using not just matrices, but structures consisting of a matrix and row-specific equivalence relations on the columns of the matrix. We call such structures pseudo-models.

Definition 6.7. A pseudo model is a pair $\langle (X_{ij}), \{\sim_i\}_i \rangle$, where:

1. (X_{ij}) is a matrix of maximal consistent sets of formulae,
2. $\sim_i$ is an equivalence relation on the columns that satisfies the following conditions for each i_1, i_2, j_1, j_2:
   (a) $\mathsf{K}\varphi \in X_{i_1 j_1}$ iff $\mathsf{K}\varphi \in X_{i_1 j_2}$, where $j_1 \sim_{i_1} j_2$,
   (b) $\varphi \in X_{i_2 j_1}$ iff $\varphi \in X_{i_2 j_2}$, where $\mathsf{A}\varphi \in X_{i_1 j_1}$ and $j_1 \sim_{i_1} j_2$.

By the *size* of a pseudo model $\langle (X_{ij}), \{\sim_i\}_i \rangle$ we mean the size of the matrix (X_{ij}). We say that a pseudo model is *finite* if the matrix (X_{ij}) is finite.

Definition 6.8. For any ordinals $m, m', n, n' \leq \omega$ and any two pseudo models $\langle (X_{ij}), \{\sim_i\}_i \rangle$ and $\langle (X'_{ij}), \{\sim'_i\}_i \rangle$ of sizes $m \times n$ and $m' \times n'$, respectively, let $\langle (X_{ij}), \{\sim_i\}_i \rangle \sqsubseteq \langle (X'_{ij}), \{\sim'_i\}_i \rangle$ if:

1. $m \leq m'$ and $n \leq n'$,
2. $X_{ij} = X'_{ij}$ for each $i < \min(m, m')$ and each $j < \min(n, n')$,
3. $j_1 \sim_i j_2$ iff $j_1 \sim'_i j_2$ for each $i < \min(m, m')$ and $j_1, j_2 < \min(n, n')$,

The next three lemmas represent the base and the recursive cases of building the sequence of pseudo models discussed earlier. We start with the simplest of these lemmas that constructs the first element of the sequence.

Lemma 6.9. *If $Y \nvdash \varphi$, then there is a finite pseudo model $\langle (X_{ij}), \{\sim_i\}_i \rangle$ such that $Y \cup \{\neg\varphi\} \subseteq X_{00}$.*

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PROOF. The set $Y \cup \{\neg\varphi\}$ is consistent because $Y \nvdash \varphi$. By Lemma 4.5, it has a maximal consistent extension X_{00}. Let (X_{00}) be the matrix of the size 1×1 whose only cell contains the maximal consistent set X_{00} and $\sim_0$ be the equivalence relation $\{(0,0)\}$ on the singleton set $\{0\}$. Then, the pair $((X_{00}), \{\sim_0\})$ is a pseudo model by Definition 6.7. ⊣

The next lemma specifies one of two recursive steps in constructing the sequence of pseudo models. Recall from our earlier discussion that this lemma is an analogue of constructing a new maximal consistent set (world) in the traditional completeness proof.

LEMMA 6.10. *For any finite pseudo model $\langle (X_{ij}), \{\sim_i\}_i \rangle$ and any formula $\varphi \in \Phi$, if $\mathsf{K}\varphi \notin X_{i_0 j_0}$ for some i_0, j_0, then there is a finite pseudo model $\langle (X'_{ij}), \{\sim'_i\}_i \rangle$ such that*:

1. $\langle (X_{ij}), \{\sim_i\}_i \rangle \sqsubseteq \langle (X'_{ij}), \{\sim'_i\}_i \rangle$,
2. $\neg\varphi \in X'_{i_0 j'}$ *for some* j' *such that* $j_0 \sim'_{i_0} j'$.

PROOF. Consider the set of formulae

$$Y = \{\neg\varphi\} \cup \{\psi \mid \mathsf{K}\psi \in X_{i_0 j_0}\} \cup \{\chi \mid \chi, \mathsf{A}\chi \in X_{i_0 j_0}\}. \tag{16}$$

The assumption $\mathsf{K}\varphi \notin X_{i_0, j_0}$ of the lemma implies that $X_{i_0, j_0} \nvdash \mathsf{K}\varphi$ because X_{i_0, j_0} is a maximal consistent set of formula. Then, set Y is consistent by Lemma 6.1. Let Y' be the maximal consistent extension of set Y. Such an extension exists by Lemma 4.5.

CLAIM 6.11. $\mathsf{K}\psi \in X_{i_0 j_0}$ *iff* $\mathsf{K}\psi \in Y'$.

PROOF OF CLAIM. $(\Rightarrow)$ If $\mathsf{K}\psi \in X_{i_0 j_0}$, then $X_{i_0 j_0} \vdash \mathsf{K}\mathsf{K}\psi$ by Lemma 4.3. Thus, $\mathsf{K}\mathsf{K}\psi \in X_{i_0 j_0}$ because $X_{i_0 j_0}$ is a maximal consistent set. Hence, $\mathsf{K}\psi \in Y \subseteq Y'$ by equation (16) and the choice of Y' as an extension of Y.

$(\Leftarrow)$ Suppose that $\mathsf{K}\psi \notin X_{i_0 j_0}$. Then, $\neg\mathsf{K}\psi \in X_{i_0 j_0}$ because X is a maximal consistent set of formulae. Thus, $X_{i_0 j_0} \vdash \mathsf{K}\neg\mathsf{K}\psi$ by the Negative Introspection axiom and the Modus Ponens inference rule. Hence, $\mathsf{K}\neg\mathsf{K}\psi \in X_{i_0 j_0}$ because $X_{i_0 j_0}$ is a maximal consistent set. Hence, $\neg\mathsf{K}\psi \in Y \subseteq Y'$ by equation (16) and the choice of Y' as an extension of set Y. Therefore, $\mathsf{K}\psi \notin Y'$ because set Y' is consistent. ⊣

Assume that the finite matrix (X_{ij}) has the size $m \times n$. Define $m \times (n+1)$ matrix

$$(X'_{ij}) = \begin{pmatrix} X_{0,0} & \dots & X_{0,j_0} & \dots & X_{0,n-1} & X_{0,j_0} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ X_{i_0-1,0} & \dots & X_{i_0-1,j_0} & \dots & X_{i_0-1,n-1} & X_{i_0-1,j_0} \\ X_{i_0,0} & \dots & X_{i_0,j_0} & \dots & X_{i_0,n-1} & Y' \\ X_{i_0+1,0} & \dots & X_{i_0+1,j_0} & \dots & X_{i_0+1,n-1} & X_{i_0+1,j_0} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ X_{m-1,0} & \dots & X_{m-1,j_0} & \dots & X_{m-1,n-1} & X_{m-1,j_0} \end{pmatrix}. \tag{17}$$

In addition, let the relation $\sim'_{i_0}$ be the transitive, reflexive, and symmetric closure of the relation $\sim_{i_0} \cup \{(j_0, n)\}$ on the set $\{0, 1, \dots, n\}$ and, for $i \neq i_0$, relation $\sim'_i$ be the reflexive closure of the relation $\sim_i$ on the set $\{0, 1, \dots, n\}$.

CLAIM 6.12. *$\langle (X'_{ij}), \{\sim'_i\}_i \rangle$ is a finite pseudo model.*

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Proof of Claim. It suffices to verify conditions 2(a) and 2(b) of Definition 6.7.

Condition 2(a). Consider any i, j_1, j_2 such that $j_1 \sim'_i j_2$. It suffices to show that $\mathsf{K}\psi \in X'_{ij_1}$ iff $\mathsf{K}\psi \in X'_{ij_2}$ for any formula $\psi \in \Phi$. Without loss of generality, assume that $j_1 \le j_2$. We consider the following three cases separately:

Case 1: $j_1 \le j_2 < n$. Then, $j_1 \sim_i j_2$ by the assumption $j_1 \sim'_i j_2$ and the choice of the relation $\sim'_i$. Hence, $\mathsf{K}\psi \in X_{ij_1}$ iff $\mathsf{K}\psi \in X_{ij_2}$ by item 2(a) of Definition 6.7 and the assumption of the lemma that $\langle (X_{ij}), \{\sim_i\}_i \rangle$ is a pseudo model. Note also that $X'_{ij_1} = X_{ij_1}$ and $X'_{ij_2} = X_{ij_2}$ by equation (17) and the assumption $j_1 \le j_2 < n$ of the case. Thus, $\mathsf{K}\psi \in X'_{ij_1}$ iff $\mathsf{K}\psi \in X'_{ij_2}$.

Case 2: $j_1 = j_2 = n$. Then, $\mathsf{K}\psi \in X'_{ij_1}$ iff $\mathsf{K}\psi \in X'_{ij_2}$.

Case 3: $j_1 < j_2 = n$. Then, by the choice of the relation $\sim'_i$, the assumption $j_1 \sim'_i j_2$ implies that $j_1 \sim_i j_0$ and

$$i = i_0. \tag{18}$$

Observe that $\mathsf{K}\psi \in X_{i_0 j_1}$ iff $\mathsf{K}\psi \in X_{i_0 j_0}$ by item 2(a) of Definition 6.7, the statement $j_1 \sim_i j_0$ and the assumption $j_1 < n$ of the case. Thus, it follows that $\mathsf{K}\psi \in X'_{i_0 j_1}$ iff $\mathsf{K}\psi \in X'_{i_0 j_0}$ by equation (17) and the assumption $j_1 < n$. Hence, $\mathsf{K}\psi \in X'_{i_0 j_1}$ iff $\mathsf{K}\psi \in Y'$ by Claim 6.11. Then, $\mathsf{K}\psi \in X'_{i_0 j_1}$ iff $\mathsf{K}\psi \in X'_{i_0 j_2}$ by equation (17) and the assumption $j_2 = n$ of the case. Therefore, $\mathsf{K}\psi \in X'_{ij_1}$ iff $\mathsf{K}\psi \in X'_{ij_2}$ by equation (18).

Condition 2(b). Consider any i_1, i_2, j_1, j_2 and any formula $\mathsf{A}\psi \in X'_{i_1 j_1}$ such that $j_1 \sim'_{i_1} j_2$. It suffices to show that $\psi \in X'_{i_2 j_1}$ iff $\psi \in X'_{i_2 j_2}$ for any formula $\psi \in \Phi$. We consider the following four cases separately:

Case 1: $j_1, j_2 < n$. Then, by equation (17), the assumption $\mathsf{A}\psi \in X'_{i_1 j_1}$ implies that $\mathsf{A}\psi \in X_{i_1 j_1}$. In addition, the assumption $j_1 \sim'_{i_1} j_2$ and the assumption $j_1, j_2 < n$ of the case imply that $j_1 \sim_{i_1} j_2$. Thus, $\psi \in X_{i_2 j_1}$ iff $\psi \in X_{i_2 j_2}$ by item 2(b) of Definition 6.7. Hence, $\psi \in X'_{i_2 j_1}$ iff $\psi \in X'_{i_2 j_2}$ by equation (17) and the same assumption $j_1, j_2 < n$ of the case.

