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Ordinal computations

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The notion of *ordinal computability* is defined by generalising standard Turing computability on tapes of length ω to computations on tapes of arbitrary ordinal length. The fundamental theorem on ordinal computability states that a set x of ordinals is ordinal computable from ordinal parameters if and only if x is an element of the constructible universe **L**. In this paper we present a new proof of this theorem that makes use of a theory SO axiomatising the class of sets of ordinals in a model of set theory. The theory SO and the standard Zermelo–Fraenkel axiom system ZFC can be canonically interpreted in each other. The proof of the fundamental theorem is based on showing that the class of sets that are ordinal computable from ordinal parameters forms a model of SO.

1. Introduction

In Koepke (2005), the first author defined *ordinal Turing machines* and the associated notion of *ordinal computability*. The main result of Koepke (2005) asserts that ordinal computability is an absolute and stable notion.

Theorem 1. A set x of ordinals is ordinal computable from a finite set of ordinal parameters if and only if it is an element of the constructible universe **L**.

In the present paper we give a new proof of the theorem that does not depend on coding the elements of **L** into an ordinal Turing machine. To connect ordinal computability and constructibility, the proof introduces two notions that might be of further interest:

— a recursive truth function $T : \mathrm{Ord} \to 2$;
— a theory SO satisfied by the class of sets of ordinals in a model of ZFC.

In Section 2 we recall the basic definitions of ordinal computability. The theory SO and some consequences are presented in Section 3. In Section 4 we define a structure $(\mathbb{P}, \equiv, \blacktriangleleft)$ for the language of set theory within SO. We show in Section 5 that $(\mathbb{P}, \equiv, \blacktriangleleft)$ is actually a model of ZFC. Section 6 defines the recursive truth function T, which is ordinal computable. The function T codes so much information that one can read off a

		Space									
		0	1	2	3	4	5	6	7	...	...
	0	**1**	0	0	1	1	1	0	0	0	...
T	1	0	**0**	0	1	1	1	0	0		
i	2	0	0	**0**	1	1	1	0	0		
m	3	0	**0**	1	1	1	1	0	0		
e	4	0	1	**1**	1	1	1	0	0		
	⋮										
	n	1	1	1	1	**0**	1	1	1		
	$n+1$	1	1	1	1	1	**1**	1	1		
	⋮										

Fig. 1. A standard Turing computation. Head positions are indicated by boxes.

model of SO from T (Section 7). All the components are put together in Section 8 to prove the main theorem.

Our work was inspired by the *infinite time* Turing machines introduced by Joel D. Hamkins, Jeff Kidder and Andy Lewis (Hamkins and Lewis 2000). The theory SO was studied in the the second author's diploma thesis (Koerwien 2001).

2. Ordinal Turing machines

One can visualise a standard Turing computation as a time-like sequence of elementary *read*-*write*-*move* operations carried out by one or more 'heads' on 'tapes'. The sequence of actions is determined by the initial tape contents and by a finite Turing *program*. We may assume that Turing machines act on tapes whose cells are indexed by the set ω ($= \mathbb{N}$) of *natural numbers* $0, 1, \ldots$ and contain 0's or 1's. A standard Turing computation is depicted in Figure 1.

An obvious generalisation from the perspective of transfinite ordinal theory is to extend such calculations to tapes whose cells are indexed by the class Ord of all *ordinal numbers*. At *limit* ordinals the tape contents, program states and head positions are defined as *inferior limits*. This is depicted in Figure 2.

Such *ordinal Turing machines* are formalised by the following definitions.

Definition 1.

(a) A *command* is a 5-tuple $C = (s, c, c', m, s')$ where $s, s' \in \omega$ and $c, c', m \in \{0, 1\}$; the natural number s is the *state* of the command C. The effect of the command C is that if the machine is in state s and reads the symbol c under its read-write head, it then writes the symbol c', moves the head left if $m = 0$ or right if $m = 1$, and goes into state s'. States correspond to the 'line numbers' or labels of some programming languages.

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		Ordinal Space												
		0	1	2	3	4	5	6	7	...	ω	...	α	...
O	0	**1**	1	0	1	0	0	1	1	...	1	...	1	...
r	1	0	**1**	0	1	0	0	1	1	...	1	...	1	...
d	2	0	0	**0**	1	0	0	1	1	...	1	...	1	...
i	3	0	0	0	**1**	0	0	1	1	...	1	...	1	...
n	4	0	0	0	0	**0**	0	1	1	...	1	...	1	...
a	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮		⋮		⋮	
l	n	1	1	1	1	**0**	1	0	1	...	1	...	1	...
	$n+1$	1	1	1	**1**	1	1	0	1	...	1	...	1	...
T	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮		⋮		⋮	
i	ω	0	0	1	0	0	0	1	1	...	**1**	...	1	...
m	$\omega+1$	**0**	0	1	0	0	0	1	1	...	0	...	1	...
e	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮		⋮		⋮	
	ϑ	1	0	0	1	1	1	1	0	...	0	...	**0**	...
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮		⋮		⋮	

Fig. 2. An ordinal computation. Head positions are indicated by boxes.

(b) A *program* is a finite set P of commands satisfying the following structural conditions:

(i) If $(s, c, c', m, s') \in P$, there is $(s, d, d', n, t') \in P$ with $c \neq d$; thus, in state s the machine can react to reading a '0' as well as to reading a '1'.

(ii) If $(s, c, c', m, s') \in P$ and $(s, c, c'', m', s'') \in P$, then $c' = c'', m = m', s' = s''$; this means that the course of the computation is completely determined by the sequence of program states and the initial cell contents.

(c) For a program P, let

$$\text{states}(P) = \{s \mid \exists c, c', m, s' : (s, c, c', m, s') \in P\}$$

be the set of *program states*.

The intended semantics of a program is to modify the contents of the transfinite Turing tape according to the program commands. At a given time, zeros and ones on the tape can be described as a function $c : \text{Ord} \to 2$. In our applications the content function c will always have a bounded carrier, that is, $\{\xi \in \text{Ord} \mid c(\xi) \neq 0\}$ is a set. This allows us to code the collection of all (relevant) tape contents as one definable class $\mathbb{T}$. We could, for example, define

$$\mathbb{T} = \{d \mid \exists \vartheta \in \text{Ord}\, d : \vartheta \to 2\} = {}^{<\text{Ord}}2.$$

In the following we shall, however, treat tape contents as if they were defined for all ordinals.

