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ON $C^{(n)}$-EXTENDIBLE CARDINALS

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Abstract. The hierarchies of $C^{(n)}$-cardinals were introduced by Bagaria in [1] and were further studied and extended by the author in [18] and in [20]. The case of $C^{(n)}$-extendible cardinals, and of their $C^{(n)+}$-extendibility variant, is of particular interest since such cardinals have found applications in the areas of category theory, of homotopy theory, and of model theory (see [2], [3], and [4], respectively). However, the exact relation between these two notions had been left unclarified. Moreover, the question of whether the Generalized Continuum Hypothesis (GCH) can be forced while preserving $C^{(n)}$-extendible cardinals (for $n > 1$) also remained open. In this note, we first establish results in the direction of exactly controlling the targets of $C^{(n)}$-extendibility embeddings. As a corollary, we show that every $C^{(n)}$-extendible cardinal is in fact $C^{(n)+}$-extendible; this, in turn, clarifies the assumption needed in some applications obtained in [3]. At the same time, we underline the applicability of our arguments in the context of $C^{(n)}$-ultrahuge cardinals as well, as these were introduced in [20]. Subsequently, we show that $C^{(n)}$-extendible cardinals carry their own Laver functions, making them the first known example of $C^{(n)}$-cardinals that have this desirable feature. Finally, we obtain an alternative characterization of $C^{(n)}$-extendibility, which we use to answer the question regarding forcing the GCH affirmatively.

§1. Introduction.

The machinery of elementary embeddings is ubiquitous in the context of large cardinals, having been very intensively used and studied for several decades. However, and despite the fact that we have a rich theory regarding the *critical point*—usually denoted by κ—of such embeddings,[1] the general question of what kind of properties are (or can be) satisfied by the *image* of the critical point—usually denoted by $j(\kappa)$—remains quite elusive and widely open.[2]

In the direction of imposing *some* structure on the aforementioned image $j(\kappa)$, one possibility is to consider *reflection* properties that this ordinal may satisfy. This path was initiated, in its generality, by Bagaria, who introduced the so-called $C^{(n)}$-cardinals in [1]. These are strengthenings of the usual large cardinals, adding to each standard definition the extra requirement that the image $j(\kappa)$ of the embedding in question is an ordinal that is Σ_n-correct in the universe. Bagaria developed the theory of the various $C^{(n)}$-hierarchies and, moreover, showed that such notions are closely related to the general theme of reflection for the set-theoretic universe.

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[1] It being, arguably, the main focus of study, since it is, typically, the large cardinal in question.

[2] Of course, there are various special cases of large cardinal embeddings for which we do have some information regarding properties that the image of the critical point satisfies. Nevertheless, no general account has emerged so far, certainly nothing comparable to the rich available theory that focuses on the critical point itself.

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For instance, he established in [1] a level-by-level correspondence between the $C^{(n)}$-extendible cardinals and *Vopěnka's Principle* (VP), where the latter is a well-known reflection principle, having high consistency strength. Subsequently, the hierarchies of $C^{(n)}$-cardinals were further studied and extended by the author in [18] and in [20] (see also [15] for another related work). Among these hierarchies, the notion of $C^{(n)}$-extendibility is of particular interest since it has found several applications in other mathematical contexts, such as category and homotopy theory (see [2] and [3]), as well as model theory (see the recent [4]).

Nevertheless, many set-theoretic questions regarding the various $C^{(n)}$-cardinals have remained unanswered. For instance, and except for some special cases (see [18] and [20]), it is not generally clear what kind of forcing constructions preserve or destroy a given $C^{(n)}$-cardinal, such as a $C^{(n)}$-supercompact (for $n \geqslant 1$) or a $C^{(n)}$-extendible (for $n > 1$).

For example, the following question concludes [18]:

QUESTION 1.1. *Suppose that κ is $C^{(n)}$-extendible, for some $n > 1$. Can we force the* GCH *while preserving the $C^{(n)}$-extendibility of κ?*

Furthermore, Bagaria has also considered a variant of $C^{(n)}$-extendibility, called $C^{(n)+}$-extendibility, that has served as an assumption in some of the applications obtained in [3].[3] However, the annoying issue of whether these two notions coincide had remained open. Some progress towards its resolution was made by Bagaria and Brooke-Taylor (see, e.g., Propositions 14 and 15 in [2]), but the following question was stated in [2] as an open problem, for $n > 1$:

QUESTION 1.2. *Is it consistent to have a $C^{(n)}$-extendible cardinal that is not $C^{(n)+}$-extendible?*

In this present note, we start by giving the necessary preliminaries, together with an overview of earlier related work, in Section 2. In Section 3, we first establish results in the direction of exactly controlling (properties of) the targets of $C^{(n)}$-extendibility embeddings; we view this as an advance towards building some relevant theory regarding the *images* of appropriate large cardinal embeddings. In particular, we completely resolve Question 1.2 (negatively) by showing that, in fact, the two hierarchies coincide. At the same time, we adapt our arguments in the context of $C^{(n)}$-ultrahuge cardinals as well. Subsequently, we give a characterization of $C^{(n)}$-extendible cardinals in terms of elementary embeddings between the H_λ's. This brings us to Section 4, where we show that $C^{(n)}$-extendible cardinals carry their own Laver functions (while we also hint at a similar result for $C^{(n)}$-ultrahugeness). This is the first known instance of a $C^{(n)}$-cardinal notion having this desired feature. In Section 5, we use the aforementioned characterization of $C^{(n)}$-extendibility in order to answer Question 1.1 affirmatively, arguing that, after forcing with the standard class iteration that forces the global GCH in the universe, every $C^{(n)}$-extendible cardinal is preserved. Finally, in Section 6, we briefly give some (easy) observations regarding the issue of separating levels of $C^{(n)}$-extendibility.

[3]Note, though, that there is a slight divergence in terminology: $C^{(n)+}$-extendibility is called $C(n)$-extendibility in [3]. Nevertheless, this latter term was later abandoned by Bagaria in his general study of $C^{(n)}$-cardinals, and the term $C^{(n)+}$-extendibility has been in use ever since.

§2. Preliminaries.

2.1. Notation. Our notation and terminology are mostly standard; we refer the reader to [12] or [14] for an account of all undefined set-theoretic notions. We write ON for the class of all ordinals. For any set x, we write $\mathrm{rk}(x)$ for the rank of x. If κ is an (infinite) cardinal, we let H_κ stand for the collection of all sets whose transitive closure has size less than κ. We denote by GCH the *Generalized Continuum Hypothesis*; i.e., the assertion that, for every infinite cardinal κ, we have that $2^\kappa = \kappa^+$.

For every natural number n, we let $C^{(n)}$ denote the closed and unbounded proper class of ordinals α that are Σ_n-correct in V, that is, ordinals α such that V_α is a Σ_n-elementary substructure of V (denoted by $V_\alpha \prec_n V$). Note that $C^{(0)} = ON$. For every $n \geqslant 1$, the statement "$\alpha \in C^{(n)}$" is expressible by a Π_n-formula (but *not* by any Σ_n-formula). This is proven by induction on n, with the base case arising from a characterization of $\alpha \in C^{(1)}$ as those uncountable cardinals α for which $V_\alpha = H_\alpha$ (see [1]).

For any set of ordinals A, we let $\sup(A)$ denote its supremum and we let $\mathrm{Lim}(A)$ denote the collection of its limit points, that is, $\mathrm{Lim}(A) = \{\xi : \sup(A \cap \xi) = \xi\}$. Given a limit ordinal α with $\mathrm{cf}(\alpha) > \omega$ and some $C \subseteq \alpha$, we say that C is a *club* in α if $\sup(C) = \alpha$ and $\alpha \cap \mathrm{Lim}(C) \subseteq C$; moreover, we say that C is a β-*club* in α, for some regular $\beta < \mathrm{cf}(\alpha)$, if $\sup(C) = \alpha$ and $\{\xi \in \alpha \cap \mathrm{Lim}(C) : \mathrm{cf}(\xi) = \beta\} \subseteq C$. Likewise, if $I \subseteq \mathrm{cf}(\alpha)$ is an ordinal interval, then C is called I-*club* in α if it is β-club in α, for all regular $\beta \in I$.

Given any function f and any $A \subseteq dom(f)$, we let $f \restriction A$ denote the restriction of f to A; moreover, we let $f``A$ denote the pointwise image of $A$ under $f$, i.e., $f``A = \{f(x) : x \in A\}$. If $\kappa \leqslant \lambda$ are (infinite) cardinals, we let $\mathcal{P}_\kappa\lambda = \{x \subseteq \lambda : |x| < \kappa\}$. We use the three-dot notation in order to indicate partial functions, that is, $f \vdots X \longrightarrow Y$ means that $dom(f) \subseteq X$, with the inclusion possibly being proper.

If $\mathbb{P}$ is a forcing poset, we write $V^{\mathbb{P}}$ for the universe of $\mathbb{P}$-names. If κ, λ are regular cardinals, we let $\mathrm{Add}(\kappa, \lambda)$ denote the poset consisting of partial functions $p \vdots \lambda \times \kappa \longrightarrow 2$ with $|p| < \kappa$; as usual, the ordering is given by reversed inclusion.

If j is a nontrivial elementary embedding, we write $\mathrm{cp}(j)$ for its critical point. Following the standard practice, whenever we lift embeddings to forcing extensions we use the same letter j for the lifted version of the embedding.

Finally, we will need the following standard facts regarding definability and correctness of supercompact and of extendible cardinals. The statement "κ is supercompact" is Π_2-definable (see the discussion after Exercise 22.8 in [14]); moreover, if κ is supercompact, then $\kappa \in C^{(2)}$ (i.e., every supercompact cardinal is Σ_2-correct—see Proposition 22.3 in [14]). The statement "κ is extendible" is Π_3-definable (see the hint of Exercise 23.9 in [14]); moreover, if κ is extendible, then $\kappa \in C^{(3)}$ (i.e., every extendible cardinal is Σ_3-correct—see Proposition 23.10 in [14]).

