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# GROUPS WHOSE NONNORMAL SUBGROUPS ARE METAHAMILTONIAN

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**Abstract**

If $\mathfrak{X}$ is a class of groups, we define a sequence $\mathfrak{X}_1, \mathfrak{X}_2, \ldots, \mathfrak{X}_k, \ldots$ of group classes by putting $\mathfrak{X}_1 = \mathfrak{X}$ and choosing $\mathfrak{X}_{k+1}$ as the class of all groups whose nonnormal subgroups belong to $\mathfrak{X}_k$. In particular, if $\mathfrak{A}$ is the class of abelian groups, $\mathfrak{A}_2$ is the class of metahamiltonian groups, that is, groups whose nonnormal subgroups are abelian. The aim of this paper is to study the structure of $\mathfrak{X}_k$-groups, with special emphasis on the case $\mathfrak{X} = \mathfrak{A}$. Among other results, it will be proved that a group has a finite commutator subgroup if and only if it is locally graded and belongs to $\mathfrak{A}_k$ for some positive integer $k$.

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## 1. Introduction

It is well known that a group has only normal subgroups if and only if it is either abelian or the direct product of a quaternion group of order 8 and a periodic abelian group with no elements of order 4. The structure of groups for which the set of nonnormal subgroups is small in some sense has been studied by many authors in several different situations.

A group is called *metahamiltonian* if all its nonabelian subgroups are normal. Metahamiltonian groups were introduced and studied by Romalis and Sesekin. They proved in particular that the commutator subgroup of any soluble metahamiltonian group is finite of prime-power order (see [14–16]). Of course, any group whose commutator subgroup has prime order is metahamiltonian, and it was shown by Kuzennyi and Semko [8] that a soluble group $G$ is metahamiltonian if and only if each nonabelian subgroup of $G$ contains the commutator subgroup $G'$. On the other hand, Tarski groups, that is, infinite simple groups whose proper nontrivial subgroups have prime order, are obviously metahamiltonian, and in order to avoid pathological cases of this type it is usual to work within the large universe of locally graded groups. Here a group $G$ is said to be *locally graded* if every finitely generated nontrivial

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subgroup of $G$ contains a proper subgroup of finite index. For instance, it turns out that locally graded metahamiltonian groups are soluble of derived length at most 3 (see [4]). More information on metahamiltonian groups and related properties can be found in the recent papers [5] and [6] and in the survey article [3].

The aim of this paper is to provide a further contribution to this topic by looking at metahamiltonian groups in the general framework of group classes that can be obtained by iterating a restriction on nonnormal subgroups.

Let $\mathfrak{X}$ be a class of groups. Put $\mathfrak{X}_1 = \mathfrak{X}$, and suppose by induction that a group class $\mathfrak{X}_k$ has been defined for some positive integer $k$; then we denote by $\mathfrak{X}_{k+1}$ the class consisting of all groups in which every nonnormal subgroup belongs to $\mathfrak{X}_k$.

It is clear that a group class $\mathfrak{X}$ is not in general contained in $\mathfrak{X}_2$, as consideration of the class of simple groups shows, but this is certainly the case if $\mathfrak{X}$ is closed with respect to forming subgroups. Moreover, for each positive integer $k$ the class $\mathfrak{X}_{k+1}$ is subgroup closed and $\mathfrak{X}_{k+1} = (\mathfrak{X}_k)_2$, so that $\mathfrak{X}_h \leq \mathfrak{X}_k$ whenever $2 \leq h \leq k$. We put

$$\mathfrak{X}_\infty = \bigcup_{k\geq 1} \mathfrak{X}_k.$$

If $\mathfrak{X}$ is chosen to be the class $\mathfrak{A}$ of abelian groups, then $\mathfrak{A}_2$ is the usual class of metahamiltonian groups and $\mathfrak{A}_3$ consists of all groups whose nonnormal subgroups are metahamiltonian. Members of the class $\mathfrak{A}_k$ will be called *k-hamiltonian groups* (for any $k$). Thus the 1-hamiltonian and 2-hamiltonian groups are precisely the abelian and the metahamiltonian groups, respectively. Here we will investigate the structure of $k$-hamiltonian locally graded groups for $k \geq 3$.

