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Dependence of the magnetohydrodynamic shock thickness on the finite electrical conductivity

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Abstract. The results of magnetohydrodynamic (MHD) plane shock waves with infinite electrical conductivity are generalized for a plasma with a finite conductivity. We derive the adiabatic curves that describe the evolution of the shocked gas as well as the change in the entropy density. For a parallel shock (i.e. in which the magnetic field is parallel to the normal to the shock front) we find an expression for the shock thickness which is a function of the ambient magnetic field and of the finite electrical conductivity of the plasma. We give numerical estimates of the physical parameters for which the shock thickness is of the order of, or greater than, the mean free path of the plasma particles in a strongly magnetized plasma.

1. Introduction

Previously, shock waves in plasmas, both relativistic and non-relativistic, were studied assuming ideal magnetohydrodynamic (MHD) (see Landau et al. (1999); Priest (1982); Anile (1989)). Although this theory is suitable for studying most astrophysical shock waves, such as those in hot rarified astrophysical plasmas where the electrical conductivity is extremely high and the magnetic field is weak, it is interesting to study the effects of the simplest dissipative process in non-ideal MHD–those due to a finite value of the electrical conductivity σ. We generalize here the results for planar shock waves to non-ideal, non-relativistic MHD. The junction conditions that must be satisfied across a shock wave in a non-ideal plasma are given in Sec. 2. We then derive the adiabatic curve with corrections due to σ and find the corresponding change in the entropy density across the shock. We perform this analysis for an oblique shock (i.e. in which $\vec{B}$ is neither parallel nor perpendicular to the normal to the shock surface) as well as for a parallel shock (i.e. in which $\vec{B}$ is parallel to the normal to the shock surface). In Sec. 3, we find a closed expression for the shock thickness for a parallel shock in a strongly magnetized plasma and estimate its value for some physical situations. Our conclusions are discussed in Sec. 4.

2. Junction conditions and adiabatic curve

For non-ideal MHD, the fluxes of mass, energy and momentum are given by (2.1)–(2.4), respectively (see Landau et al. (1999); Landau and Lifshitz (1997)):

$$\vec{M} = \rho \vec{v}, \tag{2.1}$$

$$\vec{q} = \rho \vec{v} \left(\frac{1}{2} v^2 + \mathsf{w} \right) + \frac{1}{4\pi} \vec{B} \times (\vec{v} \times \vec{B}) - \frac{c^2}{16\pi^2 \sigma} \vec{B} \times (\vec{\nabla} \times \vec{B}), \tag{2.2}$$

and

$$\Pi_{ik} = \rho v_i v_k + p \delta_{ik} - \frac{1}{4\pi} \left(B_i B_k - \frac{1}{2} B^2 \delta_{ik} \right), \tag{2.3}$$

where ρ is the fluid density, $\vec{v}$ the velocity, w the enthalpy per unit mass, $\vec{B}$ the ambient magnetic field, and p is the fluid pressure. The electric field is

$$\vec{E} = \frac{c}{4\pi\sigma} (\vec{\nabla} \times \vec{B}) - \frac{\vec{v}}{c} \times \vec{B}, \tag{2.4}$$

where Ohm's law in its simplest form (Priest (1982); Spitzer (1962)) was used.

2.1. *Junction conditions*

We assume a two-dimensional, planar, shock wave in the y–z plane. The normal to the transition surface is in the $-x$ direction. The velocity field can be decomposed into perpendicular and tangential components to the surface of transition, $\vec{v} = (v_x, \vec{v}_t)$. It is assumed that all quantities vary as a function of x . Let n^x be a unit vector normal to the transition surface. We then have the hydrodynamical junction conditions (Landau and Lifshitz (1997))

$$[\rho v_x n^x] = 0 \tag{2.5}$$

$$[q_x n^x] = 0 \tag{2.6}$$

$$[\Pi_{ix} n^x] = 0 \tag{2.7}$$

where i is x (t) (normal (tangential) to the shock surface) and [ ] means the difference between the value of the corresponding quantities far upstream (which we denote by the subscript the '1') and the value at some point in the shock (no subscript). It is also assumed that both far upstream and far downstream, all gradients vanish (i.e. the fields and flows are uniform).