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Definition 2. Let P be a program. A triple

$$S : \vartheta \to \omega, H : \vartheta \to \mathrm{Ord}, T : \vartheta \to \mathbb{T}$$

is an *ordinal computation* by P if the following hold:

(a) ϑ is a successor ordinal or $\vartheta = \mathrm{Ord}$; ϑ is the *length* of the computation.
(b) $S(0) = H(0) = 0$; the machine starts in state 0 with head position 0.
(c) If $t < \vartheta$ and $S(t) \notin \mathrm{state}(P)$, then $\vartheta = t + 1$; the machine *stops* if the machine state is not a program state of P.
(d) If $t < \vartheta$ and $S(t) \in \mathrm{state}(P)$, then $t+1 < \vartheta$; choose the unique command $(s, c, c', m, s') \in P$ with $S(t) = s$ and $T(t)_{H(t)} = c$. This command is executed as follows:

$$\begin{aligned} T(t+1)_\xi &= \begin{cases} c', \text{ if } \xi = H(t) \\ T(t)_\xi \text{ , otherwise} \end{cases} \\ S(t+1) &= s' \\ H(t+1) &= \begin{cases} H(t)+1, \text{ if } m = 1 \\ H(t)-1, \text{ if } m = 0 \text{ and } H(t) \text{ is a successor ordinal} \\ 0, \text{ otherwise.} \end{cases} \end{aligned}$$

(e) If $t < \vartheta$ is a limit ordinal, the machine constellation at t is determined by taking inferior limits:

$$\begin{aligned} \forall \xi \in \mathrm{Ord}\ T(t)_\xi &= \liminf_{r \to t} T(r)_\xi \\ S(t) &= \liminf_{r \to t} S(r) \\ H(t) &= \liminf_{s \to t, S(s) = S(t)} H(s). \end{aligned}$$

The computation is obviously recursively determined by the initial tape contents $T(0)$ and the program P. We call it the *ordinal computation by P with input $T(0)$*. If the computation stops, $\vartheta = \beta + 1$ is a successor ordinal and $T(\beta)$ is the final tape content. In this case we say that P *computes* $T(\beta)$ from $T(0)$ and write $P : T(0) \mapsto T(\beta)$.

Definition 3. A partial function $F : \mathbb{T} \rightharpoonup \mathbb{T}$ is *ordinal computable* if there is a program P such that $P : T \mapsto F(T)$ for every $T \in \mathrm{dom}(F)$.

By coding, the notion of ordinal computability can be extended to other domains. We can, for example, *code* an ordinal $\delta \in \mathrm{Ord}$ by the characteristic function $\chi_{\{\delta\}} : \mathrm{Ord} \to 2$, $\chi_{\{\delta\}}(\xi) = 1$ iff $\xi = \delta$, and give the following definition.

Definition 4. A partial function $F : \mathrm{Ord} \rightharpoonup \mathrm{Ord}$ is *ordinal computable* if the function $\chi_{\{\delta\}} \mapsto \chi_{\{F(\delta)\}}$ is ordinal computable.

Definition 5. A subset $x \subseteq \mathrm{Ord}$ is *ordinal computable from a finite set of ordinal parameters* if there is a finite subset $z \subseteq \mathrm{Ord}$ and a program P such that $P : \chi_z \mapsto \chi_x$.

Obviously, ordinal computability extends standard Turing computability. Standard methods, such as coding several ordinal tapes in one ordinal tape, can also be implemented

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in ordinal Turing machines. A number of concrete ordinal programs are given in Koepke (2005); as usual, algorithms are defined in intuitive pseudo-code.

As an example, consider the Gödel pairing function for ordinals, which will be used later to code syntactical notions. It is defined recursively by

$$\begin{aligned} G(\alpha,\beta) \;=\; \{G(\alpha',\beta') \mid\ & \max(\alpha',\beta') < \max(\alpha,\beta) \text{ or} \\ & (\max(\alpha',\beta') = \max(\alpha,\beta) \text{ and } \alpha' < \alpha) \text{ or} \\ & (\max(\alpha',\beta') = \max(\alpha,\beta) \text{ and } \alpha' = \alpha \text{ and } \beta' < \beta)\}. \end{aligned}$$

We sketch an algorithm for computing $\gamma = G(\alpha,\beta)$ that can be implemented straightforwardly on a Turing machine with several tapes, each holding one of the variables. We represent the ordinal α by its characteristic function χ_α, that is, by a tape starting with α ones, having value zero otherwise.

```
Goedel_Pairing:
0  alpha':=0
1  beta':=0
2  eta:=0
3  flag:=0
4  gamma:=0
5  if alpha=alpha' and beta=beta' then print gamma, stop fi
6  if alpha'=eta and and beta'=eta and flag=0 then
     alpha'=:0, flag:=1, go to 5 fi
7  if alpha'=eta and and beta'=eta and flag=1 then
     eta:=eta+1, alpha':=eta, beta':=0, gamma:=gamma+1, go to 5 fi
8  if beta'<eta and flag=0 then
     beta':=beta'+1, gamma:=gamma+1, go to 5 fi
9  if alpha'<eta and flag=1 then
     alpha':=alpha'+1, gamma:=gamma+1, go to 5 fi
```

Observe that by the computation rules in Definition 2 (e) this algorithm will always cycle to Command 5 at limit times. The inverse functions G_0 and G_1 satisfying

$$\forall\gamma\gamma = G(G_0(\gamma), G_1(\gamma))$$

are also ordinal computable. To compute $G_0(\gamma)$, compute $G(\alpha,\beta)$ for $\alpha,\beta \leqslant \gamma$ until you find α,β with $G(\alpha,\beta) = \gamma$, then set $G_0(\gamma) = \alpha$.

3. The theory SO of sets of ordinals

Ordinal Turing computations do not directly produce highly hierarchical sets but ordinals and sets of ordinals. It is well known that a model of Zermelo–Fraenkel set theory with the axiom of choice (ZFC) is determined by its sets of ordinals (Jech 2003, Theorem 13.28). This motivates the formulation of a theory SO that axiomatises the sets of ordinals in a model of ZFC. The theory SO is two-sorted, where the intended interpretations are *ordinals* and *sets of ordinals*. Let L_{SO} be the language

$$L_{\mathrm{SO}} := \{\mathrm{On}, \mathrm{SOn}, <, =, \in, g\}$$

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where On and SOn are unary predicate symbols, $<, =$ and $\in$ are binary predicate symbols, and g is a two-place function. The intended standard interpretation of g is given by the Gödel pairing function G. To simplify the notation, we use lower-case Greek letters to range over elements of On and lower case Roman letters to range over elements of SOn.

Definition 6. The theory SO is formulated in the first-order language L_{SO} and consists of the following axioms:

1 *Well-ordering axiom* (WO):
$\forall \alpha, \beta, \gamma(\neg \alpha < \alpha \wedge (\alpha < \beta \wedge \beta < \gamma \to \alpha < \gamma) \wedge (\alpha < \beta \vee \alpha = \beta \vee \beta < \alpha)) \wedge$
$\forall a(\exists \alpha(\alpha \in a) \to \exists \alpha(\alpha \in a \wedge \forall \beta(\beta < \alpha \to \neg \beta \in a)))$.

2 *Axiom of infinity* (INF – existence of a limit ordinal):
$\exists \alpha(\exists \beta(\beta < \alpha) \wedge \forall \beta(\beta < \alpha \to \exists \gamma(\beta < \gamma \wedge \gamma < \alpha)))$.

3 *Axiom of extensionality* (EXT):
$\forall a, b(\forall \alpha(\alpha \in a \leftrightarrow \alpha \in b) \to a = b)$.

4 *Initial segment axiom* (INI):
$\forall \alpha \exists a \forall \beta(\beta < \alpha \leftrightarrow \beta \in a)$.

5 *Boundedness axiom* (BOU):
$\forall a \exists \alpha \forall \beta(\beta \in a \to \beta < \alpha)$.

