Robotica (1997) volume 15, pp 355–360. Printed in the United Kingdom

Congruence of circular cylinders on three given points

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SUMMARY

A method to determine the two parameter set of circular cylinders, whose surfaces contain three given points, is presented in the context of an efficient algorithm, based on the set of two parameter projections of the points onto planar sections, to compute radius and a point where the axes intersect the plane of the given points. The geometry of the surface of points, whose position vectors represent cylinder radius, *r*, and axial orientation, is revealed and described in terms of symmetry and singularity inherent in the triangle with vertices on the given points. This strongly suggests that, given one constraint on the axial orientation of the cylinder, there are up to six cylinders of identical radius on the three given points. A bivariate function, in two of the three line direction Plücker coordinates, is derived to prove this. By specifying *r* and an axis direction, say, perpendicular to a given direction, one obtains a sixth order univariate polynomial in one of the line coordinates which yields six axis directions. These ideas are needed in the design of parallel manipulators.

KEYWORDS: Cylinder; Radius; Axis; Orientation; Symmetry; Singularity; Pose; Recognition.

INTRODUCTION

The relevance of this subject, as regards cylindrical parts assembly and collision avoidance in the context of parallel manipulators, was introduced by Zsombor-Murray[1]. Furthermore the origin of the theoretical problem was attributed therein to the works of Schaal[2,3] and Strobel[4,5] who dealt extensively with cylinders of revolution on four points. These of course constitute not a congruence but a one parameter set or series. As regards the relevance to parallel mechanisms, consider the single closed loop—C—S—C—positioning manipulator treated by Schaal[3] and illustrated in Figure 1. Note that C stands for *cylindrical joint* and S for *spherical.* One may sum up the design space of such a device, intended to guide the end effector, EE on —S—, through three specified points, as a selection of any pair of cylindrical elements on these points. In this case, any convenient way of mathematically representing the congruence which is the subject of this article will provide us with a potent yet simple design tool. Let us now explore a formulation of the problem and examine some of the interesting geometric properties which are revealed.

ANALYSIS

In Figure 2 one sees how the surface of a typical cylinder of revolution contains three, typical given points. Let $\mathbf{p}_i$ be the position vector of point, P_i where $i = 1, 2, 3$ represents the given points.

To solve the problem P_i are projected perpendicularly as $P_{i\pi}$, whose position vectors are $\mathbf{p}_{i\pi}$, onto transaxial plane, Π. Furthermore let P_1 be on the origin, O. Notice that Π is any one of a two parameter set of planes with outward normal unit vector, $\mathbf{n}$, which makes angles α, β, γ with respect to axes x, y, z, respectively. These axes are fixed to the rigid body on P_i. Note that $\Pi = \Pi(O, \alpha, \beta, \gamma)$ where $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. Moreover choose $\mathbf{p}_2 = \{1, 0, 0\}^T$, where P_1P_2 is the longest side of the triangle whose vertices are P_i. Clearly $\mathbf{n} = \{\cos\alpha, \cos\beta, \cos\gamma\}^T$. We obtain the projected points with equation $\mathbf{p}_{i\pi} + k_i\mathbf{n} = \mathbf{p}_i$, where k_i is the length of the projector joining $P_{i\pi}$ to P_i, and with the perpendicularity condition between $\mathbf{n}$ and all lines in Π, expressed as $\mathbf{p}_{i\pi} \cdot \mathbf{n} = 0$. This can be expanded as the matrix multiplication below.

$$\begin{bmatrix} 1 & 0 & 0 & \cos\alpha \\ 0 & 1 & 0 & \cos\beta \\ 0 & 0 & 1 & \cos\gamma \\ \cos\alpha & \cos\beta & \cos\gamma & 0 \end{bmatrix} \begin{bmatrix} x_{i\pi} \\ y_{i\pi} \\ z_{i\pi} \\ k_i \end{bmatrix} = \begin{bmatrix} x_i \\ y_i \\ z_i \\ 0 \end{bmatrix} \quad (1)$$