In the MHD case that we are considering, (2.5)–(2.7) must be supplemented with the electromagnetic junction conditions, i.e. that the normal component of the magnetic field and the tangential component of the electric field must be constant across the shock surface:

$$[B_n] = 0, \tag{2.8}$$

$$[\vec{E}_t] = \left[\frac{c}{4\pi\sigma} (\partial_x \times \vec{B})_t - \frac{v_x}{c} \vec{B}_t + \frac{B_n}{c} \vec{v}_t \right] = 0. \tag{2.9}$$

Equation (2.5) states that the mass flux along x is conserved, i.e. $\rho v_x = j =$ constant. We write $\rho = 1/V$, where V is the specific volume, and replace $v_x = jV$

in the other junction conditions, obtaining

$$j\left[\frac{1}{2}j^2V^2 + \frac{1}{2}v_t^2 + \mathsf{w}\right] + \frac{1}{4\pi}j\left[VB_t^2\right] - \frac{1}{4\pi}B_x[\vec{v}_t.\vec{B}_t] - \frac{c^2}{16\pi^2\sigma}\left[\partial_x B_t^2\right] = 0, \quad (2.10)$$

$$j^2[V] + [p] + \frac{1}{8\pi}\left[B_t^2\right] = 0, \quad (2.11)$$

$$j[\vec{v}_t] - \frac{1}{4\pi}B_n[\vec{B}_t] = 0, \quad (2.12)$$

$$\frac{c}{4\pi\sigma}[(\partial_x \times \vec{B}_t)] - \frac{j}{c}[V\vec{B}_t] + \frac{B_n}{c}[\vec{v}_t] = 0. \quad (2.13)$$

2.2. Adiabatic curve and entropy density change

We derive the expression for the adiabatic curve and the corresponding entropy density change. From Eqs. (2.12) and (2.13), we obtain

$$\frac{1}{4\pi}B_n^2[\vec{B}_t] = j^2[V\vec{B}_t] - \frac{jc^2}{4\pi\sigma}[(\partial_x \times \vec{B}_t)_t]. \quad (2.14)$$

From (2.12), we have $[\vec{v}_t] = B_n[\vec{B}_t]/4\pi j$. We can therefore complete the squares in (2.10), obtaining

$$[\mathsf{w}] + \frac{1}{2}j^2[V^2] + \frac{1}{2}\left[\left(v_t - \frac{1}{4\pi}\frac{B_n}{j}\vec{B}_t\right)^2\right] - \frac{1}{32\pi^2}\frac{B_n^2}{j^2}\left[B_t^2\right] + \frac{1}{4\pi j}B_n[\vec{v}_t.\vec{B}_t]$$
$$+ \frac{1}{4\pi}\left[VB_t^2\right] - \frac{1}{4\pi j}B_n[\vec{v}_t.\vec{B}_t] - \frac{c^2}{16\pi^2 j\sigma}\left[\partial_x\left(B_t^2\right)\right] = 0. \quad (2.15)$$

From (2.12), the third term of (2.15) is zero and we are left with

$$[\mathsf{w}] + \frac{1}{2}j^2[V^2] - \frac{1}{32\pi^2}\frac{B_n^2}{j^2}\left[B_t^2\right] + \frac{1}{4\pi}\left[VB_t^2\right] - \frac{c^2}{16\pi^2 j\sigma}\left[\partial_x\left(B_t^2\right)\right] = 0. \quad (2.16)$$

Using (2.14), we can write the third term of (2.16) as

$$\frac{1}{32\pi^2}\frac{B_n^2}{j^2}\left[B_t^2\right] = \frac{1}{8\pi}\left[VB_t^2\right] + \frac{1}{8\pi}(V - V_1)\vec{B}.\vec{B}_1 - \frac{c^2}{32\pi^2 j\sigma}(\vec{\nabla} \times \vec{B})(\vec{B}_t + \vec{B}_{1t}). \quad (2.17)$$

From momentum conservation, we obtain

$$j^2 = -\frac{(p - p_1)}{(V - V_1)} - \frac{1}{8\pi}\frac{\left(B_t^2 - B_{1t}^2\right)}{(V - V_1)}. \quad (2.18)$$