6 *Pairing axiom* (GPF – Gödel Pairing Function):
$\forall \alpha, \beta, \gamma(g(\beta, \gamma) \leqslant \alpha \leftrightarrow \forall \delta, \epsilon((\delta, \epsilon) <^* (\beta, \gamma) \to g(\delta, \epsilon) < \alpha))$.
Here $(\alpha, \beta) <^* (\gamma, \delta)$ stands for
$\exists \eta, \vartheta(\eta = \max(\alpha, \beta) \wedge \vartheta = \max(\gamma, \delta) \wedge$
$(\eta < \vartheta \vee (\eta = \vartheta \wedge \alpha < \gamma) \vee (\eta = \vartheta \wedge \alpha = \gamma \wedge \beta < \delta)))$,
where $\gamma = \max(\alpha, \beta)$ abbreviates $(\alpha > \beta \wedge \gamma = \alpha) \vee (\alpha \leqslant \beta \wedge \gamma = \beta)$.

7 *Surjectivity of pairing* (SUR):
$\forall \alpha \exists \beta, \gamma(\alpha = g(\beta, \gamma))$.

8 *Axiom schema of separation* (SEP):
For all L_{SO}-formulae $\varphi(\alpha, P_1, \ldots, P_n)$, where $P_1, \ldots, P_n$ are variables for ordinals or sets of ordinals, postulate:
$\forall P_1, \ldots, P_n \forall a \exists b \forall \alpha(\alpha \in b \leftrightarrow \alpha \in a \wedge \varphi(\alpha, P_1, \ldots, P_n))$.

9 *Axiom schema of replacement* (REP):
For all L_{SO}-formulae $\varphi(\alpha, \beta, P_1, \ldots, P_n)$, where $P_1, \ldots, P_n$ are variables for ordinals or sets of ordinals, postulate:
$\forall P_1, \ldots, P_n(\forall \xi, \zeta_1, \zeta_2(\varphi(\xi, \zeta_1, P_1, \ldots, P_n) \wedge \varphi(\xi, \zeta_2, P_1, \ldots, P_n) \to \zeta_1 = \zeta_2) \to$
$\forall a \exists b \forall \zeta(\zeta \in b \leftrightarrow \exists \xi \in a \varphi(\xi, \zeta, P_1, \ldots, P_n)))$.

10 *Powerset axiom* (POW):
$\forall a \exists b(\forall z(\exists \alpha(\alpha \in z) \wedge \forall \alpha(\alpha \in z \to \alpha \in a) \to \exists \xi \forall \beta(\beta \in z \leftrightarrow g(\beta, \xi) \in b)))$.

In a model of ZFC, the class of sets of ordinals together with the standard relations $<, =$ and $\in$, and the Gödel pairing function G constitutes a model of SO. Note that the powerset axiom of SO requires the axiom of choice since it stipulates the existence of *well-ordered* powersets. Thus, we have the following theorem.

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Theorem 2. The theory SO can be interpreted in the theory ZFC.

For the converse direction, which will be proved in the two subsequent sections, we first indicate that all basic mathematical notions can be reasonably formalised within the system SO. Beyond the specific requirements of the present paper, this also shows that the theory SO might have some wider interest as a foundational theory.

For the formalisation of mathematics within SO we make use of the familiar class term notation $A = \{\alpha \mid \varphi(\alpha)\}$ to denote *classes* of ordinals. If $A = \{\alpha \mid \varphi(\alpha)\}$ is a non-empty class of ordinals, that is, $\forall \alpha \in A\, \mathrm{On}(\alpha)$, let $\min(A)$ denote the *minimal element* of A. The existence of a unique *minimum* follows from the axioms INI, SEP and WO. BOU ensures the existence of an *upper bound* for each set a, the least of which will be denoted $\mathrm{lub}(a)$. By INI, the classes $\iota_\alpha := \{\beta \mid \beta < \alpha\}$ are sets. Using SEP and INI, one sees that the *union* and *intersection* of two sets are again sets. Finite sets are denoted by $\{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}$. Their existence is implied by INI and SEP. We write $\mathrm{POW}(b, a)$ for b being a set satisfying POW for a; note that in SO the set b is *not* uniquely determined by a. ω denotes the least element of the class of limit numbers, which by INF is not empty. Finally, let $0 := \min(\{\alpha \mid \mathrm{On}(\alpha)\})$, $1 := \mathrm{lub}(\{0\})$, and so on.

The inverse functions G_1, G_2 of G are defined via the properties $\alpha = G_1(\beta) \leftrightarrow \exists\gamma(\beta = G(\alpha, \gamma))$ and $\alpha = G_2(\beta) \leftrightarrow \exists\gamma(\beta = G(\gamma, \alpha))$. The axioms GPF and SUR imply the well-known properties of the Gödel pairing function and its projections, such as the bijectivity and monotonicity properties. To simplify the notation, we write $(\alpha, \beta) := G(\alpha, \beta)$. Every set can be regarded as a set of pairs $a = \{(\alpha, \beta) \mid (\alpha, \beta) \in a\}$, or more generally as a set of n-tuples. In this way, n-ary *relations* and *functions* on ordinals can be represented by sets. We now define some further notions connected with relations and functions.

Definition 7. For sets or classes R, X, Y, f, we define the following notions in SO:

$$
\begin{aligned}
\emptyset &:= \iota_0 \\
\mathrm{dom}(R) &:= \{\alpha \mid \exists\beta((\alpha, \beta) \in R)\} \\
\mathrm{ran}(R) &:= \{\beta \mid \exists\alpha((\alpha, \beta) \in R)\} \\
\mathrm{fun}(f) &:= \forall\alpha, \beta_1, \beta_2((\alpha, \beta_1) \in f \wedge (\alpha, \beta_2) \in f \rightarrow \beta_1 = \beta_2) \\
f : X \rightarrow Y &:= \mathrm{fun}(f) \wedge \mathrm{dom}(f) = X \wedge \mathrm{ran}(f) \subset Y \\
\alpha = f(\beta) &:= (\alpha, \beta) \in f \\
\alpha R \beta &:= (\alpha, \beta) \in R \\
X \times Y &:= \{\gamma \mid G_1(\gamma) \in X \wedge G_2(\gamma) \in Y\} \\
X \upharpoonright Y &:= \{(\alpha, \beta) \in X \mid \alpha \in Y\}.
\end{aligned}
$$

The axioms of SO imply that these notions have their usual basic properties. We can now prove transfinite induction and recursion in SO.

Theorem 3 (SO). Let $\varphi(\alpha, X_1, \ldots, X_n)$ be an L_{SO}-formula. Then for all $X_1, \ldots, X_n$,

$$\forall\alpha((\forall\beta < \alpha\varphi(\beta, X_1, \ldots, X_n)) \rightarrow \varphi(\alpha, X_1, \ldots, X_n))$$

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implies

$$\forall \alpha \varphi(\alpha, X_1, \ldots, X_n).$$

Proof. If the statement were not true, by WO, there would be a minimal counterexample α, contradicting the assumption. □

Theorem 4 (SO). Let $R : \mathrm{On} \times \mathrm{SOn} \to \mathrm{On}$ be a function defined by some formula $\varphi(\alpha, f, \beta, X_1, \ldots, X_n)$. Then there exists a unique function $F : \mathrm{On} \to \mathrm{On}$ defined by a formula $\psi(\alpha, \beta, X_1, \ldots, X_n)$ such that

$$\forall \alpha (F(\alpha) = R(\alpha, F \upharpoonright \imath_\alpha)). \tag{$*$}$$

Proof. The proof is similar to the proof of the recursion theorem in ZF: we define the notion of *approximation functions*, which are set-functions defined on proper initial segments of Ord, satisfying ($*$) on their domain. Then we obtain F as the union of all of these approximation functions. □

As in ZF, this result can be generalised from the relation $<$ to arbitrary set-like well-founded relations. One could now develop further mathematical notions – numbers, spaces, first-order syntax and semantics – in SO in much the same way as one does in standard set theory.