Using the first three equations, to eliminate $x_{i\pi}$, $y_{i\pi}$, $z_{i\pi}$ from the fourth, yields k_i.

$$k_i = \cos\alpha x_i + \cos\beta y_i + \cos\gamma z_i \quad (2)$$

so

$$\mathbf{p}_{i\pi} = \begin{bmatrix} \sin^2\alpha x_i - \cos\alpha(\cos\beta y_i + \cos\gamma z_i) \\ \sin^2\beta y_i - \cos\beta(\cos\gamma z_i + \cos\alpha x_i) \\ \sin^2\gamma z_i - \cos\gamma(\cos\alpha x_i + \cos\beta y_i) \end{bmatrix} \quad (3)$$

Each plane Π will cut one cylinder perpendicular to its axis, which is parallel to $\mathbf{n}$, so that the centre, M, with position vector $\mathbf{m}$, of a circle with circumference on the point projections, $P_{i\pi}$, will define any desired cylinder's axis location.

$$\mathcal{M}(x, y, z) = \mathcal{M}(P_{i\pi}) \quad (4)$$

One way to characterize this surface is by plotting three sets of curves on it, *i.e.*, characteristics of constant α, β, (γ), r where r is a cylinder radius. Alternately, one might project this characteristic "contour map" back onto the fixed plane on P_i. Another possible characterization is to represent the spheropolar surface with longitude angle, θ, latitude, ϕ, and local radius, r, the radius of the

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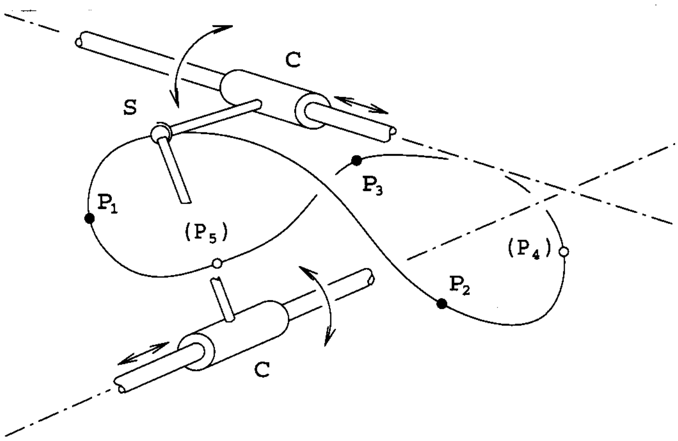

Fig. 1. A —C—S—C— mechanism to position the EE at —S—

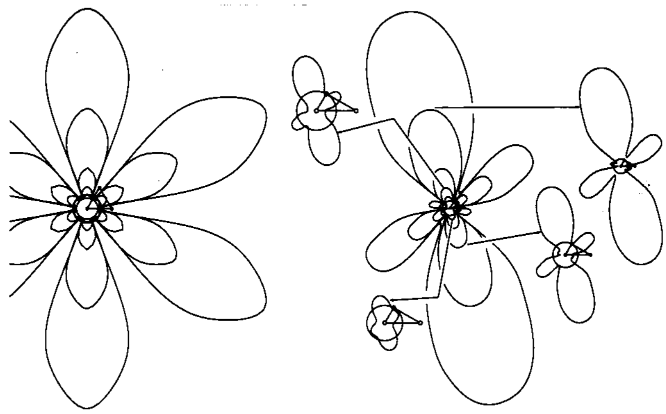

Fig. 3. Cylinder radii for equilateral and 60°/30° triangles, axes at $\phi = 1°, 2°, 5°, 10°, 20°$

particular cylinder whose axis orientation is specified by these two angles. This characterization can be easily converted to planopolar form on a (r, θ) plane showing contours of successive latitude, ϕ. This latter convention was used to generate Figure 3.

