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RATIONAL CUP PRODUCT AND ALGEBRAIC K_0-GROUPS OF RINGS OF CONTINUOUS FUNCTIONS

HIROSHI KIHARA[1]* AND NOBUYUKI ODA[2]

[1] *Center for Mathematical Sciences, University of Aizu, Tsuruga, Ikki-machi, Aizu-Wakamatsu City, 965-8580 Fukushima, Japan* (kihara@u-aizu.ac.jp)
[2] *Department of Applied Mathematics, Faculty of Science, Fukuoka University, Fukuoka 814-0180, Japan* (odanobu@cis.fukuoka-u.ac.jp)

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Abstract A connected space is called a C_0-space if its rational cup product is trivial. A characterizing property of C_0-spaces is obtained. This property is used to calculate the algebraic K_0-group $K_0(C_{\mathbb{F}}(X))$ of the ring of continuous functions for infinite-dimensional complexes X.

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1. Introduction and main results

The notion of a co-H-space, which is dual to that of an H-space, plays an essential part in homotopy theory. One of the fundamental properties of a co-H-space is that the cup product with any coefficients vanishes [**1**, Proposition 2.3].

In this paper, our primary concern is focused on spaces whose rational cup product is trivial. A characterizing property of such spaces is obtained. We use this property and refine the arguments in [**7**] to calculate the algebraic K_0-group $K_0(C_{\mathbb{F}}(X))$ of the ring of $\mathbb{F}$-valued continuous functions for infinite-dimensional complexes X, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$.

Let us begin with the definition of a C_0-space.

Definition 1.1. A connected space X is called a C_0-space if the cup product on $H^{>0}(X; \mathbb{Q}) = \bigoplus_{i>0} H^i(X; \mathbb{Q})$ is trivial.

Remark 1.2. (1) A space is called a co-H_0-space if its rationalization is homotopy equivalent to a wedge of rational spheres.

* Corresponding author.

The following implications hold:

$$\begin{array}{ccccc} & & \text{co-}H\text{-space} & & \\ & \nearrow\!\!\!\nearrow & & \searrow\!\!\!\searrow & \\ \text{1-connected co-}H\text{-space} & & & & C_0\text{-space.} \\ & \searrow\!\!\!\searrow & & \nearrow\!\!\!\nearrow & \\ & & \text{co-}H_0\text{-space} & & \end{array}$$

(See [**11**, p. 72] for the implication '1-connected co-H-space $\Rightarrow$ co-H_0-space'. See also Example 2.4 for a 1-connected C_0-space which is not a co-H_0-space.) Thus, the class of C_0-spaces is a large class of spaces which properly contains that of co-H-spaces and that of co-H_0-spaces.

(2) The notion of a C_0-space and the dual notion of a W_0-space have been discussed in [**5**]; it is shown that X is a C_0-space if and only if the rationalization of ΣX is a homotopy-cocommutative co-H-space. See [**6**] for another characterizing property of C_0-spaces.

Definition 1.3. The condition (P) for a pointed connected CW-complex X is defined as follows.
(P) Let $\pi_1, \pi_2, \ldots,$ and $\pi'_1, \pi'_2, \ldots$ be $\mathbb{Q}$-modules. Let $\varphi : \prod_i K(\pi_i, i) \to \prod_i K(\pi'_i, i)$ be a pointed map between the weak products of Eilenberg–MacLane complexes. Then the induced map

$$\varphi_\sharp : \left[X, \prod_i K(\pi_i, i)\right]_* \longrightarrow \left[X, \prod_i K(\pi'_i, i)\right]_*$$

on the pointed homotopy sets is identified with $\prod_i H^i(X; \pi_i(\varphi))$ under the standard identifications of $[X, \prod_i K(\pi_i, i)]_*$ and $[X, \prod_i K(\pi'_i, i)]_*$ with $\prod_i H^i(X; \pi_i)$ and $\prod_i H^i(X; \pi'_i)$, respectively.

Remark 1.4. Refer to [**13**, p. 28] for a weak product of countably many (pointed) spaces. Note that the weak product of Eilenberg–MacLane complexes $\{K(\pi_i, i)\}_{i \geqslant 0}$ is a CW-complex, and that the canonical inclusion of the weak product of Eilenberg–MacLane complexes $\{K(\pi_i, i)\}_{i \geqslant 0}$ into the (ordinary) product of $\{K(\pi_i, i)\}_{i \geqslant 0}$ is a weak homotopy equivalence (cf. [**13**, p. 208]). Thus, the weak product of $\{K(\pi_i, i)\}_{i \geqslant 0}$ is a (categorical) product of $\{K(\pi_i, i)\}_{i \geqslant 0}$ in the homotopy category of pointed CW-complexes.

C_0-spaces are characterized by the condition (P).

Theorem 1.5. *Let X be a pointed connected CW-complex. Then X is a C_0-space if and only if X satisfies the condition* (P).

Theorem 1.5 leads us to the following calculational result of the algebraic K_0-group $K_0(C_{\mathbb{F}}(X))$, which is a generalization of Theorem 1.5 in [**7**].

Theorem 1.6. *Let X be a connected CW-complex which is a C_0-space. Suppose that X satisfies the condition that the pointed mapping space* $\mathrm{map}_*(X, \widehat{Y})$ *from X to the*

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profinite completion $\widehat{Y}$ of any nilpotent finite complex Y is weakly contractible. Then there exists a natural group isomorphism

$$K_0(C_{\mathbb{F}}(X)) \cong \mathbb{Z} \oplus \bigoplus_{l>0} H^{sl-1}(X; \widehat{\mathbb{Z}}/\mathbb{Z}),$$

where $\widehat{\mathbb{Z}}$ is the profinite completion of the integers $\mathbb{Z}$, and $s = 2$ if $\mathbb{F} = \mathbb{C}$ or $s = 4$ if $\mathbb{F} = \mathbb{R}, \mathbb{H}$.