Using (2.17) and (2.18) in (2.16) and $\mathsf{w} = \varepsilon + pV$, we have

$$\varepsilon - \varepsilon_1 + \frac{1}{2}(p + p_1)(V - V_1) + \frac{1}{16\pi}(\vec{B}_t - \vec{B}_{1t})^2(V - V_1)$$
$$+ \frac{c^2}{32\pi^2 j\sigma}\left[(\vec{\nabla} \times \vec{B})(\vec{B}_t + \vec{B}_{1t}) - 2\partial_x\left(B_t^2\right)\right] = 0. \quad (2.19)$$

The first four terms are found in the equation for ideal MHD (e.g. Landau et al. (1999), while the last two are due to the finite electrical conductivity. The first term in the square brackets is due to the fact that $B_n \neq 0$; the second is the contribution from the tangential component of the magnetic field.

To obtain the entropy density change, we follow the procedure found in the standard literature (Landau and Lifshitz (1997)) and develop $V - V_1$ in powers of $(p-p_1)$. We also expand $(\mathsf{w}-\mathsf{w}_1)$ in powers of $(p-p_1)$ and to first order in powers of $(\mathsf{s}-\mathsf{s}_1)$. The resulting expression is

$$T(\mathsf{s}-\mathsf{s}_1) = \frac{1}{12}\left(\frac{\partial^2 V}{\partial p^2}\right)_{\mathrm{s}}(p-p_1)^3 - \frac{1}{16\pi}\left(\frac{\partial V}{\partial p}\right)_{\mathrm{s}}(B_t - B_{1t})^2(p-p_1)$$
$$-\frac{c^2}{32\pi^2 j\sigma}\left[(\vec{\nabla}\times\vec{B}).(\vec{B}_t+\vec{B}_{1t}) - 2\partial_x\left(B_t^2\right)\right]. \tag{2.20}$$

The first two terms are found in the equation for ideal MHD and the last two are the corrections due to a finite σ. In the following section, we repeat the calculations for a perpendicular shock and find that in the expressions for the adiabatic curve and the entropy density change, the only term present which depends on the conductivity is the last one.

2.3. *Perpendicular shock*

Shock waves in a plasma permeated with a magnetic field show several features, of which the most well known is related to the orientation of the magnetic field with respect to the shock plane. Although perpendicular shocks can be considered to be a special case of oblique shocks, it is interesting to write the simplified expressions for the junction conditions explicitly, and re-derive the adiabatic curve and the entropy density change for this case.

2.3.1. Hydrodynamical and electromagnetic junction conditions. For perpendicular shocks $B_n = 0$, so that the junction conditions now read

$$[\rho v_x] = 0, \tag{2.21}$$

$$\left[\rho v_x\left(\frac{1}{2}v^2 + \mathsf{w}\right) + \frac{1}{4\pi}v_x B_t^2 - \frac{c^2}{16\pi^2\sigma}\partial_x\left(B^2\right)\right] = 0, \tag{2.22}$$

$$\left[\rho v_x^2 + p + \frac{1}{8\pi}B_t^2\right] = 0, \tag{2.23}$$

$$[\rho\vec{v}_t v_x] = 0 \Rightarrow \vec{v}_t = \vec{v}_{t1}, \tag{2.24}$$

$$\frac{c}{4\pi\sigma}[(\partial_x \times \vec{B}_t)] - \frac{1}{c}[v_x\vec{B}_t] = 0. \tag{2.25}$$

2.3.2. Adiabatic curve and entropy density change. Proceeding as for an oblique shock and defining

$$\varepsilon^* = \varepsilon + \frac{B_t^2 V}{8\pi}, \quad p^* = p + \frac{1}{8\pi}B_t^2, \tag{2.26}$$

we obtain

$$\mathsf{w} - \mathsf{w}_1 - \frac{1}{2}[p^*](V+V_1) + \frac{1}{4\pi}\left[VB_t^2\right] - \frac{c^2}{16j\pi^2\sigma}\left[\partial_x\left(B_t^2\right)\right] = 0 \tag{2.27}$$

and

$$\varepsilon^* - \varepsilon_1^* + \frac{1}{2}(p^* + p_1^*)(V - V_1) - \frac{c^2}{16j\pi^2\sigma}\partial_x\left(B_t^2\right) = 0. \tag{2.28}$$

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There is now only one term that depends on the dissipative properties of the plasma, while for the oblique case we had two such terms. The missing term is related to the normal component of the magnetic field.