2.2. $C^{(n)}$-extendible cardinals. The following definition is due to Bagaria.[4] As usual in the context of $C^{(n)}$-cardinals, this is actually a schema of definitions, one for each meta-theoretic natural number $n \geqslant 1$.

[4] For a comprehensive treatment of $C^{(n)}$-extendible cardinals, see [1] and [18].

Definition 2.1 ([1]). We say that a cardinal κ is *λ-$C^{(n)}$-extendible*, for some $\lambda > \kappa$, if there exists some θ and an elementary embedding $j : V_\lambda \longrightarrow V_\theta$ with $\operatorname{cp}(j) = \kappa$, $j(\kappa) > \lambda$ and $j(\kappa) \in C^{(n)}$. Moreover, we say that κ is *$C^{(n)}$-extendible*, if it is λ-$C^{(n)}$-extendible for all $\lambda > \kappa$.

We note that, by Proposition 3.3 in [1], a cardinal is extendible if and only if it is $C^{(1)}$-extendible. In terms of definability of this hierarchy, for every $n \geqslant 1$, the statement "κ is λ-$C^{(n)}$-extendible" is Σ_{n+1}-expressible; hence, for every $n \geqslant 1$, the statement "κ is $C^{(n)}$-extendible" is Π_{n+2}-expressible (see Section 3 in [1]). In terms of correctness, for every $n \geqslant 1$, if the cardinal κ is $C^{(n)}$-extendible then $\kappa \in C^{(n+2)}$ (cf. Proposition 3.4 in [1]). Moreover, the hierarchy of $C^{(n)}$-extendible cardinals is proper; for every $n \geqslant 1$, the least $C^{(n)}$-extendible cardinal is below the least $C^{(n+1)}$-extendible cardinal, assuming both exist (cf. Proposition 3.5 in [1]).

As an indication of the strength of these notions, we mention that if there is a $C^{(2)}$-extendible cardinal, then there are unboundedly many supercompacts in the universe. To see this, let κ be $C^{(2)}$-extendible and note that $\kappa \in C^{(4)}$ (by Proposition 3.4 in [1]). Meanwhile, the statement "there are unboundedly many supercompact cardinals" is easily seen to be Π_4-expressible (given the Π_2-definability of supercompactness). By standard facts (see, for instance, Propositions 23.6 and 23.7 in [14]), this statement holds in V_κ and, thus, holds in the universe as well, due to the correctness of κ.

In a similar manner, using the analogous definability and correctness properties, we have that if there is a $C^{(n+2)}$-extendible cardinal (for $n \geqslant 1$), then there are unboundedly many $C^{(n)}$-extendibles in the universe (cf. Proposition 3.6 in [1]).

The following variant of $C^{(n)}$-extendibility is of particular interest:

Definition 2.2 ([1]). We say that a cardinal κ is *λ-$C^{(n)+}$-extendible*, for some $\lambda > \kappa$ with $\lambda \in C^{(n)}$, if there is some $\theta \in C^{(n)}$ and an elementary embedding $j : V_\lambda \longrightarrow V_\theta$ with $\operatorname{cp}(j) = \kappa$, $j(\kappa) > \lambda$ and $j(\kappa) \in C^{(n)}$. Moreover, we say that κ is *$C^{(n)+}$-extendible*, if it is λ-$C^{(n)+}$-extendible for all $\lambda > \kappa$ with $\lambda \in C^{(n)}$.

Every $C^{(n)+}$-extendible cardinal is $C^{(n)}$-extendible; see the relevant discussion in Section 4 of [2]. As noted in [1], the two notions coincide when $n = 1$. In Section 3, we shall generalize this to all n, showing that the two hierarchies completely coincide.

Note that $C^{(n)}$-extendibility, following the traditional definition of usual extendibility, is witnessed locally by set embeddings between rank initial segments of the universe. Building on Bagaria's work, we further studied $C^{(n)}$-extendible cardinals in [18]. In particular, we obtained a characterization of $C^{(n)}$-extendibility in terms of class elementary embeddings, as follows.

Definition 2.3 ([18]). A cardinal κ is called *jointly λ-supercompact and θ-superstrong*, for some $\lambda, \theta \geqslant \kappa$, if there is an elementary embedding $j : V \longrightarrow M$ with M transitive, $\operatorname{cp}(j) = \kappa$, $j(\kappa) > \lambda$, ${}^\lambda M \subseteq M$ and $V_{j(\theta)} \subseteq M$.

For the global notion, we say that κ is jointly supercompact and θ-superstrong, for some fixed $\theta \geqslant \kappa$, if it is jointly λ-supercompact and θ-superstrong, for every $\lambda \geqslant \kappa$; moreover, we say that κ is jointly supercompact and superstrong if it is jointly λ-supercompact and λ-superstrong, for every $\lambda \geqslant \kappa$.

The following fact has been mentioned before (see, e.g., the discussion after Definition 2.24 in [18]); for completeness, let us now provide a proof of it.

FACT 2.4. *If κ is the least supercompact, then κ is not jointly λ-supercompact and κ-superstrong, for any λ.*

PROOF. Let κ be the least supercompact and, aiming for a contradiction, suppose that, for some λ, there is an elementary embedding $j : V \longrightarrow M$ with M transitive, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$, ${}^{\lambda}M \subseteq M$ and $V_{j(\kappa)} \subseteq M$. The fact that j is κ-superstrong (i.e., the fact that $V_{j(\kappa)} \subseteq M$) implies that $j(\kappa) \in C^{(1)}$ (see Proposition 2.2 in [1]). Hence, the supercompactness of κ, which is a Π_2-expressible statement, reflects from V down to $V_{j(\kappa)}$. Then, by elementarity, we get that there is $\alpha < \kappa$ such that $V_\kappa \models$ "α is supercompact". It follows that α is an actual supercompact cardinal (i.e., one in V), again due to the Π_2-definability of supercompactness and the correctness of κ. This contradicts the minimality of κ and concludes the proof. ⊣

The $C^{(n)}$-version of the previous definition is obtained in a straightforward manner, by appending the additional requirement that $j(\kappa) \in C^{(n)}$, both for the local and for the global notion. Then, for every $n \geqslant 1$:

THEOREM 2.5 ([18]). *A cardinal κ is $C^{(n)}$-extendible if and only if it is jointly $C^{(n)}$-supercompact and κ-superstrong if and only if it is jointly $C^{(n)}$-supercompact and superstrong.*

The previous theorem is stated as Corollary 2.31 in [18] (see also its subsequent remarks, as well as Theorem 2.28 in [18] for a level-by-level correspondence).

As we will be mainly working with the above (alternative) characterization of $C^{(n)}$-extendibility, let us now say a few words regarding the complexity of describing embeddings that are jointly λ-$C^{(n)}$-supercompact and λ-superstrong via appropriate (long) extenders. For every fixed $n \geqslant 1$:

LEMMA 2.6. *The statement "κ is jointly λ-$C^{(n)}$-supercompact and λ-superstrong" is Σ_{n+1}-expressible using extenders.*

PROOF. First of all, we note that there are various known ways in which the existence of a jointly λ-$C^{(n)}$-supercompact and λ-superstrong embedding can be captured, i.e., formalized, via the existence of appropriate extenders. For example, Corollary 2.32 (and its subsequent remarks) in [18] gives one such way, using ordinary (but long) extenders. For another example, the detailed discussion appearing in Section 5 of [1] explains how one can use extenders of the Martin-Steel form in order to capture λ-$C^{(n)}$-supercompactness—in fact, a formal characterization is given there in terms of such extenders (see also the statement of Theorem 2.20 in [18]).

Now, given any such extender E that is jointly λ-$C^{(n)}$-supercompact and λ-superstrong for κ (i.e., such that its associated embedding j_E is), we may verify this fact about E inside V_μ, for some large enough cardinal $\mu \in C^{(n)}$ (e.g., we may pick $\mu \in C^{(n)}$ with $\mathrm{cf}(\mu)$ sufficiently above all the relevant information). Such V_μ correctly verifies the fact that E is an extender whose associated embedding j_E is jointly λ-$C^{(n)}$-supercompact and λ-superstrong for κ. More concretely, let us fix some formula $\chi(\kappa, \lambda, E)$ asserting that "the extender E is jointly λ-supercompact and λ-superstrong for κ" (using, for instance, the formal characterization given in Section

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5 of [1]). Then, for any $\lambda > \kappa$, the statement "κ is jointly λ-$C^{(n)}$-supercompact and λ-superstrong" can be expressed, e.g., as follows:

$$(\exists \mu \in C^{(n)})(\exists E \in V_\mu)(\mathrm{cf}(\mu) > \beth_{\mathrm{rk}(E)} + \beth_\lambda \wedge V_\mu \models (\chi(\kappa, \lambda, E) \wedge j_E(\kappa) \in C^{(n)})),$$

which is easily seen to be Σ_{n+1}-expressible (in the parameters κ and λ). We note that the crucial contribution to the complexity of this statement comes from the requirement "$\mu \in C^{(n)}$", which is Π_n-expressible. ⊣

The above lemma gives us, in particular, an alternative and very useful way of expressing (levels of) $C^{(n)}$-extendibility (mindful of the level-by-level correspondence given by Theorem 2.28 in [18]) via the existence of appropriate extenders.

Let us also recall that for $n = 1$, that is, for ordinary extendibility, we have (yet) another characterization in terms of elementary embeddings between the H_λ's:

Theorem 2.7 ([17]). *A cardinal κ is extendible if and only if for all $\lambda = \beth_\lambda \geqslant \kappa$, there exists some cardinal μ and an elementary embedding $j : H_{\lambda^+} \longrightarrow H_{\mu^+}$ with* $\mathrm{cp}(j) = \kappa$ *and* $j(\kappa) > \lambda + 1$.

In [17], this theorem is stated as Corollary 1.4 and it is subsequently used in order to show that every extendible cardinal is preserved by the standard class iteration that forces the global GCH in the universe (cf. Theorem 2.2 in [17]).