The class of connected CW-complexes satisfying the weak contractibility condition in Theorem 1.6 contains Postnikov spaces with locally finite fundamental group, classifying spaces of compact Lie groups, and infinite loop spaces with torsion fundamental groups [**7**, Example 1.6], and this class is closed under suspension [**7**, Lemma 5.2]; a space X is called a *Postnikov space* if $\pi_n(X) = 0$ for sufficiently large n. Thus, Theorem 1.5 in [**7**] was applied to the (iterated) suspensions of such spaces. However, Theorem 1.6 enables us to give calculation examples of $K_0(C_{\mathbb{F}}(X))$ for infinite-dimensional complexes X which are not suspended.

Let $M(\pi, n)$ denote the Moore space whose nth homology is π. For a connected CW-complex K, $Q(K)$ denotes the infinite loop space defined by $Q(K) = \operatorname{colim}_n \Omega^n \Sigma^n K$. For a connected CW-complex A, $A^{(n)}$ denotes the Postnikov n-stage of A.

Corollary 1.7. *Suppose that X is one of the following spaces:*

(1) $K(\pi, 2n+1)$,

(2) $Q(M(\pi, 2n+1))$,

(3) $M(\pi, 2n+1)^{(l)}$ *with* $l \geqslant 2n+1$,

(4) $\Omega\, M(\pi, 2n+2)^{(m)}$ *with* $4n+3 > m \geqslant 2n+2$,

where π is an abelian group whose rationalization is isomorphic to $\mathbb{Q}$, and n is a positive integer. Then

$$K_0(C_{\mathbb{F}}(X)) \cong \begin{cases} \mathbb{Z} \oplus \widehat{\mathbb{Z}}/\mathbb{Z} & \text{if } \mathbb{F} = \mathbb{C} \text{ or } n \text{ is odd,} \\ \mathbb{Z} & \text{otherwise.} \end{cases}$$

In particular, Corollary 1.7 applies to $X = K(\mathbb{Z}, 2n+1), (S^{2n+1})^{(l)}$ for $n > 0$ and $l > 2n+1$.

We prove Theorem 1.5 in § 2, and Theorem 1.6 and Corollary 1.7 in § 3.

2. Characterization of C_0-spaces

For the proof of Theorem 1.5, let us give a convenient construction of Eilenberg–MacLane complexes and their (weak) products.

Let Ab denote the category of abelian groups, and let $Kom_{\geqslant 0}(Ab)$ and sAb denote the category of non-negatively graded chain complexes of abelian groups and the category of simplicial abelian groups, respectively. There is an equivalence of categories

$$E : Kom_{\geqslant 0}(Ab) \longrightarrow sAb,$$

which is called the Dold–Kan correspondence [**12**, p. 270].

Let $\mathcal{S}$ and $\mathcal{K}$ denote the category of simplicial sets and the category of compactly generated Hausdorff spaces, respectively. The realization functor $|\ |: \mathcal{S} \to \mathcal{K}$ is a left adjoint of the singular functor $S : \mathcal{K} \to \mathcal{S}$ [**3**, p. 7]. Since the realization functor $|\ |$ preserves finite limits [**2**, p. 49], the functors $|\ |$ and S restrict to

$$Ab(\mathcal{S}) \underset{S}{\overset{|\ |}{\rightleftarrows}} Ab(\mathcal{K}),$$

which also form an adjoint pair, where $Ab(\mathcal{C})$ denotes the category of abelian group objects in a category $\mathcal{C}$. Note that $sAb = Ab(\mathcal{S})$, and that if we adopt $\mathcal{K}$ as our category of topological spaces, abelian group objects in $\mathcal{K}$ are just topological abelian groups.

Define the functor K to be the composite

$$Kom_{\geqslant 0}(Ab) \xrightarrow{E} Ab(\mathcal{S}) \xrightarrow{|\ |} Ab(\mathcal{K}) \xrightarrow{U} \mathcal{K}_*,$$

where $\mathcal{K}_*$ denotes the category of pointed compactly generated Hausdorff spaces and U denotes the forgetful functor.

Lemma 2.1.

(1) *The functor K coincides with the composite*

$$Kom_{\geqslant 0}(Ab) \xrightarrow{E} Ab(\mathcal{S}) \xrightarrow{U'} \mathcal{S}_* \xrightarrow{|\ |} \mathcal{K}_*,$$

where $\mathcal{S}_*$ denotes the category of pointed simplicial sets and U' denotes the forgetful functor.

(2) *The three functors E, U', and $|\ |$ in part 1, and hence the functor K, preserve finite limits and filtered colimits.*

Proof.

(1) The assertion follows from the commutativity of the diagram

$$\begin{array}{ccc} Ab(\mathcal{S}) & \xrightarrow{|\ |} & Ab(\mathcal{K}) \\ {\scriptstyle U'}\downarrow & & \downarrow{\scriptstyle U} \\ \mathcal{S}_* & \xrightarrow{|\ |} & \mathcal{K}_*. \end{array}$$

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(2) Since E is an equivalence of categories, the assertion for E is obvious. The functor $|\ |: \mathcal{S}_* \to \mathcal{K}_*$ preserves colimits since it is a left adjoint of $S : \mathcal{K}_* \to \mathcal{S}_*$, and $|\ |$ also preserves finite limits (cf. [**2**, p. 49]).

For a category $\mathcal{C}$, $s\mathcal{C}$ denotes the category of simplicial objects in $\mathcal{C}$ [**9**, p. 4]. Note that limits and colimits in $s\mathcal{C}$ are constructed degreewise, and that the forgetful functor $Ab \to Set_*$ creates limits and filtered colimits (see [**8**, pp. 212–213]), where Set_* denotes the category of pointed sets. Thus $U' : Ab(\mathcal{S}) = sAb \to sSet_* = \mathcal{S}_*$ preserves limits and filtered colimits. □

Recall that the simplicial homotopy groups $\pi_i(X)$ are canonically isomorphic to the topological homotopy groups $\pi_i(|X|)$ for a Kan complex X [**3**, p. 60], and that any simplicial group is a Kan complex [**3**, p. 12]. Recall also that the realization $|X|$ of a simplicial set X is a CW-complex whose cells bijectively correspond to the non-degenerate simplices of X [**3**, p. 8].