3. Thickness of the shock wave

The calculation of a general expression for the shock thickness is very difficult, if not impossible. However, for a perpendicular shock it is possible to calculate the shock thickness exactly. We then have $B_n = 0$, $\vec{v}_t = 0$ and consider a coordinate system in which the only non-zero component of the magnetic field is $B_y = B$ (Landau et al. (1999)). In this case, the equation $\vec{\nabla} \cdot \vec{B} = 0$ is satisfied identically. The unidimensional ideal MHD equations are

$$\frac{\partial B}{\partial t} = \frac{\partial}{\partial x}(v_x B), \tag{3.1}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(v_x \rho) = 0, \tag{3.2}$$

$$\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + \frac{1}{8\pi\rho}\frac{\partial B^2}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}. \tag{3.3}$$

From the first two equations, it is easy to see that the ratio $B/\rho \equiv \beta$ satisfies the equation $\partial\beta/\partial t + v_x \partial\beta/\partial x = 0$ or $d\beta/dt = 0$ (Landau et al. (1999)). Hence, if the fluid is homogeneous at some initial instant, so that $\beta =$ constant, then it will remain so at all subsequent times. Substituting $B = \rho\beta$ in the third equation, we obtain

$$\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} = -\frac{1}{\rho}\frac{\partial}{\partial x}\left[p + \frac{\beta^2\rho^2}{8\pi}\right]. \tag{3.4}$$

Thus, the magnetic field has been eliminated from the equations. The equation for the velocity field, (3.4), is formally identical to that for the ideal fluid case, provided we define the 'true pressure' as $p^* = p + \beta^2\rho^2/8\pi$. We can now proceed to evaluate the thickness, following Landau and Lifshitz (1997). We write

$$\frac{\partial}{\partial t}\delta p^* - v_s^* \frac{\partial}{\partial x}\delta p^* - \alpha_p^* \delta p^* \frac{\partial}{\partial x}\delta p^* = cL^* \frac{\partial^2}{\partial x^2}\delta p^*, \tag{3.5}$$

where $v_s^{*2} \equiv (\partial p^*/\partial \rho)_s$ and L^* is the damping length, given by (A 24),

$$L^* = \frac{c}{8\pi\sigma}\frac{B_{0y}^2}{\left(B_{0y}^2 + 4\pi\rho_0 v_s^2\right)}. \tag{3.6}$$

From Landau and Lifshitz (1997) we have

$$\alpha_p^* = \frac{1}{2} v_s^{*3} \rho^2 \left[\frac{\partial^2}{\partial p^{*2}}\left(\frac{1}{\rho}\right)\right]_s. \tag{3.7}$$

Equation (3.5) can be solved using the procedure in Landau and Lifshitz (1997), obtaining the thickness of the shock wave as

$$\delta^* = \frac{4cL^*}{\alpha_p^*(p_2^* - p_1^*)}, \tag{3.8}$$

where p_2^* and p_1^* are the 'true pressures' far downstream and far upstream, respectively. Equation (3.8) is quantitatively valid for sufficiently small differences

$(p_2^* - p_1^*)$. However, we can use it qualitatively to estimate the order of magnitude of the thickness in cases where the difference $(p_2^* - p_1^*)$ is of the same order of magnitude as p_2^* and p_1^* themselves. The velocity of sound in the gas, v_s (not v_s^*), is of the same order of magnitude as the thermal velocity v. Let λ be the mean free path of the atoms in the plasma. Then from dimensional analysis, the electric conductivity can be estimated as $\sigma \sim \gamma v/\lambda \sim \gamma v_s/\lambda$, where γ takes into account anomalous effects and can have a value $10^{-6} \leqslant \gamma \leqslant 1$†.

