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Existence and uniqueness of a thermoelastic problem with variable parameters

P. BARRAL[1], M. C. NAYA-RIVEIRO[2] and P. QUINTELA[1]

[1]*Department of Applied Mathematics, Faculty of Mathematics, Universidade de Santiago de Compostela,* 15782 *Santiago de Compostela, Spain*
email: patricia.barral@usc.es, peregrina.quintela@usc.es
[2]*Department of Pedagogy and Didactics, Faculty of Educational Studies, Universidade da Coruña,* 15071 *A Coruña, Spain*
email: cristina.naya@udc.es

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The aim of this article is to study the existence and uniqueness of solution for a quasistatic fully coupled thermoelastic problem arising from some metallurgical processes. We consider mixed boundary conditions for both submodels, and a Robin boundary condition for the thermal one. Furthermore, the reference temperature, the thermal conductivity and the Lamé's parameters are assumed to depend on the material point.

Key words: Thermoelasticity; Quasistatic problem; Galerkin's method; Non-homogeneous materials

1 Introduction

In this paper, we carry out a mathematical analysis of a thermoelastic coupled problem for non-homogeneous materials arising from metallurgical industry where materials are processed under strong temperature gradients. An important example is the thermomechanical modelling of an aluminium electrolytic cell, where aluminium is produced by reduction of alumina (see Bermúdez *et al.* [5]); the electrolytic cell consists of a rectangular steel shell with an inner covering of different insulating and refractory materials, and the study of its thermomechanical deformations is essential in order to determine the cell life. Another important example is the thermomechanical modelling of slabs during direct chill castings of alloys (see Barral and Quintela [4]); during this process large thermal stresses develop inside the slab due to thermal gradients and solidification shrinkage; these deformations can disrupt the casting process and influence the slab quality. Recently also the thermomechanical modelling of structures exposed to fire is becoming extremely important for security (see Flint *et al.* [19]). In all these processes it is very important to take into account the mechanical heat dissipation, the non-homogeneity of the materials and the non-linearity of the boundary conditions. The mathematical analysis of these problems becomes very difficult due to all these aspects, the coupling between the motion and energy conservation equations and the lack of regularity of the real data.

As a first step, in Barral *et al.* [2, 3] we studied the existence, uniqueness and regularity of the mechanical problem when the behaviour law is of Maxwell–Norton type with temperature-dependent coefficients. In the present work, we deal with the coupling with the thermal problem, assuming, as a first approach, that the material is linear elastic and non-homogeneous; we consider the boundary conditions found in industrial applications like the aforementioned ones, that is, mixed boundary conditions in both submodels and also a Robin type boundary condition for the thermal one. Specifically, we prove the existence and uniqueness of the solution. Some regularity properties in space and time of this solution will be given in a forthcoming paper.

In the literature, there exist several existence and uniqueness results for thermoelastic problems. The equations of the coupled thermoelasticity were considered for the first time by Duhamel [12] in 1837 and we refer the reader to papers [8, 10, 13–15, 21, 25–32, 36] for the existence and uniqueness results for the dynamic case. On the other hand, in 1960 Boley and Weiner [6] studied the theory for the quasistatic case for the first time, and later several results of existence and uniqueness were published in the papers [1, 9, 11, 34]. We highlight the work of Viaño [35], where the existence and uniqueness of solution for a quasistatic thermoelastic problem considering a contact condition was proved; later, in Figueiredo and Trabucho [17, 18] this result was extended to three different behaviour laws to the dynamic case. Following these works, in this paper we apply the Galerkin's method, in order to prove the existence of a solution for a thermoelastic problem, where the main difficulties are:

- the problem is assumed to be quasistatic,
- a convection heat transfer boundary condition is considered in the thermal submodel,
- the reference temperature, the thermal conductivity and the Lamé's parameters depend on the material point, and
- the lack of regularity hypotheses on the data, in order to analyse the models arising from the metallurgical industrial processes previously cited.

The uniqueness of solution for this problem is proved via Gronwall's lemma, following the works of Gawinecki [20, 22–24] and Gawinecki *et al.* [26].

The outline of this paper is as follows. First, in Section 2 we will describe the mathematical model. After introducing in Section 3 the appropriate functional framework and proposing a weak formulation, in Section 4, we will prove the existence of solution to the problem. We will obtain the uniqueness of such solution in Section 5 and finally, some conclusions will be given in Section 6.

2 Mathematical model

2.1 Domain and notation

Let $\Omega \subset \mathbb{R}^3$ be an open and bounded set with smooth boundary. We refer the motion of the body to a fixed system of rectangular Cartesian axes $Op_1p_2p_3$.

Let $g(p,t)$ be a scalar function; we represent by $g(t)$ the function $p \longrightarrow g(p,t)$ and ∇g its gradient with respect to p.

If $\mathbf{u}$, $\mathbf{v}$ are vector fields in $\mathbb{R}^3$, their scalar product is represented by $\mathbf{u} \cdot \mathbf{v}$. Furthermore, $\nabla \mathbf{u}$ and $\mathrm{Div}\, \mathbf{u}$ denote the gradient and the divergence of $\mathbf{u}$, respectively.

We denote by S_3 the space of symmetric second-order tensors over $\mathbb{R}^3$ and by: its scalar product. Furthermore, if $\boldsymbol{\tau}$ is a tensor field, $|\boldsymbol{\tau}|$, $\mathrm{tr}(\boldsymbol{\tau})$ and $\mathrm{Div}\,\boldsymbol{\tau}$ denote the norm induced by this scalar product, its trace and its divergence, respectively.

We consider the notation ∂_t^r in order to denote the partial derivative with respect to t of order r, with $r \in \mathbb{N}$. As usual, for $r = 1$ we will omit the superscript r.

We represent by $[0, t_f]$ the time interval of interest. The thermoelastic problem consists of determining the displacement field $\mathbf{u}(p, t)$ and the temperature field $\theta(p, t)$ at each (p, t) in $\Omega \times (0, t_f]$.

2.2 Boundary conditions

Let $\Gamma = \partial\Omega$ be the boundary of Ω and $\mathbf{n}$ its outward unit normal vector. We assume that $\Gamma_{\mathbf{u},D}$, $\Gamma_{\mathbf{u},N}$, $\Gamma_{\theta,D}$, $\Gamma_{\theta,N}$ and $\Gamma_{\theta,R}$ are open subsets of Γ, such that

- $\Gamma = \overline{\Gamma}_{\mathbf{u},D} \cup \overline{\Gamma}_{\mathbf{u},N} = \overline{\Gamma}_{\theta,D} \cup \overline{\Gamma}_{\theta,N} \cup \overline{\Gamma}_{\theta,R}$,
- $\Gamma_{\mathbf{u},D} \cap \Gamma_{\mathbf{u},N} = \emptyset$, $\Gamma_{\theta,D} \cap \Gamma_{\theta,N} = \emptyset$, $\Gamma_{\theta,D} \cap \Gamma_{\theta,R} = \emptyset$, $\Gamma_{\theta,R} \cap \Gamma_{\theta,N} = \emptyset$,
- $\mathrm{meas}(\Gamma_{\mathbf{u},D}) > 0$ and $\mathrm{meas}(\Gamma_{\theta,D} \cup \Gamma_{\theta,R}) > 0$.

We consider the following boundary conditions:

- The displacement is known on the Dirichlet mechanical boundary $\Gamma_{\mathbf{u},D}$

$$\mathbf{u} = \mathbf{u}_D \text{ on } \Gamma_{\mathbf{u},D} \times (0, t_f].$$

- On $\Gamma_{\mathbf{u},N}$ we apply surface forces of density $\mathbf{g}$

$$\boldsymbol{\sigma}(\theta, \mathbf{u})\,\mathbf{n} = \mathbf{g} \text{ on } \Gamma_{\mathbf{u},N} \times (0, t_f].$$

- The convection heat transfer boundary condition is given by the thermal conductivity of the material k, the coefficient of convective heat transfer α_c and the external convection temperature θ^e on $\Gamma_{\theta,R}$

$$k\nabla\theta \cdot \mathbf{n} = \alpha_c(\theta^e - \theta) \text{ on } \Gamma_{\theta,R} \times (0, t_f].$$

- The heat flux h on the Neumann thermal boundary $\Gamma_{\theta,N}$ is known

$$k\nabla\theta \cdot \mathbf{n} = h \text{ on } \Gamma_{\theta,N} \times (0, t_f].$$

- The temperature is known on the Dirichlet thermal boundary $\Gamma_{\theta,D}$

$$\theta = \theta_D \text{ on } \Gamma_{\theta,D} \times (0, t_f].$$

2.3 Equilibrium equations and behaviour law

We consider a thermodynamic process with the following:

- small displacements and small velocities,

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- small temperature changes with respect to the reference temperature θ_r,
- a constant coefficient of thermal expansion α,
- the Lamé's parameters λ, μ, the thermal conductivity of the material κ, and the specific heat at constant deformation c_F independent of temperature.

Under these assumptions, the behaviour of the body and the evolution of its temperature are governed by the following equilibrium equations:

$$-\mathrm{Div}\,\boldsymbol{\sigma}(\theta,\mathbf{u}) = \mathbf{b} \quad \text{in} \quad \Omega \times (0,t_f],$$
$$\rho_0 c_F \partial_t \theta = -3\theta_r \alpha K\,\mathrm{Div}\,\partial_t \mathbf{u} + \mathrm{Div}\,(k\nabla\theta) + f \quad \text{in} \quad \Omega \times (0,t_f],$$

(see, for instance, Naya-Riveiro and Quintela [33]). Here

- $\boldsymbol{\sigma}(\theta,\mathbf{u})$ is the stress tensor given by the thermoelastic behaviour law

$$\boldsymbol{\sigma}(\theta,\mathbf{u}) = \Lambda^{-1} : \boldsymbol{\varepsilon}(\mathbf{u}) - 3\alpha(\theta-\theta_r)K\mathbf{I} \quad \text{in} \quad \Omega \times (0,t_f],$$

  where Λ^{-1} is the elasticity tensor defined as

$$\Lambda^{-1} : \boldsymbol{\tau} = \lambda \mathrm{tr}(\boldsymbol{\tau})\mathbf{I} + 2\mu\boldsymbol{\tau}, \quad \forall \boldsymbol{\tau} \in S_3; \tag{2.1}$$

  $\boldsymbol{\varepsilon}(\mathbf{u})$ denotes the linearized deformation tensor, $\mathbf{I}$ is the identity tensor and K is the bulk modulus of the material:

$$K = \frac{1}{3}(3\lambda + 2\mu).$$

- $\mathbf{b}$ are the body forces per unit volume at the reference configuration.
- ρ_0 is the reference density.
- f is the body heat per unit volume at the reference configuration.