Lemma 2.2. *Let $\pi = \bigoplus_{i\geqslant 0}\pi_i$ be a non-negatively graded $\mathbb{Z}$-module, and regard π as an object of $Kom_{\geqslant 0}(Ab)$ whose boundary maps are zero.*

(1) *The space $K(\pi)$ is the weak product of Eilenberg–MacLane complexes $\{K(\pi_i, i)\}_{i\geqslant 0}$.*

(2) *Suppose that π is a graded module over a ring R. Let G be an abelian group. Then there exists a natural isomorphism*

$$H_*(K(\pi); G) \cong \operatorname{colim} H_*(K(\pi_\lambda); G),$$

where π_λ ranges over all finitely generated graded R-submodules of π.

Proof.

(1) For a $\mathbb{Z}$-module M, $M(i)$ denotes the graded $\mathbb{Z}$-module such that $M(i)_i = M$ and $M(i)_j = 0$ for $j \neq i$. Since homology groups correspond to simplicial homotopy groups via the equivalence E [**12**, p. 271], $K(M(i))$ is a $K(M, i)$-complex. Further, we have

$$\begin{aligned} K(\pi) &= K\left(\operatorname*{colim}_n \bigoplus_{i<n} \pi_i(i)\right) = \operatorname*{colim}_n K\left(\bigoplus_{i<n} \pi_i(i)\right) \\ &= \operatorname*{colim}_n K\left(\prod_{i<n} \pi_i(i)\right) = \operatorname*{colim}_n \prod_{i<n} K(\pi_i(i)), \end{aligned}$$

by Lemma 2.1. Hence $K(\pi)$ is the weak product of $\{K(\pi_i, i)\}_{i\geqslant 0}$.

(2) Let us begin by recalling the definition of the homology $H_*(X; G)$ of a simplicial set X with coefficients in an abelian group G.

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The free abelian group functor $\mathbb{Z}\cdot : Set \longrightarrow Ab$ extends to the functor $\mathbb{Z}\cdot : \mathcal{S} \longrightarrow sAb$ in an obvious way. The Moore complex functor $M : sAb \longrightarrow Kom_{\geqslant 0}(Ab)$ is defined by assigning to $A \in sAb$ the chain complex

$$\cdots \xrightarrow{\partial} A_p \xrightarrow{\partial} A_{p-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} A_0 \longrightarrow 0$$

with $\partial = \sum_{i=0}^{p} (-1)^i d_i$, where the d_i are face operators of A. The composite functor

$$\mathcal{S} \xrightarrow{\mathbb{Z}\cdot} sAb \xrightarrow{M} Kom_{\geqslant 0}(Ab)$$

is also denoted by $\mathbb{Z}\cdot$. The homology $H_*(X; G)$ is defined to be the homology $H_*(\mathbb{Z}X \otimes G)$ of the chain complex $\mathbb{Z}X \otimes G$. Recall that the homology $H_*(X; G)$ of a simplicial set X is naturally isomorphic to the singular homology $H_*(|X|; G)$ of the realization $|X|$ [9, p. 63].

Next, let us show that the homology functor $H_p(\ ; G) : \mathcal{S} \longrightarrow Ab$ preserves filtered colimits.

The functor $\mathbb{Z}\cdot : Set \longrightarrow Ab$ is a left adjoint of the underlying set functor $Ab \longrightarrow Set$, and the functor $\cdot \otimes G : Ab \longrightarrow Ab$ is a left adjoint of the functor $\mathrm{Hom}(G,\ ) : Ab \longrightarrow Ab$. Since colimits in $\mathcal{S}$ and $Kom_{\geqslant 0}(Ab)$ are constructed degreewise, $\mathbb{Z}\cdot \otimes G : \mathcal{S} \longrightarrow Kom_{\geqslant 0}(Ab)$ preserves colimits. Further, from the construction of filtered colimits in Ab (see the proof of Lemma 2.1(2)), we can see that for any filtered diagram $\{C_i\}$ in $Kom_{\geqslant 0}(Ab)$, the canonical homomorphism

$$\mathrm{colim}\ H_p(C_i) \longrightarrow H_p(\mathrm{colim}\ C_i)$$

is bijective. Hence, $H_p(\ ; G)$ preserves filtered colimits.

Using this and Lemma 2.1, we have the isomorphisms

$$\begin{aligned} H_*(K(\pi); G) &\cong H_*(U'E(\pi); G) \\ &\cong H_*(\mathrm{colim}\, U'E(\pi_\lambda); G) \\ &\cong \mathrm{colim}\, H_*(U'E(\pi_\lambda); G) \\ &\cong \mathrm{colim}\, H_*(K(\pi_\lambda); G). \end{aligned}$$

□

Remark 2.3. Let us adopt $\mathcal{K}$ as our category of topological spaces. Then Eilenberg–MacLane complexes and their weak products can be constructed as topological abelian groups via the functor K.

Proof of Theorem 1.5. In the proof, Eilenberg–MacLane complexes and their weak products are constructed via the functor K.

($\Rightarrow$) We may assume $\pi'_i = 0$ $(i \neq n)$. Let (α_i) be an element of $\prod_i H^i(X; \pi_i)$ which corresponds to the homotopy class of $f : X \to \prod_i K(\pi_i, i)$. We have to compute the

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element α'_n of $H^n(X;\pi'_n)$ which corresponds to the composite

$$X \xrightarrow{f} \prod_i K(\pi_i, i) \xrightarrow{\varphi} K(\pi'_n, n).$$

Under the natural isomorphism

$$H^n(X;\pi'_n) \cong \mathrm{Hom}(H_n(X;\mathbb{Q}), \pi'_n),$$

α'_n is just the composite

$$H_n(X;\mathbb{Q}) \xrightarrow{f_*} H_n\left(\prod_i K(\pi_i, i);\mathbb{Q}\right) \xrightarrow{\varphi_*} H_n(K(\pi'_n, n);\mathbb{Q}) \cong \pi'_n.$$

