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DIFFERENTIATION OF MULTIPARAMETER SUPERADDITIVE PROCESSES

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0. Introduction. In this article our purpose is to prove a differentiation theorem for multiparameter processes which are strongly superadditive with respect to a strongly continuous semigroup of positive L_1-contractions (see Section 1 for definitions).

Recently, the differentiation theorem for superadditive processes with respect to a one-parameter semigroup of positive L_1-contractions has been proved by D. Feyel [**9**]. Another proof is given by M. A. Akçoğlu [**1**]. R. Emilion and B. Hachem [**7**] also proved the same theorem, but with an extra assumption on the process (see also [**1**]). The proof of this theorem for superadditive processes with respect to a Markovian semigroup of operators on L_1 is given by M. A. Akçoğlu and U. Krengel [**4**]. Thus [**1**] and [**9**] extend the result of [**4**] to the sub-Markovian setting. Here we will obtain the multiparameter sub-Markovian version of this theorem, namely Theorem 3.17 below.

Theorem 3.17 was proved by M. A. Akçoğlu and U. Krengel [**5**] for superadditive processes $\{F_{(u,v)}\}$ with respect to a semigroup of operators $\{U_{(t,r)}\}$ which is induced by measurable semigroup of measure preserving transformations on $(X, \mathscr{F}, \mu)$. In that paper the definition of superadditivity used is stronger than the superadditivity definition we consider in this work [**5**] but weaker than the strong superadditivity. R. Emilion and B. Hachem [**8**] proved Theorem 3.17 for strongly superadditive processes with respect to a Markovian semigroup $\{U_{(t,r)}\}$ of operators. The proof for the case that $\{F_{(u,v)}\}$ is an additive process with respect to a two-parameter semigroup of positive L_1-contractions $\{U_{(t,r)}\}$ which is strongly continuous for $(t, r) > \mathbf{O}$ was given by M. A. Akçoğlu and A. del Junco [**3**]. Hence 3.17 generalizes these theorems as well as Theorem 1.7 in [**1**].

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1. *Definitions.* Let $\mathbf{R}^2$ be the usual two dimensional real vector space,

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considered together with all its usual structure. The positive cone of $\mathbf{R}^2$ is $\mathbf{R}^2_+$ and the interior of $\mathbf{R}^2_+$ is C. In particular $\mathbf{R}^2$ is partially ordered in the usual way. Let $\mathbf{1}$, $\mathbf{O}$ and $\mathbf{k}$ denote the vectors $(1, 1)$, $(0, 0)$ and (k, k), for any real k, respectively.

Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space and $L_1 = L_1(X, \mathscr{F}, \mu)$ be the classical Banach space of real valued integrable functions on X. L_1^+ will denote the positive cone of L_1, and for any $E \in \mathscr{F}$, $L_1(E) = L_1(E, \mu)$ will denote the class of integrable functions with support in E. We shall not distinguish between the equivalence classes of functions and the individual functions. The relations below are often defined only modulo sets of measure zero; the words a.e. may or may not be omitted. For any $E \in \mathscr{F}$, χ_E will denote the characteristic function of E.

Let $\{T_t\}_{t \geqq 0}$ and $\{S_r\}_{r \geqq 0}$ be one-parameter strongly continuous semigroups of positive L_1-contractions (sub-Markovian operators) with $T_0 = S_0 = I$, the identity operator on L_1, and $T_tS_r = S_rT_t$ for each $t \geqq 0$ and $r \geqq 0$. This means that for each $t \geqq 0$ and $s \geqq 0$, T_t and S_r are both bounded linear operators on L_1 with $\|T_t\|_1 \leqq 1$ and $\|S_r\|_1 \leqq 1$ such that

(1.1) $\quad T_tL_1^+ \subset L_1^+$ and $S_rL_1^+ \subset L_1^+$,

(1.2) $\quad T_tT_s = T_{t+s}$ and $S_rS_p = S_{r+p}$ for all $p, r, t, s \geqq 0$,

(1.3) $\quad \lim_{t \to 0^+} \|T_tf - f\|_1 = 0 = \lim_{s \to 0^+} \|S_rf - f\|_1$ for all $f \in L_1$.

$\{T_t\}_{t \geqq 0}$ and $\{S_r\}_{r \geqq 0}$ are called Markovian operators if they satisfy

$$\int T_tfd\mu = \int fd\mu = \int S_rfd\mu$$

for all $t \geqq 0$, $s \geqq 0$ and for all $f \in L_1$ in addition to the conditions (1.1), (1.2) and (1.3). Consider the family

$$\{U_{(t,r)}\}_{(t,r) \in \mathbf{R}^2_+} = \{T_tS_r\}_{(t,r) \in \mathbf{R}^2_+}$$

which is a two-parameter strongly continuous semigroup of positive L_1-contractions with $U_{\mathbf{O}} = I$. So

(1.4) $\quad U_{(t,r)}L_1^+ \subset L_1^+ \quad$ for $(t, r) \in \mathbf{R}^2_+$,

(1.5) $\quad U_{(t,r)}U_{(u,v)} = U_{(t+u,r+v)}$ for each $(t, r), (u, v) \in \mathbf{R}^2_+$,

(1.6) $\quad \lim_{(t,r) \to \mathbf{O}} \|U_{(t,r)}f - f\|_1 = 0 \quad$ for each $f \in L_1$.