2.4 Initial conditions

We consider the following set of initial conditions:

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \theta(0) = \theta_0 \quad \text{in} \quad \Omega, \tag{2.2}$$

where $\mathbf{u}_0$ and θ_0 must satisfy the following compatibility conditions:

$$\begin{cases} \boldsymbol{\sigma}(\theta_0,\mathbf{u}_0) = \Lambda^{-1} : \boldsymbol{\varepsilon}(\mathbf{u}_0) - 3\alpha(\theta_0-\theta_r)K\mathbf{I} & \text{in} \quad \Omega, \\ -\mathrm{Div}\,\boldsymbol{\sigma}(\theta_0,\mathbf{u}_0) = \mathbf{b}(0) & \text{in} \quad \Omega, \\ \mathbf{u}_0 = \mathbf{u}_D(0) & \text{on} \quad \Gamma_{\mathbf{u},D}, \\ \boldsymbol{\sigma}(\theta_0,\mathbf{u}_0)\,\mathbf{n} = \mathbf{g}(0) & \text{on} \quad \Gamma_{\mathbf{u},N}, \\ \theta_0 = \theta_D(0) & \text{on} \quad \Gamma_{\theta,D}. \end{cases} \tag{2.3}$$

2.5 Problem (P)

Summing up, the problem we are going to study is the following:

Problem (P)
Find $\mathbf{u}(p,t)$ and $\theta(p,t)$ in $\Omega \times (0,t_f]$, satisfying:

$$-\mathrm{Div}\,\boldsymbol{\sigma}(\theta,\mathbf{u}) = \mathbf{b} \quad \text{in} \quad \Omega \times (0,t_f], \tag{2.4}$$

$$\rho_0 c_F \partial_t \theta = -3\theta_r \alpha K\,\mathrm{Div}\,\partial_t \mathbf{u} + \mathrm{Div}\,(k\nabla\theta) + f \quad \text{in} \quad \Omega \times (0,t_f], \tag{2.5}$$

$$\boldsymbol{\sigma}(\theta,\mathbf{u}) = \Lambda^{-1} : \boldsymbol{\varepsilon}(\mathbf{u}) - 3\alpha(\theta-\theta_r)K\mathbf{I} \quad \text{in} \quad \Omega \times (0,t_f], \tag{2.6}$$

$$\mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma_{\mathbf{u},D} \times (0,t_f], \tag{2.7}$$

$$\boldsymbol{\sigma}(\theta,\mathbf{u})\,\mathbf{n} = \mathbf{g} \quad \text{on} \quad \Gamma_{\mathbf{u},N} \times (0,t_f], \tag{2.8}$$

$$k\nabla\theta \cdot \mathbf{n} = \alpha_c(\theta^e - \theta) \quad \text{on} \quad \Gamma_{\theta,R} \times (0,t_f], \tag{2.9}$$

$$k\nabla\theta \cdot \mathbf{n} = h \quad \text{on} \quad \Gamma_{\theta,N} \times (0,t_f], \tag{2.10}$$

$$\theta = \theta_D \quad \text{on} \quad \Gamma_{\theta,D} \times (0,t_f], \tag{2.11}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \theta(0) = \theta_0 \quad \text{in} \quad \Omega. \tag{2.12}$$

3 A weak formulation

Let us assume that $(\mathbf{u},\theta)$ is a smooth enough solution to Problem (P) and θ_r is not null in Ω. Applying Green's formula to equation (2.4), using boundary condition (2.8) and thanks to expression (2.6), we can deduce

$$\int_\Omega \left(\Lambda^{-1} : \boldsymbol{\varepsilon}(\mathbf{u})\right) : \boldsymbol{\varepsilon}(\mathbf{v})\,dp - 3\int_\Omega \alpha(\theta-\theta_r)K\mathbf{I} : \boldsymbol{\varepsilon}(\mathbf{v})\,dp = \int_{\Gamma_{\mathbf{u},N}} \mathbf{g}\cdot\mathbf{v}\,d\Gamma + \int_\Omega \mathbf{b}\cdot\mathbf{v}\,dp, \tag{3.1}$$

for all $\mathbf{v} \in \mathbf{D}(\bar{\Omega}) = [\mathscr{D}(\bar{\Omega})]^3$ with $\mathbf{v} = \mathbf{0}$ on $\Gamma_{\mathbf{u},D}$ and $t \in (0,t_f]$. Analogously, considering energy equation (2.5), applying Green's formula and using boundary conditions (2.9) and (2.10), we obtain the variational equality:

$$\int_\Omega \frac{\rho_0 c_F}{\theta_r}\partial_t\theta\phi\,dp = -3\int_\Omega \alpha K\mathbf{I} : \boldsymbol{\varepsilon}(\partial_t\mathbf{u})\phi\,dp - \int_\Omega k\nabla\theta\cdot\nabla\left(\frac{\phi}{\theta_r}\right)dp$$
$$+ \int_{\Gamma_{\theta,R}} \frac{\alpha_c\theta^e}{\theta_r}\phi\,d\Gamma - \int_{\Gamma_{\theta,R}} \frac{\alpha_c\theta}{\theta_r}\phi\,d\Gamma + \int_{\Gamma_{\theta,N}} \frac{h}{\theta_r}\phi\,d\Gamma + \int_\Omega \frac{f}{\theta_r}\phi\,dp, \tag{3.2}$$

for all $\phi \in \mathscr{D}(\bar{\Omega})$ with $\phi = 0$ on $\Gamma_{\theta,D}$ and $t \in (0,t_f]$.

3.1 Functional framework

In this subsection, we introduce the spaces of admissible displacements and temperatures that ensure that equalities (3.1) and (3.2) are well defined; we also introduce some bilinear operators in order to simplify the notation of the variational formulation.

We consider $\mathbf{L}^r(\Omega) = [L^r(\Omega)]^3$, $1 \leqslant r \leqslant \infty$ and $\mathbf{H}^1(\Omega) = [H^1(\Omega)]^3$ with their usual norms.

- The admissible displacements space is

$$\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \quad \mathbf{v}|_{\Gamma_{\mathbf{u},D}} = \mathbf{0}\},$$

  which is a Hilbert space with the usual norm in $\mathbf{H}^1(\Omega)$.

- The admissible temperature space is

$$H^1_{0,\Gamma_{\theta,D}}(\Omega) = \{\phi \in H^1(\Omega) : \quad \phi|_{\Gamma_{\theta,D}} = 0\},$$

which is also a Hilbert space with the usual norm of $H^1(\Omega)$.

We denote by

- $\langle \cdot, \cdot \rangle_{\mathbf{u}}$ the duality between $\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)$ and its dual, $\mathbf{H}^{1\prime}_{0,\Gamma_{\mathbf{u},D}}(\Omega)$, and
- $\langle \cdot, \cdot \rangle_{\theta}$ the duality between $H^1_{0,\Gamma_{\theta,D}}(\Omega)$ and its dual, $H^{1\prime}_{0,\Gamma_{\theta,D}}(\Omega)$.

Let us introduce

- The bilinear form $a(\cdot, \cdot)$ defined on $\mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$ by

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega (\Lambda^{-1} : \boldsymbol{\varepsilon}(\mathbf{u})) : \boldsymbol{\varepsilon}(\mathbf{v})\, dp. \tag{3.3}$$

- The bilinear form $\kappa(\cdot, \cdot)$ on $H^1(\Omega) \times H^1(\Omega)$ such that

$$\kappa(\phi, \psi) = \int_\Omega k \nabla \phi \cdot \nabla \left(\frac{\psi}{\theta_r} \right) dp. \tag{3.4}$$

- The bilinear form $c(\cdot, \cdot)$ defined on $H^1(\Omega) \times H^1(\Omega)$ by

$$c(\phi, \psi) = \int_{\Gamma_{\theta,R}} \alpha_c \frac{\phi}{\theta_r} \psi \, d\Gamma. \tag{3.5}$$

- Finally, the bilinear form $m(\cdot, \cdot)$ on $L^2(\Omega) \times \mathbf{H}^1(\Omega)$ such that

$$m(\phi, \mathbf{v}) = \int_\Omega 3\phi\alpha K \mathbf{I} : \boldsymbol{\varepsilon}(\mathbf{v})\, dp. \tag{3.6}$$

3.2 Assumptions

From now on we will assume the following hypotheses:

(H1) The elasticity tensor $\Lambda^{-1} \in [\mathbf{L}^\infty(\Omega)]^4$ and there exists $a_{\min} > 0$ such that

$$(\Lambda^{-1} : \boldsymbol{\tau}) : \boldsymbol{\tau} \geqslant a_{\min} |\boldsymbol{\tau}|^2, \quad \forall \boldsymbol{\tau} \in S_3.$$

(H2) The reference temperature $\theta_r \in W^{1,\infty}(\Omega)$, and there exists $\theta_{r,\min} > 0$ such that $\theta_r(p) \geqslant \theta_{r,\min}$ in Ω.

(H3) The reference density $\rho_0 > 0$, the specific heat at constant deformation $c_F > 0$ and the coefficient of thermal expansion $\alpha > 0$.

(H4) The thermal conductivity coefficient $k \in W^{1,\infty}(\Omega)$, and there exists $k_{\min} > 0$ such that $k(p) \geqslant k_{\min}$ in Ω.

(H5) The body forces $\mathbf{b} \in W^{2,2}(0, t_f; \mathbf{L}^2(\Omega))$.

(H6) The body heat $f \in W^{1,2}(0,t_f;L^2(\Omega))$.

(H7) $\mathbf{u}_D$ is the restriction to $\Gamma_{\mathbf{u},D} \times (0,t_f)$ of a function called $\bar{\mathbf{u}}_D$ such that $\bar{\mathbf{u}}_D \in W^{2,2}(0,t_f;\mathbf{H}^{\frac{1}{2}}(\Gamma))$.

(H8) θ_D is the restriction to $\Gamma_{\theta,D} \times (0,t_f)$ of a function called $\bar{\theta}_D$ such that $\bar{\theta}_D \in W^{2,2}(0,t_f;H^{\frac{1}{2}}(\Gamma))$.

(H9) The surface forces $\mathbf{g} \in W^{2,2}(0,t_f;\mathbf{L}^2(\Gamma_{\mathbf{u},N}))$ and $h \in W^{1,2}(0,t_f;L^2(\Gamma_{\theta,N}))$.

(H10) The coefficient of convective heat transfer $\alpha_c \in L^\infty(\Gamma_{\theta,R})$, and there exists $\alpha_{c,\min} > 0$ satisfying $\alpha_c(p) \geqslant \alpha_{c,\min}$ *a.e.* on $\Gamma_{\theta,R}$.

(H11) The external convection temperature $\theta^e \in W^{1,2}(0,t_f;L^2(\Gamma_{\theta,R}))$.

(H12) The initial conditions $\mathbf{u}_0 \in \mathbf{H}^1(\Omega)$ and $\theta_0 \in H^1(\Omega)$.

(H13) The initial conditions $\mathbf{u}_0$ and θ_0 satisfy the following:

$$
\begin{aligned}
&a(\mathbf{u}_0,\mathbf{v}) - m(\theta_0-\theta_r,\mathbf{v}) = \int_{\Gamma_{\mathbf{u},N}} \mathbf{g}(0)\cdot\mathbf{v}\,d\Gamma + \int_\Omega \mathbf{b}(0)\cdot\mathbf{v}\,dp,\ \forall \mathbf{v} \in \mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega),\\
&\mathbf{u}_0 = \mathbf{u}_D(0) \text{ on } \Gamma_{\mathbf{u},D}, \qquad \theta_0 = \theta_D(0) \text{ on } \Gamma_{\theta,D}.
\end{aligned}
$$

Definition 3.1 *We define in $L^2(\Omega)$ the following scalar product*

$$(\phi,\psi)_2 = \int_\Omega \frac{\rho_0 c_F}{\theta_r}\phi\psi\,dp.$$

Notice that $(\cdot,\cdot)_2$ is well defined thanks to hypotheses (H2) and (H3) *and it is equivalent to the usual one; we denote by $\|\cdot\|_2$ its associated norm.*

Using expressions (3.1) and (3.2), the operators defined in Subsection 3.1, and taking into account Definition 3.1, we propose the following weak formulation of Problem (P):
Problem (VP)
Find $(\mathbf{u}(t),\theta(t)) \in \mathbf{H}^1(\Omega)\times H^1(\Omega)$ satisfying *a.e.* $t \in (0,t_f)$:

$$
\left\{
\begin{aligned}
&a(\mathbf{u}(t),\mathbf{v}) - m(\theta(t)-\theta_r,\mathbf{v}) = \int_{\Gamma_{\mathbf{u},N}} \mathbf{g}(t)\cdot\mathbf{v}\,d\Gamma + \int_\Omega \mathbf{b}(t)\cdot\mathbf{v}\,dp,\ \ \forall \mathbf{v} \in \mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega), \qquad (3.7a)\\
&(\partial_t\theta(t),\phi)_2 + \kappa(\theta(t),\phi) + m(\phi,\partial_t\mathbf{u}(t)) + c(\theta(t),\phi) = \int_\Omega \frac{f(t)}{\theta_r}\phi\,dp + c(\theta^e(t),\phi)\\
&+ \int_{\Gamma_{\theta,N}} \frac{h(t)}{\theta_r}\phi\,d\Gamma, \quad \forall \phi \in H^1_{0,\Gamma_{\theta,D}}(\Omega), \qquad (3.7b)
\end{aligned}
\right.
$$

the boundary conditions (2.7), (2.11) and the initial conditions (2.12).