Case 2: $j_1 = j_2 = n$. Then, $X'_{i_2 j_1} = X'_{i_2 j_2}$. Hence, $\psi \in X'_{i_2 j_1}$ iff $\psi \in X'_{i_2 j_2}$.

Case 3: $j_1 < j_2 = n$. Then, the assumption $j_1 \sim'_{i_1} j_2$ and the definition of the relation $\sim'$ imply that

$$i_0 = i_1 \quad \text{ and } \quad j_0 \sim_{i_1} j_1. \tag{19}$$

Also, the assumption $\mathsf{A}\psi \in X'_{i_1 j_1}$, by the assumption $j_1 < n$ of the case and equation (17), implies that

$$\mathsf{A}\psi \in X_{i_1 j_1}. \tag{20}$$

We further divide this case into the following two subcases:

Subcase 3A: $i_0 = i_1 = i_2$. The statement $\mathsf{A}\psi \in X_{i_1 j_1}$ implies $X_{i_1 j_1} \vdash \mathsf{KA}\psi$ by the Introspection of Knowing All axiom. Thus, $\mathsf{KA}\psi \in X_{i_1 j_1}$ because $X_{i_1 j_1}$ is a maximal consistent set. Hence, $\mathsf{KA}\psi \in X_{i_1 j_0}$ by item 2(a) of Definition 6.7 and part $j_0 \sim_{i_1} j_1$ of statement (19). Then, $X_{i_1 j_0} \vdash \mathsf{A}\psi$ by the Truth axiom and the Modus Ponens inference rule. Thus, because $X_{i_1 j_0}$ is a maximal consistent set,

$$\mathsf{A}\psi \in X_{i_1 j_0}. \tag{21}$$

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Hence, $\psi \in X_{i_1 j_0}$ iff $\psi \in X_{i_1 j_1}$ by item 2(b) of Definition 6.7 and the part $j_0 \sim_{i_1} j_1$ of statement (19). Then, $\psi \in X_{i_1 j_0}$ iff $\psi \in X'_{i_1 j_1}$ by the assumption $j_1 < n$ of the case and equation (17). Thus, $\psi \in X_{i_1 j_0}$ iff $\psi \in X'_{i_2 j_1}$ by the assumption $i_1 = i_2$ of the subcase.

Recall that we are proving that $\psi \in X'_{i_2 j_1}$ iff $\psi \in X'_{i_2 j_2}$. Then, it suffices to show that $\psi \in X_{i_1 j_0}$ iff $\psi \in X'_{i_2 j_2}$. Furthermore, due to the assumption $i_1 = i_2$ of the subcase, it is enough to prove that $\psi \in X_{i_1 j_0}$ iff $\psi \in X'_{i_1 j_2}$. Finally, due to equation (17), the assumption $j_2 = n$ of the case, and part $i_0 = i_1$ of statement (19), it suffices to show that $\psi \in X_{i_1 j_0}$ iff $\psi \in Y'$.

We show the two parts of this biconditional statement separately:

Assume $\psi \in X_{i_1 j_0}$. Thus, $\psi, \mathsf{A}\psi \in X_{i_1 j_0}$ due to statement (21). Hence, $\psi, \mathsf{A}\psi \in X_{i_0 j_0}$ by part $i_0 = i_1$ of statement (19). Then, $\psi \in Y \subseteq Y'$ by equation (16).

Next, suppose $\psi \notin X_{i_1 j_0}$. Then, $\neg\psi \in X_{i_1 j_0}$ because $X_{i_1 j_0}$ is a maximal consistent set. At the same time, $X_{i_1 j_0} \vdash \mathsf{A}\neg\psi$ by statement (21), the Negation axiom, and the Modus Ponens inference rule. Hence, $\mathsf{A}\neg\psi \in X_{i_1 j_0}$ because $X_{i_1 j_0}$ is a maximal consistent set. Thus, $\neg\psi, \mathsf{A}\neg\psi \in X_{i_1 j_0}$. Then, $\neg\psi, \mathsf{A}\neg\psi \in X_{i_0 j_0}$ by part $i_0 = i_1$ of statement (19). Hence, $\neg\psi \in Y \subseteq Y'$ by equation (16).

Subcase 3B: $i_0 = i_1 \neq i_2$. By part 2(b) of Definition 6.7 and part $j_0 \sim_{i_1} j_1$ of statement (19), statement (20) implies that $\psi \in X_{i_2 j_1}$ iff $\psi \in X_{i_2 j_0}$. Then, $\psi \in X'_{i_2 j_1}$ iff $\psi \in X_{i_2 j_0}$ by equation (17) and the assumption $j_1 < n$ of the case. Hence, $\psi \in X'_{i_2 j_1}$ iff $\psi \in X'_{i_2 n}$ by equation (17). Therefore, $\psi \in X'_{i_2 j_1}$ iff $\psi \in X'_{i_2 j_2}$ by the assumption $j_2 = n$ of the case.

Case 4: $j_2 < j_1 = n$. Then, the assumption $j_1 \sim'_{i_1} j_2$ and the definition of the relation $\sim'$ imply that

$$i_0 = i_1 \quad \text{and} \quad j_0 \sim_{i_0} j_2. \tag{22}$$

Then, $\mathsf{A}\psi \in X'_{i_0 j_1}$ by the assumption $\mathsf{A}\psi \in X'_{i_1 j_1}$. Hence, $X'_{i_0 j_1} \vdash \mathsf{KA}\psi$ by the Introspection of Knowing All axiom and the Modus Ponens inference rule. Thus, $\mathsf{KA}\psi \in X'_{i_0 j_1}$ because $X'_{i_0 j_1}$ is a maximal consistent sets. Then, $\mathsf{KA}\psi \in X'_{i_0 n}$ by the assumption $j_1 = n$ of the case. Hence, $\mathsf{KA}\psi \in Y'$ by equation (17). Then, $\mathsf{KA}\psi \in X_{i_0 j_0}$ by Claim 6.11. Thus, $\mathsf{KA}\psi \in X_{i_0 j_2}$ by item 2(a) of Definition 6.7 and the part $j_0 \sim_{i_0} j_2$ of statement (22). Hence, $\mathsf{KA}\psi \in X_{i_1 j_2}$ by the part $i_0 = i_1$ of statement (22). Then, $X_{i_1 j_2} \vdash \mathsf{A}\psi$ by the Truth axiom and the Modus Ponens inference rule. Thus, because $X_{i_1 j_2}$ is a maximal consistent set,

$$\mathsf{A}\psi \in X_{i_1 j_2}. \tag{23}$$

From this point, the proof continues the same way as in Case 3, except that j_1 plays the role of j_2 and j_2 plays the role of j_1. In addition, we use statements (22) and (23) instead of statements (19) and (20), respectively. This concludes the proof of the claim. $\dashv$

To finish the proof of the lemma, note that, by equation (17) and the definition of relation $\sim'$, we have $\langle (X_{ij}), \{\sim_i\}_i \rangle \sqsubseteq \langle (X'_{ij}), \{\sim'_i\}_i \rangle$. Also, $\neg\psi \in Y \subseteq Y' = X'_{i_0 n}$ by equations (16) and (17). $\dashv$

Next is the last of the three lemmas capturing the recursive construction of a sequence of pseudo models. This lemma extends the sequence when formula $\mathsf{A}\varphi$ does not belong to some maximal consistent set.

As usual in set theory, by $X \triangle Y$ we denote the symmetric difference of sets X and Y. For example, $\{1,2\} \triangle \{2,3\} = \{1,3\}$.

LEMMA 6.13. *For any finite pseudo model* $\langle (X_{ij}), \{\sim_i\}_i\rangle$ *and any formula* $\varphi \in \Phi$, *if* $\mathsf{A}\varphi \notin X_{i_0 j_0}$ *for some* i_0, j_0, *then there is a finite pseudo model* $\langle (X'_{ij}), \{\sim'_i\}_i\rangle$ *such that*:

1. $\langle (X_{ij}), \{\sim_i\}_i\rangle \sqsubseteq \langle (X'_{ij}), \{\sim'_i\}_i\rangle$,
2. *for some* i_1, j_1 *such that* $j_0 \sim_{i_0} j_1$ *and* $\varphi \in X'_{i_1 j_0} \triangle X'_{i_1 j_1}$.

PROOF. By Lemma 6.6, the assumption $\mathsf{A}\varphi \notin X_{i_0 j_0}$ implies that there are maximal consistent sets Y and Z such that

$$\varphi \in Y \quad \text{and} \quad \neg\varphi \in Z, \tag{24}$$

$$\psi \in Y \quad \text{iff} \quad \psi \in Z \quad \text{for any formula } \mathsf{A}\psi \in X_{i_0 j_0}. \tag{25}$$

Assume that matrix (X_{ij}) has the size $m \times n$. Define $(m+1) \times (n+1)$ matrix

$$(X'_{ij}) = \begin{pmatrix} X_{0,0} & \dots & X_{0,j_0} & \dots & X_{0,n-1} & X_{0,j_0} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ X_{i_0-1,0} & \dots & X_{i_0-1,j_0} & \dots & X_{i_0-1,n-1} & X_{i_0-1,j_0} \\ X_{i_0,0} & \dots & X_{i_0,j_0} & \dots & X_{i_0,n-1} & X_{i_0,j_0} \\ X_{i_0+1,0} & \dots & X_{i_0+1,j_0} & \dots & X_{i_0+1,n-1} & X_{i_0+1,j_0} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ X_{m-1,0} & \dots & X_{m-1,j_0} & \dots & X_{m-1,n-1} & X_{m-1,j_0} \\ Y & \dots & Y & \dots & Y & Z \end{pmatrix}. \tag{26}$$

In addition, let the relation $\sim'_{i_0}$ be the transitive, reflexive, and symmetric closure of the relation $\sim_{i_0} \cup \{(j_0, n)\}$ on the set $\{0,1,\dots,n\}$. Let $\sim'_m$ be the reflexive closure of the empty relation. Finally for $i \notin \{i_0, m\}$, let the relation $\sim'_i$ be the reflexive closure of the relation $\sim_i$ on the set $\{0,1,\dots,n\}$.

Observe that, for any formula $\psi \in \Phi$,

$$\psi \in X_{i_0 j_0} \ \text{ iff } \ \psi \in X'_{i_0 n} \tag{27}$$

because sets $X_{i_0 j_0}$ and $X'_{i_0 n}$ are equal by equation (26).

CLAIM 6.14. $\langle (X'_{ij}), \{\sim'_i\}_i\rangle$ *is a finite pseudo model.*

PROOF OF CLAIM. It suffices to verify conditions 2(a) and 2(b) of Definition 6.7.

Condition 2(a). Consider any i, j_1, j_2 such that $j_1 \sim'_i j_2$. It suffices to show that $\mathsf{K}\psi \in X'_{ij_1}$ iff $\mathsf{K}\psi \in X'_{ij_2}$ for any formula $\psi \in \Phi$.

If $i = m$, then the assumption $j_1 \sim'_i j_2$ implies that $j_1 = j_2$ by the definition of the relation $\sim'_m$. Hence, $\mathsf{K}\psi \in X'_{ij_1}$ iff $\mathsf{K}\psi \in X'_{ij_2}$. If $i < m$, then the proof is the same as the proof of Condition 2(a) in Claim 6.12 except that instead of Claim 6.11 it uses statement (27).