2.3. $C^{(n)}$-ultrahuge cardinals. Ultrahuge cardinals and their $C^{(n)}$-versions were recently introduced by the author, as a natural strengthening of the usual superhuge cardinals. Let us recall the relevant definition:

Definition 2.8 ([20]). A cardinal κ is called *λ-ultrahuge*, for some $\lambda \geqslant \kappa$, if there exists an elementary embedding $j : V \longrightarrow M$ with M transitive, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$, ${}^{j(\kappa)}M \subseteq M$ and $V_{j(\lambda)} \subseteq M$. Moreover, we say that κ is *ultrahuge*, if it is λ-ultrahuge for all $\lambda \geqslant \kappa$.

For any given $n \geqslant 1$, the $C^{(n)}$-version of ultrahugeness is defined accordingly, by appending—as expected—the additional requirement that $j(\kappa) \in C^{(n)}$.

Clearly, a cardinal is ultrahuge if and only if it is $C^{(1)}$-ultrahuge. As discussed in [20], if κ is $C^{(n)}$-ultrahuge then it is $C^{(n)}$-extendible and, thus, $\kappa \in C^{(n+2)}$ as well. In addition, for every $n \geqslant 1$, the statement "κ is λ-$C^{(n)}$-ultrahuge" is Σ_{n+1}-expressible and, hence, the statement "κ is $C^{(n)}$-ultrahuge" is Π_{n+2}-expressible. Moreover, we showed in [20] that the $C^{(n)}$-ultrahuge cardinals form a proper hierarchy that refines the usual large cardinal hierarchy between the well-known notions of superhugeness and almost 2-hugeness; see Theorem 4.3 (and its subsequent remarks) in [20].

Let us now turn to our current treatment of $C^{(n)}$-extendible (and of $C^{(n)}$-ultrahuge) cardinals, for $n \geqslant 1$.

§3. Controlling targets.

We first establish results in the direction of exactly controlling (properties of) the targets of $C^{(n)}$-extendibility embeddings. We start with $n = 2$.

Proposition 3.1. *Suppose that κ is $C^{(2)}$-extendible. Then, for all $\lambda > \kappa$, there is some $\theta > \lambda$ and an elementary embedding $j : V \longrightarrow M$ with M transitive,* $\mathrm{cp}(j) = \kappa$,

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$j(\kappa) > \theta$, ${}^{\theta}M \subseteq M$, $V_{j(\theta)} \subseteq M$ and such that both $j(\kappa)$ and $j(\theta)$ are supercompact cardinals. Moreover, θ may be taken to be an inaccessible cardinal that belongs to $C^{(2)}$.

Proof. Let κ be a $C^{(2)}$-extendible cardinal and fix some $\lambda > \kappa + 1$. Let $\theta > \lambda$ be any $C^{(2)}$ cardinal that is the target of some λ-$C^{(2)}$-extendibility embedding for κ. That is, let $h : V \longrightarrow N$ be an elementary embedding with N transitive, $\mathrm{cp}(h) = \kappa$, $h(\kappa) > \lambda$, ${}^{\lambda}N \subseteq N$, $V_{h(\lambda)} \subseteq N$ and $h(\kappa) = \theta \in C^{(2)}$. Of course, θ is inaccessible.

Let $\mathcal{U}$ be the usual normal measure on κ that is derived from h and note that $\mathcal{U} \in V_{\kappa+2} \subseteq V_{\lambda}$. By elementarity, $\mathcal{W} = h(\mathcal{U})$ is a normal measure on θ in the sense of N. But note that $\mathcal{W} \in V_{\theta+2} \subseteq V_{h(\lambda)} \subseteq N$, so $\mathcal{W}$ is indeed a normal measure on θ (i.e., in V). Now, a standard reflection argument shows that the set $\{\alpha < \kappa : V_{\kappa} \models \text{“}\alpha \text{ is supercompact”}\}$ belongs to $\mathcal{U}$; thus, by elementarity, the set $\{\alpha < \theta : V_{\theta} \models \text{“}\alpha \text{ is supercompact”}\}$ belongs to $\mathcal{W}$. Moreover, note that $\theta \in C^{(2)}$ both in V and in N; the latter because, by elementarity, $\theta = h(\kappa)$ is $C^{(2)}$-extendible in N and, thus, it belongs to $C^{(2)}$ (indeed $C^{(4)}$) by Proposition 3.4 in [1]. Consequently, and since being supercompact is Π_2-expressible, for every $\alpha < \theta$ we have that α is supercompact in V if and only if it is supercompact in N if and only if it is supercompact in V_{θ}.

For this choice of θ, let $j : V \longrightarrow M$ be an elementary embedding witnessing the $(\theta + 2)$-$C^{(2)}$-extendibility of κ, i.e., M is transitive, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \theta + 2$, ${}^{\theta}M \subseteq M$, $V_{j(\theta)+2} \subseteq M$ and $j(\kappa) \in C^{(2)}$. Note that $j(\theta)$ is inaccessible. Moreover, note that $\theta \in C^{(2)}$ in M as well: since being in $C^{(2)}$ is Π_2-expressible, we have that $V_{j(\kappa)} \models \theta \in C^{(2)}$; thus, and since $M \models j(\kappa) \in C^{(4)}$ (by Proposition 3.4 in [1]), it follows that $M \models \theta \in C^{(2)}$.

Since $\mathcal{W}$ is a normal measure on θ, we have that $j(\mathcal{W})$ is a normal measure on $j(\theta)$ in the sense of M. But notice that $j(\mathcal{W}) \in V_{j(\theta)+2} \subseteq M$, from which we get that $j(\mathcal{W})$ is indeed a normal measure on $j(\theta)$ in V. In addition, by elementarity and the above discussion, we have that

$$D = \{\alpha < j(\theta) : M \models \text{“}\alpha \text{ is supercompact”}\} \in j(\mathcal{W}).$$

At this point, we momentarily pause the proof in order to show the following lemma. As this lemma is intended to be used in other similar proofs further below in this note, we state it in the format that will also serve our future purposes.

Lemma 3.2. *Suppose that $h : V \longrightarrow N$ is a jointly λ-supercompact and λ-superstrong embedding for κ, for some $\lambda > \kappa + 1$. Let $\mathcal{U}$ be the usual normal measure on κ that is derived from h, let $\theta = h(\kappa)$ and let $\mathcal{W} = h(\mathcal{U})$. Furthermore, suppose that $j : V \longrightarrow M$ is a jointly $(\theta + 2)$-supercompact and $(\theta + 2)$-superstrong embedding for κ. In this situation, if $A \subseteq j(\theta)$ is a $(j(\kappa), j(\theta))$-club subset of $j(\theta)$, then we have that $A \in j(\mathcal{W})$.*

Proof of lemma. Suppose that we are in the situation of the lemma. Note that, by the closure of the models N and M (as already explained in the proof above), we have that $\mathcal{W}$ is a normal measure on the inaccessible θ, while $j(\mathcal{W})$ is a normal measure on the inaccessible $j(\theta)$ (i.e., as all these are computed in V).

Let $\varphi(\alpha, \beta, X)$ be the statement “every (α, β)-club subset of β belongs to X”. To establish the lemma, note that it is enough to verify that $\varphi(j(\kappa), j(\theta), j(\mathcal{W}))$ holds in M, since $V_{j(\theta)+2} \subseteq M$.

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By elementarity of j, we have that $M \models \varphi(j(\kappa), j(\theta), j(\mathcal{W}))$ if and only if $\varphi(\kappa, \theta, \mathcal{W})$ holds in V. But the latter is true if and only if $\varphi(\kappa, \theta, \mathcal{W})$ holds in N, because $V_{\theta+2} \subseteq V_{h(\lambda)} \subseteq N$. Therefore, by elementarity of h now, we have that $\varphi(\kappa, \theta, \mathcal{W}) = \varphi(\kappa, h(\kappa), h(\mathcal{U}))$ holds in N if and only if $S \in \mathcal{U}$, where

$$S = \{\alpha < \kappa : \varphi(\alpha, \kappa, \mathcal{U})\}.$$

Fix $\alpha < \kappa$ and let $C \subseteq \kappa$ be an (α, κ)-club in κ. It is enough to check that $C \in \mathcal{U}$ or, equivalently, that $\kappa \in h(C)$. By elementarity, $h(C)$ is an $(\alpha, h(\kappa))$-club subset of $h(\kappa)$. But note that $C \subseteq h(C)$, with C being unbounded in κ. Thus, $h(C)$ is unbounded in κ and therefore, since $h(C)$ is $(\alpha, h(\kappa))$-club in $h(\kappa)$, we get that $\kappa \in h(C)$, as desired. ⊣

Returning to the proof of the proposition, we now perform an elementary chain construction in order to build various factor embeddings of j, in such a way that each witnesses, in M, the θ-$C^{(2)}$-extendibility of κ and, moreover, is such that the image of θ is supercompact in the sense of M. For more examples of such constructions and relevant details, the interested reader may consult [18].

We fix an initial limit ordinal $\beta_0 \in (j(\lambda), j(\theta))$ and we let

$$X_0 = \{j(f)(j``\theta, x) : f \in V, f : \mathcal{P}_\kappa\theta \times V_\theta \longrightarrow V, x \in V_{\beta_0}\} \prec M.$$

For any $\xi + 1 < j(\theta)$, given β_ξ and X_ξ, we let $\beta_{\xi+1} = \sup(X_\xi \cap j(\theta)) + \omega$ and

$$X_{\xi+1} = \{j(f)(j``\theta, x) : f \in V, f : \mathcal{P}_\kappa\theta \times V_\theta \longrightarrow V, x \in V_{\beta_{\xi+1}}\} \prec M.$$

If $\xi < j(\theta)$ is limit and we have already defined β_α and X_α for every $\alpha < \xi$, we let $\beta_\xi = \sup_{\alpha<\xi} \beta_\alpha$ and $X_\xi = \bigcup_{\alpha<\xi} X_\alpha \prec M$. This concludes the description of our elementary chain.