Remark 3.2 *From hypothesis* (H1) *we can deduce that expressions* (3.3) *and* (3.6) *define a continuous form in $\mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega)$ and in $L^2(\Omega)\times\mathbf{H}^1(\Omega)$, respectively. Furthermore, the operator a is symmetric and since meas$(\Gamma_{\mathbf{u},D}) > 0$,*

$$a(\mathbf{v},\mathbf{v}) \geqslant a_{\min}\|\mathbf{v}\|^2_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)}, \quad \forall \mathbf{v} \in \mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega), \tag{3.8}$$

i.e. the bilinear form $a(\cdot,\cdot)$ *is* $\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)$*-elliptic. On the other hand, considering hypotheses* (H2) *and* (H10)*, expression* (3.5) *defines a symmetric continuous form in* $H^1(\Omega)\times H^1(\Omega)$*, and furthermore,*

$$c(\phi,\phi) \geqslant \frac{\alpha_{c,min}}{\|\theta_r\|_{L^\infty(\Omega)}}\|\phi\|^2_{L^2(\Gamma_{\theta,R})}, \quad \forall\phi\in H^1(\Omega). \tag{3.9}$$

Remark 3.3 *Taking into account hypotheses* (H2) *and* (H4)*, equality* (3.4) *defines a continuous form in* $H^1(\Omega)\times H^1(\Omega)$*. For simplicity of notation,* κ *will be considered as the sum of two bilinear forms* κ_1 *and* κ_2*, in this way*

$$\kappa(\phi,\psi) = \kappa_1(\phi,\psi)+\kappa_2(\phi,\psi) = \int_\Omega \frac{k}{\theta_r}\nabla\phi\cdot\nabla\psi\,dp - \int_\Omega k\nabla\phi\cdot\frac{\nabla\theta_r}{\theta_r^2}\psi\,dp. \tag{3.10}$$

Thanks to Poincaré's inequality in $H^1_{0,\Gamma_{\theta,D}}(\Omega)$*, the expression of* κ_1 *defines a norm of* $H^1(\Omega)$ *which is equivalent to the usual one* (*see Brezis* [7])*. Furthermore, it satisfies the following:*

$$\kappa_1(\phi,\phi) \geqslant \frac{k_{\min}}{\|\theta_r\|_{L^\infty(\Omega)}}\|\phi\|^2_{H^1_{0,\Gamma_{\theta,D}}(\Omega)}, \quad \forall\phi\in H^1_{0,\Gamma_{\theta,D}}(\Omega). \tag{3.11}$$

4 Existence of a solution to Problem (VP)

Theorem 4.1 *Under assumptions* (H1)–(H13)*, there exists a solution* $(\mathbf{u},\theta)$ *of Problem* (VP) *such that*

$$\mathbf{u}\in L^\infty\big(0,t_f;\mathbf{H}^1(\Omega)\big), \quad \partial_t\mathbf{u}\in L^2\big(0,t_f;\mathbf{H}^1(\Omega)\big), \text{ and} \tag{4.1}$$

$$\theta\in L^\infty\big(0,t_f;H^1(\Omega)\big), \quad \partial_t\theta\in L^2\big(0,t_f;L^2(\Omega)\big). \tag{4.2}$$

The proof is divided into five steps and follows the following scheme. First, we transform the problem into a homogeneous one and the Galerkin method is applied in order to derive a sequence of problems approximating the Problem (VP), for which the existence and uniqueness of solution is shown. Then, based on some *a priori* estimates, and using a limit procedure, the deduced Galerkin sequence is proved to be convergent and so, the existence of solution for the original problem is obtained.

4.1 Step I: A variable change by translation

Assumptions (H7) and (H8) imply the existence of $\underline{\mathbf{u}}$ and $\underline{\theta}$ satisfying (see Duvaut and Lions [16]):

$$\underline{\mathbf{u}}\in W^{2,2}(0,t_f;\mathbf{H}^1(\Omega)) \text{ and } \underline{\mathbf{u}}=\mathbf{u}_D \text{ on } \Gamma_{\mathbf{u},D}\times(0,t_f], \tag{4.3}$$

$$\underline{\theta}\in W^{2,2}(0,t_f;H^1(\Omega)) \text{ and } \underline{\theta}=\theta_D \text{ on } \Gamma_{\theta,D}\times(0,t_f]. \tag{4.4}$$

Hence, we deduce that $\underline{\mathbf{u}}\in C^1([0,t_f];\mathbf{H}^1(\Omega))$ and $\underline{\theta}\in C^1([0,t_f];H^1(\Omega))$.

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Let us define the following translations in the unknowns and the initial conditions:

$$\tilde{\mathbf{u}} = \mathbf{u} - \underline{\mathbf{u}}, \quad \tilde{\mathbf{u}}_0 = \mathbf{u}_0 - \underline{\mathbf{u}}(0), \quad \tilde{\theta} = \theta - \underline{\theta}, \quad \tilde{\theta}_0 = \theta_0 - \underline{\theta}(0). \tag{4.5}$$

Therefore, with respect to these new unknowns, Problem (VP) can be transformed into an equivalent one:

Problem $(\widetilde{VP})$

Find $(\tilde{\mathbf{u}}(t), \tilde{\theta}(t)) \in \mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega) \times H^1_{0,\Gamma_{\theta,D}}(\Omega)$ satisfying *a.e.* $t \in (0, t_f)$

$$\begin{cases} a(\tilde{\mathbf{u}}(t), \mathbf{v}) - m(\tilde{\theta}(t), \mathbf{v}) = \langle L_{\mathbf{u}}(t), \mathbf{v}\rangle_{\mathbf{u}}, \ \forall \mathbf{v} \in \mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega), & (4.6a) \\ (\partial_t\tilde{\theta}(t), \phi)_2 + \kappa(\tilde{\theta}(t), \phi) + m(\phi, \partial_t\tilde{\mathbf{u}}(t)) + c(\tilde{\theta}(t), \phi) & \\ = \langle L_\theta(t), \phi\rangle_\theta, \ \forall \phi \in H^1_{0,\Gamma_{\theta,D}}(\Omega), & (4.6b) \end{cases}$$

and the initial conditions

$$\tilde{\mathbf{u}}(0) = \tilde{\mathbf{u}}_0, \quad \tilde{\theta}(0) = \tilde{\theta}_0. \tag{4.7}$$

In equations (4.6a) and (4.6b), $L_{\mathbf{u}}(t)$ and $L_\theta(t)$ are the linear forms defined by the following:

$$\begin{aligned} \langle L_{\mathbf{u}}(t), \mathbf{v}\rangle_{\mathbf{u}} &= \int_{\Gamma_{\mathbf{u},N}} \mathbf{g}(t)\cdot\mathbf{v}\, d\Gamma + \int_\Omega \mathbf{b}(t)\cdot\mathbf{v}\, dp - a(\underline{\mathbf{u}}(t), \mathbf{v}) \\ &\quad + m(\underline{\theta}(t) - \theta_r, \mathbf{v}), \ \forall \mathbf{v} \in \mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega), \qquad (4.8) \\ \langle L_\theta(t), \phi\rangle_\theta &= \int_\Omega \frac{f(t)}{\theta_r}\phi\, dp + c(\theta^e(t), \phi) + \int_{\Gamma_{\theta,N}} \frac{h(t)}{\theta_r}\phi\, d\Gamma - (\partial_t\underline{\theta}(t), \phi)_2 \\ &\quad - \kappa(\underline{\theta}(t), \phi) - m(\phi, \partial_t\underline{\mathbf{u}}(t)) - c(\underline{\theta}(t), \phi), \ \forall \phi \in H^1_{0,\Gamma_{\theta,D}}(\Omega). \qquad (4.9) \end{aligned}$$

Notice that, since a, m, $(\cdot,\cdot)_2$, k and c are continuous forms and thanks to hypotheses (H2), (H5), (H6), (H9)–(H11), $L_{\mathbf{u}}(t)$ and $L_\theta(t)$ are also continuous forms for all $t \in [0, t_f]$.

Summing up, it is enough to prove the existence of a solution to Problem $(\widetilde{VP})$ satisfying

$$\begin{cases} \tilde{\mathbf{u}} \in L^\infty\big(0, t_f; \mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)\big), \ \partial_t\tilde{\mathbf{u}} \in L^2\big(0, t_f; \mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)\big), \ \text{and} \\ \tilde{\theta} \in L^\infty\big(0, t_f; H^1_{0,\Gamma_{\theta,D}}(\Omega)\big), \ \partial_t\tilde{\theta} \in L^2\big(0, t_f; L^2(\Omega)\big). \end{cases} \tag{4.10}$$

4.2 Step II: Existence and uniqueness of a solution for each approximated problem

First of all we remark that thanks to hypothesis (H13) and to variable change (4.5), we deduce

$$\begin{gathered} \tilde{\mathbf{u}}_0 \in \mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega), \ \tilde{\theta}_0 \in H^1_{0,\Gamma_{\theta,D}}(\Omega), \\ a(\tilde{\mathbf{u}}_0, \mathbf{v}) - m(\tilde{\theta}_0, \mathbf{v}) = \langle L_{\mathbf{u}}(0), \mathbf{v}\rangle_{\mathbf{u}}, \ \forall \mathbf{v} \in \mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega). \end{gathered} \tag{4.11}$$

Since the space $\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)$ (respectively $H^1_{0,\Gamma_{\theta,D}}(\Omega)$) is separable, there exists a numerable base $\{\mathbf{w}^{\mathbf{u}}_i\}_{i\in\mathbb{N}}$ (respectively $\{w^\theta_i\}_{i\in\mathbb{N}}$), such that for all $m \in \mathbb{N}$ the elements of the set $\{\mathbf{w}^{\mathbf{u}}_j\}_{j=1}^m$ (respectively $\{w^\theta_j\}_{j=1}^m$) are linearly independent, and the finite linear combinations

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of the $\mathbf{w}_j^{\mathbf{u}}$, $j \in \mathbb{N}$ are dense in $\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)$ (respectively, w_j^θ in $H^1_{0,\Gamma_{\theta,D}}(\Omega)$). In addition, we can choose a base in $H^1_{0,\Gamma_{\theta,D}}(\Omega)$ satisfying

$$(w_k^\theta, w_l^\theta)_2 = \delta_{kl}, \quad l, k \in \mathbb{N}.$$

Furthermore, we can choose $\mathbf{w}_1^{\mathbf{u}} = \tilde{\mathbf{u}}_0$ if $\|\tilde{\mathbf{u}}_0\|_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)} \neq 0$, and

$$w_1^\theta = \frac{\tilde{\theta}_0}{\|\tilde{\theta}_0\|_2} \text{ if } \|\tilde{\theta}_0\|_2 \neq 0,$$

or any other function with unitary norm in other case.

We denote by $\mathbf{H}^{\mathbf{u}\,m}_{0,\Gamma_{\mathbf{u},D}} = [\mathbf{w}_j^{\mathbf{u}}]_{j=1}^m$ and $H^{\theta\,m}_{0,\Gamma_{\theta,D}} = [w_j^\theta]_{j=1}^m$ the subspaces generated by $\{\mathbf{w}_j^{\mathbf{u}}\}_{j=1}^m$ and $\{w_j^\theta\}_{j=1}^m$, respectively.

For each $m \in \mathbb{N}$, we consider the following problem, which approximates Problem $(\widetilde{VP})$, in a sense that will be specified later:

Problem $(\widetilde{VP}_m)$

Find $(\tilde{\mathbf{u}}_m(t), \tilde{\theta}_m(t)) \in \mathbf{H}^{\mathbf{u}\,m}_{0,\Gamma_{\mathbf{u},D}} \times H^{\theta\,m}_{0,\Gamma_{\theta,D}}$ for all $t \in (0, t_f)$ satisfying

$$\begin{cases} a(\tilde{\mathbf{u}}_m(t), \mathbf{v}) - m(\tilde{\theta}_m(t), \mathbf{v}) = \langle L_{\mathbf{u}}(t), \mathbf{v}\rangle_{\mathbf{u}}, \quad \forall \mathbf{v} \in \mathbf{H}^{\mathbf{u}\,m}_{0,\Gamma_{\mathbf{u},D}}, & (4.12a) \\ (\partial_t \tilde{\theta}_m(t), \phi)_2 + \kappa(\tilde{\theta}_m(t), \phi) + m(\phi, \partial_t \tilde{\mathbf{u}}_m(t)) + c(\tilde{\theta}_m(t), \phi) & \\ = \langle L_\theta(t), \phi\rangle_\theta, \quad \forall \phi \in H^{\theta\,m}_{0,\Gamma_{\theta,D}}, & (4.12b) \end{cases}$$

and the initial conditions

$$\tilde{\mathbf{u}}_m(0) = \tilde{\mathbf{u}}_0, \quad \tilde{\theta}_m(0) = \tilde{\theta}_0. \tag{4.13}$$

Lemma 1 *Under assumptions* (H1)–(H13), *for each* $m \in \mathbb{N}$, *there exists a unique solution* $(\tilde{\mathbf{u}}_m, \tilde{\theta}_m)$ *of Problem* $(\widetilde{VP}_m)$ *such that*

$$\tilde{\mathbf{u}}_m \in C^1\big([0, t_f]; \mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)\big) \text{ and } \tilde{\theta}_m \in C^1\big([0, t_f]; H^1_{0,\Gamma_{\theta,D}}(\Omega)\big). \tag{4.14}$$