Let $k$ be any positive integer, and let $p_1, \ldots, p_k$ be pairwise distinct odd prime numbers. If $A$ is a cyclic group of order $p_1 \cdots p_k$ and $x$ is the automorphism of $A$ which inverts all elements, the semidirect product $\langle x\rangle \ltimes A$ is a $(k + 1)$-hamiltonian group which is not $k$-hamiltonian. Therefore $\mathfrak{A}_k$ is a proper subclass of $\mathfrak{A}_{k+1}$ for each positive integer $k$.

It is clear that $\mathfrak{X}_k \leq \mathfrak{Y}_k$ for all $k$, whenever $\mathfrak{X}$ and $\mathfrak{Y}$ are arbitrary group classes such that $\mathfrak{X} \leq \mathfrak{Y}$. Notice also that if $\mathfrak{D}$ is the class of groups in which all subgroups are normal, then the classes $\mathfrak{A}_2$ and $\mathfrak{D}_2$ do not coincide, as consideration of the direct product of two copies of the quaternion group of order 8 shows.

Groups with a finite commutator subgroup have been characterised by Neumann [10] as those groups which have boundedly finite conjugacy classes of elements; he also proved that a group is finite-by-abelian if and only if each of its subgroups has finite index in its normal closure. The final result of the paper provides a further characterisation of the class $\mathfrak{F}\mathfrak{A}$ consisting of all groups with a finite commutator subgroup. In fact, it shows that $\mathfrak{F}\mathfrak{A}$ coincides with the class of locally graded $\mathfrak{A}_\infty$-groups. Therefore the class $\mathfrak{F}\mathfrak{A}$ is saturated by $k$-hamiltonian locally graded groups, when $k$ ranges over all positive integers, and hence nothing more can be said about the size of the commutator subgroup of locally graded groups in the class $\mathfrak{A}_\infty$.

Most of our notation is standard and can be found in [12].

## 2. Statements and proofs

As we pointed out in the introduction, the class $\mathfrak{X}_k$ is closed with respect to subgroups for every $k > 1$, even if the class $\mathfrak{X} = \mathfrak{X}_1$ is not subgroup closed. We will begin this section by considering some closure properties that are inherited from $\mathfrak{X}$ to the group classes $\mathfrak{X}_k$.

First of all, notice that if $\mathfrak{X}$ is a group class which is closed with respect to homomorphic images, the same property obviously holds also for every $\mathfrak{X}_k$. Recall that a group class $\mathfrak{X}$ is *local* if a group $G$ belongs to $\mathfrak{X}$ whenever each of its finite subsets is contained in an $\mathfrak{X}$-subgroup. Clearly, a subgroup closed group class $\mathfrak{X}$ is local if and only if it contains all groups whose finitely generated subgroups belong to $\mathfrak{X}$. Our first lemma shows that the property of being local is inherited by the classes $\mathfrak{X}_k$.

Lemma 2.1. *Let $\mathfrak{X}$ be a local group class. Then, for each positive integer $k$, the class $\mathfrak{X}_k$ is also local.*