First, let us observe that the second component of

$$f_* : H_n(X;\mathbb{Q}) \longrightarrow H_n\left(\prod_i K(\pi_i, i);\mathbb{Q}\right) \cong H_n(K(\pi_n, n);\mathbb{Q}) \bigoplus H_n\left(\prod_{i<n} K(\pi_i, i);\mathbb{Q}\right)$$

is zero. For $\zeta \in H_n(X;\mathbb{Q})$, we write $f_*\zeta = \epsilon + \eta$ according to the above decomposition of $H_n(\prod_i K(\pi_i, i);\mathbb{Q})$. Suppose that $\eta \neq 0$. By part 2 of Lemma 2.2, there is a finite-dimensional graded $\mathbb{Q}$-submodule π_λ of $\pi_{<n} := \bigoplus_{i<n}\pi_i$ and an element η_λ of $H_n(\prod_{i<n} K(\pi_{\lambda i}, i);\mathbb{Q})$ such that η comes from η_λ. Thus, there is an element γ_λ of $H^n(\prod_{i<n} K(\pi_{\lambda i}, i);\mathbb{Q})$ such that $\gamma_\lambda(\eta_\lambda) \neq 0$. By choosing a left inverse of the inclusion $\pi_\lambda \hookrightarrow \pi$, we have a retract diagram

$$\prod_{i<n} K(\pi_{\lambda i}, i) \overset{\iota}{\hookrightarrow} \prod_i K(\pi_i, i) \xrightarrow{\sigma} \prod_{i<n} K(\pi_{\lambda i}, i).$$

Set $\gamma = \sigma^*\gamma_\lambda$. Then we have

$$\begin{aligned} \gamma(f_*\zeta) &= \gamma_\lambda(\sigma_* f_*\zeta) = \gamma_\lambda(\sigma_*\epsilon + \sigma_*\eta) \\ &= \gamma_\lambda(\sigma_*\eta) = \gamma_\lambda(\sigma_*\iota_*\eta_\lambda) = \gamma_\lambda(\eta_\lambda) \neq 0, \end{aligned}$$

and hence $f^*\gamma \neq 0$. On the other hand, since $H^*(\prod_{i<n} K(\pi_{\lambda i}, i);\mathbb{Q})$ is a free graded commutative algebra on the dual basis of a homogeneous basis of $\pi_\lambda = \bigoplus_{i<n}\pi_{\lambda i}$, γ is a linear combination of cup products of elements of degree $< n$. Thus the assumption on X implies that $f^*\gamma = 0$, which is a contradiction.

Note that the restriction of φ_* to the first summand

$$H_n(K(\pi_n, n);\mathbb{Q}) \hookrightarrow H_n\left(\prod_i K(\pi_i, i);\mathbb{Q}\right) \xrightarrow{\varphi_*} H_n(K(\pi'_n, n);\mathbb{Q})$$

is identified with $\pi_n(\varphi) : \pi_n \to \pi'_n$ in a canonical way. Thus, the observation above implies that $\alpha'_n = H^n(X;\pi_n(\varphi))(\alpha_n)$.

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($\Leftarrow$) Let $\varphi_{pq} : K(\mathbb{Q}, p) \times K(\mathbb{Q}, q) \to K(\mathbb{Q}, p+q)$ be a map corresponding to the cup product

$$\cup : H^p(\ ;\mathbb{Q}) \times H^q(\ ;\mathbb{Q}) \longrightarrow H^{p+q}(\ ;\mathbb{Q}).$$

Suppose that X is not a C_0-space. Then there are positive integers p and q such that

$$\varphi_{pq\sharp} : [X, K(\mathbb{Q}, p) \times K(\mathbb{Q}, q)]_* \longrightarrow [X, K(\mathbb{Q}, p+q)]_*$$

is non-zero. But $\prod_{i>0} H^i(X; \pi_i(\varphi_{pq})) = 0$. $\square$

We end this section by giving a construction of C_0-spaces.

Example 2.4. Let us construct a (finite- or infinite-dimensional) CW-complex $X = e^0 \cup e^{p_1} \cup e^{p_2} \cup \cdots$ such that $1 < p_1,\ 2p_1 < p_2,\ 2p_2 < p_3, \ldots$. X is a C_0-space for dimensional reasons. If some attaching map $\varphi : S^{p_i - 1} \to X^{p_{i-1}}$ of the p_i-cell e^{p_i} is not a torsion element in $\pi_{p_i - 1}(X^{p_{i-1}})$, then X is not a co-H_0-space because it is not rationally a wedge of spheres.

3. Calculations of $K_0(C_{\mathbb{F}}(X))$

In this section, we prove Theorem 1.6 and Corollary 1.7 using Theorem 1.5 and results of [7].

3.1. The group $K_0(C_{\mathbb{F}}(X))$

We study the algebraic K_0-group $K_0(C_{\mathbb{F}}(X))$ of the ring of continuous $\mathbb{F}$-valued functions. Let us begin by recalling fundamental results on $K_0(C_{\mathbb{F}}(X))$ from [7].

Let $\mathbb{F}$ be $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, the field of real or complex numbers or the quaternion algebra, respectively. Let $G_n(\mathbb{F}^N)$ denote the finite Grassmannian consisting of n-dimensional subspaces in $\mathbb{F}^N$, which is endowed with the standard CW-complex structure. Moreover, the spaces $G_n(\mathbb{F})$ and $G_\infty(\mathbb{F})$ are defined by

$$G_n(\mathbb{F}) = \varinjlim_N G_n(\mathbb{F}^N); \qquad G_\infty(\mathbb{F}) = \varinjlim_n G_n(\mathbb{F}),$$

which also have the standard CW-complex structures. Then the product space $\mathbb{Z} \times G_\infty(\mathbb{F})$ has the standard H-group structure which corresponds to direct sum operation.

See [7, Definitions 1.3 and 3.6] for the definitions of (unpointed and pointed) compactly bounded homotopy sets.

Proposition 3.1. *Let X be a pointed connected CW-complex. Then there is a natural isomorphism*

$$K_0(C_{\mathbb{F}}(X)) \cong \mathbb{Z} \oplus [X, G_\infty(\mathbb{F})]_{CB*}$$

of abelian groups, and a natural isomorphism

$$[X, G_\infty(\mathbb{F})]_{CB*} \cong \operatorname*{colim}_n [X, G_n(\mathbb{F}^{2n})]_*$$

of pointed sets.