A family of functions $(F_{(u,v)}\}_{(u,v) \in C}$ is called a superadditive process (with respect to $\{U_{(t,r)}\}_{(t,r) \in \mathbf{R}^2_+}$) if it is superadditive with respect to each parameter separately [**4**], [**10**], [**13**], [**6**]; i.e.,

(1.7) $\quad F_{(u,v)} \in L_1 \quad$ for each $(u, v) \in C$.

(1.8) For each $(t, r) \in \mathbf{R}^2_+$ and $(u, v) \in C$ with $\mathbf{O} \leqq (t, r) \leqq (u, v)$

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$$\text{a)}\quad F_{(u,v)} \geqq F_{(u,r)} + U_{(0,r)}F_{(u,v-r)} \quad \text{if } 0 < r < v,$$

$$\text{b)}\quad F_{(u,v)} \geqq F_{(t,v)} + U_{(t,0)}F_{(u-t,v)} \quad \text{if } 0 < t < u.$$

If $\{-F_{(u,v)}\}$ is superadditive, then $\{F_{(u,v)}\}$ is called subadditive (with respect to $\{U_{(t,r)}\}$); and if both $\{F_{(u,v)}\}$ and $\{-F_{(u,v)}\}$ are superadditive, then $\{F_{(u,v)}\}$ is called additive [**4**], [**3**].

A family of functions $\{F_{(u,v)}\}_{(u,v)\in C}$ is called a strongly superadditive process (with respect to $\{U_{(t,r)}\}_{(t,r)\in \mathbf{R}_+^2}$[**13**] if it satisfies (1.7) and

(1.9) if $(t, r), (u, v) \in C$ with $\mathbf{O} < (t, r) < (u, v)$,

then

$$F_{(t,r)} \leqq F_{(u,v)} - U_{(t,0)}F_{(u-t,v)} - U_{(0,r)}F_{(u,v-r)} + U_{(t,r)}F_{(u-t,v-r)}.$$

Any strongly superadditive process $\{F_{(u,v)}\}$ which satisfies

(1.10) $F_{(u,0)} = F_{(0,v)} \equiv 0, u > 0, v > 0$

is necessarily a superadditive process [**13**]. Below, when we mention a strongly superadditive process, we will mean a process satisfying (1.7), (1.9) and (1.10).

Let $D = \{m2^{-k}:m, k = 1, 2, \dots\}$ be the set of positive binary numbers, and let $D \times D = B$. A family of functions $\{F_{(u,v)}\}_{(u,v)\in B}$ defined on B will also be called superadditive process if $F_{(u,v)} \in L_1$ for each $(u, v) \in B$ and (1.8) is satisfied for each $(t, r), (u, v) \in B$. Similar definitions apply to subadditive and additive processes on B.

Throughout this paper only the two parameter case is considered and the extension of the results to arbitrary n-parameter case, $n \geqq 1$, is straightforward. By

$$q - \lim_{(u,v)\to\mathbf{O}}$$

we shall mean that the limit is taken as u and v approach to zero through the positive rational numbers [**4**], [**3**].

2. Positive superadditive processes. In this section will show that if $\{F_{(u,v)}\}$ is a superadditive process with

$$\sup_{(u,v)\in C} \frac{1}{uv}\int F^-_{(u,v)}d\mu < \infty,$$

where

$$F^-_{(u,v)} = \max(0, -F_{(u,v)}),$$

then it can be assumed to be a positive superadditive process with the further property that if $\{G_{(u,v)}\}$ is an additive process such that

$$0 \leqq G_{(u,v)} \leqq F_{(u,v)} \quad \text{for each } (u, v) \in C,$$

then $G_{(u,v)}$ is identically zero.

A family $\{F_{(u,v)}\}_{(u,v)\in C}$ of L_1-functions is called continuous if $(u, v) \mapsto F_{(u,v)}$ is a continuous function from C to L_1 with the norm topology of L_1. Observe that if $f \in L_1$, then $\{U_{(t,r)}f\}$ is a continuous family. Hence

$$I_{(t,r)}f = \int_0^t \int_0^r U_{(s_1,s_2)}f ds_2 ds_1$$

can be defined in the usual way as the L_1-limit of the corresponding Riemann sums. For convenience, we will consider a particular type of Riemann sums as in [**1**]. If α is a real number, let $[\alpha]$ be the largest integer which is strictly less than α. For a pair $(t, r) \in C$ and an integer $k \geqq 1$, let