4. Assembling sets along well-founded relations

In standard set theory, a set z can be represented as a *point in a well-founded relation*: consider the $\in$-relation on the transitive closure $\mathrm{TC}(\{z\})$ with the distinguished element $z \in \mathrm{TC}(\{z\})$. By the Mostowski isomorphism theorem, z is uniquely determined by the isomorphism type of the pair $(z, \in \upharpoonright \mathrm{TC}(\{z\}))$. So one may represent z by some pair (x, R_x) such that R_x is a well-founded relation *on ordinals* and $(x, R_x) \cong (z, \in \upharpoonright \mathrm{TC}(\{z\}))$.

Working in the theory SO, we can assume that the relation R_x is a set of ordinals and that the point (x, R_x) is coded as a set of ordinals by defining the *ordered pair* of an ordinal and a set of ordinals as

$$(x, R_x) = \{G(x, \alpha) \mid \alpha \in R_x\}.$$

So we *assume* SO *for the following construction*. We shall eventually define a model of ZFC within SO. Constructions of set-theoretic structures from collections of well-founded relations are common in higher-order recursion theory, such as, for example, in the construction of the admissible set $\mathbf{L}_{\omega_1^{\mathrm{CK}}}$ from the hyperarithmetic reals (Sacks 1990).

Definition 8. An ordered pair $x = (x, R_x)$ is a *point* if R_x is a well-founded relation on ordinals and $x \in \mathrm{dom}(R_x)$. Let $\mathbb{P}$ be the class of all points. Unless specified otherwise, we use R_x to denote the well-founded relation of the point x.

Note that, according to our previous considerations, one can reasonably define the class $\mathbb{P}$ in SO as well as in ZFC. In ZFC, $(x, \in \upharpoonright \mathrm{TC}(\{x\}))$ is a point, and any point $x = (x, R_x)$ can be interpreted as a standard set $I(x)$: define recursively

$$I_x : \mathrm{dom}(R_x) \to V, \text{ by } I_x(u) = \{I_x(v) \mid v R_x u\}.$$

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Then let $I(x) = I_x(x)$ be the *interpretation* of x. Note that for points x and y

$$I_x(u) = I_y(v) \text{ iff } \{I_x(u') \mid u'R_xu\} = \{I_x(v') \mid v'R_yv\}$$
$$\text{iff } (\forall u'R_xu\exists v'R_yvI_x(u') = I_y(v')) \wedge (\forall v'R_yv\exists u'R_xuI_x(u') = I_y(v')).$$

This means that the relation $I_x(u) = I_y(v)$ in the variables u and v can be defined recursively without actually forming the interpretations $I_x(u)$ and $I_x(v)$. Hence this relation can be defined in SO.

Definition 9. With R_x and R_y well-founded relations on ordinals, define a relation $\equiv$ on points $(u, R_x), (v, R_y)$ by induction on the product well-order $R_x \times R_y$:

$$(u, R_x) \equiv (v, R_y) \text{ iff } \forall u'R_xu\exists v'R_yv\ \ (u', R_x) \equiv (v', R_y)$$
$$\wedge\forall v'R_yv\exists u'R_xu\ \ (u', R_x) \equiv (v', R_y).$$

Lemma 1 (SO). $\equiv$ is an equivalence relation on $\mathbb{P}$.

Proof.

Reflexivity: Consider a point $x = (x, R_x)$. We show by induction on R_x that for all $u \in \mathrm{dom}(R_x)$ we have $(u, R_x) \equiv (u, R_x)$. Assume that the claim holds for all vR_xu. Consider some vR_xu. By the induction assumption, $(v, R_x) \equiv (v, R_x)$. This implies

$$\forall vR_xu\exists wR_xu\ \ (v, R_x) \equiv (w, R_x).$$

By symmetry, we also have

$$\forall wR_xu\exists vR_xu\ \ (v, R_x) \equiv (w, R_x).$$

Together these imply $(u, R_x) \equiv (u, R_x)$.
In particular, $x = (x, R_x) \equiv (x, R_x) = x$.

Symmetry: Consider points $x = (x, R_x)$ and $y = (y, R_y)$. We show by induction on the well-founded relation $R_x \times R_y$ that

$$(u, R_x) \equiv (v, R_y) \text{ iff } (v, R_y) \equiv (u, R_x).$$

Assume that the claim holds for all (u', v') with $u'R_xu$ and $v'R_yv$. Assume that $(u, R_x) \equiv (v, R_y)$. To show that $(v, R_y) \equiv (u, R_x)$, consider $v'R_yv$. By assumption, take $u'R_xu$ such that $(u', R_x) \equiv (v', R_y)$. By the induction assumption on symmetry, $(v', R_y) \equiv (u', R_x)$. Hence

$$\forall v'R_yv\exists u'R_xu\ \ (v', R_y) \equiv (u', R_x).$$

Similarly,

$$\forall u'R_xu\exists v'R_yv\ \ (v', R_y) \equiv (u', R_x),$$

and thus $(v, R_y) \equiv (u, R_x)$. This shows

$$(u, R_x) \equiv (v, R_y) \rightarrow (v, R_y) \equiv (u, R_x).$$

By the symmetry of the situation, the implication from right to left also holds and

$$(u, R_x) \equiv (v, R_y) \leftrightarrow (v, R_y) \equiv (u, R_x).$$

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In particular, for $x = (x, R_x)$ and $y = (y, R_y)$

$$x \equiv y \leftrightarrow y \equiv x.$$

Transitivity: Consider points $x = (x, R_x)$, $y = (y, R_y)$ and $z = (z, R_z)$. We show by induction on the well-founded relation $R_x \times R_y \times R_z$ that

$$(u, R_x) \equiv (v, R_y) \wedge (v, R_y) \equiv (w, R_z) \rightarrow (u, R_x) \equiv (w, R_z).$$

Assume that the claim holds for all (u', v', w') with $u' R_x u$, $v' R_y v$ and $w' R_z w$. Assume also that

$$(u, R_x) \equiv (v, R_y) \wedge (v, R_y) \equiv (w, R_z).$$

To show that $(u, R_x) \equiv (w, R_z)$, consider $u' R_x u$. By $(u, R_x) \equiv (v, R_y)$, take $v' R_y v$ such that $(u', R_x) \equiv (v', R_y)$. By $(v, R_y) \equiv (w, R_z)$, take $w' R_z w$ such that $(v', R_y) \equiv (w', R_z)$. By the inductive assumption, $(u', R_x) \equiv (v', R_y)$ and $(v', R_y) \equiv (w', R_z)$ imply that $(u', R_x) \equiv (w', R_z)$. Thus,

$$\forall u' R_x u \exists w' R_z w \ \ (u', R_x) \equiv (w', R_z).$$

Similarly,

$$\forall w' R_z w \exists u' R_x u \ \ (u', R_x) \equiv (w', R_z).$$

So, $(u, R_x) \equiv (w, R_z)$. In particular, for $x = (x, R_x)$, $y = (y, R_y)$ and $z = (z, R_z)$

$$x \equiv y \wedge y \equiv z \rightarrow x \equiv z. \qquad \square$$

We now define a membership relation for points.