The position vector, $\mathbf{m}$, of a point on $\mathcal{M}$, can be conveniently obtained by intersecting two right bisecting planes of the segments $P_{i\pi}P_{i+1,\pi}$ with any plane Π. An equivalent process is represented by intersecting right bisecting lines of two sides of the projected triangle.

$$[\mathbf{m} - (2\mathbf{p}_{i\pi} + \mathbf{p}_{i,i+1,\pi})/2] \cdot \mathbf{p}_{i,i+1,\pi} = 0 \tag{5}$$

where $i = 1,\ 2$ and $\mathbf{p}_{i,i+1,\pi} \equiv \mathbf{p}_{i+1,\pi} - \mathbf{p}_{i\pi}$. Recall that $\mathbf{m} \cdot \mathbf{n} = 0$.

Since the triangle $P_1P_2P_3$ is normalized to $\|\mathbf{p}_2 - \mathbf{p}_1\| = 1$, which is the longest side, the other two may be nondimensionalized as $\rho_1 = \|\mathbf{p}_3 - \mathbf{p}_2\|$ and $\rho_2 = \|\mathbf{p}_1 - \mathbf{p}_3\|$, $\rho_2 \le \rho_1 \le 1$ which gives

$$\mathbf{p}_3 = \{(\rho_2^2 - \rho_1^2 + 1)/2,\ \sqrt{\rho_2^2 - x_3^2},\ 0\}^T = \{x_3, y_3, z_3\}^T \tag{6}$$

In this way the multiplicative effects of point pattern size, hand and circulation, i.e. up/down normals, $\mathbf{n}$, are eliminated from the mapping of $\mathcal{M}$ which can proceed with the solution below. First, note that $\mathbf{p}_{1\pi} = \{0, 0, 0\}^T$ and that from equation (3) expressions for $\mathbf{p}_{2\pi}$ and $\mathbf{p}_{3\pi}$ simplify to

$$\mathbf{p}_{2\pi} = \begin{bmatrix} x_{2\pi} \\ y_{2\pi} \\ z_{2\pi} \end{bmatrix} = \begin{bmatrix} 1 - a^2 \\ -ab \\ -ca \end{bmatrix} \tag{7}$$

and

$$\mathbf{p}_{3\pi} = \begin{bmatrix} x_{3\pi} \\ y_{3\pi} \\ z_{3\pi} \end{bmatrix} = \begin{bmatrix} (1 - a^2)x - aby \\ (1 - b^2)y - abx \\ -c(ax - by) \end{bmatrix} \tag{8}$$

The two equations, equation (5), $i = 2, 3$, simplify to

$$(\mathbf{m} - \mathbf{p}_{i\pi}/2) \cdot \mathbf{p}_{i\pi} = 0 \tag{9}$$

Together with the condition $\mathbf{m} \cdot \mathbf{n} = 0$ these provide three simultaneous equations which can be solved for $\mathbf{m} = \{x_m, y_m, z_m\}^T$.

$$x_{2\pi}x_m + y_{2\pi}y_m + z_{2\pi}z_m = (x_{2\pi}^2 + y_{2\pi}^2 + z_{2\pi}^2)/2 \tag{10}$$

$$x_{3\pi}x_m + y_{3\pi}y_m + z_{3\pi}z_m = (x_{3\pi}^2 + y_{3\pi}^2 + z_{3\pi}^2)/2 \tag{11}$$

$$ax_m + by_m + cz_m = 0 \tag{12}$$

where $a = \cos\alpha$, $b = \cos\beta$, $c = \cos\gamma$, $x = x_3$ and $y = y_3$. Clearly, cylinder radius, r, can be obtained immediately as $r^2 = x_m^2 + y_m^2 + z_m^2$. It might be convenient to compute $\mathcal{M}$ with the spherical coordinate pair, (θ, ϕ), where $a = \cos\theta\cos\phi$, $b = \sin\theta\cos\phi$ and $c = \sin\phi$. Note that θ and ϕ correspond to angles of meridian and latitude, respectively, which define the direction of cylinder axes.