Proof The proof is based on transforming Problem $(\widetilde{VP}_m)$ into an equivalent one expressed as a differential system, whose existence and uniqueness of solution is easily proved. For this purpose, we take into account that if there exists a solution, it will admit the expression

$$\tilde{\mathbf{u}}_m(t) = \sum_{i=1}^m \tilde{g}_{im}(t)\mathbf{w}_i^{\mathbf{u}}, \quad \tilde{\theta}_m(t) = \sum_{i=1}^m \tilde{h}_{im}(t) w_i^\theta,$$
$$\text{and } \tilde{\mathbf{u}}_m(0) = \sum_{i=1}^m \tilde{g}_{im}(0)\mathbf{w}_i^{\mathbf{u}} = \tilde{\mathbf{u}}_0, \quad \tilde{\theta}_m(0) = \sum_{i=1}^m \tilde{h}_{im}(0) w_i^\theta = \tilde{\theta}_0.$$

Then, Problem $(\widetilde{VP}_m)$ is equivalent to the following differential system:

$$\begin{cases}[A_m]^t\{\tilde{g}_m(t)\} - [M_m]^t\{\tilde{h}_m(t)\} = \{L_{\mathbf{u}m}(t)\},\\ \{\partial_t\tilde{h}_m(t)\} + [K_m]^t\{\tilde{h}_m(t)\} + [M_m]\{\partial_t\tilde{g}_m(t)\} + [H_m]^t\{\tilde{h}_m(t)\} = \{L_{\theta m}(t)\},\end{cases} \tag{4.15}$$

with the initial conditions

$$\{\tilde{g}_m(0)\} = \begin{cases}\mathbf{e}_1 & \text{if } \tilde{\mathbf{u}}_0 \neq \mathbf{0},\\ \mathbf{0} & \text{if } \tilde{\mathbf{u}}_0 = \mathbf{0},\end{cases} \tag{4.16}$$

$$\{\tilde{h}_m(0)\} = \begin{cases}\|\tilde{\theta}_0\|_2\mathbf{e}_1 & \text{if } \tilde{\theta}_0 \neq 0,\\ \mathbf{0} & \text{if } \tilde{\theta}_0 = 0,\end{cases} \tag{4.17}$$

where $\mathbf{e}_1$ is the first vector of the canonic base, and in system (4.15) we use the superscript t to denote the transpose of a matrix.

In system (4.15), we have used the following notation:

$[A_m]_{ij} = a(\mathbf{w}_i^{\mathbf{u}}, \mathbf{w}_j^{\mathbf{u}})$, $[M_m]_{ij} = m(w_i^{\theta}, \mathbf{w}_j^{\mathbf{u}})$, $[K_m]_{ij} = \kappa(w_i^{\theta}, w_j^{\theta})$, and
$[H_m]_{ij} = c(w_i^{\theta}, w_j^{\theta})$ with $1 \leqslant i, j \leqslant m$.
$\{\tilde{g}_m(t)\} = (\tilde{g}_{1m}(t), \ldots, \tilde{g}_{mm}(t))^t$ and $\{\tilde{h}_m(t)\} = (\tilde{h}_{1m}(t), \ldots, \tilde{h}_{mm}(t))^t$.
$\{L_{\mathbf{u}m}(t)\} = (\langle L_{\mathbf{u}}(t), \mathbf{w}_1^{\mathbf{u}}\rangle_{\mathbf{u}}, \ldots, \langle L_{\mathbf{u}}(t), \mathbf{w}_m^{\mathbf{u}}\rangle_{\mathbf{u}})^t$ and
$\{L_{\theta m}(t)\} = (\langle L_{\theta}(t), w_1^{\theta}\rangle_{\theta}, \ldots, \langle L_{\theta}(t), w_m^{\theta}\rangle_{\theta})^t$.

Differentiating with respect to time the first equation of system (4.15), and substituting $\{\partial_t\tilde{g}_m(t)\}$ in the second equation, we obtain the following equivalent problem:
Problem $(\widetilde{VP}_m)_a$
Find $\{\tilde{g}_m\}$, $\{\tilde{h}_m\} \in C^1([0, t_f]; \mathbb{R}^m)$ satisfying

$$\begin{cases}\left[[I_m] + [M_m][A_m]^{-1}[M_m]^t\right]\{\partial_t\tilde{h}_m(t)\} + \left[[K_m]^t + [H_m]^t\right]\{\tilde{h}_m(t)\} = \{L_m(t)\}, & \text{(4.18a)}\\ \{\tilde{g}_m(t)\} = [A_m]^{-1}\left[\{L_{\mathbf{u}m}(t)\} + [M_m]^t\{\tilde{h}_m(t)\}\right], & \text{(4.18b)}\end{cases}$$

with the initial conditions (4.16) and (4.17). In equation (4.18a) L_m is given by

$$\{L_m(t)\} = \{L_{\theta m}(t)\} - [M_m][A_m]^{-1}\{\partial_t L_{\mathbf{u}m}(t)\}. \tag{4.19}$$

We deduce from the coercivity of operator a, given by relation (3.8), that the matrix $[A_m]$ is symmetric and positive defined and so also its inverse

$$([A_m]^{-1}\mathbf{p}, \mathbf{p})_{\mathbb{R}^m} \geqslant \gamma\|\mathbf{p}\|^2, \quad \forall\mathbf{p} \in \mathbb{R}^m, \quad (\gamma > 0). \tag{4.20}$$

Taking into account Remarks 3.2 and 3.3, we deduce that

$$L_{\mathbf{u}m} \in C^1\big([0, t_f]; \mathbb{R}^m\big) \quad \text{and} \quad L_{\theta m} \in C\big([0, t_f]; \mathbb{R}^m\big).$$

Hence, Problem $(\widetilde{VP}_m)$ is equivalent to Problem $(\widetilde{VP}_m)_a$. In order to prove the existence and uniqueness of solution for this former problem, we notice that equation (4.18a) with the initial condition (4.17) admits a unique solution, since $L_m \in C([0, t_f]; \mathbb{R}^m)$ and the matrix $[S_m] = [I_m] + [M_m][A_m]^{-1}[M_m]^t$ is invertible. Indeed, -1 is not a proper value of $[M_m][A_m]^{-1}[M_m]^t$ because if there existed any $\mathbf{p} \in \mathbb{R}^m$, $\mathbf{p} \neq \mathbf{0}$, $\|\mathbf{p}\| = 1$ such that $[M_m][A_m]^{-1}[M_m]^t\mathbf{p} = -\mathbf{p}$, replacing it into inequality (4.20), we would obtain:

$$-1 = (-\mathbf{p}, \mathbf{p}) = ([M_m][A_m]^{-1}[M_m]^t\mathbf{p}, \mathbf{p}) = ([A_m]^{-1}\,[M_m]^t\mathbf{p},\,[M_m]^t\mathbf{p}) \geqslant \gamma\|\,[M_m]^t\mathbf{p}\|^2 \geqslant 0,$$

which is a contradiction.

In consequence, there exists a unique $\{\tilde{h}_m\}$ satisfying expressions (4.17) and (4.18a). Finally, we can define $\{\tilde{g}_m\}$ through relation (4.18b), which satisfies equality (4.16) thanks to compatibility property (4.11). □

4.3 Step III: *A priori* estimates

Our aim is to obtain the limit of the sequences $\{\tilde{\mathbf{u}}_m\}_{m\in\mathbb{N}}$ and $\{\tilde{\theta}_m\}_{m\in\mathbb{N}}$ as $m \to \infty$. For that purpose, it is necessary to obtain some *a priori* estimates independent of m. From now on, c_l, $l \geqslant 1$, will denote a positive constant.

Lemma 2 (*A priori* estimates I) *Under assumptions* (H1)–(H13), *the sequences* $\{\tilde{\mathbf{u}}_m\}_{m\in\mathbb{N}}$ *and* $\{\tilde{\theta}_m\}_{m\in\mathbb{N}}$ *given by Lemma 1, satisfy*

(a) the sequence $\{\tilde{\mathbf{u}}_m\}_{m\in\mathbb{N}}$ *is bounded in* $L^\infty(0, t_f; \mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega))$,

(b) the sequence $\{\tilde{\theta}_m\}_{m\in\mathbb{N}}$ *is bounded in* $L^\infty(0, t_f; L^2(\Omega))$ *and in* $L^2(0, t_f; H^1_{0,\Gamma_{\theta,D}}(\Omega))$.

Proof Considering Problem $(\widetilde{VP}_m)$ and taking as test functions $\mathbf{v} = \partial_t\tilde{\mathbf{u}}_m(t)$ and $\phi = \tilde{\theta}_m(t)$, we obtain for any $t \in (0, t_f]$:

$$\begin{cases} a(\tilde{\mathbf{u}}_m(t), \partial_t\tilde{\mathbf{u}}_m(t)) - m(\tilde{\theta}_m(t), \partial_t\tilde{\mathbf{u}}_m(t)) = \langle L_{\mathbf{u}}(t), \partial_t\tilde{\mathbf{u}}_m(t)\rangle_{\mathbf{u}}, \\ (\partial_t\tilde{\theta}_m(t), \tilde{\theta}_m(t))_2 + \kappa(\tilde{\theta}_m(t), \tilde{\theta}_m(t)) + m(\tilde{\theta}_m(t), \partial_t\tilde{\mathbf{u}}_m(t)) + c(\tilde{\theta}_m(t), \tilde{\theta}_m(t)) \\ = \langle L_\theta(t), \tilde{\theta}_m(t)\rangle_\theta. \end{cases}$$

Adding these equations, taking into account definitions (3.3) and (3.10), Remark 3.2 and integrating over $(0, t)$, we can deduce

$$\begin{aligned} &\frac{1}{2}a(\tilde{\mathbf{u}}_m(t), \tilde{\mathbf{u}}_m(t)) + \frac{1}{2}\|\tilde{\theta}_m(t)\|_2^2 + \int_0^t \kappa_1(\tilde{\theta}_m(s), \tilde{\theta}_m(s))ds + \int_0^t c(\tilde{\theta}_m(s), \tilde{\theta}_m(s))ds \\ &\quad = \frac{1}{2}a(\tilde{\mathbf{u}}_0, \tilde{\mathbf{u}}_0) + \frac{1}{2}\|\tilde{\theta}_0\|_2^2 - \int_0^t \kappa_2(\tilde{\theta}_m(s), \tilde{\theta}_m(s))\,ds + \langle L_{\mathbf{u}}(t), \tilde{\mathbf{u}}_m(t)\rangle_{\mathbf{u}} - \langle L_{\mathbf{u}}(0), \tilde{\mathbf{u}}_0\rangle_{\mathbf{u}} \\ &\qquad - \int_0^t \langle \partial_t L_{\mathbf{u}}(s), \tilde{\mathbf{u}}_m(s)\rangle_{\mathbf{u}} ds - \int_0^t \langle L_\theta(s), \tilde{\theta}_m(s)\rangle_\theta ds. \end{aligned} \tag{4.21}$$

Since a, $L_{\mathbf{u}}$, L_θ, $\partial_t L_{\mathbf{u}}$ are continuous, and taking into account hypotheses (H2), (H4), (H10), Remark 3.2, Hölder's inequality and definition of κ_2 from expression (3.10), we

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obtain the following:

$$\begin{aligned}
&\frac{1}{2}a(\tilde{\mathbf{u}}_m(t),\tilde{\mathbf{u}}_m(t))+\frac{1}{2}\|\tilde{\theta}_m(t)\|_2^2+\int_0^t \kappa_1(\tilde{\theta}_m(s),\tilde{\theta}_m(s))ds+\frac{\alpha_{c,\min}}{\|\theta_r\|_{L^\infty(\Omega)}}\int_0^t \|\tilde{\theta}_m(s)\|^2_{L^2(\Gamma_{\theta,R})}\,ds\\
&\leqslant \frac{1}{2}a_{\max}\|\tilde{\mathbf{u}}_0\|^2_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)}+\frac{1}{2}\|\tilde{\theta}_0\|_2^2+\frac{\|k\|_{L^\infty(\Omega)}\|\nabla\theta_r\|_{\mathbf{L}^\infty(\Omega)}}{\theta^2_{r,\min}}\int_0^t\|\nabla\tilde{\theta}_m(s)\|_{\mathbf{L}^2(\Omega)}\|\tilde{\theta}_m(s)\|_{L^2(\Omega)}ds\\
&+\|L_{\mathbf{u}}(t)\|_{\mathbf{H}^{1\prime}_{0,\Gamma_{\mathbf{u},D}}(\Omega)}\|\tilde{\mathbf{u}}_m(t)\|_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)}+\|L_{\mathbf{u}}(0)\|_{\mathbf{H}^{1\prime}_{0,\Gamma_{\mathbf{u},D}}(\Omega)}\|\tilde{\mathbf{u}}_0\|_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)}\\
&+\int_0^t\|\partial_t L_{\mathbf{u}}(s)\|_{\mathbf{H}^{1\prime}_{0,\Gamma_{\mathbf{u},D}}(\Omega)}\|\tilde{\mathbf{u}}_m(s)\|_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)}ds\\
&+\int_0^t\|L_\theta(s)\|_{H^{1\prime}_{0,\Gamma_{\theta,D}}(\Omega)}\|\tilde{\theta}_m(s)\|_{H^1_{0,\Gamma_{\theta,D}}(\Omega)}ds,
\end{aligned} \tag{4.22}$$