For any $\gamma < j(\theta)$ with $\mathrm{cf}(\gamma) > \theta$, let us consider $\beta_\gamma = \sup_{\alpha<\gamma} \beta_\alpha$ and the corresponding structure $X_\gamma = \bigcup_{\alpha<\gamma} X_\alpha$, that is,

$$X_\gamma = \{j(f)(j``\theta, x) : f \in V, f : \mathcal{P}_\kappa\theta \times V_\theta \longrightarrow V, x \in V_{\beta_\gamma}\} \prec M.$$

The inaccessibility of $j(\theta)$ implies that $\beta_\gamma < j(\theta)$, where note that $\mathrm{cf}(\beta_\gamma) = \mathrm{cf}(\gamma) > \theta$. We then let $\pi_\gamma : X_\gamma \cong M_\gamma$ be the Mostowski collapse and consider the composed map $j_\gamma = \pi_\gamma \circ j : V \longrightarrow M_\gamma$, producing a commutative diagram of elementary embeddings as usual (with $k_\gamma = \pi_\gamma^{-1}$).

Now, for any such γ, the embedding j_γ is (a factor of the initial j and) jointly θ-supercompact and θ-superstrong for κ. To see this, we employ a straightforward adaptation of (the proof of) Proposition 2.18 in [18]; namely, in a totally analogous manner, we initially get the following representation of the model M_γ:

$$M_\gamma = \{j_\gamma(f)(j_\gamma``\theta, x) : f \in V, f : \mathcal{P}_\kappa\theta \times V_\theta \longrightarrow V, x \in V_{\beta_\gamma}\}.$$

From this, we can now deduce that ${}^\theta M_\gamma \subseteq M_\gamma$, i.e., that j_γ is θ-supercompact for κ. This closure under θ-sequences, which is proven exactly as in the proof of Proposition 2.18 in [18], essentially comes from the fact that the initial j was θ-supercompact and that $\mathrm{cf}(\beta_\gamma) = \mathrm{cf}(\gamma) > \theta$, by choice of γ. Moreover, we have that $\mathrm{cp}(j_\gamma) = \kappa$, $j_\gamma(\kappa) = j(\kappa)$ and

$$j_\gamma(\theta) = \mathrm{cp}(k_\gamma) = \sup(X_\gamma \cap j(\theta)) = \beta_\gamma.$$

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Finally, from the above representation of M_γ and the fact that the initial j was θ-superstrong, it easily follows that j_γ is θ-superstrong for κ as well (i.e., $V_{j_\gamma(\theta)} \subseteq M_\gamma$).

Furthermore, again by the inaccessibility of $j(\theta)$, for every $\alpha < j(\theta)$ we have that $j_\gamma(\alpha) < j(\theta)$; hence, the relevant (either Martin-Steel or ordinary but long) extender E that is derived from j_γ and that witnesses its joint θ-supercompactness and θ-superstrongness actually belongs to $V_{j(\theta)} \subseteq M$. Indeed, M certainly thinks that "E is jointly θ-supercompact and θ-superstrong for κ" and, moreover, it correctly computes the values $j_E(\kappa) = j(\kappa)$ and $j_E(\theta) = j_\gamma(\theta) = \beta_\gamma$.

To summarize, for every $\gamma < j(\theta)$ with $\mathrm{cf}(\gamma) > \theta$, we can construct an embedding j_γ that is a factor of j, that is jointly θ-supercompact and θ-superstrong for κ, that is witnessed by some (long) extender inside M and, moreover, whose target $j_\gamma(\theta)$ we can sufficiently control, as explained above.

Now consider the collection of all possible targets $j_\gamma(\theta)$ arising as above (this collection is included in $j(\theta)$), for the various choices of $\gamma < j(\theta)$ with $\mathrm{cf}(\gamma) > \theta$. It is easy to see that this collection is actually a $[\theta^+, j(\theta))$-club in $j(\theta)$, i.e., it is closed under sequences of length ξ, for every (regular) $\xi \in [\theta^+, j(\theta))$.[5] Therefore, appealing to Lemma 3.2, we get that this collection in fact belongs to $j(\mathcal{W})$ and, hence, it has nonempty intersection with the set D displayed before that lemma. That is, there must exist some $\gamma < j(\theta)$ with $\mathrm{cf}(\gamma) > \theta$ for which the corresponding target $j_\gamma(\theta)$ is supercompact in the sense of M.

We now have all the necessary ingredients for concluding the proof. First of all, recalling (the proof of) Lemma 2.6, note that the statement "there exists an extender E that witnesses the joint θ-supercompactness and θ-superstrongness of κ and such that both $j_E(\kappa)$ and $j_E(\theta)$ are supercompact cardinals" is Σ_3-expressible in the parameters κ and θ. This is because supercompactness is Π_2-expressible, which means that, in the setting of Lemma 2.6, it is enough to require that $\mu \in C^{(2)}$ (and with sufficiently large cofinality) in order to correctly verify, inside V_μ, that $j_E(\kappa)$ and $j_E(\theta)$ are supercompact cardinals. Moreover, by all the previous discussion, this statement is true in M.

But since $j(\kappa)$ is Σ_3-correct in M (in fact, it is even Σ_4-correct in M), it follows that the aforementioned Σ_3-expressible statement reflects to $V_{j(\kappa)}$. Then, since $j(\kappa)$ is Σ_2-correct in V, we get that the same actually holds in V, i.e., there exists some (extender) embedding j_E that is jointly θ-supercompact and θ-superstrong for κ and such that the targets $j_E(\kappa)$ and $j_E(\theta)$ are both supercompact cardinals. $\dashv$

In particular, the previous proposition confirms the (known) fact that if there exists a $C^{(2)}$-extendible cardinal, then there are unboundedly many supercompact cardinals in the universe.[6] Moreover, it leads us to the following result:

COROLLARY 3.3. *If κ is $C^{(2)}$-extendible, then it is $C^{(2)+}$-extendible.*

---

[5] See the analogous Proposition 2.8 in [18] for more details.

[6] This was already explained in the preliminaries. On the other hand, note that we cannot get unboundedly many extendibles: if κ is $C^{(2)}$-extendible and $\lambda > \kappa$ is the least extendible above κ, then $V_\lambda \models$ "κ is $C^{(2)}$-extendible" (since this is a Π_4-expressible statement that reflects from V down to the Σ_3-correct cardinal λ) and, also, $V_\lambda \models$ "κ is the maximum extendible" (due to the Σ_3-correctness of λ and the fact that it is the least extendible above κ). In other words, it is consistent that there exists a $C^{(2)}$-extendible cardinal without any extendibles above it.

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PROOF. Let κ be $C^{(2)}$-extendible and fix some $\lambda > \kappa$ with $\lambda \in C^{(2)}$. By Proposition 3.1, fix some inaccessible and Σ_2-correct $\theta > \lambda$ and an elementary embedding $j : V \longrightarrow M$ with M transitive, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \theta$, ${}^{\theta}M \subseteq M$, $V_{j(\theta)} \subseteq M$ and such that $j(\kappa)$ and $j(\theta)$ are both supercompact cardinals.

Now consider the restricted embedding $j \restriction V_\theta : V_\theta \longrightarrow V_{j(\theta)}$, which is clearly θ-extendible for κ. Moreover, since $V_\theta \models \lambda \in C^{(2)}$, it follows by elementarity that $V_{j(\theta)} \models j(\lambda) \in C^{(2)}$. But the latter must be true in V, since $j(\theta)$ is supercompact and, thus, Σ_2-correct in V. Consequently, the (even more) restricted embedding $j \restriction V_\lambda : V_\lambda \longrightarrow V_{j(\lambda)}$ witnesses the λ-$C^{(2)+}$-extendibility of κ. ⊣

Thinking of Proposition 3.1 as our "base case", we now generalize. For $n \geqslant 1$:

THEOREM 3.4. *Suppose that κ is $C^{(n+2)}$-extendible. Then, for all $\lambda > \kappa$, there is some $\theta > \lambda$ and an elementary embedding $j : V \longrightarrow M$ with M transitive, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \theta$, ${}^{\theta}M \subseteq M$, $V_{j(\theta)} \subseteq M$ and such that both $j(\kappa)$ and $j(\theta)$ are $C^{(n)}$-extendible cardinals. Moreover, θ may be taken to be an inaccessible cardinal that belongs to $C^{(n+2)}$.*

PROOF. We follow a similar strategy as in the proof of Proposition 3.1. Suppose that κ is $C^{(n+2)}$-extendible, for some $n \geqslant 1$, and fix $\lambda > \kappa + 1$. Let $\theta > \lambda$ be any $C^{(n+2)}$ cardinal that is the target of some λ-$C^{(n+2)}$-extendibility embedding for κ. That is, let $h : V \longrightarrow N$ be an elementary embedding with N transitive, $\mathrm{cp}(h) = \kappa$, $h(\kappa) > \lambda$, ${}^{\lambda}N \subseteq N$, $V_{h(\lambda)} \subseteq N$ and $h(\kappa) = \theta \in C^{(n+2)}$. Of course, θ is inaccessible.

Let $\mathcal{U}$ be the usual normal measure on κ that is derived from h and note that $\mathcal{U} \in V_{\kappa+2} \subseteq V_\lambda$. Let $\mathcal{W} = h(\mathcal{U})$ and observe that $\mathcal{W} \in V_{\theta+2} \subseteq V_{h(\lambda)}$. Similarly to Proposition 3.1, we now get that $\mathcal{W}$ is a normal measure on θ (i.e., in V) and that the set $\{\alpha < \theta : V_\theta \models \text{"}\alpha \text{ is } C^{(n)}\text{-extendible"}\}$ belongs to $\mathcal{W}$. Moreover, note that $\theta \in C^{(n+2)}$ both in V and in N; the latter because, by elementarity, $\theta = h(\kappa)$ is $C^{(n)}$-extendible in N and, thus, it belongs to $C^{(n+2)}$ by Proposition 3.4 in [1]. Consequently, and since being $C^{(n)}$-extendible is Π_{n+2}-expressible, for every $\alpha < \theta$ we have that α is $C^{(n)}$-extendible in V if and only if it is $C^{(n)}$-extendible in N if and only if it is $C^{(n)}$-extendible in V_θ.