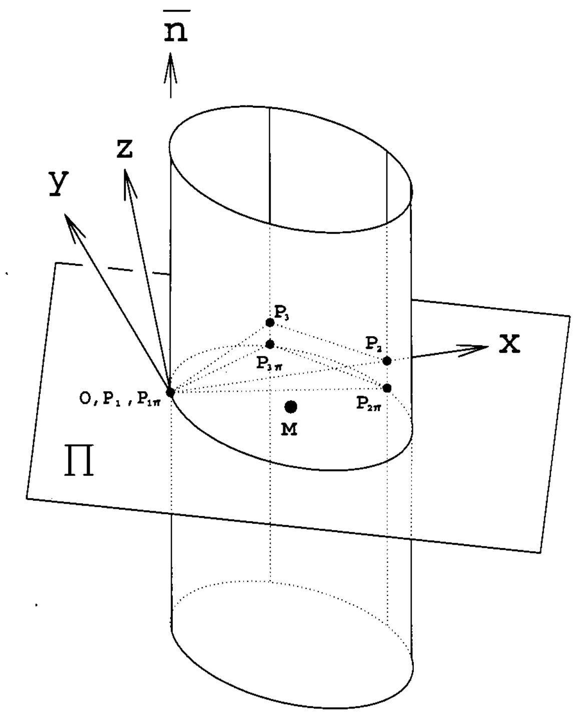

Fig. 2. Projection of given points on plane of cylinder right section

COMPUTATION

To show that the solution is computationally efficient, a *BASIC* program containing twelve scant lines of code, which generates points in the sequence necessary to form a family of characteristics in θ, is tabulated below.