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Condition 2(b). Consider any i_1, i_2, j_1, j_2 and any formula $\mathsf{A}\psi \in X'_{i_1 j_1}$ such that $j_1 \sim'_{i_1} j_2$. It suffices to show that $\psi \in X'_{i_2 j_1}$ iff $\psi \in X'_{i_2 j_2}$. We consider the following four cases separately:

Case 1: $j_1, j_2 < n$. If $i_2 = m$, then $X'_{i_2 j_1} = Y = X'_{i_2 j_2}$ by equation (26). Hence, $\psi \in X'_{i_2 j_1}$ iff $\psi \in X'_{i_2 j_2}$. If $i_2 < m$, then the proof is the same as the proof in Case 1 of Condition 2(b) in Claim 6.12.

Case 2: $j_1 = j_2 = n$. Then, $X'_{i_2 j_1} = X'_{i_2 j_2}$. Hence, $\psi \in X'_{i_2 j_1}$ iff $\psi \in X'_{i_2 j_2}$.

Case 3: $j_1 < j_2 = n$. Then, the assumption $j_1 \sim'_{i_1} j_2$ and the definition of the relation $\sim'$ imply that

$$i_0 = i_1 \quad \text{and} \quad j_0 \sim_{i_1} j_1. \tag{28}$$

Also, the assumption $\mathsf{A}\psi \in X'_{i_1 j_1}$, by the assumption $j_1 < n$ of the case and equation (26), implies that

$$\mathsf{A}\psi \in X_{i_1 j_1}. \tag{29}$$

We further divide this case into the following two subcases:

Subcase 3A: $i_2 \neq m$. Then, $\psi \in X_{i_2 j_1}$ iff $\psi \in X_{i_2 j_0}$ by item 2(b) of Definition 6.7 and statements (28) and (29). Hence, by equation (26), the assumption $i_2 \neq m$ of the subcase, and the assumption $j_1 < n$ of the case, $\psi \in X'_{i_2 j_1}$ iff $\psi \in X'_{i_2 j_n}$. Thus, $\psi \in X'_{i_2 j_1}$ iff $\psi \in X'_{i_2 j_2}$ by the assumption $j_2 = n$ of the case.

Subcase 3B: $i_2 = m$. Note that $\mathsf{A}\psi \in X_{i_0 j_1}$ by statement (29) and the part $i_0 = i_1$ of statement (28). Thus, $X_{i_0 j_1} \vdash \mathsf{KA}\psi$ by the Introspection of Knowing All axiom and the Modus Ponens inference rule. Hence, because $X_{i_0 j_1}$ is a maximal consistent set, $\mathsf{KA}\psi \in X_{i_0 j_1}$. Then, $\mathsf{KA}\psi \in X_{i_0 j_0}$ by part 2(a) of Definition 6.7 and both parts of statement (28). Hence, $X_{i_0 j_0} \vdash \mathsf{A}\psi$ by the Truth axiom and the Modus Ponens inference rule. Thus, $\mathsf{A}\psi \in X_{i_0 j_0}$ because $X_{i_0 j_0}$ is a maximal consistent set. Then, $\psi \in Y$ iff $\psi \in Z$ by statement (25). Therefore, $\psi \in X'_{i_2 j_1}$ iff $\psi \in X'_{i_2 j_2}$ by equation (26), the assumption $i_2 = m$ of the subcase, and the assumption $j_1 < j_2 = n$ of the case.

Case 4: $j_2 < j_1 = n$. Similar to how we proved statements (22) and (23) in Case 4 of the proof of Claim 6.12, we can show that

$$i_0 = i_1 \quad \text{and} \quad j_0 \sim_{i_0} j_2, \tag{30}$$

$$\mathsf{A}\psi \in X_{i_1 j_1}. \tag{31}$$

From this point, the proof continues the same way as in Case 3 of the current proof, except that j_1 plays the role of j_2 and j_2 plays the role of j_1. In addition, we use statements (30) and (31) instead of statements (28) and (29), respectively. This concludes the proof of the claim. $\dashv$

To finish the proof of the lemma, note that item 1 of this lemma is true by the choice of the pseudo model $\langle (X'_{ij}), \{\sim'_i\}_i \rangle$. To prove item 2, observe that $\varphi \in Y \triangle Z$ by statement (24). Thus, $\varphi \in X'_{m j_0} \triangle X'_{mn}$ by equation (26). Also, recall that $j_0 \sim'_{i_0} n$ by the definition of relation $\sim'$. $\dashv$

The previous lemma concludes the description of the three steps of the recursive construction of a sequence of pseudo models. The next definition captures the

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property of the pseudo model at which the construction terminates after a finite number of steps or, as we will see later, the property of the pseudo model obtained at the limit of the construction.

Definition 6.15. A pseudo model $\langle (X_{ij}), \{\sim_i\}_i \rangle$ is *closed* when for any formula $\varphi \in \Phi$ and any indices i_0, j_0,

1. if $\neg \mathsf{K}\varphi \in X_{i_0 j_0}$, then there is an index j_1 such that $j_0 \sim_{i_0} j_1$ and $\neg\varphi \in X_{i_0 j_1}$,
2. if $\neg \mathsf{A}\varphi \in X_{i_0 j_0}$, then there are indices i_1, j_1 such that $j_0 \sim_{i_0} j_1$ and $\varphi \in X_{i_1 j_0} \triangle X_{i_1 j_1}$.

For any sequence of pairs of sets $(S_0, T_0), (S_1, T_1), (S_2, T_2), \dots,$ by union $\bigcup_i (S_i, T_i)$ we mean the pair $(\bigcup_i S_i, \bigcup_i T_i)$. Recall that, formally, by a matrix of size $m \times n$ we mean a function defined on the Cartesian product of ordinals m and n. Note that a family of equivalence relations $\{\sim_i\}_i$ can also be viewed as a function that assigns a relation $\sim_i$ to each index i. Hence, any pseudo model is a pair of functions. Thus, keeping in mind that functions (as all relations) are sets, for any sequence $M_0, M_1, M_2, \dots$ of pseudo models, one can consider the union $\bigcup_i M_i$.

Lemma 6.16. *For any infinite chain $M_0 \sqsubseteq M_1 \sqsubseteq M_2 \sqsubseteq \dots$ of pseudo models, $\bigcup_i M_i$ is a pseudo model and $M_i \sqsubseteq \bigcup_i M_i$.*

Proof. The statement of the lemma follows from Definitions 6.7 and 6.8. ⊣

Lemma 6.17. *For any finite pseudo model M there is a closed (possibly infinite) pseudo model M' such that $M \sqsubseteq M'$.*

Proof. Let $(i_1, j_1, \sigma_1), \dots, (i_k, j_k, \sigma_k), \dots$ be any enumeration of all such triples that i_k and j_k are non-negative integers and σ_k is a formula of the form either $\mathsf{K}\psi$ or $\mathsf{A}\psi$. Construct an infinite chain of pseudo models $M_0 \sqsubseteq M_1 \sqsubseteq M_2 \sqsubseteq \dots$ using the following infinite recursive procedure:

Base Case: Let $M_0 = M$. Label all elements of the sequence (i_1, j_1, σ_1), ..., $(i_k, j_k, \sigma_k), \dots$ as "*unfulfilled*."

Recursive Step: Suppose that chain $M_0 \sqsubseteq M_1 \sqsubseteq M_2 \sqsubseteq M_{k-1}$ is constructed and some of the elements of the sequence $(i_1, j_1, \sigma_1), \dots, (i_k, j_k, \sigma_k), \dots$ are already labeled as "fulfilled."

Do the following steps to construct pseudo model M_k:

1. Consider the smallest ℓ such that model M_{k-1} contain cells (i_ℓ, j_ℓ), the element $(i_\ell, j_\ell, \sigma_\ell)$ is labeled as "*unfulfilled*," and $\sigma_\ell \notin X_{i_\ell, j_\ell}$. If such ℓ does not exist, define $M_k = M_{k-1}$ and skip the next two steps.
2. Use either Lemma 6.10 (when σ_ℓ has the form $\mathsf{K}\psi$) or Lemma 6.13 (when formula σ_ℓ has the form $\mathsf{A}\psi$) to extend the pseudo model M_{k-1} to a new pseudo model M_k.
3. Label the element $(i_\ell, j_\ell, \sigma_\ell)$ as "fulfilled."

Let $M' = \bigcup_i M_i$. Note that M' is a pseudo model by Lemma 6.16. Pseudo model M' is closed by Definition 6.15. ⊣

6.3. Completeness: final steps. So far, we have discussed pseudo models of our logical system. We are now ready to describe how any closed pseudo model can be converted into a model as described in Definition 2.1.

DEFINITION 6.18. For any pseudo model $M = \langle (X_{ij}), \{\sim_i\}_i \rangle$, where (X_{ij}) is a (possibly infinite) matrix of size $m \times n$, let M^* be the epistemic model

$$(\{i \in \omega \mid i < m\}, \{j \in \omega \mid j < n\}, \{\sim_i\}_i, \pi),$$

where $\pi(p)$ is equal to the set $\{(i, j) \mid i < m, j < n, p \in X_{ij}\}$ for each propositional variable p.

The next lemma connects a closed pseudo model M with the corresponding epistemic model M^*.

LEMMA 6.19. *For any closed pseudo model $M = \langle (X_{ij}), \{\sim_i\}_i \rangle$ and any formula $\varphi \in \Phi$, if $\Vdash$ is the satisfaction relation for the epistemic model M^*, then $(i, j) \Vdash \varphi$ iff $\varphi \in X_{ij}$ for all i, j.*

PROOF. We prove the statement of the lemma by structural induction on the formula φ.

Suppose that formula φ is a propositional variable p. Then, $(i, j) \Vdash p$ iff $(i, j) \in \pi(p)$ by item 1 of Definition 2.2. At the same time, $(i, j) \in \pi(p)$ iff $p \in X_{ij}$ by Definition 6.18.

If formula φ is a negation or an implication, then the statement of the lemma follows from the induction hypothesis, Definition 2.2, and the assumption that X_{ij} is a maximal consistent set in the standard way.

Suppose that formula φ has the form $\mathsf{K}\psi$.

$(\Rightarrow)$: Assume that $\mathsf{K}\psi \notin X_{ij}$. Thus, $\neg\mathsf{K}\psi \in X_{ij}$ because X_{ij} is a maximal consistent set. Then, by the assumption of the lemma that model M is closed and item 1 of Definition 6.15, there exists an index j' such that $j \sim_i j'$ and $\neg\varphi \in X_{ij'}$. Thus, $\psi \notin X_{ij'}$ because the set $X_{ij'}$ is consistent. Hence, $(i, j') \nVdash \psi$ by the induction hypothesis. Therefore, $(i, j) \nVdash \mathsf{K}\psi$ by item 4 of Definition 2.2 and the statement $j \sim_i j'$.