$$I^k_{(t,r)} = 2^{-2k} \sum_{i=0}^{[t2^k]} \sum_{j=0}^{[r2^k]} S^j_{2^{-k}} T^i_{2^{-k}}$$

$$I^k_{\mathbf{O}} = 0.$$

Then

$$\lim_{k\to\infty} I^k_{(t,r)}f = I_{(t,r)}f$$

exists in L_1-norm for each $(t, r) \in C$ and each $f \in L_1$. This defines $I_{(t,r)}$ as a positive linear operator on L_1 with norm

$$\|I_{(t,r)}\| \leqq tr.$$

If ϕ is a bounded linear function on L_1, then

$$\phi(I_{(t,r)}f) = \int_0^t \int_0^r \phi(U_s f)ds, \quad f \in L_1.$$

Here we note that if $h \in L_1^+$ is a nonzero function, then $I_{(t,r)}h$ is also nonzero for each $(t, r) \in C$, which follows from the fact that $U_{(t,r)}h$ converges to h as $(t, r) \to \mathbf{O}^+$.

LEMMA 2.1. *Let $\{F(u, v)\}_{(u,v)\in B}$ be a superadditive process on B. Then*

$$I^{k+1}_{(u,v)}(4^{k+1}F_{\mathbf{2}^{-(k+1)}}) \leqq I^k_{(u,v)}(4^k F_{\mathbf{2}^{-k}}) \leqq F_{(u,v)}$$

for every $(u, v) \in B$ and for each sufficiently large integer $k \geqq 0$ such that $2^k u$ and $2^k v$ are integers.

Proof. Let $s = 2^{-(k+1)}$, and $u = 2m_1 s$, $v = 2m_2 s$. Then

$$I^{k+1}_{(u,v)}(4^{k+1}F_{\mathbf{2}^{-(k+1)}})$$
$$= \sum_{i=0}^{2m_1-1} \sum_{j=0}^{2m_2-1} T^i_{2^{-(k+1)}} S^j_{2^{-(k+1)}} F_{\mathbf{2}^{-(k+1)}}$$

$$= \sum_{i=0}^{2m_1-1} \sum_{j=0}^{2m_2-1} T_s^i S_s^j F_s$$

$$= \sum_{i=0}^{2m_1-1} T_i^s \left[\sum_{j=0}^{m_2-1} S_{2s}^j (F_s + S_s F_s) \right]$$

$$= \sum_{j=0}^{m_2-1} S_{2s}^j \left[\sum_{i=0}^{m_1-1} T_{2s}^i (F_s + T_s F_s) + S_s \sum_{i=0}^{m_1-1} T_{2s}^i (F_s + T_s F_s) \right]$$

$$\leqq \sum_{j=0}^{m_2-1} S_{2s}^j \left[\sum_{i=0}^{m_1-1} T_{2s}^i (F_{(2s,s)} + S_s F_{(2s,s)}) \right]$$

$$\leqq \sum_{j=0}^{m_2-1} \sum_{i=0}^{m_1-1} S_{2s}^j T_{2s}^i F_{(2s,2s)} \quad \text{by (1.8)(a) and (b)}$$

$$= I_{(u,v)}(4^k F_{2^{-k}}).$$

Now by superadditivity we see that, by induction,

$$I_{(u,v)}(4^k F_{2^{-k}}) \leqq F_{(u,v)}$$

giving the result desired.

LEMMA 2.2 *Let* $\{F_{(u,v)}\}_{(u,v) \in B}$ *be a positive superadditive process on B. Let*

$$f = \text{a.e.} \liminf_{\substack{(u,v) \to \mathbf{O} \\ (u,v) \in B}} \frac{1}{uv} F_{(u,v)}.$$

Then: a) *If* $h \in L_1^+$ *and* $h \leqq f$, *then*

$$I_{(u,v)} h \leqq F_{(u,v)} \quad \textit{for each } (u, v) \in B.$$

b) $f < \infty$ a.e. *and*

$$F_{(u,v)} \downarrow 0 \textit{ as } (u, v) \downarrow \mathbf{O} \textit{ in } B.$$

Proof. Let

$$f_n = \inf_{\substack{s_1,s_2 \leqq 2^{-n} \\ (s_1,s_2) \in B}} \frac{1}{s_1 s_2} F_{(s_1,s_2)}$$

and let

$$h_n = \min(h, f_n).$$

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Then $f_n \leqq f_{n+1}$ for each positive integer n. Thus $h_n \uparrow h$ as $n \to \infty$, that is why it is enough to show that

$$I_{(u,v)}h_n \leqq F_{(u,v)}.$$

If $k \geqq n$ is an integer such that both $2^k u$ and $2^k v$ are also integers, then

$$I^k_{(u,v)}h_n \leqq I^k_{(u,v)}(4^k F_{\mathbf{2}^{-k}}) \leqq F_{(u,v)}$$

since

$$f_n \leqq \frac{1}{s_1 s_2} F_{(s_1,s_2)}$$

for every $(s_1, s_2) \in B$ with $s_i \leqq 2^{-n}$, $i = 1, 2$. Thus this implies in turn that

$$I_{(u,v)}h_n \leqq F_{(u,v)}$$

giving (a).