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```
100 DTR=3.141592654/180:INPUT R1,R2:
    X=(R2*R2-R1*R1+1)/2:Y=SQR(R2*R2-X*X)
110 FOR T=0 TO 71
120 TH=5*T*DR:CT=COS(TH):ST=SIN(TH)
130 FOR P=1 TO 17
140 PH=5*P*DTR:CP=COS(PH):W=SIN(PH):
    U=CP*CT:V=CP*ST:REM U=a, V=b, W=c
150 A=1-U*U:B=-U*V:C=-W*U:D=A*X-U*V*Y:
    E=(1-V*V)*Y-U*V*X:F=-W*(U*X+V*Y)
160 K=(A*A+B*B+C*C)/2:L=(D*D+E*E+F*F)/2
170 DM=A*E*W+B*F*U+C*D*V-U*E*C-V*F*A-
    W*D*B:
    IF DM=0 THEN GOTO 200
180 XN=K*E*W+C*L*V-V*F*K-W*L*B:
    YN=A*L*W+K*F*U-U*L*C-W*D*K:XM=XN/DM:
    YM=YN/DM
190 ZN=B*L*U+K*D*V-U*E*K-V*L*A:ZM=ZN/DM:
    R=SQR(XM*XM+YM*YM+ZM*ZM)
200 XMH=XM-ZM*U/V:YMH=YM-ZM*V/W:NEXT P
210 NEXT T
```

To form a characteristic family in ϕ one merely interchanges the nested FOR/NEXT loops. So one parameter families of cylinder radii and axes, characterized by directions on circles of latitude and meridian with respect to an equatorial plane on $\{P_i\}$, may be readily provided. Similarly it is easy to compute the cylinders whose axes are parallel to the spokes of any great circle whose normal direction is (θ_w, ϕ_w). Here the angle parameter of the spokes, ψ, will span an arc subtending π on the spoke diameter parallel to the equatorial plane. The algorithm above is used with a single loop iterated in ψ where θ and ϕ are computed as follows.

$$\sin\phi = -\sin\psi\cos\phi_w, \quad \cos\phi = \sqrt{1-\sin^2\phi}$$

$$\cos\theta = (\sin\psi\cos\theta_w\sin\phi_w + \cos\psi\sin\theta_w)/\cos\phi$$

$$\sin\theta = (\sin\psi\sin\theta_w\sin\phi_w - \cos\psi\cos\theta_w)/\cos\phi$$

SINGULARITY

Figure 4 shows various views of P_i, chosen arbitrarily. A projection of the equatorial plane appears as P_i^H where one sees a section of the cylinder circumscribing these point projections. This view represents a singularity because here, at $\phi = \pi/2$, the section and hence the cylinder radius remain constant for all values of θ. Below this is a front view where the points P_i^F appear in a line and the circumscribing circle's axis lies parallel to the equatorial plane and the cylinder has infinite radius. This

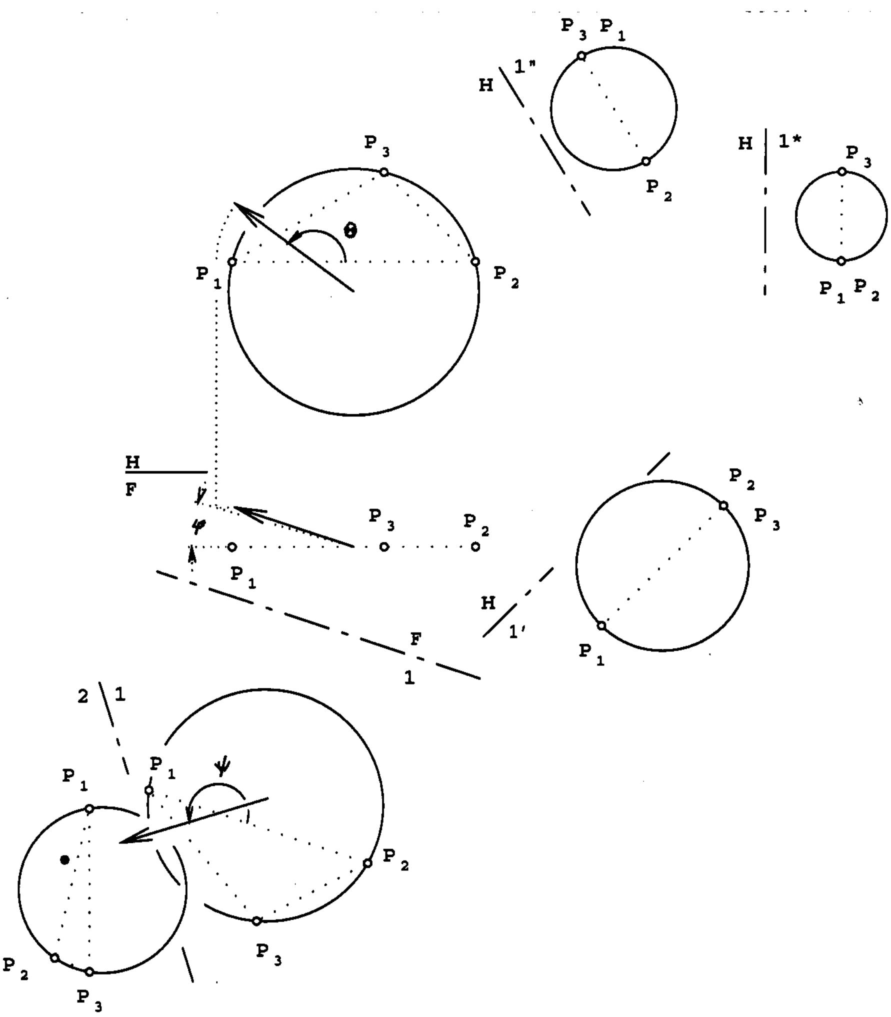

Fig. 4. Two general and five singular cylinder sections

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occurs in *all* axial directions parallel to the equatorial plane except those shown as P_i', P_i'' and P_i^* where the circumscribing circles may appear as having diameters of the three triangular heights. Of course a circle of infinite radius is also a valid solution in these three cases as well. The arrow direction represents an arbitrary axial direction which produces the cylindrical section on P_i^2. The axis of this circle can be projected to locate M in the reference frame of P_i, *i.e.*, P_i^H, P_i^F. This construction is equivalent to computing a point on one of the two sets of five radius characteristics at constant ϕ shown in Figure 3. The section shown as P_i^1, on the other hand, corresponds to evaluating a point on one of the curves in Figure 5, where the cylinder axes are perpendicular to a given direction. Returning to the projections P_i', P_i'' and P_i^*, we see the three cylinders of minimum radius and these views must be separated by maxima. Therefore it seems that any trajectory of axis directions on the sphere which monotonically spans $-\pi/2 \le \theta \le \pi/2$ should be associated with a cylinder radius function of sixth order.