$(\Leftarrow)$: Suppose that $\mathsf{K}\psi \in X_{ij}$. Consider any j' such that $j \sim_i j'$. By item 4 of Definition 2.2, it suffices to show that $(i, j') \Vdash \psi$. Indeed, the assumptions $\mathsf{K}\psi \in X_{ij}$ and $j \sim_i j'$, by item 2(a) of Definition 6.7, imply that $\mathsf{K}\psi \in X_{ij'}$. Hence, $X_{ij'} \vdash \psi$ by the Truth axiom and the Modus Ponens inference rule. Thus, $\psi \in X_{ij'}$ because $X_{ij'}$ is a maximal consistent set. Therefore, by the induction hypothesis, $(i, j') \nVdash \psi$.

Suppose that formula φ has the form $\mathsf{A}\psi$.

$(\Rightarrow)$: The assumption $\mathsf{A}\psi \notin X_{ij}$ implies $\neg\mathsf{A}\psi \in X_{ij}$ because X_{ij} is a maximal consistent set. Hence, by the assumption of the lemma that pseudo model M is closed and item 2 of Definition 6.15, there is an agent i' and a world j' such that $j \sim_i j'$ and $\psi \in X_{i'j} \triangle X_{i'j'}$. Thus, by the induction hypothesis, *exactly one* of the following statements is true: $(i', j) \Vdash \psi$ or $(i', j') \Vdash \psi$. Then, $(i, j) \nVdash \mathsf{A}\psi$ by the assumption $j \sim_i j'$ and item 5 of Definition 2.2.

$(\Leftarrow)$: Let $\mathsf{A}\psi \in X_{ij}$. Towards the proof of $(i, j) \Vdash \mathsf{A}\psi$, consider an arbitrary agent i' and arbitrary worlds j', j'' such that $j \sim_i j', j \sim_i j''$, and $(i', j') \Vdash \psi$. By item 5 of Definition 2.2, it suffices to show that $(i', j'') \Vdash \psi$.

First, by the induction hypothesis, the assumption $(i', j') \Vdash \psi$ implies $\psi \in X_{i'j'}$. Thus, by item 2(b) of Definition 6.7, the assumption $\mathsf{A}\psi \in X_{ij}$ and the assumption $j \sim_i j'$ imply that $\psi \in X_{i'j}$.

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Second, again by item 2(b) of Definition 6.7, the assumption $\mathsf{A}\psi \in X_{ij}$ and the assumption $j \sim_i j''$ imply that $\psi \in X_{i'j''}$. Therefore, $(i', j'') \Vdash \psi$ by the induction hypothesis. ⊣

We are now ready to state and prove the strong completeness theorem for our logical system.

THEOREM 6.20. *For any set of formulae $Y \subseteq \Phi$ and any formula $\varphi \in \Phi$, if $Y \nvdash \varphi$, then there is an agent a and a world w of an epistemic model such that $(a, w) \Vdash \gamma$ for each $\gamma \in Y$ and $(a, w) \nVdash \varphi$.*

PROOF. By Lemma 6.9, there is a finite pseudo model $M = \langle (X_{ij}), \{\sim_i\}_i \rangle$ such that $Y \cup \{\neg\varphi\} \subseteq X_{00}$. By Lemma 6.17, there is a closed pseudo model $M' = \langle (X'_{ij}), \{\sim'_i\}_i \rangle$ such that $M \sqsubseteq M'$. Note that $X_{00} = X'_{00}$ by item 2 of Definition 6.8. Hence, $Y \cup \{\neg\varphi\} \subseteq X'_{00}$. Thus, $\gamma \in X'_{00}$ for each formula $\gamma \in Y$ and, because set X'_{00} is consistent, $\varphi \notin X'_{00}$. Therefore, $(0, 0) \Vdash \gamma$ for each $\gamma \in Y$ and $(0, 0) \nVdash \varphi$ in epistemic model M^* by Lemma 6.19. ⊣

§7. Decidability.

In this section, we prove the decidability of our logical system. The standard way to show decidability in modal logic is to prove completeness with respect to the class of finite models. This usually can be done using the filtration technique in one of two ways: on-the-fly filtration or post-filtration. The on-the-fly filtration restricts formulae in all maximal consistent sets to subformulae of a given formula. This technique is often more efficient, but it only proves completeness instead of strong completeness. Post-filtration is applied after strong completeness is shown. It consists of collapsing the infinite model into a finite one by merging "similar" possible worlds.

It appears that our matrix construction is too complex for either an on-the-fly filtration or a post-filtration to be feasible. Instead, we use a different approach suggested by an anonymous reviewer. We introduce neighborhood semantics for our logic which is very different and is much simpler than the one given in Definitions 2.1 and 2.2. Then, we use on-the-fly filtration to show the completeness of our logical system with respect to the class of finite neighborhood models. Note that, unlike our original semantics, the neighborhood semantics does not capture the intended meaning of modality A as "know how to tell apart." This semantics is only used to prove the decidability of our logical system. Because strong completeness is not required to prove decidability, the fact that we do not show strong completeness for the neighborhood semantics is not significant.

7.1. Neighborhood semantics.

In this subsection, we define a new and simple semantics for our logical system. As described above, this semantics will be used later to prove the decidability of the system. By $\mathcal{P}(X)$ we denote the power set of set X.

DEFINITION 7.1. A neighborhood model is a tuple $(W, \sim, \{\mathcal{N}_w\}_{w \in W}, \pi)$, where:

1. W is a (possibly empty) set of worlds,
2. $\sim$ is an equivalence relation on set W,
3. $\mathcal{N}_w \subseteq \mathcal{P}(W)$ is a family of "neighborhoods" of world $w \in W$ such that:
   (a) $W \in \mathcal{N}_w$ for each world $w \in W$,
   (b) $X \cap Y \in \mathcal{N}_w$ for any world $w \in W$ and any $X, Y \in \mathcal{N}_w$,

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(c) $W \setminus X \in \mathcal{N}_w$ for any world $w \in W$ and any $X \in \mathcal{N}_w$,
(d) $\mathcal{N}_w = \mathcal{N}_u$ for any worlds $w, u \in W$ such that $w \sim u$,
(e) if $w \in X$, $X \in \mathcal{N}_w$, and $w \sim u$, then $u \in X$,

4. $\pi(p) \subseteq W$ for each propositional variable p.

Definition 7.2. For any formula $\varphi \in \Phi$ and any world $w \in W$ of a neighborhood model $(W, \sim, \{\mathcal{N}_w\}_{w \in W}, \pi)$, the satisfaction relation $w \Vdash \varphi$ is defined as follows:

1. $w \Vdash p$ if $w \in \pi(p)$.
2. $w \Vdash \neg\varphi$ if $w \nVdash \varphi$.
3. $w \Vdash \varphi \to \psi$ if either $w \nVdash \varphi$ or $w \Vdash \psi$.
4. $w \Vdash \mathsf{K}\varphi$ if $u \Vdash \varphi$ for each world $u \in W$ such that $u \sim w$.
5. $w \Vdash \mathsf{A}\varphi$ if $\{u \in W \mid u \Vdash \varphi\} \in \mathcal{N}_w$.

The proof of the next theorem is straightforward.

Theorem 7.3 (Soundness). *If $\vdash \varphi$, then $w \Vdash \varphi$ for each world $w \in W$ of each neighborhood model $(W, \sim, \{\mathcal{N}_w\}_{w \in W}, \pi)$.*

7.2. Capturing sets. In the next several subsections, we prove the completeness of our logical system with respect to neighborhood semantics. Towards this proof, in this subsection, we introduce the notion of capturing sets. This notion will be used in the next subsection to define the worlds of the canonical neighborhood model. Informally, a set of formulae Y *captures* a set of formulae X if set Y is at least as expressive as set X.

Definition 7.4. A set $Y \subseteq \Phi$ captures a set $X \subseteq \Phi$, written as $X \preceq Y$, if for each formula $\chi \in X$, there is a formula $\gamma \in Y$ such that $\vdash \chi \leftrightarrow \gamma$.

It is easy to see that $\preceq$ is a reflexive and transitive relation on $\mathcal{P}(\Phi)$. More importantly, it has the following *right upward monotonicity* property:

Lemma 7.5. *If $X \preceq Y$ and $Y \subseteq Z$, then $X \preceq Z$.*

Definition 7.6. For any set of formula X, let $\widehat{X}$ be the set of all Boolean combinations of formulae from X.

Let us now consider arbitrary Boolean expressions built from a fixed finite set of propositional variables $x_1, \dots, x_n$. Although there are infinitely many such *expressions*, there are only 2^{2^n} Boolean *functions* that these expressions represent. Any two Boolean expressions that represent the same Boolean function must be provably equivalent in propositional logic due to the completeness theorem for propositional calculus. Thus, one can identify a "basis" of 2^{2^n} Boolean expressions such that all Boolean expressions are propositionally equivalent to one of the expressions in the basis.

We will now apply the observation from the previous paragraph to an arbitrary finite set X of formulae in language Φ. We can think of formulae in X as atomic propositions. Formulae in set $\widehat{X}$ are arbitrary Boolean combinations of these atomic propositions. Thus, set $\widehat{X}$ must contain a finite subset (basis) Y such that each formula in set $\widehat{X}$ is provably equivalent to one of the formulae in Y. Note that "provably" here refers to provability from tautologies using the Modus Ponens inference rule only. However, such provability implies provability in our logical system. Thus, the following lemma holds.

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LEMMA 7.7. *For any finite set X, there is a finite set Y such that $\widehat{X} \preceq Y$.*

In other words, if set X is finite, then the (infinite!) set $\widehat{X}$ can be captured by some *finite* set.

7.3. Canonical model. In this subsection, for any nonempty *finite* set Γ of formulae, we construct a finite canonical neighborhood model. The model is finite in the sense that the set of possible worlds is finite. This implies that all neighborhoods and all sets of neighborhoods are also finite.

By Lemma 7.7, there is a finite set Γ_1 such that $\widehat{\Gamma} \preceq \Gamma_1$. By Lemma 7.5, we can assume that $\Gamma \subseteq \Gamma_1$. By Definition 7.4, for any formula $\gamma \in \widehat{\Gamma}$ there is a formula $\gamma^* \in \Gamma_1$ such that

$$\vdash \gamma \leftrightarrow \gamma^*. \tag{32}$$

Throughout the rest of this section, we fix a specific function $* : \widehat{\Gamma} \to \Gamma_1$ that satisfies statement (32). The next lemma follows from statement (32) and the Substitution inference rule.

LEMMA 7.8. $\vdash \mathsf{A}\gamma \leftrightarrow \mathsf{A}(\gamma^*)$ *for any formula $\gamma \in \widehat{\Gamma}$.*

DEFINITION 7.9. $\Gamma_2 = \Gamma_1 \cup \{\mathsf{A}\gamma, \neg\mathsf{A}\gamma, \mathsf{K}\gamma, \neg\mathsf{K}\gamma, \neg\gamma \mid \gamma \in \Gamma_1\}$.

DEFINITION 7.10. W is the set of maximal consistent subsets of set Γ_2.

Note that sets Γ_2 and W are finite due to set Γ_1 being finite.