If $f = \infty$ on a set of positive measure, then there is a nonzero $h \in L_1^+$ such that $Mh \leqq f$ for each constant $M \geqq 0$. Hence

$$MI_{(u,v)}h \leqq F_{(u,v)} \text{ for each } M \geqq 0$$

by (a). This is a contradiction since $I_{(u,v)}h$ is a nonzero function and $F_{(u,v)} \in L_1$. Now we observe that

$$F_{(u_1,v_1)} \geqq F_{(u_2,v_2)} \text{ if } u_1 \geqq u_2 \text{ and } v_1 \geqq v_2$$

where $(u_i, v_i) \in B$, $i = 1, 2$, by superadditivity and the positivity of $\{U_{(t,r)}\}$. If $F_{(u,v)}$ does not decrease to 0 a.e. as $(u, v) \to \mathbf{O}$, then f would be ∞ on a set of positive measure.

LEMMA 2.3. *Let* $\{G_{(u,v)}\}_{(u,\,v) \in B}$ *be a positive additive process on B. Then there exists a unique continuous additive process* $\{G'_{(u,v)}\}_{(u,v) \in C}$ *that extends* $\{G_{(u,v)}\}_{(u,v) \in B}$.

Proof. It is known that [**3**]

$$q - \lim_{\substack{u \to 0 \\ u \in D}} \frac{1}{u^2} G_{\mathbf{u}} \text{ exists a.e.}$$

Then by the previous lemma $G_{(u,v)} \downarrow 0$ a.e. and in L_1-norm as $(u, v) \to \mathbf{O}$, $(u, v) \in B$. Therefore if $(u_1, v_1), (u_2, v_2) \in B$ with $(u_1, v_1) < (u_2, v_2)$ then

$$\begin{aligned} &\|G_{(u_2,v_2)} - G_{(u_1,v_1)}\| \\ &= \|T_{u_1}G_{(u_2-u_1,v_1)} + S_{v_1}G_{(u_1,v_2-v_1)} \\ &+ T_{u_1}S_{v_1}G_{(u_2-u_1,v_2-v_1)}\| \end{aligned}$$

$$\leqq \|G_{(u_2-u_1,v_1)}\| + \|G_{(u_1,v_2-v_1)}\| + \|G_{(u_2-u_1,v_2-v_1)}\|$$

which implies that $\{G_{(u,v)}\}_{(u,v)\in B}$ is continuous on B. Here we also used the fact that for any fixed v (or u) the additive process $G_{(u,v)} \downarrow 0$ as $u \to 0^+$ (or $v \to 0^+$ resp.) through dyadic rationals [**1**]. Hence $\{G_{(u,v)}\}_{(u,v)\in B}$ has a unique continuous extension

$$\{G'_{(u,v)}\}_{(u,v)\in \mathbf{R}^2_+}.$$

Additivity of this extension is straightforward.

Let $\{F_{(u,v)}\}_{(u,v)\in C}$ be a positive subadditive process (that is $\{-F_{(u,v)}\}$ is superadditive). For a pair $(u, v) \in B$ and an integer $k \geqq 0$ such that both $2^k u$ and $2^k v$ are also integer, let

$$G^k_{(u,v)} = I^k_{(u,v)}(4^k F_{\mathbf{2}^{-k}}) = \sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1} S^j_{2^{-k}} T^i_{2^{-k}} F_{\mathbf{2}^{-k}},$$

where $(m_1, m_2) = (2^k u, 2^k v)$. Thus if

$$\sup_{(u,v)\in C} \frac{1}{uv}\int F_{(u,v)}d\mu = \alpha < \infty$$

then

$$\int G^k_{(u,v)}d\mu \leqq m_1 m_2 \int F_{\mathbf{2}^{-k}}d\mu \leqq (uv)\alpha.$$

Moreover by Lemma 2.1 (applied to $\{-F_{(u,v)}\}$) we have

$$F_{(u,v)} \leqq G^k_{(u,v)} \leqq G^{k+1}_{(u,v)}.$$

Hence

$$G^k_{(u,v)} \uparrow G_{(u,v)} \in L_1 \quad \text{as } k \to \infty.$$

Obviously $\{G^k_{(u,v)}\}_{(u,v)\in B}$ is an additive process for every $k \geqq 0$. Therefore whenever k is sufficiently large we obtain a positive additive process $\{G_{(u,v)}\}_{(u,v)\in B}$ such that

$$F_{(u,v)} \leqq G_{(u,v)} \quad \text{for each } (u, v) \in B.$$

Now extend $\{G_{(u,v)}\}_{(u,v)\in B}$ to $\mathbf{R}^2_+$ by Lemma 2.3 and denote it by

$$\{G_{(u,v)}\}_{(u,v)\in C}.$$

Let $(u, v) \in C$ be fixed and let

$$(u, v) = (t, r) + (x, y)$$

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for $(t, r) \in B$ and $(x, y) \in C$. Then