POLAR PLOTS OF CYLINDER RADIUS

Having dealt with directions at $\phi = 0$, plots of cylinder radius, $r = r(\theta)$, for an equilateral and a 60°/30° triangle at $\phi = 1°, 2°, 5°, 10°, 20°$ and for spoke directions, ψ at $\theta_w = 30°$, $\phi_w = 60°$, are shown in Figure 5.

SOLUTION WITH SPECIFIED RADIUS

The sixth order nature of the cylinder radius as a function of axis orientation is shown clearly in Figure 3 and less clearly in Figure 5. Furthermore, it may be seen that at latitudinal directions $\phi > 1°$, approximately, a semicircle of constant radius will intersect a one parameter axial directions characteristic less than six times, maybe not at all. Hence it would be useful to be able to compute axial direction as a function of some given value of r.

Therefore before embarking upon the development of avoidance/insertion procedures for cylindrical objects there remains this crucial characterization task. It is not hard to formulate the problem in a simple-minded way, using ordinary point position and parametric line vector equations. Say that P_i, r and a vector $\mathbf{w}$, in the direction

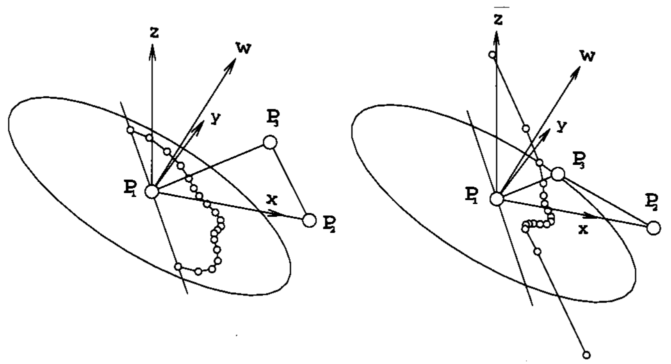

Fig. 5. Cylinder radii for equilateral and 60°/30° triangles, axes perpendicular to w, $\theta_w = 30°$, $\phi_w = 60°$

w perpendicular to a desired axis, are specified. Then the following ten equations can be written.

$$\mathbf{u} = \mathbf{s} + u(\mathbf{t} - \mathbf{s}),\ \mathbf{s}^2 = r^2,\ (\mathbf{p}_2 - \mathbf{t})^2 = r^2,$$
$$(\mathbf{p}_3 - \mathbf{u})^2 = r^2$$
$$(\mathbf{t} - \mathbf{s}) \cdot \mathbf{s} = 0,\ (\mathbf{t} - \mathbf{s}) \cdot (\mathbf{p}_2 - \mathbf{t}) = 0,$$
$$(\mathbf{t} - \mathbf{s}) \cdot (\mathbf{p}_3 - \mathbf{u}) = 0,\ (\mathbf{t} - \mathbf{s}) \cdot \mathbf{w} = 0$$

Notice that the first equation expands to three, in x, y, z, which specifies the colinearity of three points on the axis, given by position vectors $\mathbf{s}$, $\mathbf{t}$, $\mathbf{u}$, in terms of the scalar parameter u. This approach, unfortunately, yields a univariate polynomial of 64th degree; not very satisfactory.

Let us now consider a homogeneous line coordinate formulation with a spherotangential line complex kernel. By using line geometry, a sixth degree multivariate polynomial, which describes the congruence of right circular cylinders whose surfaces are on three given points, is obtained. The polynomial contains the point coordinates, the cylinder radius and two axis orientation parameters. The geometric implication, that if one of these parameters is specified there may be in the congruence as many as six cylinders of some given radius, is confirmed. For example, given the radius of the cylinder and the direction of any vector perpendicular to its axis, one can then obtain a sixth degree univariate since the constraint imposed by the direction perpendicular to a cylinder axis results in a linear equation in the remaining, unknown direction parameter.

