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UNIFORM CONVERGENCE TO A LAW CONTAINING GAUSSIAN AND CAUCHY DISTRIBUTIONS

JOSE H. BLANCHET
Industrial Engineering and Operations Research, Columbia University, 1255 Amsterdam Ave. Mail Code 4403, New York, NY
E-mail: jose.blanchet@columbia.edu

CARLOS G. PACHECO-GONZÁLEZ
Departamento de Matematicas, CINVESTAV-IPN, A. Postal 14-740, Mexico D.F. 07000, Mexico
E-mail: cpacheco@math.cinvestav.mx

A source of light is placed d inches apart from the center of a detection bar of length $L \geq d$. The source spins very rapidly, while shooting beams of light according to, say, a Poisson process with rate λ. The positions of the beams, relative to the center of the bar, are recorded for those beams that actually hit the bar. Which law best describes the time-average position of the beams that hit the bar given a fixed but long time horizon t? The answer is given in this paper by means of a uniform weak convergence result in L, d as $t \to \infty$. Our approximating law includes as particular cases the Cauchy and Gaussian distributions.

1. INTRODUCTION

The following physical mechanism is motivated by a construction of the Cauchy distribution that is discussed in Ross [6, Section 5.6.3] (see also Feller [2]). A source of light (think of a laser) is to be placed at d inches of distance from the center of a screen (or a bar) of L inches of length. Whenever the source is well calibrated, it spins very rapidly, and at exponential inter-arrival times the source of light shoots beams of light. We think that the source of light is not far apart from the bar (or screen), thus it is natural to ask that $0 < d \leq L$. The exponential times have mean $1/\lambda$ and occur one after another in an i.i.d. (independently and identically distributed) fashion and

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independently of the light generation. The light travels in straight line and is detected only when it hits the screen. At this point, the exact location of the hit—as measured by the distance to the center of the screen—is recorded. So, for instance, if an observer is behind the screen, facing the source of light, and a beam is detected 3 inches to the right of the center, then the location is equal to 3 and if a beam is detected 3 inches to the left of the center then the location is equal to -3.

Suppose, for the sake of an argument, that an experimenter is interested in testing if the source is well calibrated. In order to do this he plans to run the experiment up to a *long time* t so that a large number of measurements are recorded. Note that without loss of generality, by taking the distance to the center as the length units, we can work with the normalized length of the bar $M := L/d \geq 1$, so that we can take now $d = 1$. If the source is well calibrated, the distribution of the average position will be centered around zero. The experimenter's job is simply to test the hypothesis that the source is well calibrated. One possibility for the experimenter is to choose a confidence interval around the mean to test this hypothesis. In such case, how should he choose a distribution to compute the confidence interval? Owing to the Central Limit Theorem (CLT), one choice could be normal. Nevertheless, if the size of the bar M is very large (which could easily occur if the experimenter places the source of light very close to the center of the bar) the Gaussian approximation implied by the CLT is not a good choice. To see this in an extreme case simply think of the case $M = \infty$, in this situation the recorded positions are distributed Cauchy and therefore the CLT is not applicable.

Our main contribution is to characterize a family of random variables that approximates in distribution such average position regardless of the size of the bar M as long as the time t is chosen large enough. The precise mathematical statement is given in Theorem 1. We show that the correct family of approximating distributions depends on the relative sizes of the time horizon, t, and the length of the bar M (or equivalently the distance to the center of the bar). Suitable relationships between various parameters in our family allow to recover the Gaussian or the Cauchy distribution.

The rest of the paper is organized as follows. In Section 2, we present the precise mathematical description of our model and introduce our approximating family of distributions. The main result and its proof are in Section 3.

2. THE MODEL AND MAIN RESULT

We first provide a mathematical description of the model under consideration. Then, we will introduce our family of approximating distributions together with some of its basic properties. Finally, we present the main result and its proof.

2.1. The Model

We now proceed to describe a mathematical model for the experiment assuming that the source of light is well calibrated. According to the description given in the

Introduction we see that the source shoots beams of light according to a Poisson process with rate λ. Equivalently, one can think of the source of light shooting a generic beam according to the following mechanism. First, pick a point, say Θ, uniformly at random from the circumference of radius one, centered at the source of light and then shoot the beam to the selected point. The beam will reach the bar if and only if $\Theta \in (-\pi/2, \pi/2)$ and if $\mathrm{Tan}(\Theta) \in [-M, M]$—the range of Θ has been taken assuming that the observer looks the experiment from the top with the bar on the right-hand side and the source to the left. A beam of light reaches the bar with probability

$$\begin{aligned}\alpha_M &:= P(\Theta \in (-\pi/2, \pi/2), \mathrm{Tan}(\Theta) \in [-M, M]) \\ &= P(\mathrm{Tan}(\Theta) \in [-M, M] | \Theta \in (-\pi/2, \pi/2)) \times 1/2 \\ &= P(|Z| \leq M) \times 1/2,\end{aligned}$$

where the random variable Z, corresponding to the value of $\mathrm{Tan}(\Theta)$ given that $\Theta \in (-\pi/2, \pi/2)$, is distributed Cauchy. Throughout the rest of our discussion we shall use Z to denote a generic Cauchy random variable.