Usually, in canonical models for epistemic logics, two worlds are called indistinguishable if they contain the same K-formulae. If the indistinguishable worlds are maximal consistent sets of formulae, then the Introspection of Knowing All axiom and the Truth axiom would imply that these sets will also have the same A-formulae. Indeed, if one world contains a formula $\mathsf{A}\varphi$, then, by the Introspection of Knowing All axiom, it also contains formula $\mathsf{KA}\varphi$. Then, the other world contains a formula $\mathsf{KA}\varphi$ and, by the Truth axiom, it also contains the formula $\mathsf{A}\varphi$. However, because we are constructing a finite model, we define the worlds as maximal consistent *subsets* of Γ_2. As a result, formula $\mathsf{KA}\varphi$, from the above argument, might not belong to a world even if formula $\mathsf{A}\varphi$ does. To account for this, we add item 2 to the definition below.

DEFINITION 7.11. For any worlds $w, u \in W$, let $w \sim u$ when for each formula $\gamma \in \Gamma_1$,

1. $\mathsf{K}\gamma \in w$ iff $\mathsf{K}\gamma \in u$,
2. $\mathsf{A}\gamma \in w$ iff $\mathsf{A}\gamma \in u$.

We now proceed to define the families of neighborhoods $\mathcal{N}_w$. We start by defining a neighborhood $N(\gamma) \subseteq W$ for each formula $\gamma \in \Gamma_1$. After that, we specify to which of the families each neighborhood belongs to.

DEFINITION 7.12. $N(\gamma) = \{w \in W \mid \gamma \in w\}$ for any formula $\gamma \in \Gamma_1$.

DEFINITION 7.13. $\mathcal{N}_w = \{N(\gamma) \mid \gamma \in \Gamma_1, \mathsf{A}\gamma \in w\}$.

Intuitively, the above definition is chosen to match Definition 7.12 and item 5 of Definition 7.2.

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DEFINITION 7.14.

$$\pi(p) = \begin{cases} N(p), & \text{if } p \in \Gamma, \\ \varnothing, & \text{otherwise.} \end{cases}$$

7.4. Properties of neighborhoods. In the previous subsection, we have defined the canonical neighborhood model. In the next subsection, we show that it satisfies properties 3(a)–(e) of Definition 7.1. Towards this goal, in this subsection, we establish three auxiliary properties of the neighborhoods.

LEMMA 7.15. *$(\gamma \to \gamma)^* \in \Gamma_1$ and $N((\gamma \to \gamma)^*) = W$ for any $\gamma \in \Gamma$.*

PROOF. Note that $\gamma \to \gamma \in \widehat{\Gamma}$ by Definition 7.6. Hence, $(\gamma \to \gamma)^* \in \Gamma_1$ because function $*$ maps set $\widehat{\Gamma}$ into set Γ_1.

To prove the other part of the lemma, consider any world $w \in W$. It suffices to show that $w \in N((\gamma \to \gamma)^*)$. Indeed, $\gamma \to \gamma$ is a tautology. Thus, $\vdash (\gamma \to \gamma)^*$ due to statement (32). Thus, $(\gamma \to \gamma)^* \in w$ because $(\gamma \to \gamma)^* \in \Gamma_1 \subseteq \Gamma_2$ and w is a maximal consistent subset of Γ_2. Therefore, $w \in N((\gamma \to \gamma)^*)$ by Definition 7.12. ⊣

LEMMA 7.16. *$N(\gamma_1) \cap N(\gamma_2) = N((\gamma_1 \wedge \gamma_2)^*)$ for any formulae $\gamma_1, \gamma_2 \in \Gamma_1$.*

PROOF. Consider any world $w \in W$. It suffices to prove that the statements $w \in N(\gamma_1) \cap N(\gamma_2)$ and $w \in N((\gamma_1 \wedge \gamma_2)^*)$ are equivalent.

By the definition of the intersection, the statement $w \in N(\gamma_1) \cap N(\gamma_2)$ is equivalent to the conjunction of the statements

$$w \in N(\gamma_1) \text{ and } w \in N(\gamma_2).$$

By Definition 7.12, the conjunction of the above two statements is equivalent to the conjunction of the statements

$$\gamma_1 \in w \text{ and } \gamma_2 \in w.$$

Because $\gamma_1, \gamma_2 \in \Gamma_1 \subseteq \Gamma_2$ and w is a maximal consistent subset of Γ_2, the conjunction of the above two formulae is equivalent to the conjunction of the statements

$$w \vdash \gamma_1 \text{ and } w \vdash \gamma_2.$$

By the laws of propositional logic, the conjunction of the two above statements is equivalent to

$$w \vdash \gamma_1 \wedge \gamma_2.$$

Note that $\gamma_1 \wedge \gamma_2 \in \widehat{\Gamma}$ because $\gamma_1, \gamma_2 \in \Gamma_1 \subseteq \widehat{\Gamma}$ and set $\widehat{\Gamma}$ is closed with respect to Boolean operations. Hence, by statement (32), the above formula is equivalent to

$$w \vdash (\gamma_1 \wedge \gamma_2)^*.$$

Observe that $(\gamma_1 \wedge \gamma_2)^* \in \Gamma_1 \subseteq \Gamma_2$ because function $*$ maps set $\widehat{\Gamma}$ to Γ_1. Since w is a maximal consistent subset of Γ_2, the above formula is equivalent to

$$(\gamma_1 \wedge \gamma_2)^* \in w.$$

The last statement is equivalent to the statement $w \in N((\gamma_1 \wedge \gamma_2)^*)$ by Definition 7.12. ⊣

LEMMA 7.17. *$W \setminus N(\gamma) = N((\neg\gamma)^*)$ for any formula $\gamma \in \Gamma_1$.*

PROOF. Consider any world $w \in W$. It suffices to show that $w \notin N(\gamma)$ iff $w \in N((\neg\gamma)^*)$. By Definition 7.12, the statement $w \notin N(\gamma)$ is equivalent to the statement

$$\gamma \notin w.$$

Note that $\gamma \in \Gamma_1$ by the assumption of the lemma. Thus, $\neg\gamma \in \Gamma_2$ by Definition 7.9. Then, because set w is a maximal consistent subset of Γ_2, the above statement is equivalent to

$$w \vdash \neg\gamma.$$

Observe that $\gamma \in \Gamma_1$ also implies $\gamma \in \Gamma_1 \subseteq \widehat{\Gamma}$. Then, $\neg\gamma \in \widehat{\Gamma}$ due to set $\widehat{\Gamma}$ being closed with respect to Boolean operations. Thus, by statement (32), the above statement is equivalent to

$$w \vdash (\neg\gamma)^*.$$

Note that $(\neg\gamma)^* \in \Gamma_1 \subseteq \Gamma_2$ because function $*$ maps set $\widehat{\Gamma}$ into set γ_2. Then, because set w is a maximal consistent subset of Γ_2, the above statement is equivalent to

$$(\neg\gamma)^* \in w.$$

The last statement is equivalent to $w \in N((\neg\gamma)^*)$ by Definition 7.12. ⊣

Note that the proof above is somewhat convoluted because, for an arbitrary $\gamma \in \Gamma_1$, formula $\neg\gamma$ belongs to sets $\widehat{\Gamma}$ and Γ_2, but not necessarily to Γ_1.

7.5. Well-definedness of canonical model. In this subsection, we prove that the canonical model $(W, \sim, \{\mathcal{N}_w\}_{w\in W}, \pi)$ is a neighborhood model as specified in Definition 7.1. For this, we verify items 3(a)–(e) of the definition.

LEMMA 7.18. *$W \in \mathcal{N}_w$ for each world $w \in W$.*

PROOF. Recall the assumption in the beginning of Section 7.3 that set Γ is nonempty. Let γ be any formula from Γ. Note that $\gamma \to \gamma$ is a propositional tautology. Thus, $\vdash \mathsf{A}(\gamma \to \gamma)$ by the Necessitation inference rule. Note that $\gamma \to \gamma \in \widehat{\Gamma}$. Hence, by Lemma 7.8 and propositional reasoning

$$\vdash \mathsf{A}((\gamma \to \gamma)^*). \tag{33}$$

Note that $(\gamma \to \gamma)^* \in \Gamma_1$ by Lemma 7.15. Thus, $\mathsf{A}((\gamma \to \gamma)^*) \in \Gamma_2$ by Definition 7.9. Then, $\mathsf{A}((\gamma \to \gamma)^*) \in w$ by statement (33) and because w is a maximal consistent subset of Γ_2. Hence, $N((\gamma \to \gamma)^*) \in \mathcal{N}_w$ by Definition 7.13 and the part $(\gamma \to \gamma)^* \in \Gamma_1$ of Lemma 7.15. Therefore, $w \in \mathcal{N}_w$ by the part $N((\gamma \to \gamma)^*) = W$ of Lemma 7.15. ⊣

LEMMA 7.19. *For any formulae $\gamma_1, \gamma_2 \in \Gamma_1$ and any world $w \in W$, if $N(\gamma_1), N(\gamma_2) \in \mathcal{N}_w$, then $N(\gamma_1) \cap N(\gamma_2) \in \mathcal{N}_w$.*

PROOF. The assumption $N(\gamma_1), N(\gamma_2) \in \mathcal{N}_w$ implies $\mathsf{A}\gamma_1, \mathsf{A}\gamma_2 \in w$ by Definition 7.13. Thus, $w \vdash \mathsf{A}(\gamma_1 \wedge \gamma_2)$ by the Conjunction axiom and propositional reasoning. Note that $\gamma_1 \wedge \gamma_2 \in \widehat{\Gamma}$. Hence, $w \vdash \mathsf{A}((\gamma_1 \wedge \gamma_2)^*)$ by Lemma 7.8. Observe that $\mathsf{A}((\gamma_1 \wedge \gamma_2)^*) \in \Gamma_2$ by Definition 7.9 and because the range of function $*$ is Γ_1. Then, $\mathsf{A}((\gamma_1 \wedge$

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$\gamma_2)^*) \in w$ because w is a maximal consistent subset of Γ_2. Thus, $N((\gamma_1 \wedge \gamma_2)^*) \in \mathcal{N}_w$ by Definition 7.13. Therefore, $N(\gamma_1) \cap N(\gamma_2) \in \mathcal{N}_w$ by Lemma 7.16. ⊣

LEMMA 7.20. *For any $\gamma \in \Gamma_1$, if $N(\gamma) \in \mathcal{N}_w$, then $W \setminus N(\gamma) \in \mathcal{N}_w$.*

PROOF. The assumption $N(\gamma) \in \mathcal{N}_w$ implies $\mathsf{A}\gamma \in w$ by Definition 7.13. Thus, $w \vdash \mathsf{A}\neg\gamma$ by the Negation axiom and the Modus Ponens inference rule. Hence, $w \vdash \mathsf{A}((\neg\gamma)^*)$ by Lemma 7.8 and propositional reasoning. Note that $\mathsf{A}((\neg\gamma)^*) \in \Gamma_2$ by Definition 7.9 and because the range of function $*$ is Γ_1. Then, $\mathsf{A}((\neg\gamma)^*) \in w$ because w is a maximal consistent subset of Γ_2. Thus, $N((\neg\gamma)^*) \in \mathcal{N}_w$ by Definition 7.13. Therefore, $W \setminus N(\gamma) \in \mathcal{N}_w$ by Lemma 7.17. ⊣

LEMMA 7.21. *For any formula $\gamma \in \Gamma_1$, if $N(\gamma) \in \mathcal{N}_w$ and $w \sim u$, then $N(\gamma) \in \mathcal{N}_u$.*