$a_{\max}$ being the constant of continuity of $a(\cdot,\cdot)$. Using the inequality

$$2\sqrt{\alpha}\sqrt{\beta}ab\leqslant \alpha a^2+\beta b^2,\quad \text{where } \alpha,\ \beta>0, \tag{4.23}$$

with $\alpha=\frac{1}{2}$ and $\beta=\frac{1}{2}$ for the fifth term on the right-hand side of inequality (4.22) and taking into account properties (3.8) and (3.11), we rewrite expression (4.22) in the following form:

$$\begin{aligned}
&\frac{a_{\min}}{2}\|\tilde{\mathbf{u}}_m(t)\|^2_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)}+\frac{1}{2}\|\tilde{\theta}_m(t)\|_2^2+\frac{k_{\min}}{\|\theta_r\|_{L^\infty(\Omega)}}\int_0^t\|\tilde{\theta}_m(s)\|^2_{H^1_{0,\Gamma_{\theta,D}}(\Omega)}ds\\
&+\frac{\alpha_{c,\min}}{\|\theta_r\|_{L^\infty(\Omega)}}\int_0^t\|\tilde{\theta}_m(s)\|^2_{L^2(\Gamma_{\theta,R})}\,ds\\
&\leqslant\frac{1}{2}\left[(a_{\max}+1)\|\tilde{\mathbf{u}}_0\|^2_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)}+\|\tilde{\theta}_0\|_2^2+\|L_{\mathbf{u}}(0)\|^2_{\mathbf{H}^{1\prime}_{0,\Gamma_{\mathbf{u},D}}(\Omega)}\right]\\
&+\frac{\|k\|_{L^\infty(\Omega)}\|\nabla\theta_r\|_{\mathbf{L}^\infty(\Omega)}}{\theta^2_{r,\min}}\int_0^t\|\tilde{\theta}_m(s)\|_{H^1_{0,\Gamma_{\theta,D}}(\Omega)}\|\tilde{\theta}_m(s)\|_{L^2(\Omega)}ds\\
&+\|L_{\mathbf{u}}(t)\|_{\mathbf{H}^{1\prime}_{0,\Gamma_{\mathbf{u},D}}(\Omega)}\|\tilde{\mathbf{u}}_m(t)\|_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)}+\int_0^t\|\partial_t L_{\mathbf{u}}(s)\|_{\mathbf{H}^{1\prime}_{0,\Gamma_{\mathbf{u},D}}(\Omega)}\|\tilde{\mathbf{u}}_m(s)\|_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)}ds\\
&+\int_0^t\|L_\theta(s)\|_{H^{1\prime}_{0,\Gamma_{\theta,D}}(\Omega)}\|\tilde{\theta}_m(s)\|_{H^1_{0,\Gamma_{\theta,D}}(\Omega)}ds.
\end{aligned} \tag{4.24}$$

Now, we apply again inequality (4.23) to the following terms on the right-hand side of inequality (4.24):

- the second term with $\alpha=\dfrac{k_{\min}}{4\|\theta_r\|_{L^\infty(\Omega)}}$ and $\beta=\dfrac{\|k\|^2_{L^\infty(\Omega)}\|\nabla\theta_r\|^2_{\mathbf{L}^\infty(\Omega)}\|\theta_r\|_{L^\infty(\Omega)}}{\theta^4_{r,\min}k_{\min}}$,
- the third term with $\alpha=\dfrac{1}{a_{\min}}$ and $\beta=\dfrac{a_{\min}}{4}$,
- the fourth term with $\alpha=\dfrac{1}{4}$ and $\beta=1$, and
- the fifth term with $\alpha=\dfrac{k_{\min}}{4\|\theta_r\|_{L^\infty(\Omega)}}$ and $\beta=\dfrac{\|\theta_r\|_{L^\infty(\Omega)}}{k_{\min}}$.

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Hence, we get

$$\frac{a_{\min}}{2}\|\tilde{\mathbf{u}}_m(t)\|^2_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)} + \frac{1}{2}\|\tilde{\theta}_m(t)\|_2^2 + \frac{k_{\min}}{\|\theta_r\|_{L^\infty(\Omega)}}\int_0^t \|\tilde{\theta}_m(s)\|^2_{H^1_{0,\Gamma_{\theta,D}}(\Omega)}ds$$
$$+ \frac{\alpha_{c,\min}}{\|\theta_r\|_{L^\infty(\Omega)}}\int_0^t \|\tilde{\theta}_m(s)\|^2_{L^2(\Gamma_{\theta,R})}\,ds$$
$$\leqslant \frac{1}{2}\left[(a_{\max}+1)\|\tilde{\mathbf{u}}_0\|^2_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)} + \|\tilde{\theta}_0\|_2^2 + \|L_{\mathbf{u}}(0)\|^2_{\mathbf{H}^{1'}_{0,\Gamma_{\mathbf{u},D}}(\Omega)}\right]$$
$$+ \frac{k_{\min}}{4\|\theta_r\|_{L^\infty(\Omega)}}\int_0^t \|\tilde{\theta}_m(s)\|^2_{H^1_{0,\Gamma_{\theta,D}}(\Omega)}ds + \frac{\|\theta_r\|_{L^\infty(\Omega)}}{k_{\min}}\int_0^t \|L_\theta(s)\|^2_{H^{1'}_{0,\Gamma_{\theta,D}}(\Omega)}ds$$
$$+ \frac{\|k\|^2_{L^\infty(\Omega)}\|\nabla\theta_r\|^2_{\mathbf{L}^\infty(\Omega)}\|\theta_r\|_{L^\infty(\Omega)}}{\theta^4_{r,\min}k_{\min}}\int_0^t \|\tilde{\theta}_m(s)\|^2_{L^2(\Omega)}ds$$
$$+ \frac{1}{a_{\min}}\|L_{\mathbf{u}}(t)\|^2_{\mathbf{H}^{1'}_{0,\Gamma_{\mathbf{u},D}}(\Omega)} + \frac{a_{\min}}{4}\|\tilde{\mathbf{u}}_m(t)\|^2_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)} + \frac{1}{4}\int_0^t \|\partial_t L_{\mathbf{u}}(s)\|^2_{\mathbf{H}^{1'}_{0,\Gamma_{\mathbf{u},D}}(\Omega)}ds$$
$$+ \int_0^t \|\tilde{\mathbf{u}}_m(s)\|^2_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)}ds + \frac{k_{\min}}{4\|\theta_r\|_{L^\infty(\Omega)}}\int_0^t \|\tilde{\theta}_m(s)\|^2_{H^1_{0,\Gamma_{\theta,D}}(\Omega)}ds. \tag{4.25}$$

From equality $L_{\mathbf{u}}(t) = L_{\mathbf{u}}(0) + \int_0^t \partial_t L_{\mathbf{u}}(s)ds$, and applying inequality (4.23), we deduce

$$\|L_{\mathbf{u}}(t)\|^2_{\mathbf{H}^{1'}_{0,\Gamma_{\mathbf{u},D}}(\Omega)} \leqslant 2\|L_{\mathbf{u}}(0)\|^2_{\mathbf{H}^{1'}_{0,\Gamma_{\mathbf{u},D}}(\Omega)} + 2t_f\int_0^t \|\partial_t L_{\mathbf{u}}(s)\|^2_{\mathbf{H}^{1'}_{0,\Gamma_{\mathbf{u},D}}(\Omega)}ds. \tag{4.26}$$

Then, thanks to the previous expression and taking into account Definition 3.1 in the second term on the left-hand side of inequality (4.25), this can be rewritten as follows:

$$c_1\left(\|\tilde{\mathbf{u}}_m(t)\|^2_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)} + \|\tilde{\theta}_m(t)\|^2_{L^2(\Omega)} + \int_0^t \|\tilde{\theta}_m(s)\|^2_{H^1_{0,\Gamma_{\theta,D}}(\Omega)}ds + \int_0^t \|\tilde{\theta}_m(s)\|^2_{L^2(\Gamma_{\theta,R})}\,ds\right)$$
$$\leqslant c_2\left[\|\tilde{\mathbf{u}}_0\|^2_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)} + \|\tilde{\theta}_0\|_2^2 + \|L_{\mathbf{u}}(0)\|^2_{\mathbf{H}^{1'}_{0,\Gamma_{\mathbf{u},D}}(\Omega)}\right]$$
$$+ c_3\left[\int_0^t \|\tilde{\mathbf{u}}_m(s)\|^2_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)}ds + \int_0^t ||\tilde{\theta}_m(s)||^2_{L^2(\Omega)}ds\right]$$
$$+ c_4\left[\int_0^t \|\partial_t L_{\mathbf{u}}(s)\|^2_{\mathbf{H}^{1'}_{0,\Gamma_{\mathbf{u},D}}(\Omega)}ds + \int_0^t \|L_\theta(s)\|^2_{H^{1'}_{0,\Gamma_{\theta,D}}(\Omega)}ds\right], \tag{4.27}$$

with

$$c_1 = \min\left\{\frac{a_{\min}}{4}, \frac{\rho_0 c_F}{2\|\theta_r\|_{L^\infty(\Omega)}}, \frac{k_{\min}}{2\|\theta_r\|_{L^\infty(\Omega)}}, \frac{\alpha_{c,\min}}{\|\theta_r\|_{L^\infty(\Omega)}}\right\},\ c_2 = \max\left\{\frac{a_{\max}+1}{2}, \frac{a_{\min}+4}{2a_{\min}}\right\},$$
$$c_3 = \max\left\{\frac{\|k\|^2_{L^\infty(\Omega)}\|\nabla\theta_r\|^2_{\mathbf{L}^\infty(\Omega)}\|\theta_r\|_{L^\infty(\Omega)}}{\theta^4_{r,\min}k_{\min}}, 1\right\} \text{ and } c_4 = \max\left\{\frac{a_{\min}+8t_f}{4a_{\min}}, \frac{\|\theta_r\|_{L^\infty(\Omega)}}{k_{\min}}\right\}.$$

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If we introduce the notation

$$|||L|||^2 = \int_0^{t_f} \left(\|\partial_t L_{\mathbf{u}}(s)\|^2_{\mathbf{H}^{1\prime}_{0,\Gamma_{\mathbf{u},D}}(\Omega)} + \|L_\theta(s)\|^2_{H^{1\prime}_{0,\Gamma_{\theta,D}}(\Omega)} \right) ds,$$
$$\Phi_m(t) = \|\tilde{\mathbf{u}}_m(t)\|^2_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)} + \|\tilde{\theta}_m(t)\|^2_{L^2(\Omega)},$$

expression (4.27) can be rewritten as follows:

$$\Phi_m(t) + \int_0^t \|\tilde{\theta}_m(s)\|^2_{H^1_{0,\Gamma_{\theta,D}}(\Omega)} ds + \int_0^t \|\tilde{\theta}_m(s)\|^2_{L^2(\Gamma_{\theta,R})} ds$$
$$\leqslant \frac{c_2}{c_1}\left[\|\tilde{\mathbf{u}}_0\|^2_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)} + \|\tilde{\theta}_0\|^2_2 + \|L_{\mathbf{u}}(0)\|^2_{\mathbf{H}^{1\prime}_{0,\Gamma_{\mathbf{u},D}}(\Omega)} \right] + \frac{c_3}{c_1}\int_0^t \Phi_m(s) ds + \frac{c_4}{c_1}|||L|||^2. \quad (4.28)$$

Since $\|\tilde{\mathbf{u}}_0\|_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)}$, $\|\tilde{\theta}_0\|_2$ and $|||L|||^2$ are bounded data, we deduce

$$\Phi_m(t) \leqslant c_5 + \frac{c_3}{c_1}\int_0^t \Phi_m(s) ds.$$

In consequence, thanks to Gronwall's lemma we obtain a bound for $\Phi_m(t)$ with $t \in [0, t_f]$. Therefore, there exists $M_{1,\mathbf{u},\theta}$ independent of m, such that

$$\|\tilde{\mathbf{u}}_m(t)\|^2_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)} \leqslant M_{1,\mathbf{u},\theta}, \quad ||\tilde{\theta}_m(t)||^2_{L^2(\Omega)} \leqslant M_{1,\mathbf{u},\theta}, \quad 0 \leqslant t \leqslant t_f. \quad (4.29)$$

This concludes the proof taking into account again expression (4.28). □

Lemma 3 (*A priori* estimates II) *Under assumptions* (H1)–(H13)*, the sequences* $\{\tilde{\mathbf{u}}_m\}_{m\in\mathbb{N}}$ *and* $\{\tilde{\theta}_m\}_{m\in\mathbb{N}}$ *given by Lemma 1, satisfy:*

(a) *the sequence* $\{\partial_t \tilde{\mathbf{u}}_m\}_{m\in\mathbb{N}}$ *is bounded in* $L^2(0, t_f; \mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega))$,

(b) *the sequence* $\{\partial_t \tilde{\theta}_m\}_{m\in\mathbb{N}}$ *is bounded in* $L^2(0, t_f; L^2(\Omega))$,

(c) *the sequence* $\{\tilde{\theta}_m\}_{m\in\mathbb{N}}$ *is bounded in* $L^\infty(0, t_f; H^1_{0,\Gamma_{\theta,D}}(\Omega))$.