Using the Thinning Theorem (see, e.g., [5]), we can model the arrival process of the beams that do hit the bar according to a Poisson process $(N(t) : t \geq 0)$ with parameter $\lambda\alpha_M$. Since the source spins very rapidly we can regard each of the beams of light as i.i.d. and independent of the Poisson process. If we use X_k^M to denote the position of the kth beam detected on the screen, then for all $x \in \mathbb{R}$, $P(X_k^M \leq x) = P(Z \leq x | Z \in [-M, M])$. In particular, the density of X_k^M is given by

$$f_{X_k^M}(x) = \frac{1}{\pi\left(1 + x^2\right)} \frac{I(|x| \leq M)}{\arctan(M) - \arctan(-M)}. \tag{1}$$

In [4] one can find properties of this truncated random variable, such as the moments. Another study of scaled sums of truncated random variables can be found in [1].

Now, let Y_t be the accumulated sum of locations observed, that is,

$$Y_t := \sum_{k=1}^{N(t)} X_k^M, \tag{2}$$

hence the time average position, which is our object of interest, takes the form $\overline{Y}_t = Y_t/t$.

2.2. The Approximating Law

From the CLT for compound Poisson process (see, e.g., Whitt [9]) we have that

$$\frac{Y_t}{t^{1/2}\lambda^{1/2}\alpha_M^{1/2}\sigma_M} = \frac{t^{1/2}\overline{Y}_t}{\lambda^{1/2}\alpha_M^{1/2}\sigma_M} \Longrightarrow \mathcal{N}(0,1) \quad \text{as } t \to \infty, \tag{3}$$

where $\sigma_M^2 := \mathrm{Var}\left(X_1^M\right)$ (here $\Longrightarrow$ stands for weak convergence or convergence in distribution). Hence, under the conditions mentioned above, the normal distribution would be appropriate to approximate the distribution of $\overline{Y}_t$ for large values of t. In particular, the result stated in (3) provides rigorous support for the formal approximation

$$\overline{Y}_t \overset{D}{\approx} \frac{\lambda^{1/2}\alpha_M^{1/2}\sigma_M}{t^{1/2}}\mathcal{N}(0,1),$$

where $\mathcal{N}(0,1)$ denotes a standard Gaussian random variable.

On the other hand, if $M=\infty$, Y_t is the compound Poisson sum of i.i.d. Cauchy random variables and one can easily verify that $\overline{Y}_t \Longrightarrow \lambda Z$ as $t\to\infty$, which provides rigorous support for the approximation $\overline{Y}_t \overset{D}{\approx} \lambda Z$.

So, how can one characterize a family of distributions that can approximate the distribution of $\overline{Y}_t$ uniformly in M as long as t is large? Our main result provides an answer to this question. The approximation is given in (7) below. However, we first need to define our approximating family of distributions, which comes from the limiting behavior of a suitable scaled version of (2).

PROPOSITION 1: *Consider the time average*

$$\overline{Y}_t := \frac{\sum_k^{N(t)} X_k^{M(t)}}{t}, \tag{4}$$

where, for each $t\geq 0$, *the set* $\{X_1^{M(t)}, X_2^{M(t)}\ldots\}$ *is a sequence of truncated Cauchy random variables as in* (1) *such that* $M(t)=\kappa t$ *with* $0<\kappa<\infty$, *and* $\{N(t),\ t\geq 0\}$ *is a Poisson process with parameter* $\lambda/2$. *We have that* $\overline{Y}_t \Longrightarrow Z(\kappa)$ *as* $t\to\infty$, *where* $Z(\kappa)$ *is a symmetric random variable with characteristic function given by*

$$E\exp(i\theta Z(\kappa)) = \exp\left(\frac{2\lambda}{\pi}\int_{-\kappa}^{\kappa}\left(\frac{\cos(\theta y)-1}{y^2}\right)dy\right). \tag{5}$$