$$\begin{aligned} &F_{(u,v)} - G_{(u,v)} \\ &\leqq [F_{(t,r)} - G_{(t,r)}] + T_t[F_{(x,r)} - G_{(x,r)}] \\ &+ S_r[F_{(t,y)} - G_{(t,y)}] + T_tS_r[F_{(x,y)} - G_{(x,y)}] \\ &\leqq T_t[F_{(x,r)} - G_{(x,r)}] + S_r[F_{(t,y)} - G_{(t,y)}] \\ &+ T_tS_r[F_{(x,y)} - G_{(x,y)}] \end{aligned}$$

since

$$F_{(t,r)} - G_{(t,r)} \leqq 0.$$

On the other hand,

$$\begin{aligned} \|F_{(u,v)} - G_{(u,v)}\| &= \|F_{(x,r)} - G_{(x,r)}\| + \|F_{(t,y)} - G_{(t,y)}\| \\ &+ \|F_{(x,y)} - G_{(x,y)}\| \\ &\leqq 2(x + y + xy)\alpha < \infty. \end{aligned}$$

Thus $\|F_{(u,v)} - G_{(u,v)}\|$ can be made arbitrarily small. This together with the above inequality implies that

$$G_{(u,v)} \geqq F_{(u,v)} \text{ for each } (u, v) \in C.$$

So we have obtained:

Fact 2.4 Let $\{F_{(u,v)}\}_{(u,v)\in C}$ be a positive subadditive process. If

$$\sup_{(u,v)\in C} \frac{1}{uv} \int F_{(u,v)} d\mu = \alpha < \infty,$$

then there is a positive additive process $\{G_{(u,v)}\}_{(u,v)\in C}$ such that

$$F_{(u,v)} \leqq G_{(u,v)} \quad \text{for each } (u, v) \in C.$$

Secondly let $\{F_{(u,v)}\}_{(u,v)\in C}$ be a positive superadditive process. For $(u, v) \in B$ and sufficiently large integer $k \geqq 0$ again let

$$G^k_{(u,v)} = I^k_{(u,v)}(4^k F_{2^{-k}}).$$

Assume that $\{G'_{(u,v)}\}_{(u,v)\in C}$ is an additive process satisfying

$$0 \leqq G'_{(u,v)} \leqq F_{(u,v)}, \quad (u, v) \in C.$$

Consequently $F_{2^{-k}} \geqq G'_{2^{-k}}$, and hence

$$G^k_{(u,v)} \geqq I^k_{(u,v)}(4^k G'_{2^{-k}}) = G'_{(u,v)}$$

by the additivity of $\{G'_{(u,v)}\}$. Also by Lemma 2.1, we have

$$F_{(u,v)} \geqq G^k_{(u,v)} \geqq G^{k+1}_{(u,v)} \geqq 0.$$

Hence $G^k_{(u,v)} \downarrow G_{(u,v)}$ exists as $k \to \infty$, and satisifies

$$G'_{(u,v)} \leqq G_{(u,v)} \leqq F_{(u,v)} \quad \text{for each } (u, v) \in B.$$

Additivity of $\{G_{(u,v)}\}_{(u,v)\in B}$ is obvious. Hence, by continuity, it can be extended to all $(u, v) \in C$. Moreover for a fixed $(u, v) \in C$ let $(t, r) \in B$ and $(x, y) \in C$ such that

$$(u, v) = (t, r) + (x, y).$$

Then

$$\begin{aligned} &G_{(u,v)} - F_{(u,v)} \\ &\leqq [G_{(t,r)} - F_{(t,r)}] + T_t[G_{(x,r)} - F_{(x,r)}] \\ &+ S_r[G_{(t,y)} - F_{(t,y)}] + T_tS_r[G_{(x,y)} - F_{(x,y)}] \\ &\leqq T_t[G_{(x,r)} - F_{(x,r)}] + S_r[G_{(t,y)} - F_{(t,y)}] \\ &+ T_tS_r[G_{(x,y)} - F_{(x,y)}] \end{aligned}$$

since

$$G_{(t,r)} - F_{(t,r)} \leqq 0.$$

By Lemma 2.2 both $\|G_{(x,y)}\|$ and $\|F_{(x,y)}\|$ decrease to 0 as $(x, y) \to \mathbf{O}$. The same holds for $\|G_{(x,r)}\|$, $\|G_{(t,y)}\|$, $\|F_{(x,r)}\|$ and $\|F_{(t,y)}\|$ as x or y tend to 0^+. Consequently we have

$$G_{(u,v)} \leqq F_{(u,v)}.$$

Thus

$$G_{(u,v)} \leqq F_{(u,v)} \quad \text{for each } (u, v) \in C.$$

This gives:

Fact 2.5. Given a positive superadditive process

$$\{F_{(u,v)}\}_{(u,v)\in C}.$$

Then there is a maximal additive process $\{G_{(u,v)}\}_{(u,v)\in C}$ such that

$$0 \leqq G_{(u,v)} \leqq F_{(u,v)} \quad \text{for each } (u, v) \in C$$

and such that if $\{G'_{(u,v)}\}_{(u,v)\in C}$ is another process with

$$0 \leqq G'_{(u,v)} \leqq F_{(u,v)}$$

then also

$$G'_{(u,v)} \leqq G_{(u,v)}.$$

THEOREM 2.6. *Let* $\{F_{(u,v)}\}_{(u,v)\in C}$ *be a superadditive process such that*

$$\sup_{(u,v)\in C} \frac{1}{uv} \int F^{-}_{(u,v)} d\mu < \infty. \tag{2.7}$$

Then there are two positive additive processes $\{G^{i}_{(u,v)}\}_{(u,v)\in C}$, $i = 1, 2$, *such that*

$$\{F_{(u,v)} + G^{1}_{(u,v)} - G^{2}_{(u,v)}\}_{(u,v)\in C}$$

is a positive superadditive process that does not dominate any nonzero positive additive process.

Proof. $\{F^{-}_{(u,v)}\}_{(u,v)\in C}$ is a positive subadditive process. Hence by Fact 2.4 we can find a positive additive process

$$\{G^{1}_{(u,v)}\}_{(u,v)\in C}$$

such that

$$G^{1}_{(u,v)} \geqq F^{-}_{(u,v)} \quad \text{for each } (u, v) \in C.$$

Then $\{F_{(u,v)} + G^{1}_{(u,v)}\}$ becomes a positive superadditive process. Then applying Fact 2.5 we get a maximal additive process

$$\{G^{2}_{(u,v)}\}_{(u,v)\in C}$$

such that

$$0 \leqq G^{2}_{(u,v)} \leqq F_{(u,v)} + G^{1}_{(u,v)}.$$

Hence $\{F_{(u,v)} + G^{1}_{(u,v)} - G^{2}_{(u,v)}\}$ is the process with desired properties.

Remark 2.8. For any positive process $\{G_{(u,v)}\}_{(u,v)\in C}$ if the limit

$$g = q - \lim_{(u,v)\to \mathbf{O}} \frac{1}{uv} G_{(u,v)}$$

exists a.e., then it is finite a.e. by Lemma 2.2(b). Since we know that the limits

$$g_i = q - \lim_{u\to 0^+} \frac{1}{u^2} G^{i}_{\mathbf{u}}, \quad i = 1, 2,$$

exist and are finite a.e. [**3**], Theorem 2.6 shows that given any superadditive process

$$\{F_{(u,v)}\}_{(u,v)\in C}$$

with (2.7), we can assume without loss of generality that it is a positive superadditive process that does not dominate any nonzero positive additive process.

3. Almost everywhere convergence. Given a strongly continuous semigroup $\{K_t\}_{t \geqq 0}$ of positive L_1-contractions with $K_0 = I$. In [**1**] a set $E \in \mathscr{F}$ is called bounded if there exists a positive constant $\lambda < \infty$ and $t > 0$ such that

(3.1) $$\int K_t f d\mu \geqq \lambda \int f d\mu, \text{ for each } f \in L_1^+(E).$$

The following lemma which we will use here is due to M. A. Akçoğlu [**1**].

LEMMA 3.2. *Given any $g \in L_1^+$ and $\epsilon > 0$. Then there exists a bounded set $E \in \mathscr{F}$ such that*

$$\int_{E^c} g d\mu < \epsilon.$$

Proof. Let K_t^* be the adjoint transformation of K_t. Then a bounded set can be characterized by the fact that

$$K_t^* 1 \geqq \lambda \text{ a.e. on } E.$$

Since $K_t^* 1 \leqq K_s^* 1 \leqq 1$, whenever $0 \leqq s \leqq t$, (1.3) implies that

$$q - \lim_{t \to 0} K_t^* 1 = 1 \text{ a.e.}$$

Then the proof follows.

LEMMA 3.3. *For any $A \in \mathscr{F}$ and $s > 0$,*

$$\lim_{s \downarrow 0} \frac{1}{s} \int_0^s K_r^* \chi_A dr = \chi_A \text{ a.e.}$$