Definition 10. Let $x = (x, R_x)$ and $y = (y, R_y)$ be points. Then set

$$x \blacktriangleleft y \text{ iff } \exists v R_y y \ \ x \equiv (v, R_y).$$

Lemma 2 (SO). The equivalence relation $\equiv$ is a congruence relation with respect to $\blacktriangleleft$, that is,

$$x \blacktriangleleft y \wedge x \equiv x' \wedge y \equiv y' \rightarrow x' \blacktriangleleft y'.$$

Proof. Let $x \blacktriangleleft y \wedge x \equiv x' \wedge y \equiv y' \rightarrow x' \blacktriangleleft y'$. Take $v R_y y$ such that $x \equiv (v, R_y)$. By $y \equiv y'$, take $v' R_{y'} y'$ such that $(v, R_y) \equiv (v', R_{y'})$. Since $\equiv$ is an equivalence relation, the equivalences $x \equiv x'$, $x \equiv (v, R_y)$ and $(v, R_y) \equiv (v', R_{y'})$ imply $x' \equiv (v', R_{y'})$. Hence $x' \blacktriangleleft y'$. $\square$

5. The class of points satisfies ZFC

We show that the class $\mathbb{P}$ of points with the relations $\equiv$ and $\blacktriangleleft$ satisfies the axioms ZFC of Zermelo–Fraenkel set theory with the axiom of choice. For the existence axioms of ZFC we prove a lemma about combining points into a single point.

Lemma 3 (SO). Let $(x_i \mid i \in A)$ be a set-sized definable sequence of points, that is, A is a set of ordinals and the function $i \mapsto x_i \in \mathbb{P}$ is definable. Then there is a point $y = (y, R_y)$

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such that for all points x

$$x \blacktriangleleft y \text{ iff } \exists i \in A \ \ x \equiv x_i.$$

Proof. For $i \in A$ let $x_i = (x_i, R_i)$. Define points $x_i' = (x_i', R_i')$ by 'colouring' every element of $\mathrm{dom}(R_i)$ by the 'colour' i:

$$x_i' = (i, x_i) \text{ and } R_i' = \{((i,\alpha),(i,\beta)) \mid (\alpha,\beta) \in R_i\}.$$

The points (x_i', R_i') and (x_i, R_i) are isomorphic, so $(x_i', R_i') \equiv (x_i, R_i)$. We may thus assume that the domains of the well-founded relations R_i are pairwise disjoint. Take some $y \notin \bigcup_{i \in A} \mathrm{dom}(R_i)$ and define the point $y = (y, R_y)$ by

$$R_y = \bigcup_{i \in A} R_i \cup \{(x_i, y) \mid i \in A\}.$$

Consider $i \in A$. If $x \in \mathrm{dom}(R_i)$, the iterated R_i-predecessors of x are equal to the iterated R_y-predecessors of x. Hence $(x, R_i) \equiv (x, R_y)$.

Assume now that $x \blacktriangleleft y$. Take $vR_y y$ such that $x \equiv (v, R_y)$. Take $i \in A$ such that $v = x_i$. By the previous remark,

$$x \equiv (v, R_y) = (x_i, R_y) \equiv (x_i, R_i) = x_i.$$

Conversely, consider $i \in A$ and $x \equiv x_i$. Then $x \equiv x_i = (x_i, R_i) \equiv (x_i, R_y)$ and $x_i R_y y$. This implies $x \blacktriangleleft y$. □

We are now able to canonically interpret the theory ZFC within SO.

Theorem 5 (SO). $\mathbb{P} = (\mathbb{P}, \equiv, \blacktriangleleft)$ is a model of ZFC.

Proof.

Extensionality: We need to show the axiom of *extensionality* holds in $\mathbb{P}$:

$$\forall x \forall y (\forall z (z \blacktriangleleft x \leftrightarrow z \blacktriangleleft y) \rightarrow x \equiv y).$$

Consider points x and y such that $\forall z (z \blacktriangleleft x \leftrightarrow z \blacktriangleleft y)$, and consider $uR_x x$. Then $(u, R_x) \blacktriangleleft (x, R_x) = x$. By assumption, $(u, R_x) \blacktriangleleft (y, R_y)$. By definition, take $vR_y y$ such that $(u, R_x) \equiv (v, R_y)$. Thus

$$\forall u R_x x \exists v R_y y \ \ (u, R_x) \equiv (v, R_y).$$

By exchanging x and y, we also get

$$\forall v R_y y \exists u R_x x \ \ (u, R_x) \equiv (v, R_y).$$

Hence $x \equiv y$.

Pairing: We need to show the axiom of *pairing* holds in $\mathbb{P}$:

$$\forall x \forall y \exists z \forall w (w \blacktriangleleft z \leftrightarrow (w \equiv x \vee w \equiv y)).$$

Consider points $x = (x, R_x)$ and $y = (y, R_y)$. By the comprehension lemma, Lemma 3, there is a point $z = (z, R_z)$ such that for all points w

$$w \blacktriangleleft z \leftrightarrow (w \equiv x \vee w \equiv y).$$

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Unions: We need to show the axiom of *unions* holds in $\mathbb{P}$:

$$\forall x \exists y \forall z (z \blacktriangleleft y \leftrightarrow \exists w (w \blacktriangleleft x \wedge z \blacktriangleleft w)).$$

Consider a point $x = (x, R_x)$. Let

$$A = \{i \in \mathrm{dom}(R_x) \mid \exists u \in \mathrm{dom}(R_x) \ \ iR_x u R_x x\}.$$

For $i \in A$ define the point $x_i = (i, R_x)$. By the comprehension lemma, Lemma 3, there is a point $y = (y, R_y)$ such that for all points z

$$z \blacktriangleleft y \leftrightarrow \exists i \in A \ \ z \equiv x_i.$$

To show the axiom, consider some $z \blacktriangleleft y$. Take $i \in A$ such that $z \equiv x_i$. Take $u \in \mathrm{dom}(R_x)$ such that $iR_x u R_x x$. Then $z \equiv x_i = (i, R_x) \blacktriangleleft (u, R_x) \blacktriangleleft (x, R_x) = x$, that is, $\exists w (z \blacktriangleleft w \blacktriangleleft x)$.

Conversely, assume that $\exists w (z \blacktriangleleft w \blacktriangleleft x)$ and take w such that $z \blacktriangleleft w \blacktriangleleft x$. Take $uR_x x$ such that $w \equiv (u, R_x)$. Then $z \blacktriangleleft (u, R_x)$. Take $iR_x u$ such that $z \equiv (i, R_x) = x_i$. Then $z \blacktriangleleft y$.