In (3.6), we take $B_{0y}^2/(B_{0y}^2 + 4\pi\rho_0 v_s^2) \sim B_{0y}^2/p^*$ and, in (3.7), $\alpha_p^*(p_2^* - p_1^*) \sim v_s^{*2} \sim p^*/\rho$. Using these relations in (3.8) we obtain

$$\delta \sim c^2 \frac{\rho^2 B_{0y}^2}{\gamma p^{*2} p} \lambda. \tag{3.9}$$

The shock thickness is larger than the mean free path when $c^2\rho^2 B_{0y}^2 \geqslant \gamma p^{*2} p$. Let us first assume that $B_{0y}^2 \geqslant p$. We then have $p^{*2} \sim B_{0y}^4$ and $p \leqslant B_{0y}^2 \leqslant c^2\rho^2/\gamma p$. As a specific numerical example, consider $\rho \sim 10^2$ g cm^{-3} and $T \sim 10^8$ K (characteristic parameters at the center of a massive star before collapse). Assuming hydrogen gas, we have $n \sim 10^{26}$ cm^{-3} and $p \sim nT \sim 10^{18}$ erg cm^{-3} (for iron nuclei, the pressure would be two orders of magnitude smaller). For the above densities and pressures, the shock thickness is larger than the mean free path of the particles if the magnetic field is in the interval 10^9 G $\leqslant B_{0y} \leqslant \gamma^{-1/2} 10^{14}$ G. If we now assume that $B_{0y}^2 \leqslant p$, the shock thickness is larger than the mean free path if $\gamma p^3/c^2\rho^2 \leqslant B_{0y}^2$. For the above parameters we have $\gamma^{1/2} 10^5$ G $\leqslant B_{0y} \leqslant 10^{10}$ G‡. If neither of the two conditions above are fulfilled, the shock thickness is smaller than the mean free path. This means that the MHD approach breaks down and kinetic theory is needed to study the structure of the shock.

4. Conclusions

In this article, we extended the results of shock waves treated in ideal MHD to the non-ideal case, in which the electrical conductivity is finite. We considered Ohm's law in its simplest form (Spitzer (1962)), but took into account phenomenological (through the parameter γ) plasma effects that can modify the classical Spitzer electrical conductivity (e.g. turbulence). The expressions for the adiabatic curve and the entropy density change across the shock were generalized. Finally, we

† We know that in accretion disks, protostars, galactic nuclei and neutron X-ray sources, for example, the plasma cannot have ideal Spitzer values for the conductivity and viscosity in order to obtain the observed accretion rates. Therefore, it is generally assumed that these quantities are highly anomalous (due to turbulence, for example). Another example where the assumption of anomalous resistivity is used is in the treatment of solar flares, which are generally assumed to be due to magnetic reconnection. If ideal Spitzer values are used for the plasma in solar flares, reconnection times are $\sim 10^6$ times longer than the observed time scales for the flares. In general, plasmas near shocks are expected to be highly anomalous (i.e. $\gamma \ll 1$) due to turbulence.

‡ For $\gamma \sim 10^{-6}$, the magnetic field range is 10^2 G $\leqslant B_{0y} \leqslant 10^{10}$ G. A magnetic field $B \geqslant 10^2$ G can easily be present at the center of a massive star. The magnetic field increases in the collapse of the core of a massive star to $B \geqslant 10^6$ (for the collapse to a white dwarf) and to $B \geqslant 10^9$ (for the collapse to a neutron star). Thus, at the center of a massive star, we may expect that the magnetic field varies from 10^2 to 10^{10} G during the collapse of its core and the start of a supernova explosion.

derived the expression for the shock thickness of a finite conductivity in the case of a parallel shock in strongly magnetized plasmas. The conditions that the ambient magnetic field must satisfy for the thickness to be of the order of the particle mean free path were estimated. We found that these conditions can be fulfilled for the plasma expected in the origin of a supernova explosion (Thompson and Murray (2001); Akimaya and Wheeler (2002)). Extensions to the results presented in this paper, using a more general Ohm's law, as well as to relativistic shocks, are presently under investigation.