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Proof. We have the natural isomorphisms of abelian groups

$$\begin{aligned} K_0(C_{\mathbb{F}}(X)) &\cong [X, \mathbb{Z} \times G_\infty(\mathbb{F})]_{CB} \\ &\cong \mathbb{Z} \oplus [X, G_\infty(\mathbb{F})]_{CB} \\ &\cong \mathbb{Z} \oplus [X, G_\infty(\mathbb{F})]_{CB*} \end{aligned}$$

by [7, Theorem 1.4 and Proposition 3.9(ii)].

The second isomorphism is obtained by the definition of a pointed compactly bounded homotopy set and the filtration $G_\infty(\mathbb{F}) = \bigcup_n G_n(\mathbb{F}^{2n})$. □

We would like to interpret the group structure of $[X, G_\infty(\mathbb{F})]_{CB_*}$ in terms of the direct system $\{[X, G_n(\mathbb{F}^{2n})]_*\}$ (cf. Proposition 3.1); the interpretation is used to identify the group $[X, G_\infty(\mathbb{F})]_{CB_*}$ with the direct sum $\bigoplus_{l>0} H^{sl-1}(X; \widehat{\mathbb{Z}}/\mathbb{Z})$ under the assumption of Theorem 1.6. For this purpose, we need the following categorical lemmas.

Lemma 3.2. *Let $\mathcal{C}$ be a category, and I and J small categories. Let $A : I \to \mathcal{C}$ and $B \colon J \to \mathcal{C}$ be functors. Consider the following conditions on the category $\mathcal{C}$.*

(a) *$\mathcal{C}$ is a cocomplete, Cartesian closed category.*

(b) *$\mathcal{C}$ is the undercategory $(U \downarrow \mathcal{D})$ for some cocomplete, Cartesian closed category $\mathcal{D}$ and some object U of $\mathcal{D}$.*

(c) *$\mathcal{C}$ is a cocomplete, additive category.*

If $\mathcal{C}$ satisfies the condition (a) or (c), or if $\mathcal{C}$ satisfies the condition (b) and I and J are filtered, then

$$\operatorname*{colim}_{i,j} A_i \times B_j \cong \operatorname*{colim}_{i} A_i \times \operatorname*{colim}_{j} B_j$$

holds.

Proof. First, suppose that $\mathcal{C}$ satisfies the condition (a). Recall that $\operatorname*{colim}_{i,j} A_i \times B_j \cong \operatorname*{colim}_{i} \operatorname*{colim}_{j} A_i \times B_j$ holds (cf. [8, p. 231]). Since $\mathcal{C}$ is Cartesian closed, $K \times \bullet$ has a right adjoint for $K \in \mathcal{C}$. Thus, we have

$$\begin{aligned} \operatorname*{colim}_{i,j} A_i \times B_j &\cong \operatorname*{colim}_{i} \operatorname*{colim}_{j} A_i \times B_j \\ &\cong \operatorname*{colim}_{i} \left(A_i \times \operatorname*{colim}_{j} B_j\right) \\ &\cong \operatorname*{colim}_{i} A_i \times \operatorname*{colim}_{j} B_j. \end{aligned}$$

Second, suppose that $\mathcal{C}$ satisfies the condition (b) and I and J are filtered. Since the forgetful functor from $(U \downarrow \mathcal{D})$ to $\mathcal{D}$ creates finite products and filtered colimits (cf. [8, p. 112]), the result follows from that in the case where (a) is satisfied.

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Third, suppose that $\mathcal{C}$ satisfies the condition (c). In an additive category, a product is a biproduct, and hence a coproduct [8, § 2 in Chapter VIII]. Thus we have

$$\begin{aligned} \operatorname*{colim}_{i,j} A_i \oplus B_j &\cong \operatorname*{colim}_{i} \operatorname*{colim}_{j} A_i \oplus B_j \\ &\cong \operatorname*{colim}_{i} (A_i \oplus \operatorname*{colim}_{j} B_j) \\ &\cong \operatorname*{colim}_{i} A_i \oplus \operatorname*{colim}_{j} B_j \end{aligned}$$

(cf. [8, p. 231]). □

Remark 3.3. The category Set of sets and the category $\mathcal{K}$ of compactly generated Hausdorff spaces satisfy the condition (a) of Lemma 3.2. Hence, the categories Set_* and $\mathcal{K}_*$ satisfy the condition (b) of Lemma 3.2. The category Ab satisfies the condition (c) of Lemma 3.2.

Lemma 3.4. *Let $\mathcal{C}$ be a category satisfying one of the conditions (a), (b), or (c) in Lemma 3.2. Let $\mathbb{N}$ be the linearly ordered set of natural numbers, and $A : \mathbb{N} \to \mathcal{C}$ a functor. Let $\{f_{n,n} : A_n \times A_n \to A_{2n}\}_{n\in\mathbb{N}}$ be a family of morphisms in $\mathcal{C}$ such that the diagram*

$$\begin{array}{ccc} A_n \times A_n & \xrightarrow{f_{n,n}} & A_{2n} \\ {\scriptstyle \phi_{n,n'} \times \phi_{n,n'}} \downarrow & & \downarrow {\scriptstyle \phi_{2n,2n'}} \\ A_{n'} \times A_{n'} & \xrightarrow{f_{n',n'}} & A_{2n'} \end{array}$$

commutes, where $\phi_{k,l} = A(k \leqslant l)$. Then there exists a unique morphism $f : \operatorname*{colim}_n A_n \times \operatorname*{colim}_n A_n \longrightarrow \operatorname*{colim}_n A_n$ such that the diagram

$$\begin{array}{ccc} A_n \times A_n & \xrightarrow{\qquad f_{n,n} \qquad} & A_{2n} \\ {\scriptstyle \phi_n \times \phi_n} \downarrow & & \downarrow {\scriptstyle \phi_{2n}} \\ \operatorname*{colim}_n A_n \times \operatorname*{colim}_n A_n & \xrightarrow{f} & \operatorname*{colim}_n A_n \end{array}$$

commutes, where ϕ_n denotes the canonical morphism $A_n \to \operatorname*{colim}_n A_n$.