PROOF: We notice first that

$$E\exp\left(i\theta\overline{Y}_t\right) = \exp\left(2\lambda t\left(\phi_{M(t)}\left(\frac{i\theta}{t}\right)-1\right)\right),$$

where $\phi_{M(t)}(i\theta) := E\exp\left(i\theta X_k^{M(t)}\right)$. However,

$$\begin{aligned}
2\lambda t\left(\phi_M(i\theta/t)-1\right) &= 2\lambda\int_{-\kappa t}^{\kappa t}\left(\exp\left(\frac{i\theta x}{t}\right)-1\right)\frac{t\,dx}{\pi\left(1+x^2\right)p_{\kappa t}} \\
&= 2\lambda\int_{-\kappa}^{\kappa}\frac{(\exp(i\theta y)-1)}{\pi\left(1+y^2t^2\right)p_{\kappa t}}t^2dy = 2\lambda\int_{-\kappa}^{\kappa}\frac{(\exp(i\theta y)-1)}{\pi\left(t^{-2}+y^2\right)p_{\kappa t}}dy \\
&= \frac{2\lambda}{p_{\kappa t}}\int_{-\kappa}^{\kappa}\frac{\cos(\theta y)-1}{\pi\left(t^{-2}+y^2\right)}dy.
\end{aligned}$$

Notice that the integrand is bounded by $(\cos(\theta y) - 1)/y^2$ which is integrable. By the Dominated Convergence theorem

$$2\lambda t\left(\phi_M\left(i\theta/t\right) - 1\right) \longrightarrow 2\lambda \int_{-\kappa}^{\kappa} \left(\frac{\cos\left(\theta y\right) - 1}{\pi y^2}\right) dy$$

and therefore

$$E \exp\left(i\theta \overline{Y}_t\right) \longrightarrow \exp\left(2\lambda \int_{-\kappa}^{\kappa} \left(\frac{\cos\left(\theta y\right) - 1}{\pi y^2}\right) dy\right).$$

By Lévy's continuity theorem $\overline{Y}_t \Longrightarrow Z(\kappa)$, where $Z(\kappa)$ is a random variable with characteristic function given in (5).

Since the characteristic function is real valued, then Z is symmetric. ■

DEFINITION 1: *We say that W is distributed according to an averaged conditioned Cauchy with changing threshold distribution with parameter $\beta_0, \beta_1 > 0$, denoted via $W := W(\beta_0, \beta_1) \sim \mathrm{ACT}(\beta_0, \beta_1)$, if*

$$E \exp(i\theta W) = \exp\left(\beta_0 \int_{-\beta_1}^{\beta_1} [\cos(\theta y) - 1]/y^2 dy\right).$$

Remark 1: The random variable W appears to be a particular case of a family introduced in [7], which treats dimensions 2 and higher. The authors in [7] refer to these types of random variables as truncated stable distributions. Our terminology is motivated by the result in Proposition 1.

We now show some properties of the ACT r.v.

PROPOSITION 2: *Suppose that $W(\beta_0, \beta_1) \sim \mathrm{ACT}(\beta_0, \beta_1)$. Then,*
(i) W is a symmetric infinitely divisible random variable.
(ii) $W(\beta_0, \beta_1) \Longrightarrow \beta_0 Z$ as $\beta_1 \to \infty$.
(iii) If $\beta_0 = 1/(2\beta_1)$ then $W(\beta_0, \beta_1) \Longrightarrow \mathcal{N}(0, 1)$ as $\beta_1 \to 0$.
(iv) W has an infinitely differentiable density. Moreover, if $f_W^{(k)}(\cdot)$ denotes the k-th derivative of the density of W, then we have that

$$\left|\frac{1}{\sqrt{2\pi}} f^{(k)}(x)\right| \leq 2 \int_{1/\beta_1}^{\infty} \exp(-2\beta_0 \theta c_1) |\theta|^k \, d\theta + 2 \int_0^{1/\beta_1} \exp(-2\beta_0 \beta_1 \theta^2 c_2) |\theta|^k \, d\theta,$$

where $c_1 = \int_0^1 [(1 - \cos(y))/y^2] dy \in (0, \infty)$ and $c_2 = \inf_{0 \leq z \leq 1} (1/z) \int_0^z [(1 - \cos(u))/u^2] du \in (0, \infty)$.

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PROOF: (i) From the proof of Proposition 1 we also obtain, when taking $t \to \infty$, that

$$E\exp(i\theta W) = \exp\left(\beta_0 \int_{-\beta_1}^{\beta_1} [\exp(i\theta y) - 1]/y^2 dy\right). \quad \textbf{(6)}$$

The measure $\nu(dy) := dy/y^2$ is such that $\int_{\mathbb{R}} \min(y^2, 1)\nu(dy) < \infty$, hence (6) corresponds to the characteristic function of a infinitely divisible random variable, see, for example, [8, Section 1.2.4]. Such infinitely divisible random variable is symmetric.