Proof. Since $\mathfrak{X}_1 = \mathfrak{X}$, the statement is obvious if $k = 1$. Suppose now that the class $\mathfrak{X}_k$ is local for some positive integer $k$. As $\mathfrak{X}_{k+1}$ is subgroup closed, it is enough to prove that a group $G$ belongs to $\mathfrak{X}_{k+1}$ provided that all its finitely generated subgroups are $\mathfrak{X}_{k+1}$-groups. Let $X$ be any subgroup of $G$ which is not in $\mathfrak{X}_k$, and let $\mathcal{W}_X$ be the set of all finitely generated subgroups of $X$ which are not contained in an $\mathfrak{X}_k$-subgroup of $G$. Then $\mathcal{W}_X$ is not empty, because the class $\mathfrak{X}_k$ is local. If $g$ is any element of $G$ and $U \in \mathcal{W}_X$, the subgroup $\langle g, U\rangle$ belongs to $\mathfrak{X}_{k+1}$, whence $U^g = U$. It follows that all elements of $\mathcal{W}_X$ are normal in $G$. Moreover, $\langle x, U\rangle \in \mathcal{W}_X$ for all $x \in X$, and so

$$X = \langle V \mid V \in \mathcal{W}_X\rangle$$

is likewise normal in $G$. Therefore $G$ belongs to $\mathfrak{X}_{k+1}$, and hence $\mathfrak{X}_{k+1}$ is a local class. □

Since the class of abelian groups is obviously local, the choice $\mathfrak{X} = \mathfrak{A}$ in the above statement gives the following interesting special case, which generalises a well-known property of metahamiltonian groups.

Corollary 2.2. *For each positive integer $k$, the class of $k$-hamiltonian groups is local.*

Let $\mathfrak{X}$ be a group class. A subgroup $X$ of a group $G$ is said to be *compressed* by $\mathfrak{X}$ if it contains a normal subgroup $N$ of $G$ such that $G/N$ is an $\mathfrak{X}$-group; in this case, such a subgroup $N$ will be called an *$\mathfrak{X}$-compressor* for $X$ in $G$. Of course, if the class $\mathfrak{X}$ is closed with respect to homomorphic images, the core $X_G$ of an $\mathfrak{X}$-compressed subgroup $X$ is an $\mathfrak{X}$-compressor for $X$ in $G$. It is also clear that in any group the class of finite groups compresses all subgroups of finite index.

Lemma 2.3. *Let $\mathfrak{X}$ be a group class, and let $X$ and $Y$ be subgroups of a group $G$ such that $X$ is not normal in $G$ and $Y \leq X$. If $Y$ is compressed in $G$ by the class $\mathfrak{X}_k$ for some integer $k > 1$, then $Y$ is compressed in $X$ by $\mathfrak{X}_{k-1}$.*

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Proof. Let $N$ be an $\mathfrak{X}_k$-compressor for $Y$ in $G$. Then $X/N$ is a nonnormal subgroup of the $\mathfrak{X}_k$-group $G/N$, and hence it belongs to $\mathfrak{X}_{k-1}$, which means that $Y$ is compressed by $\mathfrak{X}_{k-1}$ in $X$. □

We shall say that a subgroup closed group class $\mathfrak{X}$ is a *Robinson class* if every finitely generated hyper-(abelian or finite) group, whose subgroups of finite index are compressed by $\mathfrak{X}$, belongs to $\mathfrak{X}$ and is polycyclic-by-finite. Recall here that a group is *hyper-(abelian or finite)* if it has an ascending normal series whose factors are either abelian or finite. If the class $\mathfrak{X}$ is closed also with respect to homomorphic images, then $\mathfrak{X}$ is a Robinson class if and only if every finitely generated hyper-(abelian or finite) group whose finite homomorphic images belong to $\mathfrak{X}$ is polycyclic-by-finite and belongs to $\mathfrak{X}$. The most relevant group class of this type is that of nilpotent groups, a result that was proved by Robinson [11]. On the other hand, although any polycyclic-by-finite group whose finite homomorphic images are supersoluble is likewise supersoluble (see [1]), it is easy to see that supersoluble groups do not form a Robinson class. It follows easily from Robinson's theorem and from the fact that polycyclic groups are residually finite that the class $\mathfrak{N}_c$ of all nilpotent groups of class at most $c$ has the Robinson property. Thus $\mathfrak{A}$ is a Robinson class. As shown in [2], the class $\mathfrak{A}_2$ of metahamiltonian groups also has the Robinson property. Our next theorem proves that $k$-hamiltonian groups form a Robinson class, for each positive integer $k$.