Replacement schema: We need to show the *replacement schema* holds in $\mathbb{P}$, that is, for every first-order formula $\varphi(u, v)$ in the language of $\equiv$ and $\blacktriangleleft$ the following is true in $\mathbb{P}$:

$$\forall u, v, v' ((\varphi(u, v) \wedge \varphi(u, v')) \rightarrow v \equiv v') \rightarrow \forall x \exists y \forall z (z \blacktriangleleft y \leftrightarrow \exists u (u \blacktriangleleft x \wedge \varphi(u, z))).$$

Note that the formula φ may contain further free parameters, which we do not mention for the sake of simplicity. Assume that $\forall u, v, v' ((\varphi(u, v) \wedge \varphi(u, v')) \rightarrow v \equiv v')$ and let $x = (x, R_x)$ be a point. Let $A = \{i \mid iR_x x\}$. For each $i \in A$ we have the point $(i, R_x) \blacktriangleleft (x, R_x) = x$. Using replacement and choice in SO, we can pick for each $i \in A$ a point $z_i = (z_i, R_{z_i})$ such that $\varphi((i, R_x), z_i)$ holds if such a point exists. By the comprehension lemma, Lemma 3, there is a point $y = (y, R_y)$ such that for all points z

$$z \blacktriangleleft y \leftrightarrow \exists i \in A \ \ z \equiv z_i.$$

To show the instance of the replacement schema under consideration, assume that $z \blacktriangleleft y$. Take $i \in A$ such that $z \equiv z_i$. Then $(i, R_x) \blacktriangleleft (x, R_x) = x$, $\varphi((i, R_x), z_i)$ and $\varphi((i, R_x), z)$. Hence $\exists u (u \blacktriangleleft x \wedge \varphi(u, z))$.

Conversely, assume that $\exists u (u \blacktriangleleft x \wedge \varphi(u, z))$. Take $u \blacktriangleleft x$ such that $\varphi(u, z)$. Take $iR_x x$, $i \in A$ such that $u \equiv (i, R_x)$. Then $\varphi((i, R_x), z)$. By the definition of z_i, we have $\varphi((i, R_x), z_i)$. The functionality of the formula φ implies $z \equiv z_i$. Hence $\exists i \in A z \equiv z_i$ and $z \blacktriangleleft y$.

Separation schema: The replacement schema also implies the *separation* schema.

Powersets: We need to show the axiom of *powersets* holds in $\mathbb{P}$:

$$\forall x \exists y \forall z (z \blacktriangleleft y \leftrightarrow \forall w (w \blacktriangleleft z \rightarrow w \blacktriangleleft x)).$$

By the separation schema, it suffices to show that

$$\forall x \exists y \forall c (\forall w (w \blacktriangleleft c \rightarrow w \blacktriangleleft x) \rightarrow c \blacktriangleleft y).$$

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Consider a point $x = (x, R_x)$. Let $F = \mathrm{dom}(R_x) \cup \mathrm{ran}(R_x)$ be the *field* of R_x. By the powerset axiom of SO, choose some set P such that $\mathrm{Pow}(P, F)$:

$$\forall z(\exists \alpha(\alpha \in z) \wedge \forall \alpha(\alpha \in z \to \alpha \in F) \to \exists \xi \forall \beta(\beta \in z \leftrightarrow (\beta, \xi) \in P)).$$

Choose two large ordinals δ and y such that

$$\forall \alpha \in F \alpha < \delta \text{ and } \forall \xi(\xi \in \mathrm{ran}(P) \to (\delta, \xi) < y).$$

Define a point $y = (y, R_y)$ by

$$R_y = R_x \cup \{(\beta, (\delta, \xi)) \mid (\beta, \xi) \in P\} \cup \{((\delta, \xi), y) \mid \xi \in \mathrm{ran}(P)\}.$$

To show the axiom, consider some point $c = (c, R_c)$ such that $\forall w(w \blacktriangleleft c \to w \blacktriangleleft x)$. Define a corresponding subset z of F by

$$z = \{\beta \in F \mid \exists v R_c c \ (v, R_c) \equiv (\beta, R_x)\}.$$

We may assume for simplicity that $z \neq \emptyset$. By the powerset axiom of SO, choose $\xi \in \mathrm{ran}(P)$ such that

$$\forall \beta(\beta \in z \leftrightarrow (\beta, \xi) \in P).$$

We claim that $((\delta, \xi), R_y) \equiv c$ and thus $c \blacktriangleleft y$.
Consider $\beta R_y(\delta, \xi)$. By the definition of R_y, we have $(\beta, \xi) \in P$, so $\beta \in z$. By the definition of z, choose $v R_c c$ such that $(v, R_c) \equiv (\beta, R_x) \equiv (\beta, R_y)$.
Conversely, consider $v R_c c$. Then $(v, R_c) \blacktriangleleft (c, R_c) = c$. The subset property implies $(v, R_c) \blacktriangleleft (x, R_x) = x$. Take $\beta R_x x$ such that $(v, R_c) \equiv (\beta, R_x) \equiv (\beta, R_y)$. But, by definition, $\beta \in z$, $(\beta, \xi) \in P$ and $\beta R_y(\delta, x)$.

Axiom of choice: We need to show the *axiom of choice* holds in $\mathbb{P}$:

$$\forall x((\forall y, z((y \blacktriangleleft x \wedge z \blacktriangleleft x) \to (\exists u\ u \blacktriangleleft y \wedge (\neg y \equiv z \to \neg \exists u(u \blacktriangleleft y \wedge u \blacktriangleleft z))))) \to \exists w \forall y(y \blacktriangleleft x \to \exists u((u \blacktriangleleft w \wedge u \blacktriangleleft y) \wedge \forall v((v \blacktriangleleft w \wedge v \blacktriangleleft y) \to u \equiv v)))).$$

Let $x = (x, R_x) \in \mathbb{P}$ be a point such that

$$\forall y, z((y \blacktriangleleft x \wedge z \blacktriangleleft x) \to (\exists u\ u \blacktriangleleft y \wedge (\neg y \equiv z \to \neg \exists u(u \blacktriangleleft y \wedge u \blacktriangleleft z)))).$$

Choose an ordinal $\alpha \in \mathrm{dom}(R_x)$ and define the point $w = (\alpha, R_w)$ by letting its 'elements' be least ordinals in the 'elements' of x:

$$R_w = R_x \cup \{(\xi, \alpha) \mid \exists \zeta(\xi R_x \zeta R_x x \wedge (\forall \xi' < \xi \forall \zeta'((\zeta, R_x) \equiv (\zeta', R_x) \to \neg(\xi R_x \xi' R_x \zeta))))\}.$$

To show that w witnesses the axiom of choice for x, consider a point y with $y \blacktriangleleft x$. We may assume that y is of the form $y = (\zeta, R_x)$ where $\zeta R_x x$. By the assumption on x, there exists $u \blacktriangleleft y$. Take some ξ such that $(\xi, R_x) \equiv u$. We may assume that ζ and ξ with these properties are chosen so that ξ is minimal in the ordinals. Then

$$\xi R_x \zeta R_x x \wedge (\forall \xi' < \xi \forall \zeta'((\zeta, R_x) \equiv (\zeta', R_x) \to \neg(\xi R_x \xi' R_x \zeta))), \qquad (+)$$

so $\xi R_w \alpha$. Thus $u \blacktriangleleft w$. To show the uniqueness of this u with $u \blacktriangleleft w \wedge u \blacktriangleleft y$, consider some v with $v \blacktriangleleft w \wedge v \blacktriangleleft y$. We may assume that v is of the form $v = (\xi', R_w)$ with $\xi' R_w \alpha$. By the definition of R_w, we choose some ζ' such that

$$\xi' R_x \zeta' R_x x \wedge (\forall \xi'' < \xi' \forall \zeta''((\zeta', R_x) \equiv (\zeta'', R_x) \to \neg(\xi' R_x \xi'' R_x \zeta'))). \qquad (\times)$$