Proof. Define the functor $A \times A : \mathbb{N} \times \mathbb{N} \to \mathcal{C}$ by $A \times A(n, m) = A_n \times A_m$. Define the functors $\mathit{diag} : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ and $2 \times \bullet : \mathbb{N} \to \mathbb{N}$ by $\mathit{diag}\ (n) = (n, n)$ and $2 \times \bullet(n) = 2n$, respectively. Then the family $\{f_{n,n} : A_n \times A_n \to A_{2n}\}$ defines a natural transformation from $\mathbb{N} \overset{\mathit{diag}}{\to} \mathbb{N} \times \mathbb{N} \overset{A\times A}{\to} \mathcal{C}$ to $\mathbb{N} \overset{2\times\bullet}{\longrightarrow} \mathbb{N} \overset{A}{\to} \mathcal{C}$. Thus, we have the morphism

$$\operatorname*{colim}_n\ f_{n,n} : \operatorname*{colim}_n\ A_n \times A_n \to \operatorname*{colim}_n\ A_{2n}.$$

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Since $diag : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is final (cf. [8, p. 217]), we see that

$$\operatorname*{colim}_{n} A_n \times A_n \cong \operatorname*{colim}_{n,m} A_n \times A_m \cong \operatorname*{colim}_{n} A_n \times \operatorname*{colim}_{n} A_n$$

holds (cf. Lemma 3.2). Since $2 \times \cdot : \mathbb{N} \to \mathbb{N}$ is final, we have

$$\operatorname*{colim}_{n} A_{2n} \cong \operatorname*{colim}_{n} A_n.$$

Hence, $\operatorname*{colim}_{n} f_{n,n}$ defines the morphism

$$f : \operatorname*{colim}_{n} A_n \times \operatorname*{colim}_{n} A_n \to \operatorname*{colim}_{n} A_n,$$

which is the desired one. □

Let us recall the explicit description of the H-structure of $G_\infty(\mathbb{F})$ (cf. [4, Theorem 4.5 in Chapter 9]).

For $n \leqslant n'$, define the inclusion

$$\iota_{n,n'} : G_n(\mathbb{F}^{2n}) \hookrightarrow G_{n'}(\mathbb{F}^{2n'})$$

by $\iota_{n,n'}(V) = \mathbb{F}^{n'-n} \oplus V \oplus 0 \quad (\subset \mathbb{F}^{n'-n} \oplus \mathbb{F}^{2n} \oplus \mathbb{F}^{n'-n} = \mathbb{F}^{2n'})$. For $n > 0$, define the linear monomorphisms $j_o, j_e :\ \mathbb{F}^{2n} \hookrightarrow \mathbb{F}^{2\cdot 2n}$ by

$$j_o(x_1, \ldots, x_{2n}) = (x_1, 0, x_2, 0, \ldots, x_{2n}, 0),$$

$$j_e(x_1, \ldots, x_{2n}) = (0, x_1, 0, x_2, \ldots, 0, x_{2n}),$$

and define the map $\mu_{n,n} :\ G_n(\mathbb{F}^{2n}) \times G_n(\mathbb{F}^{2n}) \longrightarrow G_{2n}(\mathbb{F}^{2\cdot 2n})$ by

$$\mu_{n,n}(V, W) = j_o(V) + j_e(W).$$

The assignments

$$n \longmapsto G_n(\mathbb{F}^{2n}) \text{ and } n \leqslant n' \longmapsto \iota_{n,n'}$$

define a functor from $\mathbb{N}$ to $\mathcal{K}_*$, which is denoted by G. Since the family of pointed maps

$$\{\mu_{n,n} : G_n(\mathbb{F}^{2n}) \times G_n(\mathbb{F}^{2n}) \longrightarrow G_{2n}(\mathbb{F}^{2\cdot 2n})\}_{n\in\mathbb{N}}$$

satisfies the compatibility condition in Lemma 3.4, we have the map

$$\mu : G_\infty(\mathbb{F}) \times G_\infty(\mathbb{F}) \longrightarrow G_\infty(\mathbb{F})$$

by Lemma 3.4, which is just the H-structure classifying the reduced K-theory $\tilde{K}_{\mathbb{F}}$.

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Note that the pointed set $\operatorname*{colim}_n [X, G_n(\mathbb{F}^{2n})]_*$ is the colimit of the composite functor

$$\mathbb{N} \xrightarrow{\ G\ } \mathcal{K}_* \xrightarrow{[X,\ ]_*} Set_*,$$

which is denoted by A. Since the family of pointed maps

$$\{\mu_{n,n\sharp} : [X, G_n(\mathbb{F}^{2n})]_* \times [X, G_n(\mathbb{F}^{2n})]_* \longrightarrow [X, G_{2n}(\mathbb{F}^{2\cdot 2n})]_*\}$$

satisfies the compatibility condition in Lemma 3.4, we have the operation

$$\alpha : \operatorname*{colim}_n [X, G_n(\mathbb{F}^{2n})]_* \times \operatorname*{colim}_n [X, G_n(\mathbb{F}^{2n})]_* \longrightarrow \operatorname*{colim}_n [X, G_n(\mathbb{F}^{2n})]_*$$

by Lemma 3.4.

Lemma 3.5. *The operation α defines an abelian group structure on the pointed set $\operatorname*{colim}_n [X, G_n(\mathbb{F}^{2n})]_*$, making the natural isomorphism in Proposition 3.1 one of abelian groups.*

Proof. The result is obtained from the construction. □

We also need the following lemmas.

Lemma 3.6. *The composites*

$$G_n(\mathbb{F}^{2n}) \xrightarrow{(1,0)} G_n(\mathbb{F}^{2n}) \times G_n(\mathbb{F}^{2n}) \xrightarrow{\mu_{n,n}} G_{2n}(\mathbb{F}^{2\cdot 2n}),$$

$$G_n(\mathbb{F}^{2n}) \xrightarrow{(0,1)} G_n(\mathbb{F}^{2n}) \times G_n(\mathbb{F}^{2n}) \xrightarrow{\mu_{n,n}} G_{2n}(\mathbb{F}^{2\cdot 2n})$$

are homotopic to $\iota_{n,2n}$, where 1 and 0 denote the identify map and the trivial map onto the base point, respectively.