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Appendix A

In this appendix, we sketch the derivation of the expression for L^*, the damping length used to calculate the shock thickness. Neglecting the displacement current (which is a good approximation in non-relativistic electrodynamics), the evolution equation for the magnetic field in a medium with electrical conductivity σ moving with a velocity $\vec{v}$ is

$$\frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \times (\vec{v} \times \vec{B}) = \frac{c^2}{4\pi\sigma}\nabla^2 \vec{B}. \tag{A 1}$$

Adding the equations for the fluid, which we assume have neither viscosity nor thermal conduction, we have

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho\vec{v}) = 0, \tag{A 2}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\,\vec{\nabla})\vec{} = -\frac{1}{\rho}\vec{\nabla}p - \frac{1}{4\pi\rho}\vec{B} \times (\vec{\nabla} \times \vec{B}). \tag{A 3}$$

A.1. Hydromagnetic waves

Let us assume that $B = \vec{B}_0 + \vec{b}$, $\rho = \rho_0 + \delta\rho$, $p = p_0 + \delta p$, and $\vec{v} = \delta\vec{v}$. Replacing these terms in the above equations, keeping only terms to first order in the perturbations, expanding the density in powers of the perturbation in the pressure (i.e. $\delta\rho = \delta p/v_s^2 + (\partial^2\rho/\partial p^2)_s \delta p^2$, where v_s is the sound velocity of the medium) and taking the Fourier transform of the equations, we obtain

$$-\omega\delta p_0 + v_s^2\rho_0\vec{k}\delta\vec{v}_0 = 0, \tag{A 4}$$

$$-\left(\omega + i\frac{c^2}{4\pi\sigma}k^2\right)\vec{b}_0 = \vec{k} \times [\delta\vec{v}_0 \times \vec{B}_0], \tag{A 5}$$

$$-\omega\delta\vec{v}_0 = -\frac{\vec{k}}{\rho_0}\delta p_0 - \frac{1}{4\pi\rho_0}\vec{B}_0 \times (\vec{k} \times \vec{b}_0). \tag{A 6}$$

From (A 4), we find $\delta p_0 = v_s^2\rho_0(\vec{k}\delta\vec{v}_0)/\omega$ and using this in (A 6), we obtain

$$-\delta\vec{v}_0 = -\frac{\vec{k}}{\omega}v_s^2\left(\frac{\vec{k}}{\omega}\delta\vec{v}_0\right) - \frac{1}{4\pi\rho_0}\vec{B}_0 \times \left(\frac{\vec{k}}{\omega} \times \vec{b}_0\right). \tag{A 7}$$

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We define the scalar phase velocity as $u = \omega/k$, assuming that $\vec{k}$ is along the x-axis (i.e. $\vec{k} = k\check{x}$) and that $\vec{B}_0$ is in the x–y plane. Writing the previous equation in its components, we have

$$\delta p_0 = \frac{v_{\rm s}^2 \rho_0}{u_{\rm A}} \delta v_{0x}, \tag{A 8}$$

$$\left(u - \frac{v_{\rm s}^2}{u}\right) \delta v_{0x} = \frac{1}{4\pi\rho_0} b_{0y} B_{0y}, \tag{A 9}$$

$$u\delta v_{0y} = -\frac{1}{4\pi\rho_0} b_{0y} B_{0x}, \tag{A 10}$$

$$\left(u + i\frac{c^2}{4\pi\sigma}k\right) b_{0y} = \delta v_{0x} B_{0y} - \delta v_{0y} B_{0x}, \tag{A 11}$$

$$u\delta v_{0z} = -\frac{1}{4\pi\rho_0} b_{0z} B_{0x}, \tag{A 12}$$

$$\left(u + i\frac{c^2}{4\pi\sigma}k\right) b_{0z} = -\delta v_{0z} B_{0x}. \tag{A 13}$$

A.2. Generalized Alfven waves

Using (A 12) and (A 13), we obtain the compatibility relationship

$$u^2 + i\frac{c^2 k}{4\pi\sigma} u - \frac{B_{0x}^2}{4\pi\rho_0} = 0, \tag{A 14}$$

from which we obtain

$$u = \pm\frac{1}{2}\sqrt{\frac{B_{0x}^2}{\pi\rho_0} - \frac{c^4 k^2}{16\pi^2\sigma^2}} - i\frac{c^2 k}{8\pi\sigma}. \tag{A 15}$$

From (A 15), the phase velocity is a complex number if σ is finite. Rewriting $u_{\rm A}$ in terms of ω, we obtain the dispersion relationship:

$$\omega = \frac{1}{2}\frac{B_{0x}k}{\sqrt{\pi\rho_0}}\sqrt{1 - \frac{\rho_0}{B_{0x}^2}\frac{c^4 k^2}{16\pi\sigma^2}} - i\frac{c^2 k^2}{8\pi\sigma}. \tag{A 16}$$

We take the plus sign in (A 15) since the frequency is a positive quantity. The fact that the imaginary part is nonlinear in k means that the Alfven waves are damped and dissipated as a function of k. For $\sigma \to \infty$ we recover the known dispersion relationship for ideal MHD.