Theorem 2.4. *Let $\mathfrak{X}$ be a Robinson class of groups. Then, for each positive integer $k$, the class $\mathfrak{X}_k$ also has the Robinson property.*

Proof. The statement is obvious if $k = 1$. Suppose now by induction on $k$ that $\mathfrak{X}_k$ is a Robinson class for some positive integer $k$, and let $G$ be any finitely generated hyper-(abelian or finite) group in which all subgroups of finite index are compressed by $\mathfrak{X}_{k+1}$. If $X$ is any nonnormal subgroup of finite index of $G$, it follows from Lemma 2.3 that every subgroup of finite index of $X$ is compressed in $X$ by the Robinson class $\mathfrak{X}_k$; then the finitely generated hyper-(abelian or finite) group $X$ belongs to $\mathfrak{X}_k$, and in particular it is polycyclic-by-finite. On the other hand, if all subgroups of finite index of $G$ are normal, then every finite homomorphic image of $G$ is nilpotent, and so it follows from Robinson's theorem that $G$ itself is nilpotent and hence also polycyclic. Therefore $G$ is polycyclic-by-finite in any case.

Let $H$ be any nonnormal subgroup of $G$. Since $H$ is the intersection of a collection of subgroups of finite index of $G$ (see, for instance, [13, 5.4.16]), it is contained in a subgroup $K$ of finite index which is not normal in $G$. Thus $K$ belongs to $\mathfrak{X}_k$ and so $H$ is an $\mathfrak{X}_k$-group, because $\mathfrak{X}_k$ is subgroup closed. Therefore $G$ belongs to $\mathfrak{X}_{k+1}$ and hence $\mathfrak{X}_{k+1}$ is a Robinson class. □

Let $\mathfrak{X}$ be a class of groups. A group is said to be *minimal non-*$\mathfrak{X}$ if it is not an $\mathfrak{X}$-group but all its proper subgroups belong to $\mathfrak{X}$. We shall say that $\mathfrak{X}$ is *accessible* if every locally graded group whose proper subgroups belong to $\mathfrak{X}$ is either finite or an $\mathfrak{X}$-group, or equivalently if any locally graded minimal non-$\mathfrak{X}$ group is finite. In particular, a group class $\mathfrak{X}$ containing all finite groups is accessible if and only if there

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are no minimal non-$\mathfrak{X}$ groups in the universe of locally graded groups. It is easy to show that abelian groups form an accessible group class and $\mathfrak{A}$ shares such a property with other relevant classes of groups. Further information on accessible group classes can be found in [7].

It is known that the class of metahamiltonian groups is accessible (see [2]), and we will see that all $\mathfrak{A}_k$ are accessible classes.

Lemma 2.5. *Let $\mathfrak{X}$ be a subgroup closed group class and let $G$ be a group in the class $\mathfrak{X}_k$ for some integer $k > 1$. Then all proper subgroups of $G''$ belong to $\mathfrak{X}_{k-1}$.*

Proof. Let $X$ be any subgroup of $G$ which is not in $\mathfrak{X}_{k-1}$. Then all subgroups of $G$ containing $X$ are normal, so that $G/X$ is a Dedekind group and hence $G'' \leq X$. It follows that all proper subgroups of $G''$ belong to $\mathfrak{X}_{k-1}$. □

Theorem 2.6. *Let $\mathfrak{X}$ be a Robinson class consisting of soluble-by-finite groups, which is local and closed with respect to homomorphic images. If $\mathfrak{X}$ is accessible, then for each positive integer $k$ the class $\mathfrak{X}_k$ is also accessible.*