PROOF. The assumption $N(\gamma) \in \mathcal{N}_w$ implies that $\mathsf{A}\gamma \in w$ by Definition 7.13. Thus, $\mathsf{A}\gamma \in u$ by Definition 7.11 and the assumption $w \sim u$ of the lemma. Therefore, $N(\gamma) \in \mathcal{N}_u$ by Definition 7.13. ⊣

LEMMA 7.22. *For any formula $\gamma \in \Gamma_1$, if $N(\gamma) \in \mathcal{N}_w$, $w \in N(\gamma)$, and $w \sim u$, then $u \in N(\gamma)$.*

PROOF. The assumptions $N(\gamma) \in \mathcal{N}_w$ and $w \in N(\gamma)$ imply $\mathsf{A}\gamma \in w$ and $\gamma \in w$ by Definitions 7.13 and 7.12, respectively. Then, $w \vdash \mathsf{K}\gamma$ by the Self-Knowledge axiom and the Modus Ponens inference rule applied twice. Note that $\mathsf{K}\gamma \in \Gamma_2$ by Definition 7.9. Hence, $\mathsf{K}\gamma \in w$ because w is a maximal consistent subset of Γ_2. Thus, $\mathsf{K}\gamma \in u$ by Definition 7.11 and the assumption $w \sim u$ of the lemma. Then, $u \vdash \gamma$ by the Truth axiom and the Modus Ponens inference rule. Hence $\gamma \in u$ because $\gamma \in \Gamma_1 \subseteq \Gamma_2$ and w is a maximal consistent subset of Γ_2. Therefore, $u \in N(\gamma)$ by Definition 7.12. ⊣

7.6. Truth lemma. In this subsection, we prove the truth lemma for the neighborhood semantics. It is stated and proven as Lemma 7.27 at the end of this subsection. We start with several lemmas used in the induction step of the proof of Lemma 7.27. In this section, we assume that the finite set Γ is closed with respect to subformulae.

LEMMA 7.23. *For any world $w \in W$ and any formula $\mathsf{K}\varphi \in \Gamma$ such that $\mathsf{K}\varphi \notin w$, there is a world $u \in W$ such that $w \sim u$ and $\varphi \notin u$.*

PROOF. Consider the following set of formulae

$$X = \{\neg\varphi\} \cup \{\mathsf{K}\psi \mid \mathsf{K}\psi \in w\} \cup \{\neg\mathsf{K}\chi \mid \neg\mathsf{K}\chi \in w\} \tag{34}$$

$$\cup \{\mathsf{A}\sigma \mid \mathsf{A}\sigma \in w\} \cup \{\neg\mathsf{A}\tau \mid \neg\mathsf{A}\tau \in w\}. \tag{35}$$

CLAIM 7.24. *Set X is consistent.*

PROOF OF CLAIM. Suppose the opposite. Then, there are formulae

$$\mathsf{K}\psi_1, \dots, \mathsf{K}\psi_k, \neg\mathsf{K}\chi_1, \dots, \neg\mathsf{K}\chi_\ell, \mathsf{A}\sigma_1, \dots, \mathsf{A}\sigma_m, \neg\mathsf{A}\tau_1, \dots, \neg\mathsf{A}\tau_n \in w \tag{36}$$

such that

$$\mathsf{K}\psi_1, \dots, \mathsf{K}\psi_k, \neg\mathsf{K}\chi_1, \dots, \neg\mathsf{K}\chi_\ell, \mathsf{A}\sigma_1, \dots, \mathsf{A}\sigma_m, \neg\mathsf{A}\tau_1, \dots, \neg\mathsf{A}\tau_n \vdash \varphi.$$

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Then, by Lemma 4.2,

$$\mathsf{K}\mathsf{K}\psi_1,\dots,\mathsf{K}\mathsf{K}\psi_k,\mathsf{K}\neg\mathsf{K}\chi_1,\dots,\mathsf{K}\neg\mathsf{K}\chi_\ell,\mathsf{K}\mathsf{A}\sigma_1,\dots,\mathsf{K}\mathsf{A}\sigma_m,\\ \mathsf{K}\neg\mathsf{A}\tau_1,\dots,\mathsf{K}\neg\mathsf{A}\tau_n\vdash\mathsf{K}\varphi.$$

Thus, by Lemma 4.3 and the Modus Ponens rule applied k times,

$$\mathsf{K}\psi_1,\dots,\mathsf{K}\psi_k,\mathsf{K}\neg\mathsf{K}\chi_1,\dots,\mathsf{K}\neg\mathsf{K}\chi_\ell,\mathsf{K}\mathsf{A}\sigma_1,\dots,\mathsf{K}\mathsf{A}\sigma_m,\\ \mathsf{K}\neg\mathsf{A}\tau_1,\dots,\mathsf{K}\neg\mathsf{A}\tau_n\vdash\mathsf{K}\varphi.$$

Hence, by the Negative Introspection axiom and the Modus Ponens inference rule applied ℓ times,

$$\mathsf{K}\psi_1,\dots,\mathsf{K}\psi_k,\neg\mathsf{K}\chi_1,\dots,\neg\mathsf{K}\chi_\ell,\mathsf{K}\mathsf{A}\sigma_1,\dots,\mathsf{K}\mathsf{A}\sigma_m,\\ \mathsf{K}\neg\mathsf{A}\tau_1,\dots,\mathsf{K}\neg\mathsf{A}\tau_n\vdash\mathsf{K}\varphi.$$

Then, by the Introspection of Knowing All axiom and the Modus Ponens inference rule applied m times,

$$\mathsf{K}\psi_1,\dots,\mathsf{K}\psi_k,\neg\mathsf{K}\chi_1,\dots,\neg\mathsf{K}\chi_\ell,\mathsf{A}\sigma_1,\dots,\mathsf{A}\sigma_m,\mathsf{K}\neg\mathsf{A}\tau_1,\dots,\mathsf{K}\neg\mathsf{A}\tau_n\vdash\mathsf{K}\varphi.$$

Thus, by Lemma 4.4 and the Modus Ponens inference rule applied n times,

$$\mathsf{K}\psi_1,\dots,\mathsf{K}\psi_k,\neg\mathsf{K}\chi_1,\dots,\neg\mathsf{K}\chi_\ell,\mathsf{A}\sigma_1,\dots,\mathsf{A}\sigma_m,\neg\mathsf{A}\tau_1,\dots,\neg\mathsf{A}\tau_n\vdash\mathsf{K}\varphi.$$

Hence, $w\vdash\mathsf{K}\varphi$ by statement (36). Observe that $\mathsf{K}\varphi\in\Gamma$ by the assumption of the lemma. Hence, $\mathsf{K}\varphi\in\Gamma\subseteq\Gamma_1\subseteq\Gamma_2$. Therefore, $\mathsf{K}\varphi\in w$ because w is a maximal consistent subset of Γ_2, which contradicts the assumption of the lemma. ⊣

By Lemma 4.5, set X can be extended to a maximal consistent set X'. Let u be the set $\Gamma_2\cap X'$. Note that the assumption $\mathsf{K}\varphi\in\Gamma$ implies $\varphi\in\Gamma\subseteq\Gamma_1$ because set Γ is closed with respect to subformulae. Then, $\neg\varphi\in\Gamma_2$ by Definition 7.9. Observe also that $\neg\varphi\in X\subseteq X'$. Thus, $\neg\varphi\in\Gamma_2\cap X'=u$. Hence, $\varphi\notin u$ because u is a consistent set of formulae. To finish the proof of the lemma, we need to show $w\sim u$. To establish this, by Definition 7.11, it suffices to prove the following claim.

Claim 7.25. *For each formula* $\gamma\in\Gamma_1$,

1. $\mathsf{K}\gamma\in w$ *iff* $\mathsf{K}\gamma\in u$,
2. $\mathsf{A}\gamma\in w$ *iff* $\mathsf{A}\gamma\in u$.

Proof of Claim. By Definition 7.9, the assumption $\gamma\in\Gamma_1$ implies that $\mathsf{K}\gamma,\mathsf{A}\gamma,\neg\mathsf{K}\gamma,\neg\mathsf{A}\gamma\in\Gamma_2$.

If $\mathsf{K}\gamma\in w$, then $\mathsf{K}\gamma\in X\subseteq X'$ by statement (34). Thus, $\mathsf{K}\gamma\in X'\cap\Gamma_2=u$ because $\mathsf{K}\gamma\in\Gamma_2$.

On the other hand, if $\mathsf{K}\gamma\in u$, then $\neg\mathsf{K}\gamma\notin u$ because set u is consistent. Then, $\neg\mathsf{K}\gamma\notin w$ by statement (34). Thus, $\mathsf{K}\gamma\in w$ because w is a maximal consistent subset of Γ_2 and $\mathsf{K}\gamma,\neg\mathsf{K}\gamma\in\Gamma_2$.

The proof of the second part of the claim is similar. ⊣

This concludes the proof of the lemma. ⊣

Lemma 7.26. *For any world* $w\in W$ *and any formula* $\mathsf{A}\varphi\in\Gamma$, *if* $\mathsf{A}\varphi\notin w$, *then* $N(\varphi)\notin\mathcal{N}_w$.

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PROOF. Suppose $N(\varphi) \in \mathcal{N}_w$. Thus, by Definition 7.13, there exists a formula $\gamma \in \Gamma_1$ such that

$$\mathsf{A}\gamma \in w \tag{37}$$

and

$$N(\varphi) = N(\gamma). \tag{38}$$

Note that $\mathsf{A}\varphi \in \Gamma \subseteq \Gamma_1$ by the assumption of the lemma. Thus, $\neg\mathsf{A}\varphi \in \Gamma_2$ by Definition 7.9. Then, $\neg\mathsf{A}\varphi \in w$ because w is a maximal consistent subset of w. Hence, $\nvdash \mathsf{A}\gamma \to \mathsf{A}\varphi$ by statement (37) and consistency of set w. Thus, $\nvdash \gamma \leftrightarrow \varphi$ by the Substitution inference rule applied contrapositively. We consider the following two cases separately:

Case 1: $\nvdash \gamma \to \varphi$. Thus, the set $\{\neg\varphi, \gamma\}$ is consistent. By Lemma 4.5, it can be extended to a maximal consistent set u. Note that $\gamma \in u$ and, because set u is consistent, $\varphi \notin u$. Also $\varphi \in \Gamma \subseteq \Gamma_1$ because set Γ is closed with respect to subformulae and also $\gamma \in \Gamma_1$. Thus, $u \notin N(\varphi)$ and $u \in N(\gamma)$ by Definition 7.12, which contradicts to statement (38).

Case 2: $\nvdash \varphi \to \gamma$. The proof is similar to the previous case. $\dashv$

LEMMA 7.27. *$\varphi \in w$ iff $w \Vdash \varphi$ for each formula $\varphi \in \Gamma$.*

PROOF. We prove the statement of the lemma by induction on the structural complexity of formula φ.