Next, for this choice of θ, we let $j : V \longrightarrow M$ be an elementary embedding witnessing the $(\theta + 2)$-$C^{(n+2)}$-extendibility of κ, i.e., M is transitive, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \theta + 2$, ${}^{\theta}M \subseteq M$, $V_{j(\theta)+2} \subseteq M$ and $j(\kappa) \in C^{(n+2)}$. Note that $j(\theta)$ is inaccessible and that, once again, we are in the situation of Lemma 3.2. Moreover, note that $\theta \in C^{(n+2)}$ in M as well: since being in $C^{(n+2)}$ is Π_{n+2}-expressible, we have that $V_{j(\kappa)} \models \theta \in C^{(n+2)}$; thus, and since $M \models j(\kappa) \in C^{(n+4)}$ (by Proposition 3.4 in [1]), it follows that $M \models \theta \in C^{(n+2)}$.

As in Proposition 3.1, we have that $j(\mathcal{W})$ is a normal measure on $j(\theta)$ (i.e., in the sense of V) and also:

$$D' = \{\alpha < j(\theta) : M \models \text{"}\alpha \text{ is } C^{(n)}\text{-extendible"}\} \in j(\mathcal{W}).$$

Now, we once more perform an elementary chain construction in order to build various factor embeddings of j, in such a way that each witnesses, in M, the θ-$C^{(n+2)}$-extendibility of κ and, moreover, is such that the image of θ is $C^{(n)}$-extendible in the sense of M. The definition of the elementary chain is exactly as in the proof of Proposition 3.1, hence we will not repeat it here.

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For any $\gamma < j(\theta)$ with $\mathrm{cf}(\gamma) > \theta$, let us again consider $\beta_\gamma = \sup_{\alpha<\gamma} \beta_\alpha$ and the corresponding structure $X_\gamma = \bigcup_{\alpha<\gamma} X_\alpha$, that is,

$$X_\gamma = \{j(f)(j``\theta, x) : f \in V, f : \mathcal{P}_\kappa\theta \times V_\theta \longrightarrow V, x \in V_{\beta_\gamma}\} \prec M.$$

The inaccessibility of $j(\theta)$ again implies that $\beta_\gamma < j(\theta)$, where $\mathrm{cf}(\beta_\gamma) = \mathrm{cf}(\gamma) > \theta$. We let $\pi_\gamma : X_\gamma \cong M_\gamma$ be the Mostowski collapse and consider the composed map $j_\gamma = \pi_\gamma \circ j : V \longrightarrow M_\gamma$, producing a commutative diagram of elementary embeddings as usual (with $k_\gamma = \pi_\gamma^{-1}$). As in the proof of Proposition 3.1, we employ the arguments from Proposition 2.18 in [18] in order to conclude, again, that, for every such γ, the embedding j_γ is jointly θ-supercompact and θ-superstrong for κ (and a factor of j) where, in fact, $\mathrm{cp}(j_\gamma) = \kappa$, $j_\gamma(\kappa) = j(\kappa)$ and $j_\gamma(\theta) = \mathrm{cp}(k_\gamma) = \sup(X_\gamma \cap j(\theta)) = \beta_\gamma$.

As before, by the inaccessibility of $j(\theta)$, the relevant extender E that is derived from j_γ and that witnesses its joint θ-supercompactness and θ-superstrongness actually belongs to $V_{j(\theta)} \subseteq M$. Indeed, M certainly thinks that "E is jointly θ-supercompact and θ-superstrong for κ" and, moreover, it correctly computes the values $j_E(\kappa) = j(\kappa)$ and $j_E(\theta) = j_\gamma(\theta) = \beta_\gamma$.

We now consider again the collection of all possible targets $j_\gamma(\theta)$ arising as above (this collection is included in $j(\theta)$), for the various choices of $\gamma < j(\theta)$ with $\mathrm{cf}(\gamma) > \theta$. As before, this collection is a $[\theta^+, j(\theta))$-club in $j(\theta)$ and so, by Lemma 3.2, it actually belongs to $j(\mathcal{W})$. Hence, it has nonempty intersection with the set D' displayed above, i.e., there is some $\gamma < j(\theta)$ with $\mathrm{cf}(\gamma) > \theta$ for which the corresponding target $j_\gamma(\theta)$ is $C^{(n)}$-extendible in the sense of M.

Now, again recalling (the proof of) Lemma 2.6, the statement "there exists an extender E witnessing the joint θ-supercompactness and θ-superstrongness of κ and such that both $j_E(\kappa)$ and $j_E(\theta)$ are $C^{(n)}$-extendible cardinals" is Σ_{n+3}-expressible in the parameters κ and θ. By all the previous discussion, this statement is true in M. But since $j(\kappa)$ is Σ_{n+3}-correct in M (in fact, it is even Σ_{n+4}-correct in M), this statement reflects to $V_{j(\kappa)}$. Then, since $j(\kappa)$ is Σ_{n+2}-correct in V, we get that the same actually holds in V, i.e., there is an (extender) embedding j_E that is jointly θ-supercompact and θ-superstrong for κ and such that the targets $j_E(\kappa)$ and $j_E(\theta)$ are both $C^{(n)}$-extendible cardinals. ⊣

In particular, for every $n \geqslant 1$, the previous theorem confirms the (known) fact that if there exists a $C^{(n+2)}$-extendible cardinal, then there are unboundedly many $C^{(n)}$-extendible cardinals in the universe (see Proposition 3.6 in [1]). Moreover, it leads us to the following result that answers Question 1.2 negatively, in showing that the notions of $C^{(n)}$-extendibility and of $C^{(n)+}$-extendibility coincide. Let us remark that, independently from our results and in the context of a different study, Gitman and Hamkins have very recently reached the same conclusion; see [10].

Corollary 3.5. *For every $n \geqslant 1$, if κ is $C^{(n)}$-extendible, then it is $C^{(n)+}$-extendible.*

Proof. The case $n = 1$ is discussed in [1], while the case $n = 2$ is Corollary 3.3. So fix some $n \geqslant 3$, suppose that κ is $C^{(n)}$-extendible and fix some $\lambda > \kappa$ with $\lambda \in C^{(n)}$. By Theorem 3.4, fix some inaccessible and Σ_n-correct $\theta > \lambda$ and an elementary embedding $j : V \longrightarrow M$ with M transitive, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \theta$, ${}^\theta M \subseteq M$, $V_{j(\theta)} \subseteq M$ and such that $j(\kappa)$ and $j(\theta)$ are both $C^{(n-2)}$-extendible cardinals. In particular, by Proposition 3.4 in [1], both $j(\kappa)$ and $j(\theta)$ belong to $C^{(n)}$.

Now consider the restricted embedding $j \restriction V_\theta : V_\theta \longrightarrow V_{j(\theta)}$, which is clearly θ-extendible for κ. Moreover, since $V_\theta \models \lambda \in C^{(n)}$, it follows by elementarity that $V_{j(\theta)} \models j(\lambda) \in C^{(n)}$. But the latter must be true in V, since $j(\theta) \in C^{(n)}$. Consequently, the (even more) restricted embedding $j \restriction V_\lambda : V_\lambda \longrightarrow V_{j(\lambda)}$ witnesses the λ-$C^{(n)+}$-extendibility of κ. ⊣

Let us remark that, in the light of Corollary 3.5, it now follows that the assumption of $C^{(n)+}$-extendibility that has been used in various results appearing in [3] and elsewhere has now been rendered redundant.[7] The situation is thus clarified in the sense that we only need to consider $C^{(n)}$-extendible cardinals, without any additional requirements on their witnessing embeddings, a notion that so far appears to be a rather robust and well-behaved one.

Before we briefly turn to the context of ultrahugeness, we also give a description of $C^{(n)}$-extendibility in terms of elementary embeddings between the H_λ's. This description will be crucially used in the Section 5, where we deal with the preservation of $C^{(n)}$-extendible cardinals by the GCH forcing iteration. The following is a straightforward adaptation of our corresponding result in [17], for each $n \geqslant 1$:

PROPOSITION 3.6. *Let κ be a cardinal and let $\lambda = \beth_\lambda > \kappa$. Then, κ is $(\lambda+1)$-$C^{(n)}$-extendible if and only if there is a cardinal μ and an elementary embedding $j : H_{\lambda^+} \longrightarrow H_{\mu^+}$ with* $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda + 1$ *and* $j(\kappa) \in C^{(n)}$.

PROOF. This is totally analogous to the proof of Proposition 1.3 in [17], by just appending everywhere the additional clause "$j(\kappa) \in C^{(n)}$". ⊣

Using Theorem 3.4, we obtain the following characterization.

PROPOSITION 3.7. *For every $n \geqslant 1$, a cardinal κ is $C^{(n+2)}$-extendible if and only if, for all $\lambda = \beth_\lambda > \kappa$, there is some μ and an elementary embedding $j : H_{\lambda^+} \longrightarrow H_{\mu^+}$ with* $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$ *and such that $j(\kappa)$ is a $C^{(n)}$-extendible cardinal.*

PROOF. Suppose that κ is $C^{(n+2)}$-extendible and fix some $\lambda = \beth_\lambda > \kappa$. By Theorem 3.4, fix some $\theta > \lambda$ and an elementary embedding $h : V \longrightarrow M$ with M transitive, $\mathrm{cp}(h) = \kappa$, $h(\kappa) > \theta$, ${}^\theta M \subseteq M$, $V_{h(\theta)} \subseteq M$ and such that $h(\kappa)$ is $C^{(n)}$-extendible. Then, exactly as in the proof of Proposition 1.3 in [17] and for $\mu = h(\lambda)$, we get an elementary embedding $j : H_{\lambda^+} \longrightarrow H_{\mu^+}$ with $\mathrm{cp}(j) = \kappa$ and $j(\kappa) = h(\kappa)$; thus, $j(\kappa) > \lambda$ and $j(\kappa)$ is a $C^{(n)}$-extendible cardinal.