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Now

$$v \blacktriangleleft y \blacktriangleleft x \text{ and } v = (\xi', R_w) \blacktriangleleft (\zeta', R_w) \blacktriangleleft (x, R_x) = x.$$

Since the 'elements' of x are 'pairwise disjoint', we have $y \equiv (\zeta', R_w)$. Since $y \equiv (\zeta, R_x)$, the conditions (+) and (×) become equivalent and define the same ordinal $\xi = \xi'$. Hence

$$u \equiv (\xi, R_x) \equiv (\xi', R_w) \equiv v.$$

Foundation schema: We need to show the *foundation schema* holds in $\mathbb{P}$, that is, for every first-order formula $\varphi(u)$ in the language of $\equiv$ and $\blacktriangleleft$ the following is true in $\mathbb{P}$:

$$\exists u \varphi(u) \to \exists y(\varphi(y) \wedge \forall z(z \blacktriangleleft y \to \neg\varphi(z)).$$

Note that the formula φ may contain further free parameters, which we do not mention for the sake of simplicity. Assume that $\exists u \varphi(u)$. Take a point $x = (x, R_x)$ such that $\varphi(x)$. Since R_x is well-founded, we may take an R_x-minimal $y \in \mathrm{dom}(R_x)$ such that $\varphi((y, R_x))$. Letting y also denote the point (y, R_x), we have $\varphi(y)$. To prove the axiom, consider some point $z \blacktriangleleft y$. Take vR_xy such that $z \equiv (v, R_x)$. By the R_x-minimal choice of y, we have $\neg\varphi((v, R_x))$. Hence $\neg\varphi(z)$.

Infinity: We need to show the axiom of *infinity* holds in $\mathbb{P}$, that is,

$$\exists x((\exists y y \blacktriangleleft x) \wedge (\forall y(y \blacktriangleleft x \to \exists z(z \blacktriangleleft x \wedge \forall u(u \blacktriangleleft z \leftrightarrow (u \blacktriangleleft y \vee u \equiv y)))))).$$

In SO, let ω be the smallest limit ordinal. We show that

$$x = (\omega, < \upharpoonright (\omega+1)^2)$$

witnesses the axiom. Since $(0, < \upharpoonright (\omega+1)^2) \blacktriangleleft (\omega, < \upharpoonright (\omega+1)^2)$, we have $\exists y y \blacktriangleleft x$. Consider some $y \blacktriangleleft x$. We may assume that $y = (n, < \upharpoonright (\omega+1)^2)$ for some $n < \omega$. Set

$$z = (n+1, < \upharpoonright (\omega+1)^2).$$

It is easy to check that

$$z \blacktriangleleft x \wedge \forall u(u \blacktriangleleft z \leftrightarrow (u \blacktriangleleft y \vee u \equiv y)).$$

□

Theorem 6. In the set theoretical universe V, consider a class $\mathscr{S} \subseteq \{x \mid x \subseteq \mathrm{Ord}\}$ such that $\mathscr{S} = (\mathrm{Ord}, \mathscr{S}, <, =, \in, G)$ is a model of the theory SO. Then there is a unique inner model $(M, \in)$ of ZFC such that $\mathscr{S} = \{v \in M \mid v \subseteq \mathrm{Ord}\}$.

Proof. Define the model $\mathbb{P} = (\mathbb{P}, \equiv, \blacktriangleleft)$ from $(\mathrm{Ord}, \mathscr{S}, <, =, \in, G)$ as above. Consider a point $x = (x, R_x) \in \mathbb{P}$. Then x is also an ordinal in the sense of V. In $\mathscr{S}$, apply the recursion theorem to the well-founded relation R_x and obtain an order-preserving map

$$\sigma : (\mathrm{dom}(R_x), R_x) \to (\mathrm{Ord}, <).$$

Transfer the map σ to V by defining

$$\tilde{\sigma} = \{(\alpha, \beta) \mid \mathscr{S} \vDash \sigma(\alpha) = \beta\} : \mathrm{dom}(R_x) \to \mathrm{Ord}\,.$$

This map is order-preserving and witnesses the fact that R_x is well-founded in V. So (x, R_x) is a point in the sense of V. In V, define the interpretation function $I : \mathbb{P} \to V$

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recursively by

$$I_x : \mathrm{dom}(R_x) \to V, \text{ by } I_x(u) = \{I_x(v) \mid vR_xu\} \text{ and } I(x) = I_x(x).$$

Set

$$M = \{I(x) \mid x \in \mathbb{P}\}.$$

Transitivity:
Consider $y \in I(x) \in M$. Choose vR_xx such that $y = I_x(v)$. Then $(v, R_x) \in \mathbb{P}$ and

$$y = I_x(v) = I((v, R_x)) \in M.$$

So M is transitive.

Surjectivity:
The above definitions imply that the function $I : \mathbb{P} \to M$ is surjective and preserves $\equiv$ and $=$, and $\blacktriangleleft$ and $\in$:

$$\forall x, y \in \mathbb{P} : ((x \equiv y \leftrightarrow I(x) = I(y)) \wedge (x \blacktriangleleft y \leftrightarrow I(x) \in I(y))).$$

Inner model:
The above imply that M is a transitive $\in$-model of the ZFC-axioms, that is, M is an inner model.

$\mathscr{S} = \{v \in M \mid v \subseteq \mathrm{Ord}\}$:
Let $v \in \mathscr{S}$. We build a point that will be interpreted as v. Choose an ordinal α such that $v \subseteq \alpha$. Define a well-founded relation R_x on $\alpha + 1$ by

$$\xi R_x \zeta \text{ iff } (\xi < \zeta < \alpha \text{ or } (\zeta = \alpha \wedge \xi \in v)).$$

Then $x = (\alpha, R_x)$ is a point. Let $I_x(u) = \{I_x(v) \mid vR_xu\}$ be the recursive interpretation function for x. For $\zeta < \alpha$ we have $I_x(\zeta) = \zeta$ since we have inductively

$$I_x(\zeta) = \{I_x(\xi) \mid \xi R_x \zeta\} = \{\xi \mid \xi < \zeta\} = \zeta.$$

Therefore,

$$I(x) = I_x(\alpha) = \{I_x(\xi) \mid \xi R_x \alpha\} = \{\xi \mid \xi \in v\} = v.$$

Hence, $v = I(x) \in M$.
The previous argument also shows that one may canonically represent an ordinal ξ by the point $(\xi, < \restriction (\xi + 1)^2)$:

$$I((\xi, < \restriction (\xi + 1)^2)) = \xi.$$

For the converse inclusion, consider some $v \in M$, $v \subseteq \alpha \in \mathrm{Ord}$. Choose a point $x \in \mathbb{P}$ such that $I(x) = v$. Since $\mathscr{S}$ satisfies the separation schema,

$$v = \{\xi < \alpha \mid \xi \in v\} = \{\xi < \alpha \mid \mathscr{S} \models (\xi, < \restriction (\xi + 1)^2) \blacktriangleleft x\} \in \mathscr{S}.$$

The model M is unique since it is determined by its sets of ordinals (see Jech (2003, Theorem 13.28). □

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6. A recursive truth predicate

The Gödel pairing function G allows us to code a finite sequence $\alpha_0, \ldots, \alpha_{n-1}$ of ordinals as a single ordinal:

$$\alpha = G(\ldots G(G(\alpha_0, \alpha_1), \alpha_2) \ldots).$$

The usual operations on finite sequences, such as concatenation, cutting at a certain length and substitution, are ordinal computable using the Gödel functions G, G_0, G_1. This means we can code terms and formulae of a first-order language by single ordinals in an ordinal computable way.