First, suppose that formula φ is a propositional variable p. Note that $p \in \Gamma \subseteq \Gamma_1$ by the assumption of the lemma. Note that the statement $p \in w$ is equivalent to the statement $w \in N(p)$ by Definition 7.12. The statement $w \in N(p)$ is equivalent to the statement $w \in \pi(p)$ by Definition 7.14. Finally, the statement $w \in \pi(p)$ is equivalent to the statement $w \Vdash p$ by item 1 of Definition 7.2.

Next, assume that formula φ has the form $\neg\psi$. Note that $\varphi \in \Gamma$ by the assumption of the lemma. Hence, $\psi \in \Gamma \subseteq \Gamma_1$ because set Γ is closed with respect to subformulae. Then,

$$\psi, \neg\psi \in \Gamma_2. \tag{39}$$

$(\Rightarrow)$: Let $w \Vdash \neg\psi$. Then, $w \nVdash \psi$ by item 2 of Definition 7.2. Hence, $\psi \notin w$ by the induction hypothesis. Therefore, $\neg\psi \in w$ by statement (39) because w is a maximal consistent subset of Γ_2.

$(\Leftarrow)$: Let $\neg\psi \in w$. Then, $\psi \notin w$ because set w is consistent. Hence, $w \nVdash \psi$ by the induction hypothesis. Therefore, $w \Vdash \neg\psi$ by item 2 of Definition 7.2.

Assume that formula φ has the form $\psi_1 \to \psi_2$. Thus, $\psi_1, \psi_2 \in \Gamma \subseteq \Gamma_1$ because set Γ is closed with respect to subformulae. Then,

$$\psi_1, \psi_2, \neg\psi_1, \neg\psi_2 \in \Gamma_2. \tag{40}$$

$(\Rightarrow)$: Let $w \Vdash \psi_1 \to \psi_2$. Then, by item 3 of Definition 7.2, either $w \nVdash \psi_1$ or $w \Vdash \psi_2$. We consider these two cases separately.

Case 1: $w \nVdash \psi_1$. Thus, $\psi_1 \notin w$ by the induction hypothesis. Then, $\neg\psi_1 \in w$ by statement (40) because w is a maximal consistent subset of Γ_2. Note that $\neg\psi_1 \to (\psi_1 \to \psi_2)$ is a propositional tautology. Thus, $w \vdash \psi_1 \to \psi_2$ by the Modus Ponens inference rule. Therefore, $\psi_1 \to \psi_2 \in w$ because w is a maximal consistent subset of Γ_2 and $\psi_1 \to \psi_2 = \varphi \in \Gamma \subseteq \Gamma_1 \subseteq \Gamma_2$ by the assumption $\varphi \in \Gamma$ of the lemma.

Case 2: $w \Vdash \psi_2$. Then, $\psi_2 \in w$ by the induction hypothesis. Note that $\psi_2 \to (\psi_1 \to \psi_2)$ is a propositional tautology. Then, $w \vdash \psi_1 \to \psi_2$ by the Modus Ponens inference rule. Therefore, just like in the first case, $\psi_1 \to \psi_2 \in w$.

$(\Leftarrow)$: Let $\psi_1 \to \psi_2 \in w$. Thus, by the Modus Ponens rule, if $w \vdash \psi_1$, then $w \vdash \psi_2$. In other words, either $w \nvdash \psi_1$ or $w \vdash \psi_2$. Then, either $\psi_1 \notin w$ or $w \vdash \psi_2$. Hence, either $\psi_1 \notin w$ or $\psi_2 \in w$ by statement (40) because w is a maximal consistent subset of Γ_2. Thus, $w \nVdash \psi_1$ or $w \vdash \psi_2$ by the induction hypothesis. Therefore, $w \Vdash \psi_1 \to \psi_2$ by item 3 of Definition 7.2.

Suppose that formula φ has the form $\mathsf{K}\psi$.

$(\Rightarrow)$: Assume that $\mathsf{K}\psi \in w$. Consider any world $u \in W$ such that $w \sim u$. By item 4 of Definition 7.2, it suffices to show that $u \Vdash \psi$. Indeed, by Definition 7.11, the assumptions $\mathsf{K}\psi \in w$ and $w \sim u$ imply that $\mathsf{K}\psi \in u$. Hence $u \vdash \psi$ by the Truth axiom and the Modus Ponens inference rule. Note that $\psi \in \Gamma \subseteq \Gamma_1 \subseteq \Gamma_2$ because $\mathsf{K}\psi \in \Gamma$ and set Γ is closed with respect to subformulae. Thus, $\psi \in u$ because u is a maximal consistent subset of Γ_2. Therefore, $u \Vdash \psi$ by the induction hypothesis.

$(\Leftarrow)$: Assume that $\mathsf{K}\psi \notin w$. Then, by Lemma 7.23, there is a world $u \in W$ such that $w \sim u$ and $\psi \notin u$. Thus, $u \nVdash \psi$ by the induction hypothesis. Therefore, $w \nVdash \mathsf{K}\psi$ by item 4 of Definition 7.2.

Suppose that formula φ has the form $\mathsf{A}\psi$. Note that $\psi \in \Gamma \subseteq \Gamma_1$ because set Γ is closed with respect to subformulae and $\varphi \in \Gamma$ by the assumption of the lemma.

$(\Rightarrow)$: Assume that $\mathsf{A}\psi \in w$. Then, $N(\psi) \in \mathcal{N}_w$ by Definition 7.13. Thus, $\{u \in W \mid \psi \in u\} \in \mathcal{N}_w$ by Definition 7.12. Hence, $\{u \in W \mid u \Vdash \psi\} \in \mathcal{N}_w$ by the induction hypothesis. Therefore, $w \Vdash \mathsf{A}\psi$ by item 5 of Definition 7.2.

$(\Leftarrow)$: Assume that $\mathsf{A}\psi \notin w$. Thus, $N(\psi) \notin \mathcal{N}_w$ by Lemma 7.26. Then, $\{u \in W \mid \psi \in u\} \notin \mathcal{N}_w$ by Definition 7.12. Hence, $\{u \in W \mid u \Vdash \psi\} \notin \mathcal{N}_w$ by the induction hypothesis. Therefore, $w \nVdash \mathsf{A}\psi$ by item 5 of Definition 7.2. ⊣

7.7. Completeness and decidability. In this subsection, we finish the proof of the completeness theorem for neighborhood semantics. We say that a neighborhood model $(W, \sim, \{\mathcal{N}_w\}_{w\in W}, \pi)$ is *finite* if set W is finite. Note that this implies that the family $\mathcal{N}_w$ is finite and all its elements are finite sets.

THEOREM 7.28 (Neighborhood completeness). *If* $\nvdash \varphi$, *then* $w \nVdash \varphi$ *for some world* w *of a finite neighborhood model.*

PROOF. Let Γ be the finite set of all subformulae of formula $\neg\varphi$. Consider the canonical model defined in the previous subsection. Note that this model is finite because set Γ_2 is also finite.

The assumption $\nvdash \varphi$ implies that the set $\{\neg\varphi\}$ is consistent. By Lemma 4.5 it can be extended to a maximal consistent set X. Let w be the set $X \cap \Gamma_2$. Note that $\neg\varphi \in w$ because $\neg\varphi \in X$ and $\neg\varphi \in \Gamma \subseteq \Gamma_1 \subseteq \Gamma_2$. Thus, $\varphi \notin w$ because set w is consistent. Therefore, $w \nVdash \varphi$ by Lemma 7.27. ⊣

THEOREM 7.29 (Decidability). *Set* $\{\varphi \in \Phi \mid \vdash \varphi\}$ *is decidable.*

PROOF. Consider an algorithm that takes a formula φ as an input and executes two processes in parallel. The first process enumerates all proofs in our logical system

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and checks if any of them is a proof of formula φ. The second process enumerates all finite neighborhood models and checks if formula φ is false in at least one world of this model. By Theorem 7.3, at most one of these processes will succeed on a given input φ. By Theorem 7.28, at least one process will succeed on a given input φ. If the first process succeeds, formula φ belongs to the set $\{\varphi \in \Phi \mid \ \vdash \varphi\}$. If the second process succeeds, it does not. $\dashv$

§8. Future work. In this section, we discuss several possible extensions of our logical system.

8.1. Know one. Our "telling apart" modality $\mathsf{A}\varphi$ captures the ability of an agent to identify all possible agents with a given property φ. One can also consider the "knowing one" modality:

$(a, w) \Vdash \mathsf{O}\varphi$ *when there exists an agent* $b \in Ag$ *such that* $(b, u) \Vdash \varphi$ *for each world* $u \in W$, *where* $w \sim_a u$.

Informally, $(a, w) \Vdash \mathsf{O}\varphi$ means that in world w agent a knows at least one agent (the agent b) that has property φ. Here are some of the properties of the interplay between this new modality and the modality that we have studied in this article:

1. $\mathsf{O}\varphi \wedge \mathsf{A}\psi \to \mathsf{O}(\varphi \wedge \psi) \vee \mathsf{O}(\varphi \wedge \neg\psi)$,
2. $\mathsf{O}\varphi \to \mathsf{K}\mathsf{O}\varphi$,
3. $\mathsf{K}\varphi \to \mathsf{O}\varphi$.

Although the Necessitation rule for modality O is not sound, the Monotonicity rule

$$\frac{\varphi \to \psi}{\mathsf{O}\varphi \to \mathsf{O}\psi}$$

is sound. An interesting possible question for future work could be to study the definability of modalities O, A, and K through each other. If they are not definable through each other, then one can look for a complete axiomatization of the interplay between these modalities. We think that a variation of our matrix technique could be potentially useful for proving completeness in this new setting.

8.2. Conditional tell apart. Another possible extension of this work is to consider "conditional telling apart." For example, although in a world w an agent a, a pediatrician, might not be able to tell sick people apart, the agent can tell sick children apart. We write this as

$$(a, w) \Vdash \mathsf{A}(\text{“is sick”} \mid \text{“is a child”}).$$

Formally, conditional telling apart modality $\mathsf{A}(\ \mid\ )$ is defined as follows:

$(a, w) \Vdash \mathsf{A}(\varphi \mid \psi)$ *when for each agent* $b \in Ag$ *such that* $(b, w) \Vdash \psi$ *and any worlds* $u, u' \in W$, *if* $w \sim_a u$, $w \sim_a u'$, *and* $(b, u) \Vdash \varphi$, *then* $(b, u') \Vdash \varphi$.

Formula $\mathsf{A}(\varphi \mid \top)$ is equivalent to $\mathsf{A}\varphi$. Here is an example of a property in the language of conditional telling apart:

$$\mathsf{A}(\varphi \mid \psi_1) \wedge \mathsf{A}(\varphi \mid \psi_2) \to \mathsf{A}(\varphi \mid \psi_1 \vee \psi_2).$$

A complete axiomatization of all such properties is a possible direction for future research.