Conversely, fix some $\lambda = \beth_\lambda > \kappa$ and let $j : H_{\lambda^+} \longrightarrow H_{j(\lambda)^+}$ be an elementary embedding with $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$ and such that $j(\kappa)$ is a $C^{(n)}$-extendible cardinal. Since $j(\kappa)$ is $C^{(n)}$-extendible, we have that $j(\kappa) \in C^{(n+2)}$. Therefore, by Proposition 3.6 above, we have that κ is $(\lambda+1)$-$C^{(n+2)}$-extendible. ⊣

We remark that, given our earlier results, the case $n = 0$ in Proposition 3.7 is totally analogous: one just replaces "$j(\kappa)$ is a $C^{(n)}$-extendible cardinal" by "$j(\kappa)$ is a supercompact cardinal" in the final clause of the proposition.

Towards concluding the current section, let us now briefly turn to $C^{(n)}$-ultrahuge cardinals. As it will become clear, we shall appropriately adapt Proposition 3.1 and Theorem 3.4 in the context of $C^{(n)}$-ultrahugeness embeddings. The arguments and

[7] Recalling that $C^{(n)+}$-extendibility is called $C(n)$-extendibility in [3].

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the results are totally parallel, hence we will skip several details. Once again, we start with $n = 2$:

PROPOSITION 3.8. *Suppose that κ is $C^{(2)}$-ultrahuge. Then, for all $\lambda > \kappa$, there is some $\theta > \lambda$ and an elementary embedding $j : V \longrightarrow M$ with M transitive, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \theta$, ${}^{j(\kappa)}M \subseteq M$, $V_{j(\theta)} \subseteq M$ and such that both $j(\kappa)$ and $j(\theta)$ are supercompact cardinals. Moreover, θ may be taken to be an inaccessible cardinal that belongs to $C^{(2)}$.*

PROOF. Let κ be a $C^{(2)}$-ultrahuge cardinal and fix some $\lambda > \kappa + 1$. Let $\theta > \lambda$ be any $C^{(2)}$ cardinal that is the target of some λ-$C^{(2)}$-ultrahugeness embedding for κ. That is, let $h : V \longrightarrow N$ be an elementary embedding with N transitive, $\mathrm{cp}(h) = \kappa$, $h(\kappa) > \lambda$, ${}^{h(\kappa)}N \subseteq N$, $V_{h(\lambda)} \subseteq N$ and $h(\kappa) = \theta \in C^{(2)}$. Of course, θ is inaccessible.

Let $\mathcal{U}$ be the usual normal measure on κ that is derived from h and let $\mathcal{W} = h(\mathcal{U})$. As before, $\mathcal{W}$ is indeed a normal measure on θ (i.e., in V) and, also, the set $\{\alpha < \kappa : V_\kappa \models \text{“}\alpha \text{ is supercompact”}\}$ belongs to $\mathcal{U}$. Thus, by elementarity, the set $\{\alpha < \theta : V_\theta \models \text{“}\alpha \text{ is supercompact”}\}$ belongs to $\mathcal{W}$. Moreover, note again that $\theta \in C^{(2)}$ both in V and in N. Consequently, and since being supercompact is Π_2-expressible, for every $\alpha < \theta$ we have that α is supercompact in V if and only if it is supercompact in N if and only if it is supercompact in V_θ.

For this choice of θ, let $j : V \longrightarrow M$ be an elementary embedding witnessing the $(\theta + 2)$-$C^{(2)}$-ultrahugeness of κ, i.e., M is transitive, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \theta + 2$, ${}^{j(\kappa)}M \subseteq M$, $V_{j(\theta)+2} \subseteq M$ and $j(\kappa) \in C^{(2)}$. Note that $j(\theta)$ is inaccessible and that, once again, we are in the situation of Lemma 3.2 (in fact, here we indeed have even more closure under sequences for the models N and M). In addition, notice that $\theta \in C^{(2)}$ in M as well. As in Proposition 3.1, $j(\mathcal{W})$ is indeed a normal measure on $j(\theta)$ in V and also:

$$D = \{\alpha < j(\theta) : M \models \text{“}\alpha \text{ is supercompact”}\} \in j(\mathcal{W}).$$

We again perform an elementary chain construction in order to build various factor embeddings of j, in such a way that each witnesses, in M, the θ-$C^{(2)}$-ultrahugeness of κ and, moreover, is such that the image of θ is supercompact in the sense of M. The definition of the chain is slightly different from the one in Proposition 3.1; this is because we need to adapt to the current context, where we are interested in closure under $j(\kappa)$-sequences for the target models. So, we fix an initial limit ordinal $\beta_0 \in (j(\lambda), j(\theta))$ and we let

$$X_0 = \{j(f)(j\text{“}j(\kappa), x) : f \in V,\ f : \mathcal{P}_{j(\kappa)}j(\kappa) \times V_\theta \longrightarrow V,\ x \in V_{\beta_0}\} \prec M.$$

For any $\xi + 1 < j(\theta)$, given β_ξ and X_ξ, we let $\beta_{\xi+1} = \sup(X_\xi \cap j(\theta)) + \omega$ and

$$X_{\xi+1} = \{j(f)(j\text{“}j(\kappa), x) : f \in V,\ f : \mathcal{P}_{j(\kappa)}j(\kappa) \times V_\theta \longrightarrow V,\ x \in V_{\beta_{\xi+1}}\} \prec M.$$

If $\xi < j(\theta)$ is limit and we have already defined β_α and X_α for every $\alpha < \xi$, we let $\beta_\xi = \sup_{\alpha<\xi} \beta_\alpha$ and $X_\xi = \bigcup_{\alpha<\xi} X_\alpha \prec M$. This concludes the description of our elementary chain.[8]

For any $\gamma < j(\theta)$ with $\mathrm{cf}(\gamma) > j(\kappa)$, we consider $\beta_\gamma = \sup_{\alpha<\gamma} \beta_\alpha$ and the corresponding structure $X_\gamma = \bigcup_{\alpha<\gamma} X_\alpha$, that is,

$$X_\gamma = \{j(f)(j\text{“}j(\kappa), x) : f \in V,\ f : \mathcal{P}_{j(\kappa)}j(\kappa) \times V_\theta \longrightarrow V,\ x \in V_{\beta_\gamma}\} \prec M.$$

[8]Note here the use of the modified “seed” $j\text{“}j(\kappa)$ (which certainly belongs to M).

The inaccessibility of $j(\theta)$ again implies that $\beta_\gamma < j(\theta)$, where $\mathrm{cf}(\beta_\gamma) = \mathrm{cf}(\gamma) > j(\kappa)$. We then let $\pi_\gamma : X_\gamma \cong M_\gamma$ be the Mostowski collapse and consider the composed map $j_\gamma = \pi_\gamma \circ j : V \longrightarrow M_\gamma$, producing a commutative diagram of elementary embeddings as usual (with $k_\gamma = \pi_\gamma^{-1}$). Now, due to our modified definition of the elementary chain, it is easy to check that, for every such γ, the embedding j_γ is θ-ultrahuge for κ (and a factor of j) where, in fact, $\mathrm{cp}(j_\gamma) = \kappa$, $j_\gamma(\kappa) = j(\kappa)$ and

$$j_\gamma(\theta) = \mathrm{cp}(k_\gamma) = \sup(X_\gamma \cap j(\theta)) = \beta_\gamma.$$

In addition, again by the inaccessibility of $j(\theta)$, for every $\alpha < j(\theta)$ we have that $j_\gamma(\alpha) < j(\theta)$; hence, the relevant extender E which is derived from j_γ and which witnesses its θ-ultrahugeness actually belongs to $V_{j(\theta)} \subseteq M$. Indeed, M certainly thinks that "E is θ-ultrahuge for κ" and, moreover, it correctly computes the values $j_E(\kappa) = j(\kappa)$ and $j_E(\theta) = j_\gamma(\theta) = \beta_\gamma$.

Now consider the collection of all possible targets $j_\gamma(\theta)$ arising as above (this collection is included in $j(\theta)$), for the various choices of $\gamma < j(\theta)$ with $\mathrm{cf}(\gamma) > j(\kappa)$. In the current setting, this collection is in fact a $(j(\kappa), j(\theta))$-club in $j(\theta)$, i.e., it is closed under sequences of length ξ, for every (regular) $\xi \in (j(\kappa), j(\theta))$. Therefore, appealing to Lemma 3.2, we get that this collection actually belongs to $j(\mathcal{W})$ and, hence, it has nonempty intersection with the set D displayed above, in the current proof. That is, there must exist some $\gamma < j(\theta)$ with $\mathrm{cf}(\gamma) > j(\kappa)$ for which the corresponding target $j_\gamma(\theta)$ is supercompact in the sense of M.