(ii) Since $\cos(\theta y) = \cos(|\theta| y)$ always, by the change of variable $x = |\theta| y$,

$$\exp\left(-\beta_0 \int_{-\beta_1}^{\beta_1} [1 - \cos(\theta y)]/y^2 dy\right) = \exp\left(-\beta_0 |\theta| \int_{-\beta_1}^{\beta_1} [1 - \cos(x)]/x dx\right).$$

Hence, by letting $\beta_1 \to \infty$, the integral converges and we end up with the characteristic function of the Cauchy random variable.

(iii) Using $[\cos(\theta y) - 1]/y^2 = -\theta^2/2! + \theta^4 y^2/4! - \ldots$, if $\beta_0 = 1/(2\beta_1)$ and $\beta_1 \to 0$, we end up with the characteristic function of $N(0,1)$:

$$\beta_0 \int_{-\beta_1}^{\beta_1} [\cos(\theta y) - 1]/y^2 dy = \beta_0 \left(-2\beta_1 \frac{\theta^2}{2!} + 2\frac{\beta_1^3}{3}\frac{\theta^4}{4!} - \ldots\right) \to -\theta^2/2.$$

(iv) Due to properties of the Fourier transform (see for instance [8, Section 3.8.4, p.188]) it suffices to show that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |E\exp(i\theta W)|\, |\theta|^k\, d\theta$$
$$= \int_{-\infty}^{\infty} \exp\left(\beta_0 \int_{-\beta_1}^{\beta_1} [\cos(|\theta| y) - 1]/y^2 dy\right) |\theta|^k d\theta < \infty$$

for all $k \in \{1, 2, ...\}$. This follows easily since

$$\int_{-\infty}^{\infty} e^{\beta_0 \int_{-\beta_1}^{\beta_1} [(\cos(|\theta| y) - 1)/y^2] dy} |\theta|^k d\theta = 2\int_0^{\infty} e^{-2\beta_0 \int_0^{\beta_1} [(1-\cos(\theta y))/y^2] dy} |\theta|^k d\theta$$
$$= 2\int_0^{\infty} e^{-2\beta_0\theta \int_0^{\theta\beta_1} [1-\cos(u)]/u^2 du}\, |\theta|^k\, d\theta$$
$$= 2\int_{1/\beta_1}^{\infty} e^{-2\beta_0\theta \int_0^{\theta\beta_1} [(1-\cos(u))/u^2] du}\, |\theta|^k\, d\theta$$
$$+ 2\int_0^{1/\beta_1} e^{-2\beta_0\theta \int_0^{\theta\beta_1} [(1-\cos(u))/u^2] du}\, |\theta|^k\, d\theta.$$

Now, we have that

$$2\int_{1/\beta_1}^{\infty} \exp\left(-2\beta_0\theta \int_0^{\theta\beta_1} [1 - \cos(u)]/u^2 du\right) |\theta|^k\, d\theta \le 2\int_{1/\beta_1}^{\infty} \exp(-2\beta_0\theta c_1)\, |\theta|^k\, d\theta$$

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and

$$
\begin{aligned}
&2\int_0^{1/\beta_1} \exp\left(-2\beta_0\theta \int_0^{\theta\beta_1} [1-\cos(u)]/u^2 du\right) |\theta|^k \, d\theta \\
&\quad = 2\int_0^{1/\beta_1} \exp(-2\beta_0\beta_1\theta^2 \frac{1}{\theta\beta_1} \int_0^{\theta\beta_1} [1-\cos(u)]/u^2 du) \, |\theta|^k \, d\theta \\
&\quad \le 2\int_0^{1/\beta_1} \exp(-2\beta_0\beta_1\theta^2 c_2) \, |\theta|^k \, d\theta,
\end{aligned}
$$

which concludes the proof. ■

3. THE APPROXIMATING RESULT

In order to describe our result we introduce $g(x) = P(U \le x) = \min(\max(0, x), 1)$, where U is uniformly distributed in the interval [0, 1]. Then we let

$$
\rho_M^2 = \frac{1}{M}\int_0^M \frac{x^2}{1+x^2} dx = 1 - \frac{1}{M}\int_0^M \frac{1}{1+x^2} dx,
$$

and $\gamma := \gamma(M, t) = g(M/t)^{1/2}$. Function g is really not special, but the properties that are interesting to us are that $g(x) \ge 0$, $g(\cdot)$ is monotone increasing continuous, $g(x) \nearrow 1$ as $x \nearrow \infty$ and $g(x) \sim x$ as $x \to 0$. Due to our scaling, the behavior of $g(\cdot)$ will allow us to retrieve the CLT approximation.