Proof. Assume for a contradiction that the statement is false and let $k$ be the smallest positive integer such that there exists an infinite locally graded group $G$ which is not an $\mathfrak{X}_k$-group while all its proper subgroups belong to $\mathfrak{X}_k$. Clearly, $k > 1$ since $\mathfrak{X}$ is accessible. Moreover, $G$ is finitely generated because $\mathfrak{X}_k$ is a local class by Lemma 2.1, and so $G$ contains a proper subgroup $X$ of finite index. By an iterated application of Lemma 2.5, there is a positive integer $n$ such that the subgroup $X^{(n)}$ either is minimal non-$\mathfrak{X}_h$ for some $h < k$ or belongs to $\mathfrak{X}$. In the first case $X^{(n)}$ is finite by the minimal assumption on $k$ and so it follows that the subgroup $X$ is soluble-by-finite in any case, and hence $G$ itself is soluble-by-finite.

Since $\mathfrak{X}_k$ is a Robinson class by Theorem 2.4, the group $G$ contains a normal subgroup $N$ of finite index such that $G/N$ is not in $\mathfrak{X}_k$. On the other hand, the Frattini factor group $G/\Phi(G)$ is infinite by a result of Lennox (see [9]), and so there exists a maximal subgroup $M$ of $G$ such that $G = MN$. It follows that $G/N \simeq M/M \cap N$ is an $\mathfrak{X}_k$-group, and this contradiction completes the proof of the theorem. □

Corollary 2.7. *For each positive integer $k$ the class of $k$-hamiltonian groups is accessible.*

Lemma 2.8. *Let $\mathfrak{X}$ be a subgroup closed group class such that $\mathfrak{X}_k$ is accessible for each positive integer $k$. If $\mathfrak{X}$ consists of soluble-by-finite groups, then all groups in $\mathfrak{X}_\infty$ are soluble-by-finite.*

Proof. Assume for a contradiction that the statement is false and let $k$ be the smallest positive integer such that the class $\mathfrak{X}_k$ contains a group $G$ which is not soluble-by-finite. Then $k > 1$ and it follows from Lemma 2.5 that all proper subgroups of $G''$ belong to $\mathfrak{X}_{k-1}$. As the class $\mathfrak{X}_{k-1}$ is accessible, $G''$ either is finite or belongs to $\mathfrak{X}_{k-1}$. In any case, $G''$ is soluble-by-finite, an evident contradiction. □

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Corollary 2.9. *Let k be a positive integer and let G be a locally graded k-hamiltonian group. Then G is soluble-by-finite.*

Proof. By Corollary 2.7 the group class $\mathfrak{A}_k$ is accessible for each positive integer $k$ and it follows from Lemma 2.8 that all groups in $\mathfrak{A}_\infty$ are soluble-by-finite. The statement is proved. □

It was mentioned in the introduction that any soluble metahamiltonian group has derived length at most 3 and its commutator subgroup has prime-power order. Notice here that $GL(2,3)$ is a soluble 3-hamiltonian group of derived length 4 and its commutator subgroup has order 24.

It follows from Corollary 2.9 that $k$-hamiltonian locally soluble groups are soluble and we can now show that their derived length is bounded in terms of $k$. In our next statement $\mathfrak{A}^n$ denotes the class of soluble groups of derived length at most $n$.

Lemma 2.10. *Let G be a soluble group in the class* $(\mathfrak{A}^n)_k$, *where n and k are positive integers. Then G has derived length at most* $n + 3(k-1)$.

Proof. It can obviously be assumed that $k > 1$. Then it follows from Lemma 2.5 that all proper subgroups of $G''$ belong to the class $(\mathfrak{A}^n)_{k-1}$ and hence by induction $G'''$ has derived length at most $n + 3(k-2)$. Therefore the derived length of $G$ is at most

$$n + 3(k-2) + 3 = n + 3(k-1)$$

and the result is proved. □

Corollary 2.11. *Let k be a positive integer, and let G be a k-hamiltonian locally soluble group. Then G is soluble of derived length at most* $3k-2$.

Our next lemma shows in particular that locally graded $k$-hamiltonian groups locally satisfy the maximal condition on subgroups.