Proof. The result follows from the definitions of $\iota_{n,2n}$ and $\mu_{n,n}$. □

Lemma 3.7. *Let $\mathcal{C}$ be a cocomplete, additive category. Let $A : \mathbb{N} \to \mathcal{C}$ be a functor and set $\phi_{k,l} = A(k \leqslant l)$. Then $\{f_{n,n} := \phi_{n,2n} + \phi_{n,2n} : A_n \oplus A_n \to A_{2n}\}_{n\in\mathbb{N}}$ satisfies the compatibility condition in Lemma 3.4, and the morphism $f : \operatorname*{colim}_n A_n \oplus \operatorname*{colim}_n A_n \to \operatorname*{colim}_n A_n$ defined by Lemma 3.4 is just the addition of $\operatorname*{colim}_n A_n$.*

Proof. It is obvious that $\{f_{n,n}\}$ satisfies the compatibility condition. The morphism f is defined by the commutative diagram

$$\begin{array}{ccc} A_n \oplus A_n & \xrightarrow{f_{n,n}} & A_{2n} \\ \downarrow{\scriptstyle \phi_n \oplus \phi_n} & & \downarrow{\scriptstyle \phi_{2n}} \\ \operatorname*{colim}_n A_n \oplus \operatorname*{colim}_n A_n & \xrightarrow{f} & \operatorname*{colim}_n A_n. \end{array}$$

Since $\phi_{2n} \circ f_{n,n} = \phi_n + \phi_n$ holds, we see that f is just the addition of $\operatorname*{colim}_n A_n$. □

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3.2. The homotopy set $[X, F_Y]_*$

Let Y be a nilpotent CW-complex of finite type with finite fundamental group. Let F_Y be the homotopy fibre of the profinite completion $c_Y : Y \to \widehat{Y}$ of Y. F_Y is homotopy equivalent to the weak product of Eilenberg–MacLane complexes $\prod_{i>0} K(\pi_{i+1}(Y) \otimes \widehat{\mathbb{Z}}/\mathbb{Z}, i)$ by Roitberg and Touhey [10, Theorem 1.1]. However, the homotopy equivalence is not natural, and it is not possible to naturally identify the homotopy set $[X, F_Y]_*$ with the product of cohomology groups $\prod_{i>0} H^i(X; \pi_{i+1}(Y) \otimes \widehat{\mathbb{Z}}/\mathbb{Z})$. Nevertheless, we have the following natural isomorphism under the assumption that X is a C_0-space.

Proposition 3.8. *Let X be a CW-complex which is a C_0-space, and Y a nilpotent CW-complex of finite type with finite fundamental group. Then there exists an isomorphism of pointed sets*

$$[X, F_Y]_* \cong \prod_{i>0} H^i(X; \pi_{i+1}(Y) \otimes \widehat{\mathbb{Z}}/\mathbb{Z})$$

which is natural with respect to Y. In particular, $[X, F_\bullet]$ can be regarded as an Ab-valued functor.

Proof. Take a homotopy equivalence h from F_Y to $\prod_{i>0} K(\pi_{i+1}(Y) \otimes \widehat{\mathbb{Z}}/\mathbb{Z}, i)$ which induces the identity on their homotopy groups. Then, by Theorem 1.5, the bijection

$$[X, F_Y]_* \cong \prod_{i>0} H^i(X; \pi_{i+1}(Y) \otimes \widehat{\mathbb{Z}}/\mathbb{Z})$$

induced by h does not depend on the choice of h, and this bijection is natural with respect to Y. □

3.3. Proofs of Theorem 1.6 and Corollary 1.7

Proof of Theorem 1.6. We prove the theorem in the case of $\mathbb{F} = \mathbb{C}$; similar arguments apply to the case of $\mathbb{F} = \mathbb{H}, \mathbb{R}$ (cf. the proof of [7, Theorem 1.5]).

By Proposition 3.1 and Lemma 3.5, we have the natural isomorphisms of abelian groups

$$\begin{aligned} K_0(C_{\mathbb{C}}(X)) &\cong \mathbb{Z} \oplus [X, G_\infty(\mathbb{C})]_{CB_*} \\ &\cong \mathbb{Z} \oplus \operatorname*{colim}_n [X, G_n(\mathbb{C}^{2n})]_*. \end{aligned}$$

Thus, it is sufficient to show that there is a natural isomorphism

$$\operatorname*{colim}_n [X, G_n(\mathbb{C}^{2n})]_* \cong \bigoplus_{l>0} H^{2l-1}(X; \widehat{\mathbb{Z}}/\mathbb{Z})$$

of abelian groups.

Recall that $\operatorname*{colim}_n [X, G_n(\mathbb{C}^{2n})]_*$ is the colimit of the functor $A : \mathbb{N} \to Set_*$ defined by

$$A(n) = [X, G_n(\mathbb{C}^{2n})]_* \ \text{ and } \ A(n \leqslant n') = \iota_{n,n'\sharp},$$

and the addition α is defined by the family

$$\{\mu_{n,n\sharp} : [X, G_n(\mathbb{C}^{2n})]_* \times [X, G_n(\mathbb{C}^{2n})]_* \to [X, G_{2n}(\mathbb{C}^{2\cdot 2n})]_*\}$$

via Lemma 3.4 (cf. Lemma 3.5).

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Since $\operatorname{map}_*(X, \widehat{G_n(\mathbb{C}^{2n})})$ is weakly contractible, we have the natural identification

$$[X, G_n(\mathbb{C}^{2n})]_* \cong [X, F_{G_n(\mathbb{C}^{2n})}]_*,$$

under which $\iota_{n,n'\sharp}$ is identified with $F_{\iota_{n,n'\sharp}}$ and $\mu_{n,n\sharp}$ is identified with

$$[X, F_{G_n(\mathbb{C}^{2n})}]_* \times [X, F_{G_n(\mathbb{C}^{2n})}]_* = [X, F_{G_n(\mathbb{C}^{2n})\times G_n(\mathbb{C}^{2n})}]_* \xrightarrow{F_{\mu_{n,n\sharp}}} [X, F_{G_{2n}(\mathbb{C}^{2\cdot 2n})}]_*.$$