Our result provides rigorous support for the approximation

$$
\overline{Y}_t \overset{D}{\approx} \gamma\rho_M W(\beta_0, \beta_1), \tag{7}
$$

where $\beta_0 = \lambda/(2\pi\gamma)$ and $\beta_1 = M/(\gamma t)$. We now provide the statement of our result and its proof.

THEOREM 1: *Define* $\beta_0 := \beta_0(t, M) = \lambda/(2\pi\gamma)$ *and* $\beta_1 := \beta_1(M, t) = M/(\gamma t)$. *Then, we have that*

$$
\lim_{t\to\infty} \sup_{M\ge 1, x\in(-\infty,\infty)} \left| P(\overline{Y}_t/\gamma \le x) - P(\rho_M W(\beta_0(M,t), \beta_1(M,t)) \le x) \right| = 0.
$$

PROOF: We use characteristic functions and apply Esseen's lemma (which we quote in the Appendix for convenience). We let

$$
H(x) = P(\overline{Y}_t/\gamma \le x) \;\text{ and }\; F(x) = P(\rho_M W(\beta_0(M,t), \beta_1(M,t)) \le x).
$$

We now verify the assumptions of Esseen's lemma. First, we have that

$$\begin{aligned}\log\chi(\theta) &:= \log E\exp(i\theta Y_t/(\gamma t)) = \lambda\alpha_M t\,(\phi_M\,(i\theta/(\gamma t))-1)\\ &= \frac{\lambda}{2}\int_{-M}^{M}\left[\exp\left(\frac{i\theta x}{\gamma t}\right)-1\right]\frac{tdx}{\pi\left(1+x^2\right)}\\ &= \frac{\lambda}{2}\int_{-M}^{M}\left[\cos\left(\frac{\theta x}{\gamma t}\right)-1\right]\frac{tdx}{\pi\left(1+x^2\right)}.\end{aligned}$$

Clearly, for each $M\in[1,\infty)$, we have that $E\overline{Y}_t=0$. Now we compute the characteristic function of $\rho_M W(\beta_0(M,t),\beta_1(M,t))$. We have that

$$\begin{aligned}\log\xi(\theta) &:= \log E\exp(i\theta\rho_M W(\beta_0(M,t),\beta_1(M,t)))\\ &= \beta_0\int_{-\beta_1}^{\beta_1}\frac{(\cos(\theta\rho_M y)-1)}{y^2}dy = \frac{\lambda}{2\pi\gamma}\int_{-M/(\gamma t)}^{M/(\gamma t)}\frac{(\cos(\theta\rho_M y)-1)}{y^2}dy.\end{aligned}$$

Using the properties derived in Proposition 2 we have that the derivative $f(\cdot)$ of $F(\cdot)$ satisfies

$$\frac{1}{\sqrt{2\pi}}\,|f(x)|\le 2\int_{1/\beta_1}^{\infty}\exp(-2\beta_0\theta c_1)d\theta+2\int_0^{1/\beta_1}\exp(-2\beta_0\beta_1\theta^2 c_2)d\theta.$$

Observe that

$$\beta_1 = M/(\gamma t)\ge\gamma^2/\gamma=\gamma \text{ and } \beta_0\beta_1=\frac{\lambda}{2\pi\gamma}\beta_1\ge\frac{\lambda}{2\pi}.$$

Therefore,

$$|f(x)|\le m:=2\int_0^{\infty}\exp(-\lambda\theta c_1/\pi)d\theta+2\int_0^{\infty}\exp(-\lambda\theta^2 c_2/\pi)d\theta<\infty.$$

We can see that $\xi(\cdot)$ is continuously differentiable for each $M\in[1,\infty)$ with $\xi(0)=1$ and $\xi'(0)=0$. Indeed, note that

$$\begin{aligned}\log\xi(\theta) &= -\frac{\theta^2}{2}\times\frac{\lambda\rho_M^2 M/(\gamma t)}{\pi\gamma}+\frac{\lambda}{2\pi\gamma}\int_{-M/(\gamma t)}^{M/(\gamma t)}\frac{\left(\cos(\theta\rho_M y)-1+\theta^2\rho_M y^2/2\right)}{y^2}dy\\ &= -\frac{\theta^2}{2}\times\frac{\lambda\rho_M^2 M/(\gamma t)}{\pi\gamma}+O(\theta^4)\end{aligned}$$

as $\theta\to 0$. We now apply Esseen's lemma to obtain that

$$|H(x)-F(x)|\le\frac{1}{\pi}\int_{-T}^{T}|\chi(\theta)-\xi(\theta)|\,\theta^{-1}d\theta+\frac{24m}{\pi T}. \quad \textbf{(8)}$$