Assuming that the second term in the square root in (A 16) is much smaller than unity, the group velocity is

$$v_{\rm A} = \frac{\partial\omega}{\partial k} \simeq \frac{B_{0x}}{\sqrt{4\pi\rho_0}} - i\frac{c^2 k}{4\pi\sigma}. \tag{A 17}$$

When the electrical conductivity is infinite, we recover the known ideal MHD result, $v_{\rm AI} = B_{0x}/2\sqrt{\pi\rho_0}$.

A.3. Generalized magnetosonic waves

From (A 9), (A 10) and (A 11), we obtain the generalized dispersion relationship for magnetosonic waves,

$$\omega^4 - \left(\frac{B_0^2}{4\pi\rho_0} + v_{\rm s}^2\right) k^2\omega^2 + \frac{B_{0x}^2}{4\pi\rho_0} v_{\rm s}^2 k^4 + i\frac{c^2k^2}{4\pi\sigma}\omega^3 - i\frac{c^2v_{\rm s}^2k^4}{4\pi\sigma}\omega = 0. \tag{A 18}$$

This relationship can be inverted to obtain $\omega = \omega(k)$. However, for our purposes, it suffices to consider the dissipative terms as a correction to the ideal dispersion relationship,

$$\omega_{\rm I}^2 = \frac{1}{2}\left[\left(\frac{B_0^2}{4\pi\rho_0} + v_{\rm s}^2\right) \pm \sqrt{\left(\frac{B_0^2}{4\pi\rho_0} + v_{\rm s}^2\right)^2 - \frac{B_{0x}^2}{\pi\rho_0}v_{\rm s}^2}\right] k^2 \equiv v_{\rm g0}^2 k^2. \tag{A 19}$$

The plus sign corresponds to *fast magnetosonic* waves, while the minus sign to *slow magnetosonic* waves. Replacing $\omega_{\rm I}$ in the last two terms of (A 18), we obtain

$$\omega^4 - \left(\frac{B_0^2}{4\pi\rho_0} + v_{\rm s}^2\right) k^2\omega^2 + \frac{B_{0x}^2}{4\pi\rho_0} v_{\rm s}^2 k^4 + i\frac{c^2 v_{\rm g0}}{4\pi\sigma}\left(v_{\rm g0}^2 - v_{\rm s}^2\right)k^5 \simeq 0. \tag{A 20}$$

If the electrical conductivity is large (but not infinite), we have

$$\omega^2 \simeq \omega_{\rm I}^2 \mp \frac{i}{4}\frac{c^2 v_{\rm g0}}{\pi\sigma}\frac{\left(v_{\rm g0}^2 - v_{\rm s}^2\right)}{\left(v_{B0}^4 - v_{B0x}^2 v_{\rm s}^2\right)^{1/2}} k^3, \tag{A 21}$$

where $v_{B0}^2 = (B_0^2/4\pi\rho_0 + v_{\rm s}^2)$ and $v_{B0x}^2 = B_{0x}^2/\pi\rho_0$. We thus have

$$\omega \simeq v_{\rm g0}k - icL^* k^2, \tag{A 22}$$

with

$$L^* = \frac{c}{8\pi\sigma}\frac{\left(v_{\rm g0}^2 - v_{\rm s}^2\right)}{\left(v_{B0}^4 - v_{B0x}^2 v_{\rm s}^2\right)^{1/2}}, \tag{A 23}$$

where L^* is the damping length. For a perpendicular shock (i.e. $B_{0x} = 0$), we have

$$L^* = \frac{c}{8\pi\sigma}\frac{B_{0t}^2}{\left(B_{0t}^2 + 4\pi\rho_0 v_{\rm s}^2\right)}. \tag{A 24}$$

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