We begin by setting up a structure in the higher dimensional space of Plücker coordinates which contains all real lines. The six homogeneous coordinates of a line are expressed as follows.

$$p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}$$

The first three coordinates are the respective x, y and z components of the vector part while the next three are respective components of the moment part. In order to represent a real line, these coordinates are subject to the following constraints.

$$p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0 \quad (13)$$
$$p_{01}^2 + p_{02}^2 + p_{03}^2 \ne 0 \quad (14)$$

Equation 13 specifies that the moment part is perpendicular to the vector part and describes a four dimensional surface in a five dimensional projective space represented by the six homogeneous coordinates. The non-zero condition, Equation 14, ensures that the line's vector part has a direction and is therefore not on the plane at ∞. To formulate the constraint equations which specify the two parameter set of cylinders on three given, fixed points, consider that there is no loss in generality by defining the points with the following simplified coordinates.

$$P_i]_{i=1}^3(x_i, y_i, z_i) = P_1(0, 0, 0),\ P_2(1, 0, 0),\ P_3(x, y, 0)$$

The constraint equations, in terms of homogeneous line coordinates, which characterize the axis direction of the

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circular cylinders of radius, *r*, on these three given points can be found by intersecting three line complexes, $R_i]_{1=1}^3$, each of which represents lines tangent to a sphere of this radius and centred on one of the given points. The equation of a complex, R_i, is expressed as follows.

$$R_i{:}(p_{23}-y_ip_{03}+z_ip_{02})^2+(p_{31}-z_ip_{01}+x_ip_{03})^2 + (p_{12}-x_ip_{02}+y_ip_{01})^2-r^2(p_{01}^2+p_{02}^2+p_{03}^2)=0$$

This simply states that every line with the same vector part magnitude and tangent to a sphere exerts a moment of the same magnitude about the sphere centre.

First we choose a $p_{0j}]_{j=1}^3 \neq 0$, say, p_{03}. This is always possible since $\forall\ p_{0j}]_{j=1}^3=0$ implies a line at ∞ which cannot be tangent to any sphere of finite radius. A parametric linear subspace (*PLS*) in k and l is now introduced. This leads to the following specification.

$$p_{01}=kp_{03} \quad \text{and} \quad p_{02}=lp_{03}$$

Replacing p_{01} and p_{02} and dividing by p_{03}, reduce the line complexes and equation (13) to

R_1/p_{03}:

$$(p_{23}^2+p_{31}^2+p_{12}^2)/p_{03}-r^2p_{03}(k^2+l^2+1)=0$$

R_2/p_{03}:

$$(p_{23}^2+p_{31}^2+p_{12}^2)/p_{03}+p_{03}-2p_{12}l+2p_{31} - r^2p_{03}(k^2+l^2+1)=0$$

R_3/p_{03}:

$$(p_{23}^2+p_{31}^2+p_{12}^2)/p_{03}+p_{03}(x^2(1+l^2)+y^2(1+k^2)) - 2xp_{12}l+2xp_{31}-2yp_{23}+2yp_{12}k - 2yp_{03}lk-r^2p_{03}(k^2+l^2+1)=0$$

R_4/p_{03}:

$$p_{23}k+p_{31}l+p_{12}=0$$

Note that Equation (13) has been included as R_4 to provide a constraint that restricts solutions to the four parameter set of valid lines among all the possible combinations of six real Plücker coordinates. Inspection of these four equations reveals that R_1, R_2, and R_3 contain a common factor, $p_{23}^2+p_{31}^2+p_{12}^2$. Now the following system of homogeneous linear equations in p_{23}, p_{31}, p_{12}, p_{03} can be expressed.