The rest of the argument now proceeds exactly as in Proposition 3.1, noting that the statement "there exists an extender E witnessing the θ-ultrahugeness of κ and such that both $j_E(\kappa)$ and $j_E(\theta)$ are supercompact cardinals" is Σ_3-expressible in the parameters κ and θ. This statement is true in M, hence it reflects to $V_{j(\kappa)}$. But since $j(\kappa)$ is Σ_2-correct in V, the desired conclusion follows. ⊣

Having dealt with our "base case", we now again generalize. For $n \geqslant 1$:

THEOREM 3.9. *Suppose that κ is $C^{(n+2)}$-ultrahuge. Then, for all $\lambda > \kappa$, there is some $\theta > \lambda$ and an elementary embedding $j : V \longrightarrow M$ with M transitive, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \theta$, ${}^{j(\kappa)}M \subseteq M$, $V_{j(\theta)} \subseteq M$ and such that both $j(\kappa)$ and $j(\theta)$ are $C^{(n)}$-ultrahuge cardinals. Moreover, θ may be taken to be an inaccessible cardinal that belongs to $C^{(n+2)}$.*

PROOF. Given our previous results in this section, this argument should now be clear. We proceed exactly as in the proof of Theorem 3.4, incorporating the necessary modifications (essentially, in the definition of the elementary chain) as we did for Proposition 3.8. Of course, we now consider the set:

$$D' = \{\alpha < j(\theta) : M \models \text{“}\alpha \text{ is } C^{(n)}\text{-ultrahuge”}\} \in j(\mathcal{W}),$$

from which we obtain a (factor) θ-ultrahuge embedding j_γ such that $j_\gamma(\theta)$ is a $C^{(n)}$-ultrahuge cardinal in the sense of M. Noting that the statement "there exists an extender E that witnesses the θ-ultrahugeness of κ and such that both $j_E(\kappa)$ and $j_E(\theta)$ are $C^{(n)}$-ultrahuge cardinals" is Σ_{n+3}-expressible and true in M, the desired conclusion follows from the fact that $j(\kappa)$ is Σ_{n+2}-correct in V. ⊣

§4. $C^{(n)}$-extendibility Laver functions. Towards further enriching the accompanying machinery of $C^{(n)}$-extendible cardinals (for $n > 1$), we now show that such cardinals carry appropriate Laver functions. This generalizes the case $n = 1$ of ordinary extendible cardinals, which are known to have their own Laver functions by the work Corazza (cf. [8]) and of the author (cf. [19]).

Let us first define the term "appropriate Laver function" in the context of $C^{(n)}$-extendibility. For $n > 1$:

DEFINITION 4.1. Suppose that κ is $C^{(n)}$-extendible. A function $\ell \vdots \kappa \longrightarrow V_\kappa$ is called a $C^{(n)}$*-extendibility Laver function* for κ if, for every cardinal $\lambda \geqslant \kappa$ and any $x \in H_{\lambda^+}$, there is an (extender) elementary embedding $j : V \longrightarrow M$ that is jointly λ-$C^{(n)}$-supercompact and λ-superstrong for κ, and such that $j(\ell)(\kappa) = x$.

We now show the following, taking a similar path as in the proof of Theorem 1.7 in [19]. For every $n > 1$:

THEOREM 4.2. *Every $C^{(n)}$-extendible cardinal carries a $C^{(n)}$-extendibility Laver function.*

PROOF. Suppose that κ is a $C^{(n)}$-extendible cardinal, for some $n > 1$, and fix some well-ordering $\lhd_\kappa$ of V_κ. Towards a contradiction, assume that there is no $C^{(n)}$-extendibility Laver function for κ.

We recursively construct a (partial) function $\ell \vdots \kappa \longrightarrow V_\kappa$, as follows. Given some $\alpha < \kappa$ and $\ell \restriction \alpha$, we define $\ell(\alpha)$ only if ℓ"$\alpha \subseteq V_\alpha$ and the following condition holds: there is $\lambda \geqslant \alpha$ and $x \in H_{\lambda^+}$ such that, for every extender embedding $j : V \longrightarrow M$ that is jointly λ-$C^{(n)}$-supercompact and λ-superstrong for α, we have that $j(\ell \restriction \alpha)(\alpha) \neq x$. Before continuing with the definition of $\ell(\alpha)$ in this case, we need the following:

CLAIM 4.3. *If there is such a λ for which the aforementioned condition holds (for some $x \in H_{\lambda^+}$), then there is such a λ with $\lambda < \kappa$.*

PROOF OF CLAIM. Let $\alpha < \kappa$ and $\ell \restriction \alpha$ be given. We check that the aforementioned condition is Σ_{n+2}-expressible, using α and $\ell \restriction \alpha$ as parameters. For this, suppose that there is $\lambda \geqslant \alpha$ and $x \in H_{\lambda^+}$ such that the condition holds.

We employ a similar idea as in the proof of Lemma 2.6. Namely, given any extender E that is jointly λ-$C^{(n)}$-supercompact and λ-superstrong for α, we may correctly verify the fact that $j_E(\ell \restriction \alpha)(\alpha) \neq x$ inside V_μ, where μ is some (any) sufficiently large cardinal that belongs to $C^{(n)}$. In fact, any such V_μ correctly verifies both the fact that E is an extender whose associated embedding j_E is jointly λ-$C^{(n)}$-supercompact and λ-superstrong for α, and that $j_E(\ell \restriction \alpha)(\alpha) \neq x$.

More precisely, as in the proof of Lemma 2.6, fix a formula $\chi(\alpha, \lambda, E)$ asserting that "the extender E is jointly λ-supercompact and λ-superstrong for α". Then, the aforementioned condition on λ and x is equivalent to the following statement $\varphi(\lambda, x, \alpha, \ell \restriction \alpha)$:

$$(\forall E)(\forall \mu \in C^{(n)} \text{ with } \mathrm{cf}(\mu) > \beth_{\mathrm{rk}(E)} + \beth_\lambda)\psi(\mu, E, \lambda, x, \alpha, \ell \restriction \alpha),$$

where ψ is the statement:

$$V_\mu \models ((\chi(\alpha, \lambda, E) \wedge j_E(\alpha) \in C^{(n)}) \longrightarrow j_E(\ell \restriction \alpha)(\alpha) \neq x).$$

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It is easily seen that $\varphi(\lambda, x, \alpha, \ell \restriction \alpha)$ is a Π_{n+1}-expressible statement where, again, the main contribution to this complexity comes from the requirement "$\mu \in C^{(n)}$". It follows that the stated condition, in full, is equivalent to a statement that is Σ_{n+2}-expressible, using α and $\ell \restriction \alpha$ as parameters.

To finish the proof, we recall that $\kappa \in C^{(n+2)}$. Hence, if there is a $\lambda \geqslant \alpha$ such that the stated condition holds (for some $x \in H_{\lambda^+}$), then this fact must reflect inside V_κ, from which the conclusion follows. ⊣

Returning to the recursive construction, if we are in the above case, then we let $\lambda_\alpha < \kappa$ be the least such cardinal $\lambda \geqslant \alpha$, and we let $\ell(\alpha)$ be the $\lhd_\kappa$-minimal witness $x \in H_{\lambda_\alpha^+}$. Otherwise, we leave ℓ undefined. This concludes the recursive construction of the function $\ell \vdots \kappa \longrightarrow V_\kappa$. Note that, by the previous claim, the range of ℓ is indeed included in V_κ, i.e., $\ell``\kappa \subseteq V_\kappa$.

According to our assumption, there must exist a least (cardinal) $\lambda \geqslant \kappa$ and some $x \in H_{\lambda^+}$ such that for every jointly λ-$C^{(n)}$-supercompact and λ-superstrong extender embedding j for κ, we have that $j(\ell)(\kappa) \neq x$; i.e., every such j fails to "anticipate" the set x. Let us fix a Π_{n+1}-formula $\varphi(\lambda, x)$ asserting this fact, using κ and ℓ as parameters. Now fix some $\theta > \lambda$ with $\theta \in C^{(n+1)}$, some inaccessible $\overline{\theta} > \theta$, and an elementary embedding $j : V \longrightarrow M$ witnessing the joint $\overline{\theta}$-$C^{(n)}$-supercompactness and $(\overline{\theta}+1)$-superstrongness of κ; i.e., M is transitive, $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \overline{\theta}$, ${}^{\overline{\theta}}M \subseteq M$, $V_{j(\overline{\theta})+1} \subseteq M$ and $j(\kappa) \in C^{(n)}$. Note that $j(\overline{\theta})$ is inaccessible and that, trivially, the embedding j also witnesses the joint λ-$C^{(n)}$-supercompactness and λ-superstrongness of κ. Moreover, notice that, in M, the cardinal θ belongs to $C^{(n+1)}$.[9] It follows that, in the model M, the cardinal λ is the least μ for which φ holds for some $x \in H_{\mu^+}$; that is, the model M thinks that $\lambda = \lambda_\kappa$ in the above notation. Therefore, by elementarity, there exists $y \in H_{\lambda^+}$ such that $j(\ell)(\kappa) = y$. By definition of $j(\ell)$, we have that $M \models \varphi(\lambda, y)$, a fact that will lead us to the desired contradiction.

We now perform an elementary chain construction in order to obtain an appropriate factor embedding of j that is witnessed by some extender in M and that actually anticipates the set y. We only mention here that the elementary substructures that we consider in the current context are of the form:

$$X_i = \{j(f)(j``\lambda, x) : f \in V, f : \mathcal{P}_\kappa\lambda \times V_{\overline{\theta}} \longrightarrow V, x \in V_{\beta_i}\} \prec M.$$

We initialize the construction by choosing a limit ordinal $\beta_0 \in (j(\kappa), j(\overline{\theta}))$; moreover, we choose some $\gamma < j(\overline{\theta})$ with $\mathrm{cf}(\gamma) > \lambda$, which serves as the length of our constructed chain. Then, the desired factor embedding j_γ results from composing j with the Mostowski collapse $\pi_\gamma : X_\gamma \cong M_\gamma$, as usual.

It is now easy to see, along the lines of the proof of Theorem 1.7 in [19], that j_γ is a jointly λ-$C^{(n)}$-supercompact and λ-superstrong embedding for κ that is witnessed by some extender $E \in M$, such that M correctly computes the value $j_E(\ell)(\kappa) = j_\gamma(\ell)(\kappa)$. Finally, notice that κ, λ, H_{λ^+} and y are all fixed by the Mostowski collapse and, thus, $j_E(\ell)(\kappa) = y$, which is the desired contradiction. ⊣

[9] To see this, note that, by elementarity, the cardinal $j(\kappa)$ is Σ_{n+2}-correct in M, and that $V_{j(\kappa)} \models \theta \in C^{(n+1)}$.

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Let us remark that, with the appropriate modifications, and with an eye on the proof of Theorem 5.2 in [20], it is not difficult to show that $C^{(n)}$-ultrahuge cardinals carry their own Laver functions as well.[10] Instead of repeating the same arguments all over again, we omit the details and leave them for the interested reader to verify.