By Proposition 3.8, we have a further natural identification

$$[X, G_n(\mathbb{C}^{2n})]_* \cong \prod_{i>0} H^i(X; \pi_{i+1}(G_n(\mathbb{C}^{2n})) \otimes \widehat{\mathbb{Z}}/\mathbb{Z}),$$

under which $\iota_{n,n'\sharp}$ is identified with $\prod_{i>0} H^i(X; \pi_{i+1}(\iota_{n,n'}) \otimes \widehat{\mathbb{Z}}/\mathbb{Z})$ and $\mu_{n,n\sharp}$ is identified with

$$\prod_{i>0} H^i(X; \pi_{i+1}(G_n(\mathbb{C}^{2n})) \otimes \widehat{\mathbb{Z}}/\mathbb{Z}) \oplus \prod_{i>0} H^i(X; \pi_{i+1}(G_n(\mathbb{C}^{2n})) \otimes \widehat{\mathbb{Z}}/\mathbb{Z})$$
$$\xrightarrow{\prod_{i>0} H^i(X;\pi_{i+1}(\iota_{n,2n})\otimes\widehat{\mathbb{Z}}/\mathbb{Z}) + \prod_{i>0} H^i(X;\pi_{i+1}(\iota_{n,2n})\otimes\widehat{\mathbb{Z}}/\mathbb{Z})} \prod_{i>0} H^i(X; \pi_{i+1}(G_{2n}(\mathbb{C}^{2\cdot 2n})) \otimes \widehat{\mathbb{Z}}/\mathbb{Z})$$

(cf. Lemma 3.6). Define the functor $\tilde{A} : \mathbb{N} \longrightarrow Ab$ by

$$\tilde{A}(n) = \prod_{i>0} H^i(X; \pi_{i+1}(G_n(\mathbb{C}^{2n})) \otimes \widehat{\mathbb{Z}}/\mathbb{Z}),$$

$$\tilde{A}(n \leqslant n') = \prod_{i>0} H^i(X; \pi_{i+1}(\iota_{n,n'}) \otimes \widehat{\mathbb{Z}}/\mathbb{Z}).$$

Then, from the identifications of $[X, G_n(\mathbb{C}^{2n})]_*$ and $\iota_{n,n\sharp}$, $\tilde{A} : \mathbb{N} \longrightarrow Ab$ is a lift of $A : \mathbb{N} \longrightarrow Set_*$. Thus, we have the isomorphism of abelian groups

$$\operatorname{colim} \tilde{A} \cong \operatorname*{colim}_n\, [X, G_n(\mathbb{C}^{2n})]_*$$

from the identification of $\mu_{n,n\sharp}$ (cf. Lemma 3.7).

Recall that $G_n(\mathbb{C}^{2n})$ has non-trivial rational homotopy groups in only finitely many dimensions, and that $\operatorname*{colim}_n$ and $\bigoplus_{i>0}$ commute in Ab [8, p. 231]. Then, we obtain the natural isomorphisms of abelian groups

$$\begin{aligned}\operatorname*{colim}_n\, [X, G_n(\mathbb{C}^{2n})]_* &\cong \operatorname*{colim}_n \bigoplus_{i>0} H^i(X; \pi_{i+1}(G_n(\mathbb{C}^{2n})) \otimes \widehat{\mathbb{Z}}/\mathbb{Z}) \\ &\cong \bigoplus_{i>0} H^i(X; \pi_{i+1}(G_\infty(\mathbb{C})) \otimes \widehat{\mathbb{Z}}/\mathbb{Z}).\end{aligned}$$

Finally, we obtain the natural isomorphism

$$\bigoplus_{i>0} H^i(X; \pi_{i+1}(G_\infty(\mathbb{C})) \otimes \widehat{\mathbb{Z}}/\mathbb{Z}) \cong \bigoplus_{l>0} H^{2l-1}(X; \widehat{\mathbb{Z}}/\mathbb{Z})$$

of abelian groups by Bott periodicity. □

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Proof of Corollary 1.7. X satisfies the condition that $\mathrm{map}_*(X, \widehat{Y})$ is weakly contractible for any nilpotent finite complex Y (cf. [7, Example 1.6]). Since X is rationally equivalent to S^{2n+1}, X is a C_0-space. Thus, Theorem 1.6 applies to X. □

Let X be a space obtained by performing finitely many operations of applying Q and $\cdot^{(k)}$ $(k \geqslant 2n+1)$ on a space described in (1)–(4) in the statement of Corollary 1.7. Then, the result of Corollary 1.7 holds for such X as well (cf. the proof of Corollary 1.7).

Remark 3.9. From the implications in Remark 1.2(1), the following assertion is weaker than Theorem 1.5 but stronger than Lemma 5.3 in [7].

If X is a co-H_0-space, then X satisfies the condition (P).

The proof is much easier than that of Theorem 1.5. Let $\mathcal{P}$ denote the class of pointed connected CW-complexes satisfying the condition (P). Then it is easily seen that the class $\mathcal{P}$ has the following properties.

(i) $\mathcal{P}$ is closed under rationalization.

(ii) The spheres S^n $(n \geqslant 1)$ are in $\mathcal{P}$.

(iii) $\mathcal{P}$ is closed under arbitrary wedge sum.

From these properties, co-H_0-spaces are in $\mathcal{P}$.

Remark 3.10. The proof of Lemma 5.3 in [7] is incorrect. In fact, the decomposition of $H^n(\prod_i K(\pi_i, i); \pi'_n)$ in the proof is wrong without the finite-dimensional assumptions on the π_i or the π'_n. To decompose $H^n(\prod_i K(\pi_i, i); \pi'_n)$, the following two isomorphisms for $\mathbb{Q}$-modules were used:

(i) $(U \otimes V)^* \cong U^* \otimes V^*$,

(ii) $\mathrm{Hom}(U, V) \cong V \otimes U^*$.

The isomorphism (i) does not hold if U and V are infinite dimensional. The isomorphism (ii) does not hold either if U and V are infinite dimensional; since an element of $V \otimes U^*$ is a finite sum of the form $\sum v_i \otimes u^i$, the image of the corresponding linear map must be a finite-dimensional $\mathbb{Q}$-submodule of V (contained in *span* $\{v_i\}$).

The 'only if' part of the proof of Theorem 1.5 is a corrected version of the proof of Lemma 5.3 in [7]. As mentioned above, Remark 3.9 gives an alternative proof of Lemma 5.3 in [7].

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