In order to estimate the integrand in the previous display, let us introduce the change of variables $y = x/(\gamma t)$ and obtain

$$\begin{aligned}\log \chi(\theta) &= \frac{\lambda}{2}\gamma t \int_{-M/\gamma t}^{M/\gamma t} [\cos(\theta y) - 1] \frac{t dy}{\pi\left(1 + (\gamma t y)^2\right)} \\ &= \lambda(\gamma t)^{-1} \int_{-M/\gamma t}^{M/\gamma t} \frac{(\cos(\theta y) - 1) t dy}{\pi\left((\gamma t)^{-2} + y^2\right)} = \frac{\lambda}{2\pi\gamma} \int_{-M/\gamma t}^{M/\gamma t} \frac{(\cos(\theta y) - 1) dy}{\left((\gamma t)^{-2} + y^2\right)}.\end{aligned}$$

Next, we write

$$\begin{aligned}\log \chi(\theta) &= \frac{\lambda}{2\pi\gamma} \int_{-M/\gamma t}^{M/\gamma t} [\cos(\theta y) - 1] \left(\frac{1}{y^2} + \frac{1}{\left((\gamma t)^{-2} + y^2\right)} - \frac{1}{y^2} \right) dy \\ &= \frac{\lambda}{2\pi\gamma} \int_{-M/\gamma t}^{M/\gamma t} \frac{[\cos(\theta y) - 1]}{y^2} dy \\ &\quad - \frac{\lambda}{2\pi\gamma} \int_{-M/\gamma t}^{M/\gamma t} \frac{(\cos(\theta y) - 1 + \frac{\theta^2 y^2}{2})\,(\gamma t)^{-2}}{\left((\gamma t)^{-2} + y^2\right) y^2} dy \\ &\quad + \frac{\lambda}{2\pi\gamma} \times \frac{\theta^2}{2} \int_{-M/\gamma t}^{M/\gamma t} \frac{(\gamma t)^{-2}}{(\gamma t)^{-2} + y^2} dy \\ &= \frac{\lambda}{2\pi\gamma} \int_{-M/\gamma t}^{M/\gamma t} \frac{[\cos(\theta \rho_M y) - 1]}{y^2} dy \\ &\quad - \frac{\lambda}{2\pi\gamma} \int_{-M/\gamma t}^{M/\gamma t} \frac{\left(\cos(\theta y) - 1 + \frac{\theta^2 y^2}{2}\right)(\gamma t)^{-2}}{\left((\gamma t)^{-2} + y^2\right) y^2} dy \\ &\quad + \frac{\lambda}{2\pi\gamma} \int_{-M/\gamma t}^{M/\gamma t} \frac{[\cos(\theta y) - \cos(\theta \rho_M y)]}{y^2} dy \\ &\quad + \frac{\lambda}{2\pi\gamma} \times \frac{\theta^2}{2} \int_{-M/\gamma t}^{M/\gamma t} \frac{(\gamma t)^{-2}}{(\gamma t)^{-2} + y^2} dy \\ &= \log \xi(\theta) - \frac{\lambda}{2\pi\gamma} \int_{-M/\gamma t}^{M/\gamma t} [\cos(\theta y) - 1 + \theta^2 y^2/2] \frac{(\gamma t)^{-2}}{\left((\gamma t)^{-2} + y^2\right) y^2} dy \\ &\quad + \frac{\lambda}{2\pi\gamma} \int_{-M/\gamma t}^{M/\gamma t} \frac{[\cos(\theta y) - \cos(\theta \rho_M y)]}{y^2} dy \\ &\quad + \frac{\lambda}{2\pi\gamma} \times \frac{\theta^2}{2} \int_{-M/\gamma t}^{M/\gamma t} \frac{(\gamma t)^{-2}}{(\gamma t)^{-2} + y^2} dy.\end{aligned}$$

Using previous display we define

$$I_1 := -\frac{\lambda}{2\pi\gamma} \int_{-M/\gamma t}^{M/\gamma t} [\cos(\theta y) - 1 + \theta^2 y^2/2] \frac{(\gamma t)^{-2}}{\left((\gamma t)^{-2} + y^2\right) y^2} dy,$$