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8.3. Non-rigid names. One might argue that the logical system proposed in this article is very restricted because it cannot explicitly refer to agents. Names can be introduced into our language through an additional modality $@_n$. This allows for a very general class of non-rigid names that could be agent-specific and world-specific. By agent-specific names, we mean names, such as *ma* (mother), whose meaning depends on the agent. For example, the statement $(a, w) \Vdash @_{ma}$"is sick" means that the mother of agent a is sick in worlds w. The statement $(a, w) \Vdash @_{ma}@_{ma}$"is sick" means that agent a's *grandma* (on the mother's side) is sick. One can write

$$(a, w) \Vdash @_{ma}\mathsf{A}\text{"is sick"}$$

to express that agent a's mother knows how to tell sick people apart. At the same time, the statement

$$(a, w) \Vdash \mathsf{A}@_{ma}\text{"is sick"}$$

means that agent a knows how to tell apart *those whose mother is sick*. The names can be also world-specific. For example, in different worlds, the mother of agent a could be different. Such a setting could be used to model situations when the agent does not know who his mother is.

To give a formal semantics of the language with non-rigid names, one can extend each epistemic model $(Ag, W, \{\sim_a\}_{a\in Ag}, \pi)$ with an additional component $\{e_a^w\}_{a\in\mathcal{A}}^{w\in W}$. By e_a^w we denote an "extension function" that maps a name n into an agent $e_a^w(n)$ whom agent a calls by name n in world w. Using extension functions, the semantics of modality $@_n$ can be defined as follows:

$(a, w) \Vdash @_n\varphi$ *when* $(e_a^w(n), w) \Vdash \varphi$.

A complete axiomatization of the properties of modalities A, O, and $\mathsf{A}(\ |\ )$ in the language with non-rigid names remains another interesting question for future work.

§9. Conclusion. The contribution of this paper is twofold. First, we proposed a sound, complete, and decidable modal logic of "knowing how to tell apart" in the egocentric setting. Second, we propose a new matrix-based technique for proving completeness results for 2D semantics. We hope that this technique could be potentially extended to other logical systems, such as those that we listed in the future work section.

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Appendix A. Proofs of auxiliary lemmas To keep the work self-contained, in this appendix, we give the proofs of three standard results used in our article.

LEMMA 4.1. *If* $X, \varphi \vdash \psi$, *then* $X \vdash \varphi \to \psi$.

PROOF. Suppose that sequence $\psi_1, \dots, \psi_n$ is a proof from set $X \cup \{\varphi\}$ and the theorems of our logical system that uses the Modus Ponens inference rule only. In other words, for each $k \leq n$, either

1. $\vdash \psi_k$, or
2. $\psi_k \in X$, or
3. ψ_k is equal to φ, or
4. there are $i, j < k$ such that formula ψ_j is equal to $\psi_i \to \psi_k$.

It suffices to show that $X \vdash \varphi \to \psi_k$ for each $k \leq n$. We prove this by induction on k by considering the four cases above separately.

Case I: $\vdash \psi_k$. Note that $\psi_k \to (\varphi \to \psi_k)$ is a propositional tautology and, thus, is an axiom of our logical system. Hence, $\vdash \varphi \to \psi_k$ by the Modus Ponens inference rule. Therefore, $X \vdash \varphi \to \psi_k$.

Case II: $\psi_k \in X$. Then, $X \vdash \psi_k$.

Case III: Formula ψ_k is equal to φ. Thus, $\varphi \to \psi_k$ is a propositional tautology. Hence, $X \vdash \varphi \to \psi_k$.

Case IV: Formula ψ_j is equal to $\psi_i \to \psi_k$ for some $i, j < k$. Thus, by the induction hypothesis, $X \vdash \varphi \to \psi_i$ and $X \vdash \varphi \to (\psi_i \to \psi_k)$. Note that formula

$$(\varphi \to \psi_i) \to ((\varphi \to (\psi_i \to \psi_k)) \to (\varphi \to \psi_k))$$

is a propositional tautology. Therefore, $X \vdash \varphi \to \psi_k$ by applying the Modus Ponens inference rule twice. ⊣

LEMMA 4.2. *If* $\varphi_1, \dots, \varphi_n \vdash \psi$, *then* $\mathsf{K}\varphi_1, \dots, \mathsf{K}\varphi_n \vdash \mathsf{K}\psi$.

PROOF. By Lemma 4.1 applied n times, the assumption $\varphi_1, \dots, \varphi_n \vdash \psi$ implies that

$$\vdash \varphi_1 \to (\varphi_2 \to \dots (\varphi_n \to \psi) \dots).$$

Thus, by the Necessitation inference rule,

$$\vdash \mathsf{K}(\varphi_1 \to (\varphi_2 \to \dots (\varphi_n \to \psi) \dots)).$$

Hence, by the Distributivity axiom and the Modus Ponens inference rule,

$$\vdash \mathsf{K}\varphi_1 \to \mathsf{K}(\varphi_2 \to \dots (\varphi_n \to \psi) \dots).$$

Then, again by the Modus Ponens rule,

$$\mathsf{K}\varphi_1 \vdash \mathsf{K}(\varphi_2 \to \dots (\varphi_n \to \psi) \dots).$$

Therefore, $\mathsf{K}\varphi_1, \dots, \mathsf{K}\varphi_n \vdash \mathsf{K}\psi$ by applying the previous steps $(n-1)$ more times. ⊣

LEMMA 4.3. $\vdash \mathsf{K}\varphi \to \mathsf{K}\mathsf{K}\varphi$.

PROOF. Note that formula $\mathsf{K}\neg\mathsf{K}\varphi \to \neg\mathsf{K}\varphi$ is an instance of the Truth axiom. Thus, by contraposition,

$$\vdash \mathsf{K}\varphi \to \neg\mathsf{K}\neg\mathsf{K}\varphi. \tag{41}$$

Also,

$$\neg\mathsf{K}\neg\mathsf{K}\varphi \to \mathsf{K}\neg\mathsf{K}\neg\mathsf{K}\varphi \tag{42}$$

is an instance of the Negative Introspection axiom. Additionally, the formula

$$\neg\mathsf{K}\varphi \to \mathsf{K}\neg\mathsf{K}\varphi$$

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is also an instance of the Negative Introspection axiom. Thus, by the law of contraposition,

$$\vdash \neg\mathsf{K}\neg\mathsf{K}\varphi \to \mathsf{K}\varphi.$$

Hence, by the Necessitation inference rule,

$$\vdash \mathsf{K}(\neg\mathsf{K}\neg\mathsf{K}\varphi \to \mathsf{K}\varphi).$$

Then, by the Distributivity axiom and the Modus Ponens inference rule,

$$\vdash \mathsf{K}\neg\mathsf{K}\neg\mathsf{K}\varphi \to \mathsf{K}\mathsf{K}\varphi. \tag{43}$$

Finally, by the laws of propositional reasoning, statements (41)–(43) imply the statement of the lemma. ⊣

REFERENCES

[1] A. Baltag, *To know is to know the value of a variable*, ***Advances in Modal Logic 11, Proceedings of the 11th Conference on Advances in Modal Logic, Budapest, Hungary, August 30–September 2, 2016*** (L. D. Beklemishev, S. Demri, and A. Maté, editors), College Publications, London, 2016, pp. 135–155.

[2] Z. Christoff and J. U. Hansen, *A logic for diffusion in social networks*. ***Journal of Applied Logic***, vol. 13 (2015), no. 1, pp. 48–77.

[3] Z. Christoff, J. U. Hansen, and C. Proietti, *Reflecting on social influence in networks*. ***Journal of Logic, Language and Information***, vol. 25 (2016), no. 3, pp. 299–333.

[4] J. van Eijck, M. Gattinger, and Y. Wang, *Knowing values and public inspection*, ***Indian Conference on Logic and Its Applications***, Springer, New York, 2017, pp. 77–90.

[5] S. Epstein and P. Naumov, *Epistemic logic of know-who*. ***Proceedings of the AAAI Conference on Artificial Intelligence***, vol. 35 (2021), no. 13, pp. 11479–11486.

[6] J. Fan, Y. Wang, and H. Van Ditmarsch, *Contingency and knowing whether*. ***The Review of Symbolic Logic***, vol. 8 (2015), no. 1, pp. 75–107.

[7] R. Fervari, A. Herzig, Y. Li, and Y. Wang, *Strategically knowing how*, ***Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence***, AAAI Press, San Francisco, 2017, pp. 1031–1038.

[8] A. J. Grove, *Naming and identity in epistemic logic. Part II: A first-order logic for naming*. ***Artificial Intelligence***, vol. 74 (1995), no. 2, pp. 311–350.

[9] A. J. Grove and J. Y. Halpern, *Naming and identity in a multi-agent epistemic logic*, ***Proceedings of the 2nd International Conference on Principles of Knowledge Representation and Reasoning*** (J. F. Allen, R. Fikes, and E. Sandewall, editors), Morgan Kaufmann, San Francisco, 1991, pp. 301–312.

[10] ———, *Naming and identity in epistemic logics. Part I: The propositional case*. ***Journal of Logic and Computation***, vol. 3 (1993), no. 4, pp. 345–378.

[11] J. Jiang and P. Naumov, *The egocentric logic of preferences*, ***The 31st International Joint Conference on Artificial Intelligence***, 2022, pp. 2676–2682.

[12] S. A. Kripke, *A completeness theorem in modal logic*, this Journal, vol. 24 (1959), no. 1, pp. 1–14.

[13] E. Mendelson, ***Introduction to Mathematical Logic***, CRC Press, Boca Raton, Florida, 2009.

[14] P. Naumov and J. Tao, *Together we know how to achieve: An epistemic logic of know-how*. ***Artificial Intelligence***, vol. 262 (2018), pp. 279–300.

[15] ———, *Everyone knows that someone knows: Quantifiers over epistemic agents*. ***The Review of Symbolic Logic***, vol. 12 (2019), no. 2, pp. 255–270.

[16] A. N. Prior, *Egocentric logic*. ***Noûs***, vol. 2 (1968), pp. 191–207.

[17] L. Schroeter, *Two-dimensional semantics*, ***The Stanford Encyclopedia of Philosophy (Fall 2021 Edition)*** (E. N. Zalta, editor), Metaphysics Research Lab, Stanford University, Palo Alto, 2021. Available at https://plato.stanford.edu/archives/fall2021/entries/two-dimensional-semantics/.

[18] J. Seligman, F. Liu, and P. Girard, *Logic in the community*, ***Logic and Its Applications*** (M. Banerjee and A. Seth, editors), Springer, Boston, 2011, pp. 178–188.

[19] ———, *Facebook and the epistemic logic of friendship*, ***TARK: Theoretical Aspects of Rationality and Knowledge. Proceedings of the 14th Conference - Chennai, India*** (B. C. Schipper, editor), University of California, Davis, DAvis, 2013, pp. 229–238.

[20] Y. WANG, *Beyond knowing that: A new generation of epistemic logics*, ***Jaakko Hintikka on Knowledge and Game-Theoretical Semantics*** (H. van Ditmarsch and G. Sandu, editors), Springer, Berlin, 2018, pp. 499–533.

[21] C. XU, Y. WANG, and T. STUDER, *A logic of knowing why*. ***Synthese***, vol. 198 (2021), pp. 1259–1285.

SCHOOL OF ELECTRONICS AND COMPUTER SCIENCE
UNIVERSITY OF SOUTHAMPTON
SOUTHAMPTON, UK

E-mail: p.naumov@soton.ac.uk

DEPARTMENT OF COMPUTER SCIENCE
LAFAYETTE COLLEGE
EASTON, PA, USA

E-mail: taoj@lafayette.edu

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