Proof Thanks to Lemma 1 we can derive the first equation of Problem $(\widetilde{VP}_m)$ with respect to time, and we obtain the following:

$$\begin{cases} a(\partial_t \tilde{\mathbf{u}}_m(t), \mathbf{v}) - m(\partial_t \tilde{\theta}_m(t), \mathbf{v}) = \langle \partial_t L_{\mathbf{u}}(t), \mathbf{v}\rangle_{\mathbf{u}}, \quad \forall \mathbf{v} \in \mathbf{H}^{\mathbf{u}\,m}_{0,\Gamma_{\mathbf{u},D}}, \\ (\partial_t \tilde{\theta}_m(t), \phi)_2 + \kappa(\tilde{\theta}_m(t), \phi) + m(\phi, \partial_t \tilde{\mathbf{u}}_m(t)) + c(\tilde{\theta}_m(t), \phi) = \langle L_\theta(t), \phi\rangle_\theta, \forall \phi \in H^{\theta\,m}_{0,\Gamma_{\theta,D}}. \end{cases}$$

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Taking as test functions $\mathbf{v} = \partial_t \tilde{\mathbf{u}}_m(t)$, $\phi = \partial_t \tilde{\theta}_m(t)$, adding the previous equations, applying definition (3.10) and integrating over $(0, t)$, we get for any $t \in (0, t_f]$

$$\begin{aligned}
&\int_0^t a(\partial_t \tilde{\mathbf{u}}_m(s), \partial_t \tilde{\mathbf{u}}_m(s))ds + \int_0^t \|\partial_t \tilde{\theta}_m(s)\|_2^2 ds + \frac{1}{2}\kappa_1(\tilde{\theta}_m(t), \tilde{\theta}_m(t)) + \frac{1}{2}c(\tilde{\theta}_m(t), \tilde{\theta}_m(t)) \\
&= \frac{1}{2}\kappa_1(\tilde{\theta}_0, \tilde{\theta}_0) + \frac{1}{2}c(\tilde{\theta}_0, \tilde{\theta}_0) - \int_0^t \kappa_2(\tilde{\theta}_m(s), \partial_t \tilde{\theta}_m(s))ds + \langle \partial_t L_{\mathbf{u}}(t), \tilde{\mathbf{u}}_m(t)\rangle_{\mathbf{u}} \\
&\quad -\langle \partial_t L_{\mathbf{u}}(0), \tilde{\mathbf{u}}_0\rangle_{\mathbf{u}} - \int_0^t \langle \partial_t^2 L_{\mathbf{u}}(s), \tilde{\mathbf{u}}_m(s)\rangle_{\mathbf{u}} ds + \langle L_\theta(t), \tilde{\theta}_m(t)\rangle_\theta \\
&\quad -\langle L_\theta(0), \tilde{\theta}_0\rangle_\theta - \int_0^t \langle \partial_t L_\theta(s), \tilde{\theta}_m(s)\rangle_\theta ds.
\end{aligned}$$

Thanks to hypotheses (H1)–(H12), all terms on the left-hand side of the previous expression are all non-negative and a, $L_{\mathbf{u}}$, $\partial_t L_{\mathbf{u}}$, $\partial_t^2 L_{\mathbf{u}}$, L_θ, $\partial_t L_\theta$ are continuous. So, taking into account Remarks 3.2 and 3.3, Definition 3.1, and applying Hölder's inequality, we get

$$\begin{aligned}
&\int_0^t a(\partial_t \tilde{\mathbf{u}}_m(s), \partial_t \tilde{\mathbf{u}}_m(s))ds + \frac{\rho_0 c_F}{\|\theta_r\|_{L^\infty(\Omega)}} \int_0^t \|\partial_t \tilde{\theta}_m(s)\|_{L^2(\Omega)}^2 ds + \frac{1}{2}\kappa_1(\tilde{\theta}_m(t), \tilde{\theta}_m(t)) \\
&+ \frac{\alpha_{c,\min}}{2\|\theta_r\|_{L^\infty(\Omega)}} \|\tilde{\theta}_m(t)\|_{L^2(\Gamma_{\theta,R})}^2 \leqslant \frac{\|k\|_{L^\infty(\Omega)}}{2\theta_{r,\min}} \|\tilde{\theta}_0\|_{H^1_{0,\Gamma_{\theta,D}}(\Omega)}^2 + \frac{\|\alpha_c\|_{L^\infty(\Omega)}}{2\theta_{r,\min}} \|\tilde{\theta}_0\|_{L^2(\Gamma_{\theta,R})}^2 \\
&+ \frac{\|k\|_{L^\infty(\Omega)} \|\nabla\theta_r\|_{\mathbf{L}^\infty(\Omega)}}{\theta_{r,\min}^2} \int_0^t \|\tilde{\theta}_m(s)\|_{H^1_{0,\Gamma_{\theta,D}}(\Omega)} \|\partial_t \tilde{\theta}_m(s)\|_{L^2(\Omega)} ds \\
&+ \|\partial_t L_{\mathbf{u}}(t)\|_{\mathbf{H}^{1\prime}_{0,\Gamma_{\mathbf{u},D}}(\Omega)} \|\tilde{\mathbf{u}}_m(t)\|_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)} + \|\partial_t L_{\mathbf{u}}(0)\|_{\mathbf{H}^{1\prime}_{0,\Gamma_{\mathbf{u},D}}(\Omega)} \|\tilde{\mathbf{u}}_0\|_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)} \\
&+ \int_0^t \|\partial_t^2 L_{\mathbf{u}}(s)\|_{\mathbf{H}^{1\prime}_{0,\Gamma_{\mathbf{u},D}}(\Omega)} \|\tilde{\mathbf{u}}_m(s)\|_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)} ds + \|L_\theta(t)\|_{H^{1\prime}_{0,\Gamma_{\theta,D}}(\Omega)} \|\tilde{\theta}_m(t)\|_{H^1_{0,\Gamma_{\theta,D}}(\Omega)} \\
&+ \|L_\theta(0)\|_{H^{1\prime}_{0,\Gamma_{\theta,D}}(\Omega)} \|\tilde{\theta}_0\|_{H^1_{0,\Gamma_{\theta,D}}(\Omega)} + \int_0^t \|\partial_t L_\theta(s)\|_{H^{1\prime}_{0,\Gamma_{\theta,D}}(\Omega)} \|\tilde{\theta}_m(s)\|_{H^1_{0,\Gamma_{\theta,D}}(\Omega)} ds.
\end{aligned}$$

Using inequality (4.23), with $\alpha = 1$ and $\beta = \frac{1}{4}$, on the fourth, fifth, sixth, eighth and ninth terms on the right-hand side of the above inequality, we arrive at

$$\begin{aligned}
&\int_0^t a(\partial_t \tilde{\mathbf{u}}_m(s), \partial_t \tilde{\mathbf{u}}_m(s))ds + \frac{\rho_0 c_F}{\|\theta_r\|_{L^\infty(\Omega)}} \int_0^t \|\partial_t \tilde{\theta}_m(s)\|_{L^2(\Omega)}^2 ds + \frac{1}{2}\kappa_1(\tilde{\theta}_m(t), \tilde{\theta}_m(t)) \\
&+ \frac{\alpha_{c,\min}}{2\|\theta_r\|_{L^\infty(\Omega)}} \|\tilde{\theta}_m(t)\|_{L^2(\Gamma_{\theta,R})}^2 \leqslant \frac{\|k\|_{L^\infty(\Omega)}}{2\theta_{r,\min}} \|\tilde{\theta}_0\|_{H^1_{0,\Gamma_{\theta,D}}(\Omega)}^2 + \frac{\|\alpha_c\|_{L^\infty(\Omega)}}{2\theta_{r,\min}} \|\tilde{\theta}_0\|_{L^2(\Gamma_{\theta,R})}^2 \\
&+ \frac{\|k\|_{L^\infty(\Omega)} \|\nabla\theta_r\|_{\mathbf{L}^\infty(\Omega)}}{\theta_{r,\min}^2} \int_0^t \|\tilde{\theta}_m(s)\|_{H^1_{0,\Gamma_{\theta,D}}(\Omega)} \|\partial_t \tilde{\theta}_m(s)\|_{L^2(\Omega)} ds \\
&+ \|\partial_t L_{\mathbf{u}}(t)\|_{\mathbf{H}^{1\prime}_{0,\Gamma_{\mathbf{u},D}}(\Omega)}^2 + \frac{1}{4}\|\tilde{\mathbf{u}}_m(t)\|_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)}^2 + \|\partial_t L_{\mathbf{u}}(0)\|_{\mathbf{H}^{1\prime}_{0,\Gamma_{\mathbf{u},D}}(\Omega)}^2
\end{aligned}$$

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$$+\frac{1}{4}\|\tilde{\mathbf{u}}_0\|^2_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)}+\int_0^t\|\partial_t^2 L_{\mathbf{u}}(s)\|^2_{\mathbf{H}^{1\prime}_{0,\Gamma_{\mathbf{u},D}}(\Omega)}ds+\frac{1}{4}\int_0^t\|\tilde{\mathbf{u}}_m(s)\|^2_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)}ds$$
$$+\|L_\theta(t)\|_{H^{1\prime}_{0,\Gamma_{\theta,D}}(\Omega)}\|\tilde{\theta}_m(t)\|_{H^1_{0,\Gamma_{\theta,D}}(\Omega)}+\|L_\theta(0)\|^2_{H^{1\prime}_{0,\Gamma_{\theta,D}}(\Omega)}+\frac{1}{4}\|\tilde{\theta}_0\|^2_{H^1_{0,\Gamma_{\theta,D}}(\Omega)}$$
$$+\int_0^t\|\partial_t L_\theta(s)\|^2_{H^{1\prime}_{0,\Gamma_{\theta,D}}(\Omega)}ds+\frac{1}{4}\int_0^t\|\tilde{\theta}_m(s)\|^2_{H^1_{0,\Gamma_{\theta,D}}(\Omega)}ds. \tag{4.30}$$

Next, we consider properties (3.8) and (3.11); we apply again inequality (4.23) to the following terms on the right-hand side of expression (4.30):