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$$I_2 := \frac{\lambda}{2\pi\gamma}\int_{-M/\gamma t}^{M/\gamma t}\frac{[\cos(\theta y)-\cos(\theta\rho_M y)]}{y^2}dy + \frac{\lambda}{2\pi\gamma}\times\frac{\theta^2}{2}\int_{-M/\gamma t}^{M/\gamma t}\frac{(\gamma t)^{-2}}{(\gamma t)^{-2}+y^2}dy$$

so that

$$|\chi(\theta)-\xi(\theta)| = \xi(\theta)\,|\exp(I_1+I_2)-1|\,. \tag{9}$$

We need to show that $I_1+I_2\to 0$ as $t\to\infty$ uniformly over $M\geq 1$. In order to do this we first observe that $\gamma t \longrightarrow \infty$ as $t \nearrow \infty$ uniformly over $M\geq 1$. Note that for $t\geq 0$

$$t\gamma = t\min[(M/t)^{1/2},1] = \min[(Mt)^{1/2},t]. \tag{10}$$

Now we provide a bound for the convergence to zero of I_1. Using Eq. (10), for $t\geq 1$, see that

$$\begin{aligned}|I_1| &\leq \frac{\lambda}{2\pi\gamma\,(\gamma t)^2}\int_{-M/\gamma t}^{M/\gamma t}\frac{\left|\cos(\theta y)-1+\theta^2y^2/2\right|}{y^4}dy\\ &\leq \frac{\lambda}{2\pi t^{1/2}}\int_{-\infty}^{\infty}\frac{\left|\cos(\theta y)-1+\theta^2y^2/2\right|}{y^4}dy\\ &= \frac{\lambda\,|\theta|^3}{2\pi t^{1/2}}\int_{-\infty}^{\infty}\frac{\left|\cos(x)-1+x^2/2\right|}{x^4}dx\end{aligned} \tag{11}$$

Next, for the last term of I_2, note that if we let $y = x/(\gamma t)$, then

$$\begin{aligned}&\frac{\lambda}{2\pi\gamma}\times\frac{\theta^2}{2}\int_{-M/\gamma t}^{M/\gamma t}\frac{(\gamma t)^{-2}}{(\gamma t)^{-2}+y^2}dy\\ &=\frac{\lambda}{2\pi\gamma}\times\frac{\theta^2 M}{2\gamma t}\times\frac{1}{M}\int_{-M}^{M}\frac{1}{1+x^2}dx\\ &=\frac{\lambda}{2\pi\gamma}\times\frac{\theta^2 M}{2\gamma t}\times\frac{2}{M}\int_{0}^{M}\frac{1}{1+x^2}dx = \frac{\lambda}{2\pi\gamma}\times\frac{\theta^2}{2}\times\frac{2M}{\gamma t}\times(1-\rho_M^2).\end{aligned}$$

We continue analyzing I_2. In the next calculations, by letting $u=|\theta|\,y$ for the second equality and $x=u\rho_M$ for the fourth equality, we obtain

$$\begin{aligned}I_2 &= \frac{\lambda}{2\pi\gamma}\int_{-M/\gamma t}^{M/\gamma t}\frac{[\cos(|\theta|\,y)-\cos(|\theta|\,\rho_M y)+\theta^2y^2(1-\rho_M^2)/2]}{y^2}dy\\ &=\frac{\lambda\,|\theta|}{\pi\gamma}\int_0^{M|\theta|/\gamma t}\frac{[\cos(u)-\cos(u\rho_M)+u^2(1-\rho_M^2)/2]}{u^2}du\\ &=\frac{\lambda\,|\theta|}{\pi\gamma}\int_0^{M|\theta|/\gamma t}\frac{[\cos(u)-1+u^2/2]}{u^2}du\\ &\quad -\frac{\lambda\,|\theta|}{\pi\gamma}\int_0^{M|\theta|/\gamma t}\frac{[\cos(u\rho_M)-1+\rho_M^2u^2/2]}{u^2}du\end{aligned}$$

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$$
\begin{aligned}
&= \frac{\lambda\,|\theta|}{\pi\gamma}\int_0^{M|\theta|/\gamma t}\frac{[\cos(u)-1+u^2/2]}{u^2}du \\
&\quad - \frac{\lambda\,|\theta|\,\rho_M}{\pi\gamma}\int_0^{\rho_M M|\theta|/\gamma t}\frac{[\cos(x)-1+x^2/2]}{x^2}dx \\
&= \frac{\lambda\,|\theta|}{\pi\gamma}\int_0^{M|\theta|/\gamma t}\frac{[\cos(u)-1+u^2/2]}{u^2}du \\
&\quad - \frac{\lambda\,|\theta|\,\rho_M}{\pi\gamma}\int_0^{M|\theta|/\gamma t}\frac{[\cos(x)-1+x^2/2]}{x^2}dx \\
&\quad + \frac{\lambda\,|\theta|\,\rho_M}{\pi\gamma}\int_{\rho_M M|\theta|/\gamma t}^{M|\theta|/\gamma t}\frac{[\cos(x)-1+x^2/2]}{x^2}dx \\
&= -\frac{\lambda\,|\theta|\,(\rho_M-1)}{\pi\gamma}\int_0^{M|\theta|/\gamma t}\frac{[\cos(x)-1+x^2/2]}{x^2}dx \\
&\quad + \frac{\lambda\,|\theta|\,\rho_M}{\pi\gamma}\int_{\rho_M M|\theta|/\gamma t}^{M|\theta|/\gamma t}\frac{[\cos(x)-1+x^2/2]}{x^2}dx.
\end{aligned}
$$