$$\tilde{R}_2=R_2/p_{03}-R_1/p_{03} =0$$

$$\tilde{R}_3=R_3/p_{03}-R_1/p_{03} =0$$

$$\tilde{R}_4=R_4/p_{03} =0$$

This can be abbreviated as follows.

$$\mathbf{M}\begin{bmatrix}p_{23}\\p_{31}\\p_{12}\end{bmatrix}=p_{03}\mathbf{b} \tag{15}$$

The determinant $|\mathbf{M}|=y(1+k^2+l^2)$, states that no two points can be coincident nor can all three be collinear. Solving for p_{23}, p_{31}, p_{12} in terms of p_{03}, *l, k,* substituting these into R_1 and then multiplying the result by $|\mathbf{M}|$ will give the following sixth order multivariate polynomial.

$$\begin{aligned} &y^2-4r^2y^2-8r^2y^2l^2-8r^2y^2k^2-4r^2y^2l^4+y^2l^4\\ &+2y^2l^2+l^4y^2k^2+2ly^3k+l^2y^4+2k^2y^4+k^4y^4\\ &+l^2k^4y^4+2k^2l^2y^4+y^4+x^4+2y^2x^2-8r^2y^2l^2k^2\\ &-4r^2y^2k^4+y^2k^2-8x^3l^3yk+8x^2k^2y^2l^2\\ &+6x^2l^4k^2y^2-4y^3xlk+3x^4l^2+3x^4l^4+x^4l^6-2x^3\\ &-4x^3lyk-4x^3l^5yk-4xly^3k^3-4xl^3y^3k^3\\ &+2y^2l^4x^2+2x^2y^2k^2-4y^3l^3xk+4y^2x^2l^2\\ &+6x^2lyk-2xy^2k^2-2xy^3-6x^3l^2\\ &-6x^3l^4-4xl^2y^2+2ly^3k^3+2l^3y^3k\\ &+2l^3y^3k^3-2xlyk-4xl^3yk-2xl^4y^2\\ &+3x^2l^2+3x^2l^4-6xl^4k^2y^2+6x^2l^5yk\\ &-2x^3l^6+12x^2l^3yk+2l^2y^2k^2-2xl^5yk\\ &+x^2+x^2l^6-8xk^2y^2l^2=0 \end{aligned}$$

Now consider a case where r is given along with $\mathbf{w}$, a vector perpendicular to the cylinder axis. Let $\mathbf{e}$ be the unit vector to be determined, parallel to the cylinder axis. Then

$$\mathbf{w}\cdot\mathbf{e}=0$$

which gives

$$p_{03}\{k\ l\ 1\}^T\cdot\{w_x\ w_y\ w_z\}^T=0$$

and so

$$l=-(w_z+kw_x)/w_y$$

which can be substituted into the multivariate so as to produce a univariate in *k*. $\mathbf{e}$ can now be determined with k and l and the normalization condition $\mathbf{e}^2=1$. Once $\mathbf{e}$ is available it is only necessary to project the three given points onto a plane perpendicular to $\mathbf{e}$, find the circumscribing circle, a right section of the cylinder on these point projections, and to note that the circle centre is the point view of the cylinder axis.

CONCLUSION

In spite of their relative unfamiliarity to the engineering community and their apparent complexity, we have shown by the above example that the tools of line geometry can greatly simplify the results, if not the procedure, of kinematic analysis. The intersection of three spherotangential line complices provides a good way to compute positions of the cylinders of given radius if one other constraint is specified, *e.g.,* a direction perpendicular to the desired axis or a plane parallel to it. Still left unsolved is the problem concerning the nature of the ruled surface of tangents on three congruent spheres, *i.e.* a description of the set of valid axial directions for cylinders, of a given radius, *r,* on three points. Our work must be extended to address this shortcoming before the results become generally useful

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in the context of cylinder recognition, collision avoidance or even to design a —C—S—C— positioning mechanism to guide the EE through three, four or five points.

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