- the third term with $\alpha=\dfrac{\|k\|^2_{L^\infty(\Omega)}\|\nabla\theta_r\|^2_{\mathbf{L}^\infty(\Omega)}\|\theta_r\|_{L^\infty(\Omega)}}{2\theta^4_{r,\min}\rho_0 c_F}$ and $\beta=\dfrac{\rho_0 c_F}{2\|\theta_r\|_{L^\infty(\Omega)}}$,
- the tenth term with $\alpha=\dfrac{\|\theta_r\|_{L^\infty(\Omega)}}{k_{\min}}$ and $\beta=\dfrac{k_{\min}}{4\|\theta_r\|_{L^\infty(\Omega)}}$;

we take into account the equalities

$$\partial_t L_{\mathbf{u}}(t)=\partial_t L_{\mathbf{u}}(0)+\int_0^t\partial_t^2 L_{\mathbf{u}}(s)ds,\quad L_\theta(t)=L_\theta(0)+\int_0^t\partial_t L_\theta(s)ds;$$

we introduce the notation

$$|||\partial_t L|||^2=\int_0^{t_f}\left(\|\partial_t^2 L_{\mathbf{u}}(s)\|^2_{\mathbf{H}^{1\prime}_{0,\Gamma_{\mathbf{u},D}}(\Omega)}+\|\partial_t L_\theta(s)\|^2_{H^1_{0,\Gamma_{\theta,D}}(\Omega)}\right)ds, \tag{4.31}$$

and, finally, applying the same reasoning used in inequality (4.26), expression (4.30) can be rewritten as follows:

$$c_6\left[\int_0^t\|\partial_t\tilde{\mathbf{u}}_m(s)\|^2_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)}ds+\int_0^t\|\partial_t\tilde{\theta}_m(s)\|^2_{L^2(\Omega)}ds+\|\tilde{\theta}_m(t)\|^2_{H^1_{0,\Gamma_{\theta,D}}(\Omega)}+\|\tilde{\theta}_m(t)\|^2_{L^2(\Gamma_{\theta,R})}\right]$$
$$\leqslant c_9\left[\|\tilde{\theta}_0\|^2_{H^1_{0,\Gamma_{\theta,D}}(\Omega)}+\|\tilde{\theta}_0\|^2_{L^2(\Gamma_{\theta,R})}+\|\tilde{\mathbf{u}}_0\|^2_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)}+\|\partial_t L_{\mathbf{u}}(0)\|^2_{\mathbf{H}^{1\prime}_{0,\Gamma_{\mathbf{u},D}}(\Omega)}\right.$$
$$\left.+\|L_\theta(0)\|^2_{H^{1\prime}_{0,\Gamma_{\theta,D}}(\Omega)}+|||\partial_t L|||^2\right]+c_8\left[\|\tilde{\mathbf{u}}_m(t)\|^2_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)}+\int_0^t\|\tilde{\theta}_m(s)\|^2_{H^1_{0,\Gamma_{\theta,D}}(\Omega)}ds\right.$$
$$\left.+\int_0^t\|\tilde{\mathbf{u}}_m(s)\|^2_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)}ds\right]. \tag{4.32}$$

The constants in inequality (4.32) are such that

$$c_6=\min\left\{a_{\min},\frac{\rho_0 c_F}{2\|\theta_r\|_{L^\infty(\Omega)}},\frac{k_{\min}}{4\|\theta_r\|_{L^\infty(\Omega)}},\frac{\alpha_{c,\min}}{2\|\theta_r\|_{L^\infty(\Omega)}}\right\}\text{ and }c_9=\max\{c_7,c_8\},$$

with

$$c_7=\max\left\{\frac{2\|k\|_{L^\infty(\Omega)}+\theta_{r,\min}}{4\theta_{r,\min}},\frac{\|\alpha_c\|_{L^\infty(\Omega)}}{2\theta_{r,\min}},3,\frac{2\|\theta_r\|_{L^\infty(\Omega)}+k_{\min}}{k_{\min}}\right\}\text{ and}$$
$$c_8=\max\left\{\frac{2\|k\|^2_{L^\infty(\Omega)}\|\nabla\theta_r\|^2_{\mathbf{L}^\infty(\Omega)}\|\theta_r\|_{L^\infty(\Omega)}+\theta^4_{r,\min}\rho_0 c_F}{4\theta^4_{r,\min}\rho_0 c_F},2t_f+1,\frac{2t_f\|\theta_r\|_{L^\infty(\Omega)}+k_{\min}}{k_{\min}}\right\}.$$

Thanks to Gronwall's lemma and Lemma 2, we can deduce from expression (4.32) the existence of a constant $M_{2,\mathbf{u},\theta}$ independent of m, such that

$$\int_0^t \|\partial_t \tilde{\mathbf{u}}_m(s)\|^2_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)} ds \leqslant M_{2,\mathbf{u},\theta}, \quad \int_0^t \|\partial_t \tilde{\theta}_m(s)\|^2_{L^2(\Omega)} ds \leqslant M_{2,\mathbf{u},\theta},$$
$$\|\tilde{\theta}_m(t)\|^2_{H^1_{0,\Gamma_{\theta,D}}(\Omega)} \leqslant M_{2,\mathbf{u},\theta}, \quad a.e.\ t \in (0, t_f).$$

□

4.4 Step IV: Passage to the limit

Taking into account some well-known results of compactness, we deduce the following result from Lemmas 2 and 3:

Corollary 1 *Under assumptions* (H1)–(H13) *there exist*

$$\tilde{\mathbf{u}} \in W^{1,2}(0, t_f; \mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)) \cap L^\infty(0, t_f; \mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)),$$
$$\tilde{\theta} \in W^{1,2}(0, t_f; L^2(\Omega)) \cap L^\infty(0, t_f; H^1_{0,\Gamma_{\theta,D}}(\Omega)),$$

and subsequences (again indexed with m) such that, as $m \to \infty$

$$\{\tilde{\mathbf{u}}_m\}_{m\in\mathbb{N}} \to \tilde{\mathbf{u}} \text{ in } L^\infty(0, t_f; \mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)) \text{ weak-star},$$
$$\{\partial_t\tilde{\mathbf{u}}_m\}_{m\in\mathbb{N}} \to \partial_t\tilde{\mathbf{u}} \text{ in } L^2(0, t_f; \mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)) \text{ weak},$$
$$\{\tilde{\theta}_m\}_{m\in\mathbb{N}} \to \tilde{\theta} \text{ in } L^\infty(0, t_f; H^1_{0,\Gamma_{\theta,D}}(\Omega)) \text{ weak-star},$$
$$\{\partial_t\tilde{\theta}_m\}_{m\in\mathbb{N}} \to \partial_t\tilde{\theta} \text{ in } L^2(0, t_f; L^2(\Omega)) \text{ weak}.$$

Furthermore for each $m \in \mathbb{N}$, there exists a non-negative integer $l \geqslant m$, such that $(\tilde{\mathbf{u}}_m, \tilde{\theta}_m)$ is the unique solution to Problem $(\widetilde{VP}_l)$.

4.5 Step V: Verifying that $(\tilde{\mathbf{u}}, \tilde{\theta})$ is a solution to Problem $(\widetilde{VP})$

As consequence of Corollary 1, $(\tilde{\mathbf{u}}, \tilde{\theta})$ satisfies properties (4.10). In order to complete the proof of the existence of a solution, it is necessary to prove that $(\tilde{\mathbf{u}}, \tilde{\theta})$ satisfies Problem $(\widetilde{VP})$. For this purpose we use the methodology from Viaño [35].

Let $j \in \mathbb{N}$ be arbitrary and $m > j$. Then, thanks to Corollary 1, $(\tilde{\mathbf{u}}_m, \tilde{\theta}_m)$ is a solution to Problem $(\widetilde{VP}_l)$, with l dependent on m and $l \geqslant m > j$. We are going to prove that we can pass to the limit in order to obtain that $(\tilde{\mathbf{u}}, \tilde{\theta})$ is a solution to Problem $(\widetilde{VP})$.

Verifying the weak equation (4.6a) *for the mechanical submodel.*

Considering in equation (4.12a), the test function $\mathbf{v}_j(t) \in \mathbf{H}^{\mathbf{u}\,j}_{0,\Gamma_{\mathbf{u},D}} \subset \mathbf{H}^{\mathbf{u}\,l}_{0,\Gamma_{\mathbf{u},D}}$ such that

$$\mathbf{v}_j(t) = \xi(t)\mathbf{w}^{\mathbf{u}}_j, \quad \xi \in C^1(0, t_f), \quad \xi(t_f) = 0, \tag{4.33}$$

and integrating over $(0, t_f)$, we can pass to the limit as $l \to \infty$, thanks to Corollary

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1, and we obtain

$$\int_0^{t_f} \left[a(\tilde{\mathbf{u}}(t), \mathbf{w}_j^{\mathbf{u}}) - m(\tilde{\theta}(t), \mathbf{w}_j^{\mathbf{u}}) - \langle L_{\mathbf{u}}(t), \mathbf{w}_j^{\mathbf{u}}\rangle_{\mathbf{u}}\right] \xi(t)dt = 0, \tag{4.34}$$

for all $\xi \in C^1(0, t_f)$, $\xi(t_f) = 0$ and for all $j \in \mathbb{N}$. Using the density of the finite linear combinations of $\mathbf{w}_j^{\mathbf{u}}$ in $\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)$, we deduce

$$a(\tilde{\mathbf{u}}(t), \mathbf{v}) - m(\tilde{\theta}(t), \mathbf{v}) = \langle L_{\mathbf{u}}(t), \mathbf{v}\rangle_{\mathbf{u}}, \tag{4.35}$$

for all $\mathbf{v} \in \mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)$ in the space of distributions $\mathscr{D}'(0, t_f)$.

Verifying the weak equation (4.6b) *for the thermal submodel.*

Let us consider in equation (4.12b) the test function $\phi_j(t) \in H^{\theta\, j}_{0,\Gamma_{\theta,D}} \subset H^{\theta\, l}_{0,\Gamma_{\theta,D}}$, $l > j$, with l in the same conditions as before

$$\phi_j(t) = \zeta(t)w_j^\theta, \quad \zeta \in C^1(0, t_f), \quad \zeta(t_f) = 0. \tag{4.36}$$

Integrating over $(0, t_f)$, taking into account equality

$$(\partial_t\tilde{\theta}_l(t), \phi_j(t))_2 = \frac{d}{dt}(\tilde{\theta}_l(t), \phi_j(t))_2 - (\tilde{\theta}_l(t), \partial_t\phi_j(t))_2, \tag{4.37}$$

and applying the initial condition (4.13) for $\tilde{\theta}_l$, we can deduce

$$\int_0^{t_f} \left[-(\tilde{\theta}_l(t), \partial_t\phi_j(t))_2 + \kappa(\tilde{\theta}_l(t), \phi_j(t)) + m(\phi_j(t), \partial_t\tilde{\mathbf{u}}_l(t)) + c(\tilde{\theta}_l(t), \phi_j(t)) - \langle L_\theta(t), \phi_j(t)\rangle_\theta\right] dt = (\tilde{\theta}_0, \phi_j(0))_2, \tag{4.38}$$

since $\phi_j(t_f) = 0$. So, thanks to Corollary 1, we can pass to the limit as $l \to \infty$, and we get

$$\int_0^{t_f} -(\tilde{\theta}(t), w_j^\theta)_2\partial_t\zeta(t)dt + \int_0^{t_f} \left[\kappa(\tilde{\theta}(t), w_j^\theta) + m(w_j^\theta, \partial_t\tilde{\mathbf{u}}(t)) + c(\tilde{\theta}(t), w_j^\theta) - \langle L_\theta(t), w_j^\theta\rangle_\theta\right] \zeta(t)dt = (\tilde{\theta}_0, w_j^\theta)_2\zeta(0), \tag{4.39}$$

for all $\zeta \in C^1(0, t_f)$, $\zeta(t_f) = 0$. In particular, equation (4.39) is true for all $\zeta \in \mathscr{D}(0, t_f)$ and for all $j \in \mathbb{N}$. Using the density of the finite linear combinations of w_j^θ in $H^1_{0,\Gamma_{\theta,D}}(\Omega)$, we deduce

$$\frac{d}{dt}(\tilde{\theta}(t), \phi)_2 + \kappa(\tilde{\theta}(t), \phi) + m(\phi, \partial_t\tilde{\mathbf{u}}(t)) + c(\theta(t), \phi) = \langle L_\theta(t), \phi\rangle_\theta, \tag{4.40}$$

for all $\phi \in H^1_{0,\Gamma_{\theta,D}}(\Omega)$ in $\mathscr{D}'(0, t_f)$. As $\partial_t\tilde{\theta} \in L^2(0, t_f; L^2(\Omega))$, equality (4.40) is equivalent to

$$(\partial_t\tilde{\theta}(t), \phi)_2 + \kappa(\tilde{\theta}(t), \phi) + m(\phi, \partial_t\tilde{\mathbf{u}}(t)) + c(\theta(t), \phi) = \langle L_\theta(t), \phi\rangle_\theta, \tag{4.41}$$

for all $\phi \in H^1_{0,\Gamma_{\theta,D}}(\Omega)$ in $\mathscr{D}'(0, t_f)$.

In order to complete the proof, we must prove that $(\tilde{\mathbf{u}}, \tilde{\theta})$ satisfies the initial conditions (4.7) of Problem $(\widetilde{VP})$.