§5. **$C^{(n)}$-extendibles and the GCH.** It is a well-known set-theoretic phenomenon that forcing globally the GCH preserves many of the usual large cardinals. The first example of this phenomenon was the case of measurable cardinals, proved by Jensen (cf. [13]). Afterwards, other similar proofs followed: Menas proved it for supercompacts (cf. [16]), Hamkins proved it for I1 embeddings (cf. [11]), and Friedman did it for n-superstrong cardinals (cf. [9]). More recently, Brooke-Taylor and Friedman proved it for 1-extendible cardinals (cf. [6]), Brooke-Taylor did it for Vopěnka's Principle (cf. [5]), the author proved it for (fully) extendible cardinals (cf. [17]), and Cheng and Gitman did it for remarkable cardinals (cf. [7]).

For completeness, we recall the following standard definition:

Definition 5.1. The *canonical forcing* $\mathbb{P}$ *for global* GCH is the class-length reverse Easton iteration of $\langle \dot{\mathbb{Q}}_\alpha : \alpha \in ON \rangle$, where $\mathbb{P}_0$ is the trivial poset and, for each α, if α is an infinite cardinal in $V^{\mathbb{P}_\alpha}$, then $\dot{\mathbb{Q}}_\alpha$ is the canonical $\mathbb{P}_\alpha$-name for the poset $\mathrm{Add}(\alpha^+, 1)^{V^{\mathbb{P}_\alpha}}$; otherwise, trivial forcing is done at that stage of the iteration. Finally, $\mathbb{P}$ is the direct limit of the $\mathbb{P}_\alpha$'s, for $\alpha \in ON$.

It is well-known that $\mathbb{P}$ (preserves ZFC and) forces the global GCH (see, for example, the comments after Definition 1 in [6], or the proof of Theorem 2 in [9]). Given our previous discussion and results in Section 3, we are now ready to finally prove the following, for every $n \geqslant 1$:

Theorem 5.2. *Every $C^{(n)}$-extendible cardinal is preserved by the canonical forcing for global* GCH.

Proof. We perform a meta-theoretic induction over natural numbers $n \geqslant 1$. Recall that the case $n = 1$ already holds; i.e., ordinary extendible cardinals are preserved by the GCH forcing $\mathbb{P}$, by Theorem 2.2 in [17].

We first argue for the case $n = 2$. So, let κ be a $C^{(2)}$-extendible cardinal and fix some inaccessible $\lambda > \kappa$. By Proposition 3.7 and the remark following it, there is some μ and an elementary embedding $j : H_{\lambda^+} \longrightarrow H_{\mu^+}$ with $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$ and such that $j(\kappa)$ is a supercompact cardinal. Note that $\mu = j(\lambda)$ is inaccessible and let G be $\mathbb{P}$-generic over V.

Then, exactly as in the proof of Theorem 2.2 in [17], it follows that the embedding lifts to $j : H_{\lambda^+}^{V[G]} \longrightarrow H_{j(\lambda)^+}^{V[G]}$ in $V[G]$. Moreover, since every supercompact cardinal is preserved by $\mathbb{P}$ (cf. [16]), it follows that $j(\kappa) \in C^{(2)}$ in $V[G]$ and so the lifted embedding witnesses the $(\lambda+1)$-$C^{(2)}$-extendibility of κ in the extension. Since there

[10] Evidently, one has to define the concept of a "$C^{(n)}$-ultrahugeness Laver function" first, but this is a straightforward modification of Definition 4.1, appropriately adapted in the context of $C^{(n)}$-ultrahuge cardinals. In order to prove the existence of $C^{(n)}$-ultrahugeness Laver functions, note that the elementary chain construction should be modified along the lines of Proposition 3.8.

are unboundedly many inaccessibles $\lambda > \kappa$, we conclude that the cardinal κ remains $C^{(2)}$-extendible in $V[G]$.

Now fix some $n > 2$ and suppose that, for each $m < n$, every $C^{(m)}$-extendible cardinal is preserved by $\mathbb{P}$. Let κ be a $C^{(n)}$-extendible cardinal and fix some inaccessible $\lambda > \kappa$. By Proposition 3.7, there is some μ and an elementary embedding $j : H_{\lambda^+} \longrightarrow H_{\mu^+}$ with $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$ and such that $j(\kappa)$ is a $C^{(n-2)}$-extendible cardinal. Note again that $\mu = j(\lambda)$ is inaccessible and let G be $\mathbb{P}$-generic over V.

Once more, the embedding lifts to $j : H_{\lambda^+}^{V[G]} \longrightarrow H_{j(\lambda)^+}^{V[G]}$ in $V[G]$. In addition, by the inductive hypothesis, $j(\kappa)$ remains $C^{(n-2)}$-extendible in $V[G]$. Therefore, we have that $j(\kappa) \in C^{(n)}$ in $V[G]$, and so the lifted embedding witnesses the $(\lambda+1)$-$C^{(n)}$-extendibility of κ in the extension. Finally, by choosing unboundedly many inaccessibles $\lambda > \kappa$, we obtain that the cardinal κ remains $C^{(n)}$-extendible in $V[G]$, as desired. ⊣

§6. On separating levels of $C^{(n)}$-extendibility. The preservation of $C^{(n)}$-extendible cardinals by the GCH forcing is one happy moment in the general study of the interaction of $C^{(n)}$-cardinals with the forcing machinery; in this context, few other results are available so far (see, e.g., Section 4 in [18]). Indeed, the issue regarding what kind of forcing constructions preserve or destroy the various $C^{(n)}$-cardinals is widely open.[11] For instance, the following natural question annoyingly remains unresolved:

QUESTION 6.1. *Let κ be a $C^{(n+1)}$-extendible cardinal, for some $n \geqslant 1$. Is there a forcing notion that destroys the $C^{(n+1)}$-extendibility of κ while preserving its $C^{(n)}$-extendibility*?

We note that a similar question is open in the context of other $C^{(n)}$-cardinals; for instance, in the case of $C^{(n)}$-supercompactness (see, e.g., Question 3.8 in [18]). Although we have no clue regarding the answers to these questions, we nevertheless give below some easy observations in the direction of separating levels of $C^{(n)}$-extendibility.

FACT 6.2. *Suppose that κ is extendible. Then, there exists a* (ZFC) *model in which κ is extendible but not $C^{(2)}$-extendible.*

PROOF. Recall that if there is a $C^{(2)}$-extendible cardinal, then there must exist unboundedly many supercompact cardinals in the universe (this is explained in the preliminaries; alternatively, it follows from Proposition 3.1). So, let κ be an extendible cardinal. If κ happens to be $C^{(2)}$-extendible as well (otherwise there is nothing to show), then let $\lambda > \kappa$ be the least supercompact cardinal above κ. Then, V_λ is a (ZFC) model such that $V_\lambda \models$ "κ is extendible", because $\lambda \in C^{(2)}$ and being extendible is a Π_3-expressible statement.

However, it is clear that κ cannot be $C^{(2)}$-extendible in V_λ, because otherwise there would exist unboundedly many $\alpha < \lambda$ such that $V_\lambda \models$ "α is supercompact". But, any such α would indeed be a supercompact cardinal in V, since $\lambda \in C^{(2)}$

[11] Of course, there are obvious examples of posets that destroy any large cardinal property of κ whatsoever; e.g., collapsing the cardinal κ to become countable. The interesting questions arise in the nontrivial case in which "destroy" typically means "destroy some level of" the given large cardinal property (while preserving lower levels of it).

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and being supercompact is a Π_2-expressible statement. This would contradict the minimality of λ. ⊣

We think of the previous fact as our "base case", separating $C^{(n+1)}$-extendibility from $C^{(n)}$-extendibility when $n = 1$. We may now fully generalize, for $n > 1$:

FACT 6.3. *Suppose that κ is $C^{(n)}$-extendible. Then, there exists a* (ZFC) *model in which κ is $C^{(n)}$-extendible but not $C^{(n+1)}$-extendible.*

PROOF. Recall that if there is a $C^{(n+1)}$-extendible cardinal (for $n > 1$), then there must exist unboundedly many $C^{(n-1)}$-extendible cardinals in the universe (this follows from Proposition 3.6 in [1], or from Theorem 3.4). So, let κ be a $C^{(n)}$-extendible cardinal. If κ happens to be $C^{(n+1)}$-extendible as well (otherwise there is nothing to show), then let $\lambda > \kappa$ be the least $C^{(n-1)}$-extendible cardinal above κ. Then, V_λ is a (ZFC) model such that $V_\lambda \models$ "κ is $C^{(n)}$-extendible", because $\lambda \in C^{(n+1)}$ and being $C^{(n)}$-extendible is a Π_{n+2}-expressible statement.

However, it is clear that κ cannot be $C^{(n+1)}$-extendible in V_λ, because otherwise there would be unboundedly many $\alpha < \lambda$ such that $V_\lambda \models$ "α is $C^{(n-1)}$-extendible". But, any such α would be a $C^{(n-1)}$-extendible cardinal in V, since $\lambda \in C^{(n+1)}$ and being $C^{(n-1)}$-extendible is a Π_{n+1}-expressible statement. This would again contradict the minimality of λ. ⊣

The above facts give us an easy way to separate the $C^{(n+1)}$-extendibility from the $C^{(n)}$-extendibility of a given cardinal κ, for $n \geqslant 1$. Note that both of these facts are consequences of the strong reflective nature of $C^{(n)}$-extendible cardinals. In particular, we have crucially used that the existence of a $C^{(n+1)}$-extendible cardinal implies the existence of unboundedly many appropriate large cardinals in the universe, where, by "appropriate", we mean either supercompact (when $n = 1$) or $C^{(n-1)}$-extendible (when $n > 1$).

Let us conclude by mentioning that it remains unclear whether an analogous "easy separation" is possible in other cases, such as $C^{(n)}$-supercompactness, where no similar strong reflective properties are available so far. In those cases, one may indeed need a (perhaps class-length) forcing construction in order to achieve such a separation. At any rate, Question 6.1 still remains valid, with its answer possibly shedding more light on those other cases as well.

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