We conclude that

$$
|I_2| \le \frac{\lambda\,|\theta|\,(1-\rho_M)^{M|\theta|/\gamma t}}{\pi\gamma}r\left(\frac{M\,|\theta|}{\gamma t}\right) + \frac{\lambda\,|\theta|\,\rho_M(1-\rho_M)^{M|\theta|/\gamma t}}{\pi\gamma}r\left(\frac{M\,|\theta|}{\gamma t}\right),
$$

where

$$
r(z) = \max_{x\in[0,z]}\frac{\left|\cos(x)-1+x^2/2\right|}{x^2}.
$$

We see that

$$
\rho_M^2 = 1-\frac{1}{M}\int_0^M \frac{1}{1+x^2}dx.
$$

Therefore, $C_1 = \sup_{M\ge 1} M(1-\rho_M) < \infty$. Consequently $|I_2| \le (2\lambda\theta^2 C_1/\pi\gamma^2 t)$ $r(M|\theta|/(\gamma t))$ and evidently we have that there exists $C_2 \in (0,\infty)$ such that for all $z \ge 0$, $r(z) \le \min\{z^2, C_2\}$. Therefore, recalling that $\gamma = \sqrt{g(M/t)}$, we obtain

$$
\begin{aligned}
|I_2| &\le \frac{\lambda\theta^2 C_1}{\pi\gamma^2 t}\min\left\{\frac{M^2\,|\theta|^2}{(\gamma t)^2}, C_2\right\} = \frac{\lambda\theta^2 C_1}{\pi\gamma^2 t}\min\left\{\frac{|\theta|^2}{\min\{\frac{t}{M},\frac{t^2}{M^2}\}}, C_2\right\} \\
&= \frac{\lambda\theta^2 C_1}{\pi}\times\frac{\min\{\max\{M/t,(M/t)^2\}\,|\theta|^2, C_2\}}{\min\{M,t\}}.
\end{aligned}
$$

It follows easily that for each $T > 0$ fixed

$$
\lim_{t\to\infty}\sup_{M\ge 1,|\theta|\le T}\frac{\lambda\theta^2 C_1}{\pi}\frac{\min\{\max\{M/t,(M/t)^2\}\,|\theta|^2, C_2\}}{\min\{M,t\}} = 0
$$

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and also, regarding the bound (11) of $|I_1|$

$$\lim_{t\to\infty} \sup_{M\geq 1, |\theta|\leq T} \frac{\lambda\,|\theta|^3}{2\pi t^{1/2}} \int_{-\infty}^{\infty} \frac{\left|\cos(x) - 1 + x^2/2\right|}{x^4} dx = 0.$$

Therefore, from Eq. (8) and (9) we conclude that

$$\lim_{t\to\infty} \sup_{M\geq 1, x\in(-\infty,\infty)} |H(x) - F(x)|$$

$$\leq \lim_{t\to\infty} \sup_{M\geq 1} \frac{1}{\pi} \int_{-T}^{T} |\chi(\theta) - \xi(\theta)|\,\theta^{-1} d\theta + \frac{24m}{\pi T} \leq \frac{24m}{\pi T}.$$

Since $T > 0$ is arbitrary we have then proved the result. ■

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APPENDIX

See Feller [3, p. 538] for the next result.

LEMMA A.1 (Esseen): *Let H be a distribution function with zero expectation and with characteristic function $\chi(\cdot)$ and F another distribution function. Suppose that the difference $H - F$ vanishes at ∞ and $-\infty$ and that F has a derivative f such that $|f(x)| \leq m$ for all $x \in (-\infty, \infty)$. Finally, suppose that $f(\cdot)$ has a Fourier transform $\xi(\cdot)$ such that $\xi(0) = 1$ and $\xi'(0) = 0$. Then, for each $x \in (-\infty, \infty)$ and $T > 0$,*

$$|H(x) - F(x)| \leq \frac{1}{\pi} \int_{-T}^{T} |\chi(\theta) - \xi(\theta)|\,\theta^{-1} d\theta + \frac{24m}{\pi T}.$$