We introduce a language L_T suitable for structures of the form

$$(\alpha, <, G \cap \alpha^3, f),$$

where $G \cap \alpha^3$ is viewed as a ternary relation and $f : \alpha \to \alpha$ is a unary function. The language has variables $v_n = G(0, n)$ for $n < \omega$ and constant symbols $c_\xi = G(1, \xi)$ for $\xi \in \mathrm{Ord}$; the symbol c_ξ will be interpreted as the ordinal ξ. Terms are defined recursively: variables and constant symbols are terms; if t is a term, then $G(2, t)$ is a term as well, which stands for $f(t)$. Atomic formulae are of the form:

— $G(3, G(t_1, t_2))$ where t_1, t_2 are terms – this stands for the equality $t_1 = t_2$.
— $G(4, G(t_1, t_2))$ where t_1, t_2 are terms – this stands for the inequality $t_1 < t_2$.
— $G(5, G(G(t_1, t_2), t_3))$ where t_1, t_2, t_3 are terms – this stands for the relation $t_3 = G(t_1, t_2)$.

L_T-Formulae are defined recursively. Atomic formulae are formulae, and if φ and ψ are formulae, then the following are formulae as well:

— $G(6, \varphi)$ – this stands for the negation $\neg\varphi$.
— $G(7, G(\varphi, \psi))$ – this stands for the conjunction $(\varphi \wedge \psi)$.
— $G(8, G(v_n, \varphi))$ where v_n is a variable – this stands for the existential quantification $\exists v_n \varphi$.

Then the satisfaction relation

$$(\alpha, <, G \cap \alpha^3, f) \vDash \varphi[b]$$

for φ an L_T-formula and b an assignment of values in α can be defined as usual. If the function f is ordinal computable, this property is ordinal computable, since the recursive Tarski truth definition can be carried out by an ordinal Turing machine.

We define the truth predicate $T : \mathrm{Ord} \to \{0, 1\}$ recursively by

$$T(\alpha) = 1 \text{ iff } (\alpha, <, G \cap \alpha^3, T \upharpoonright \alpha) \vDash G_0(\alpha)[G_1(\alpha)].$$

The assignments $\alpha \mapsto T(\alpha)$ can be enumerated successively by an ordinal Turing machine. Hence T is ordinal computable.

7. T codes a model of SO

The truth predicate T contains information about a large class of sets of ordinals.

Definition 11. For ordinals μ and α, define

$$T(\mu, \alpha) = \{\beta < \mu \mid T(G(\alpha, \beta)) = 1\}.$$

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Set

$$\mathscr{S} = \{T(\mu, \alpha) \mid \mu, \alpha \in \mathrm{Ord}\}.$$

Theorem 7. $(\mathrm{Ord}, \mathscr{S}, <, =, \in, G)$ is a model of the theory SO.

Proof. Axioms 1 to 7 are obvious. The proofs of Axiom schemas 8 and 9 rest on a Levy-type reflection principle. For $\vartheta \in \mathrm{Ord}$, define

$$\mathscr{S}_\vartheta = \{T(\mu, \alpha) \mid \mu, \alpha \in \vartheta\}.$$

Then, for any L_{SO}-formula $\varphi(v_0, \ldots, v_{n-1})$ and $\eta \in \mathrm{Ord}$, there is some limit ordinal $\vartheta > \eta$ such that

$$\forall \xi_0, \ldots, \xi_{n-1} \in \vartheta \;\; ((\mathrm{Ord}, \mathscr{S}, <, =, \in, G) \models \varphi[\xi_0, \ldots, \xi_{n-1}] \\ \text{iff } (\vartheta, \mathscr{S}_\vartheta, <, =, \in, G) \models \varphi[\xi_0, \ldots, \xi_{n-1}]).$$

Since all elements of $\mathscr{S}_\vartheta$ can be defined from the truth function T and ordinals $< \vartheta$, the right-hand side can be evaluated in the structure $(\vartheta, <, G \cap \vartheta^3, T)$ by an L_T-formula φ^* that can be recursively computed from φ. Hence,

$$\forall \xi_0, \ldots, \xi_{n-1} \in \vartheta \;\; ((\mathrm{Ord}, \mathscr{S}, <, =, \in, G) \models \varphi[\xi_0, \ldots, \xi_{n-1}] \\ \text{iff } (\vartheta, <, G \cap \vartheta^3, T) \models \varphi^*[\xi_0, \ldots, \xi_{n-1}]).$$

So sets witnessing Axioms 8 and 9 can be defined over $(\vartheta, <, G \cap \vartheta^3, T)$, and are thus elements of $\mathscr{S}$.

The powerset axiom, Axiom 10, can be shown by a similar reflection argument. □

8. Ordinal computability corresponds to constructibility

Before proving the fundamental theorem on ordinal computability, recall the definition of Gödel's model **L** of *constructible sets*. It is defined as the union of a hierarchy of levels $\mathbf{L}_\alpha$:

$$\mathbf{L} = \bigcup_{\alpha \in \mathrm{Ord}} \mathbf{L}_\alpha$$

where the levels are defined recursively by $\mathbf{L}_0 = \emptyset$, $\mathbf{L}_\delta = \bigcup_{\alpha<\delta} \mathbf{L}_\alpha$ for limit ordinals δ, and $\mathbf{L}_{\alpha+1}$ is the set of all sets that are first-order definable in the structure $(\mathbf{L}_\alpha, \in)$. The model **L** is the $\subseteq$-smallest inner model of set theory. An inner model is a transitive proper class satisfying the usual Zermelo–Fraenkel axioms ZFC. The standard reference for the theory of the model **L** is the monograph Devlin (1984).

We can now prove our main theorem.

Theorem 8. A set x of ordinals is ordinal computable from a finite set of ordinal parameters if and only if it is an element of the constructible universe **L**.

Proof. Let $x \subseteq \mathrm{Ord}$ be ordinal computable by the program P from the finite set $\{\alpha_0, \ldots, \alpha_{k-1}\}$ of ordinal parameters: $P : \chi_{\{\alpha_0, \ldots, \alpha_{k-1}\}} \mapsto \chi_x$. By the simple nature of the

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computation procedure, the same computation can be carried out inside the inner model L:

$$(\mathbf{L}, \in) \models P : \chi_{\{\alpha_0,\ldots,\alpha_{k-1}\}} \mapsto \chi_x.$$

Hence, $\chi_X \in \mathbf{L}$ and $x \in \mathbf{L}$.

Conversely, consider $x \in \mathbf{L}$. Since $(\mathrm{Ord}, \mathscr{S}, <, =, \in, G)$ is a model of the theory SO, there is an inner model M of set theory such that

$$\mathscr{S} = \{z \subseteq \mathrm{Ord} \mid z \in M\}.$$

Since $\mathbf{L}$ is the $\subseteq$-smallest inner model, $\mathbf{L} \subseteq M$. Hence $x \in M$ and $x \in \mathscr{S}$. Let $x = T(\mu, \alpha)$. By the computability of the truth predicate, x is ordinal computable from the parameters μ and α. □

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