Proof. Since $K_t^* \chi_A \leqq K_t^* 1 \leqq 1$ a.e., we see that

$$\frac{1}{s} \int_0^s K_r^* \chi_A dr \leqq 1 \text{ a.e. for each } s > 0.$$

Now observe that if $u_0 \in L_1^+$ is strictly positive a.e., then

$$u = \int_0^\infty e^{-t} K_t u_0 dt$$

is an L_1^+-function and is also strictly positive a.e. with

$$e^{-t} K_t u \leqq u.$$

Therefore the operator $P_t = e^{-t} K_t^*$ is a positive contraction on $L_1(X, u d\mu)$. Moreover $\{P_t\}_{t \geqq 0}$ on $L_1(X, u d\mu)$ is also a strongly continuous semigroup of positive L_1-contractions [**12**]. Now consider the process $\{R_s\}_{s \geqq 0}$, where

$$R_s = \int_0^s P_t \chi_A dt.$$

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This is an additive process on $L_1(X, ud\mu)$ with respect to the semigroup $\{P_t\}$. Then we know that [**11**], [**2**], [**4**]

$$(3.4) \qquad q - \lim_{s \to 0} \frac{R_s}{s} = \psi \text{ exists a.e.}$$

and is finite a.e. Recalling that $P_0 = K_0^* = I$, we see that $\psi = \chi_A$. Then

$$\frac{1}{s} \int_0^s K_r^* \chi_A dr = \frac{1}{s} \int_0^s (1 - e^{-r}) K_r^* \chi_A dr + \frac{R_s}{s}.$$

Since

$$q - \lim_{s \to 0} \frac{1}{s} \int_0^s (1 - e^{-r}) K_r^* \chi_A dr = 0 \text{ a.e.},$$

we obtain by (3.4) that

$$q - \lim_{s \to 0} \frac{1}{s} \int_0^s K_r^* \chi_A dr = \chi_A \text{ a.e.}$$

Remark 3.5. In the n-parameter case, when $n > 2$, (3.4) is given by Terrel's Theorem [**14**].

COROLLARY 3.6. *Given $A \in \mathscr{F}$, $h \in L_1^+(A)$ and $\epsilon > 0$. There exists a subset B of A with $\int_B h d\mu < \epsilon$ positive constants $\beta = \beta_B < \infty$ and s' such that*

$$\int \left[\frac{1}{s} \int_0^s K_r^* \chi_A dr \right] (h) d\mu \geqq \beta \int_{A \setminus B} h d\mu$$

for each s with $0 \leqq s \leqq s'$.

Proof. The conclusion of this corollary is the same as asserting the existence of a bounded set $A \setminus B$ with constant β such that

$$\int_B h d\mu < \epsilon \quad \text{for each } h \in L_1^+(A).$$

Since

$$q - \lim_{s \to 0} \frac{1}{s} \int_0^s K_r^* \chi_A dr = \chi_A \text{ a.e.}$$

by Lemma 3.3, the result follows easily.

For convenience, in the two parameter case we will define a bounded set somewhat differently than in [**1**]:

Definition 3.7. A set $E \in \mathscr{F}$ is called a *bounded set* if there exists a positive constant $\lambda = \lambda_E$ and $u > 0$, $v > 0$ such that

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(3.8) $$\frac{1}{uv} \int I_{(u,v)} f d\mu \geqq \lambda \int_E f d\mu \quad \text{for each } f \in L_1^+.$$

LEMMA 3.9. *Given any $g \in L_1^+$ and any $\epsilon > 0$. Then there exists a bounded set $E \in \mathscr{F}$ such that*

$$\int_{E^c} g d\mu < \epsilon.$$

Proof. By Lemma 3.2 find $\alpha > 0$, $u > 0$ and a set $A \in \mathscr{F}$ with

$$\int_{A^c} f d\mu < \epsilon/2$$

such that

(3.10) $$\int T_t f d\mu \geqq \alpha \int_A f d\mu$$

for each $f \in L_1^+$ and for each t with $0 \leqq t \leqq u$. Then by Corollary 3.6 find $\beta > 0$, $v > 0$ and a subset B of A with

$$\int_B f d\mu < \epsilon/2$$

such that

(3.11) $$\int \left[\frac{1}{s} \int_0^s S_r^* \chi_A dr \right] f d\mu \geqq \beta \int_{A \setminus B} f d\mu$$

for each s with $0 \leqq s \leqq v$. Therefore if $f \in L_1^+(A \setminus B)$, then

$$\int I_{(u,v)} f d\mu = \int \left[\int_0^u \int_0^v S_r T_t f dr dt \right] d\mu$$
$$\geqq \int_0^u \int_0^v \left[\int T_t(\chi_A S_r f) d\mu \right] dr dt$$
$$\geqq \alpha \int_0^u \int_0^v \int \chi_A S_r f d\mu dr dt \quad \text{by (3.10)}$$

since $\chi_A S_r f \in L_1^+(A)$. So

$$\int I_{(u,v)} f d\mu \geqq \alpha u \int_0^v \int \chi_A S_r f d\mu dr \geqq \alpha u \left(\beta v \int f d\mu \right)$$

By (3.11) since $f \in L_1^+(A \setminus B)$. Thus for each $f \in L_1^+(A \setminus B)$ we have (3.8) where $E = A \setminus B$ and $\lambda = \alpha\beta$. Now take

$$f = \chi_{A-B} g \in \mathrm{L}_1^+(A \setminus B).$$

Moreover $E^c = A^c \cup B$ and

$$\int_{E^c} g d\mu = \int_{A^c} g d\mu + \int_B g d\mu < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