Lemma 2.12. *Let k be a positive integer, and let G be a finitely generated locally graded k-hamiltonian group. Then G is nilpotent-by-finite.*

Proof. The group $G$ is soluble-by-finite by Corollary 2.9. If every subgroup of finite index of $G$ is normal, then all finite homomorphic images of $G$ are trivially nilpotent and so $G$ itself is nilpotent. Suppose now that $G$ contains a nonnormal subgroup $X$ of finite index, so that $k > 1$ and $X$ is a finitely generated $(k-1)$-hamiltonian group. By induction on $k$, it follows that $X$ is nilpotent-by-finite and hence $G$ itself is nilpotent-by-finite. □

Our final result characterises finite-by-abelian groups as those locally graded groups that belong to some $\mathfrak{A}_k$.

Theorem 2.13. *A locally graded group G has a finite commutator subgroup if and only if it belongs to the class* $\mathfrak{A}_\infty$.

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Proof. Assume first that there exist locally graded $\mathfrak{A}_\infty$-groups with an infinite commutator subgroup, and let $k$ be the smallest positive integer such that $\mathfrak{A}_k$ contains a locally graded (and so even soluble-by-finite) group $G$ with $G'$ infinite. Of course, $k > 1$ and, by the local closure property of the class $\mathfrak{A}_{k-1}$, the group $G$ contains a finitely generated subgroup $E$ which is not $(k-1)$-hamiltonian. Then $E$ is normal in $G$ and $G/E$ is a Dedekind group, so that $G'/G' \cap E$ is finite. Since $E$ satisfies the maximal condition on subgroups, it follows that $G'$ is finitely generated. Let $T$ be the largest locally finite normal subgroup of $G$. Then $G'T/T$ is infinite, so that $G/T$ is likewise a counterexample, and hence we may suppose without loss of generality that $G$ has no nontrivial locally finite normal subgroups. By Lemma 2.5 all proper subgroups of $G''$ belong to $\mathfrak{A}_{k-1}$ and it follows from Corollary 2.7 that $G''$ is either finite or $(k-1)$-hamiltonian. Thus in any case $G'''$ is finite and hence even trivial, so that $G''$ is torsion-free abelian. Let $X$ be any nonnormal subgroup of $G'$. Then $X$ belongs to $\mathfrak{A}_{k-1}$, so that $X'$ is a finite subgroup of $G''$ and hence $X' = \{1\}$. It follows that all nonnormal subgroups of $G'$ are abelian, that is, $G'$ is metahamiltonian and $G''$ is finite, so that $G'' = \{1\}$ and $G'$ is torsion-free abelian. Similarly, if $Y$ is any nonnormal subgroup of $G$, its commutator subgroup $Y'$ is finite and so trivial. Therefore the group $G$ is metahamiltonian. This contradiction shows that all locally graded $k$-hamiltonian groups have a finite commutator subgroup.

Conversely, let $G$ be a group with a finite commutator subgroup, and let $m$ be the number of (not necessarily distinct) prime factors of the order of $G'$. Consider an arbitrary finite *nonnormal chain* of $G$, that is, a finite chain

$$X_0 < X_1 < \cdots < X_t$$

of subgroups of $G$ such that $X_i$ is not normal in $X_{i+1}$ for $i = 0, \ldots, t-1$. If $0 \le i \le j \le t$ and $X_i \cap G' = X_j \cap G'$,

$$[X_i, X_j] \le X_j \cap G' \le X_i,$$

so that $X_i$ is normal in $X_j$ and hence $i = j$. It follows that

$$X_0 \cap G' < X_1 \cap G' < \cdots < X_t \cap G'$$

and hence $t \le m$. On the other hand, it can be easily proved by induction that a group in which all (finite) nonnormal chains contain at most $m$ elements belongs to $\mathfrak{A}_{m+1}$ and so also to the class $\mathfrak{A}_\infty$. The proof is complete. □

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