Verifying the initial condition for temperature.

Considering in the weak equality (4.41) the test function given in equation (4.36), integrating over $(0, t_f)$ and taking into account equality (4.37) it results,

$$\int_0^{t_f} \left[-(\tilde{\theta}(t), \partial_t \zeta(t) w_j^\theta)_2 + \kappa(\tilde{\theta}(t), \zeta(t) w_j^\theta) + m(\zeta(t) w_j^\theta, \partial_t \tilde{\mathbf{u}}(t)) \right.$$
$$\left. + c(\tilde{\theta}(t), \zeta(t) w_j^\theta) - \langle L_\theta(t), \zeta(t) w_j^\theta \rangle_\theta \right] dt = (\tilde{\theta}(0), \zeta(0) w_j^\theta)_2,$$

for all $\zeta \in C^1(0, t_f)$, $\zeta(t_f) = 0$. Now, if we compare the previous expression with equality (4.39), we obtain

$$(\tilde{\theta}(0), w_j^\theta)_2 \zeta(0) = (\tilde{\theta}_0, w_j^\theta)_2 \zeta(0), \quad \forall \zeta \in C^1(0, t_f), \; \zeta(t_f) = 0,$$

for all non-negative integer j, hence, we conclude $\tilde{\theta}(0) = \tilde{\theta}_0$.

Verifying the initial condition for displacements.

As $L_{\mathbf{u}} \in C^1(0, t_f; \mathbf{H}^{1'}_{0,\Gamma_{\mathbf{u},D}}(\Omega))$ it is possible to derive expression (4.35) in time; furthermore, since $\partial_t \tilde{\mathbf{u}} \in L^2(0, t_f; \mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega))$ and $\partial_t \tilde{\theta} \in L^2(0, t_f; L^2(\Omega))$, we have

$$a(\partial_t \tilde{\mathbf{u}}(t), \mathbf{v}) - m(\partial_t \tilde{\theta}(t), \mathbf{v}) = \langle \partial_t L_{\mathbf{u}}(t), \mathbf{v} \rangle_{\mathbf{u}}, \quad \forall \mathbf{v} \in \mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega).$$

In particular, taking again the test function defined in equation (4.33) and integrating in time, we deduce

$$\int_0^{t_f} \left[-a(\tilde{\mathbf{u}}(t), \partial_t \xi(t) w_j^{\mathbf{u}}) + m(\tilde{\theta}(t), \partial_t \xi(t) w_j^{\mathbf{u}}) + \langle L_{\mathbf{u}}(t), \partial_t \xi(t) w_j^{\mathbf{u}} \rangle_{\mathbf{u}} \right] dt$$
$$= a(\tilde{\mathbf{u}}(0), \xi(0) w_j^{\mathbf{u}}) - m(\tilde{\theta}(0), \xi(0) w_j^{\mathbf{u}}) - \langle L_{\mathbf{u}}(0), \xi(0) w_j^{\mathbf{u}} \rangle_{\mathbf{u}}, \tag{4.42}$$

for all $\xi \in \mathscr{C}^1(0, t_f)$, $\xi(t_f) = 0$. Taking into account expression (4.35), the term on the left-hand side of expression (4.42) vanishes, and since $\tilde{\theta}(0) = \tilde{\theta}_0$, we get

$$a(\tilde{\mathbf{u}}(0), \xi(0) w_j^{\mathbf{u}}) = [m(\tilde{\theta}_0, w_j^{\mathbf{u}}) + \langle L_{\mathbf{u}}(0), w_j^{\mathbf{u}} \rangle_{\mathbf{u}}] \xi(0),$$

for all $\xi \in \mathscr{C}^1(0, t_f)$; $\xi(t_f) = 0$. Finally, compatibility condition (4.11) let us write

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this equation in the following equivalent form:

$$a(\tilde{\mathbf{u}}(0), w_j^{\mathbf{u}})\xi(0) = a(\tilde{\mathbf{u}}_0, w_j^{\mathbf{u}})\xi(0), \quad \forall \xi \in C^1(0, t_f), \quad \xi(t_f) = 0, \ \forall j \in \mathbb{N}.$$

Therefore, we can conclude that $\tilde{\mathbf{u}}(0) = \tilde{\mathbf{u}}_0$ thanks to properties of form $a(\cdot, \cdot)$.

5 Uniqueness of solution to Problem (VP)

In this section, we prove the uniqueness of solution to Problem (VP). Our proof is based on applying Gronwall's lemma following the papers of Gawinecki [20,22–24] and Gawinecki *et al.* [26].

Theorem 5.1 *Under assumptions* (H1)–(H13)*, there exists a unique solution* $(\mathbf{u}, \theta)$ *to Problem* (VP) *satisfying properties (4.1) and (4.2).*

Proof In order to establish the uniqueness of the solution to Problem (VP) and since this problem is equivalent to Problem $(\widetilde{VP})$, it is enough to prove the uniqueness of the solution to this former problem. To this end, let $(\tilde{\mathbf{u}}_1, \tilde{\theta}_1)$, $(\tilde{\mathbf{u}}_2, \tilde{\theta}_2)$ be two solutions of Problem $(\widetilde{VP})$ and let us write

$$\tilde{\mathbf{u}} = \tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2 \in L^\infty(0, t_f; \mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)) \text{ and } \tilde{\theta} = \tilde{\theta}_1 - \tilde{\theta}_2 \in L^\infty(0, t_f; H^1_{0,\Gamma_{\theta,D}}(\Omega)).$$

So, they satisfy *a.e.* $t \in (0, t_f)$

$$\begin{cases} a(\tilde{\mathbf{u}}(t), \mathbf{v}) - m(\tilde{\theta}(t), \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega), \\ (\partial_t \tilde{\theta}(t), \phi)_2 + \kappa(\tilde{\theta}(t), \phi) + m(\phi, \partial_t \tilde{\mathbf{u}}(t)) + c(\tilde{\theta}(t), \phi) = 0, \ \forall \phi \in H^1_{0,\Gamma_{\theta,D}}(\Omega), \end{cases}$$

and the initial conditions $\tilde{\mathbf{u}}(0) = \mathbf{0}, \quad \tilde{\theta}(0) = 0.$

Integrating these equations over $(0, t)$, taking as test functions $\mathbf{v} = \partial_t \tilde{\mathbf{u}}(t)$, $\phi = \tilde{\theta}(t)$, adding the resulting equations and taking into account Remark 3.3, we arrive at the following:

$$\int_0^t \frac{1}{2}\frac{d}{ds}[a(\tilde{\mathbf{u}}(s), \tilde{\mathbf{u}}(s))]ds + \int_0^t \frac{1}{2}\frac{d}{ds}\|\tilde{\theta}(s)\|_2^2 ds + \int_0^t \kappa_1(\tilde{\theta}(s), \tilde{\theta}(s))ds$$
$$+ \int_0^t c(\tilde{\theta}(s), \tilde{\theta}(s))ds = -\int_0^t \kappa_2(\tilde{\theta}(s), \tilde{\theta}(s))ds \quad a.e. \ t \in (0, t_f).$$

Since $\tilde{\mathbf{u}}(0) = \mathbf{0}$ and $\tilde{\theta}(0) = 0$, and thanks to hypotheses (H2), (H4), (H10) and Hölder's inequality, we have the following:

$$\frac{1}{2}a(\tilde{\mathbf{u}}(t), \tilde{\mathbf{u}}(t)) + \frac{1}{2}\|\tilde{\theta}(t)\|_2^2 + \int_0^t \kappa_1(\tilde{\theta}(s), \tilde{\theta}(s))ds + \frac{\alpha_{c,\min}}{\|\theta_r\|_{L^\infty(\Omega)}}\|\tilde{\theta}(t)\|^2_{L^2(\Gamma_{\theta,R})}$$
$$\leqslant \frac{c\|k\|_{L^\infty(\Omega)}\|\nabla\theta_r\|_{\mathbf{L}^\infty(\Omega)}}{\theta_{r,\min}^2}\int_0^t \|\tilde{\theta}(s)\|_{L^2(\Omega)}\|\tilde{\theta}(s)\|_{H^1_{0,\Gamma_{\theta,D}}(\Omega)}ds, \tag{5.2}$$

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with c the constant of equivalent norms. Taking into account properties (3.8), (3.11), and using inequality (4.23) with

$$\alpha = \frac{c^2\|k\|^2_{L^\infty(\Omega)}\|\nabla\theta_r\|^2_{\mathbf{L}^\infty(\Omega)}\|\theta_r\|_{L^\infty(\Omega)}}{2\theta^4_{r,\min}k_{\min}} \quad \text{and} \quad \beta = \frac{k_{\min}}{2\|\theta_r\|_{L^\infty(\Omega)}},$$

we deduce

$$\begin{aligned}
&\frac{a_{\min}}{2}\|\tilde{\mathbf{u}}(t)\|^2_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)} + \frac{\rho_0 c_F}{2\|\theta_r\|_{L^\infty(\Omega)}}\|\tilde{\theta}(t)\|^2_{L^2(\Omega)}\\
&\quad + \frac{k_{\min}}{2\|\theta_r\|_{L^\infty(\Omega)}}\int_0^t \|\tilde{\theta}(s)\|^2_{H^1_{0,\Gamma_{\theta,D}}(\Omega)}ds + \frac{\alpha_{c,\min}}{\|\theta_r\|_{L^\infty(\Omega)}}\|\tilde{\theta}(t)\|^2_{L^2(\Gamma_{\theta,R})}\\
&\quad \leqslant \frac{c^2\|k\|^2_{L^\infty(\Omega)}\|\nabla\theta_r\|^2_{\mathbf{L}^\infty(\Omega)}\|\theta_r\|_{L^\infty(\Omega)}}{2\theta^4_{r,\min}k_{\min}}\int_0^t \|\tilde{\theta}(s)\|^2_{L^2(\Omega)}ds, \qquad (5.3)
\end{aligned}$$

and we get

$$\frac{\rho_0 c_F}{2\|\theta_r\|_{L^\infty(\Omega)}}\|\tilde{\theta}(t)\|^2_{L^2(\Omega)} \leqslant \frac{c^2\|k\|^2_{L^\infty(\Omega)}\|\nabla\theta_r\|^2_{\mathbf{L}^\infty(\Omega)}\|\theta_r\|_{L^\infty(\Omega)}}{2\theta^4_{r,\min}k_{\min}}\int_0^t \|\tilde{\theta}(s)\|^2_{L^2(\Omega)}ds.$$

Thanks to Gronwall's lemma, we can conclude that $\|\tilde{\theta}(t)\|^2_{L^2(\Omega)} = 0$ *a.e.* $t \in (0, t_f)$, hence $\tilde{\theta}_1 = \tilde{\theta}_2$. Furthermore, using again inequality (5.3), we directly deduce that $\|\tilde{\mathbf{u}}(t)\|_{\mathbf{H}^1_{0,\Gamma_{\mathbf{u},D}}(\Omega)} = 0$ *a.e.* $t \in (0, t_f)$. In consequence, we conclude that $\tilde{\mathbf{u}}_1 = \tilde{\mathbf{u}}_2$, which finishes the proof. □

6 Conclusions

In this paper we have proved the existence and uniqueness of solution to a quasistatic fully coupled thermoelastic problem associated with non-homogeneous linear elastic materials. In the thermal equations we have included the term due to the mechanical heat dissipation and in the mechanical behaviour law the deformations due to thermal gradients. We have considered mixed displacement-traction boundary conditions for the mechanical submodel and mixed Dirichlet–Neumann–Robin for the thermal one. Moreover, we have assumed that the reference temperature, the thermal conductivity and the Lamé's parameters depend on the material point.

Specifically, we have achieved a unique solution $(\mathbf{u}, \theta)$ satisfying

$$\begin{aligned}
&\mathbf{u} \in L^\infty\big(0, t_f; \mathbf{H}^1(\Omega)\big), \quad \partial_t\mathbf{u} \in L^2\big(0, t_f; \mathbf{H}^1(\Omega)\big), \text{ and}\\
&\quad \theta \in L^\infty\big(0, t_f; H^1(\Omega)\big), \quad \partial_t\theta \in L^2\big(0, t_f; L^2(\Omega)\big).
\end{aligned}$$

The results obtained in this work represent an improvement on the existing literature and it will facilitate future research in other open problems arising from mathematical modelling in industrial processes, such as the analysis of the existence of the solution when this coupled thermomechanical problem also incorporates a non-linear behaviour law, when the physical parameters depend not just on the material point but also on the temperature, or when it is necessary to incorporate a contact condition in the mechanical submodel.

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