LEMMA 3.12. *Let* $\{F_{(u,v)}\}_{(u,v)\in C}$ *be a positive superadditive process, and let* E *be a bounded set. If*

$$\limsup_{(u,v)\to\mathbf{O}} \frac{1}{uv} \int_E F_{(u,v)} d\mu > 0,$$

then $\{F_{(u,v)}\}$ *dominates a nonzero positive additive process*

$$\{G_{(u,v)}\}_{(u,v)\in C}.$$

Proof. Let (α, β), $(u, v) \in C$, then

$$I_{(u,v)}\left[\frac{1}{\alpha\beta}F_{(\alpha,\beta)}\right] = \frac{1}{\alpha\beta}\int_0^u \int_0^v U_{(s_1,s_2)}F_{(\alpha,\beta)}ds_1 ds_2.$$

By superadditivity

$$I_{(u,v)}\left[\frac{1}{\alpha\beta}F_{(\alpha,\beta)}\right]$$
$$\leqq \frac{1}{\alpha\beta}\int_0^u \int_0^v [F_{(\alpha+s_1,\beta+s_2)} - T_{s_1}F_{(\alpha,s_2)}$$
$$- S_{s_2}F_{(s_1,\beta)} - F_{(s_1,s_2)}]ds_1 ds_2.$$

Since $F_{(u,v)} \geqq 0$ and S_r and T_t are positive operators, we see that

$$I_{(u,v)}\left(\frac{1}{\alpha\beta}F_{(\alpha,\beta)}\right) \leqq \frac{1}{\alpha\beta}\int_u^{u+\alpha} \int_v^{v+\beta} F_{(s_1,s_2)}ds_1 ds_2$$
$$\leqq F_{(u+\alpha,v+\beta)}$$

since $F_{(s_1,s_2)}$ is increasing with increasing (s_1, s_2). Now let $\alpha_n > 0$ and $\beta_n > 0$ be sequences such that $\alpha_n \downarrow 0$, $\beta_n \downarrow 0$ as $n \to \infty$ and such that

$$\lim_{n\to\infty} \frac{1}{\alpha_n\beta_n}\int_E F_{(\alpha_n,\beta_n)}d\mu = K > 0. \tag{3.13}$$

For each fixed $(u, v) \in C$, the sequence

$$I_{(u,v)}\left(\frac{1}{\alpha_n\beta_n}F_{(\alpha_n,\beta_n)}\right)$$

is dominated by the integrable function $F_{(u+\alpha_1,v+\beta_1)}$. Hence one can choose a subsequence of (α_n, β_n), which we will also denote by (α_n, β_n), such that

$$G_{(u,v)} = w - \lim_{n\to\infty} I_{(u,v)}\left(\frac{1}{\alpha_n\beta_n}F_{(\alpha_n,\beta_n)}\right)$$

exists for each $(u, v) \in B$. This new process

$$\{G_{(u,v)}\}_{(u,v)\in B}$$

is a positive additive process, hence extends to a continuous additive process

$$\{G_{(u,v)}\}_{(u,v)\in C}.$$

If $(u_1, v_1), (u_2, v_2) \in B$ such that $\mathbf{O} < (u_1, v_1) < (u_2, v_2)$, then we have

$$G_{(u_1,v_1)} \leqq F_{(u_2,v_2)}.$$

Hence by continuity,

$$G_{(u,v)} \leqq F_{(u,v)} \quad \text{for each } (u, v) \in B,$$

and consequently

$$0 \leqq G_{(u,v)} \leqq F_{(u,v)} \quad \text{for each } (u, v) \in C$$

as in Section 2.

Let λ be the constant associated with the bounded set E and let $(u, v) \in C$ be such that (3.8) holds. Then

$$\int I_{(u,v)}\left(\frac{1}{\alpha_n\beta_n}F_{(\alpha_n,\beta_n)}\right)d\mu \geqq \lambda uv \int_E \frac{1}{\alpha_n\beta_n}F_{(\alpha_n,\beta_n)}d\mu.$$

Since

$$\int_E \frac{1}{\alpha_n\beta_n}F_{(\alpha_n,\beta_n)}d\mu \rightarrow K$$

by (3.13), we see that

$$\int G_{(u,v)}d\mu \geqq \lambda uvK > 0$$

showing that $\{G_{(u,v)}\}$ is a nonzero process and hence proving the lemma.

Before stating the following lemma it would be convenient to introduce some notation: for a given process $\{F_{(u,v)}\}$ and $t, r \in \mathbf{R}^+$, let

$$\theta_t F_{(u,v)} = F_{(u+t,v)}, \quad \phi_r F_{(u,v)} = F_{(u,v+r)}$$

and

$$\tau_t F_{(u,v)} = (\theta_t - T_t)F_{(u,v)}, \quad \sigma_r F_{(u,v)} = (\phi_r - S_r)F_{(u,v)}.$$

Then the superadditivity conditions (1.8)(a) and (b) take the forms

(1.8′) (a′) $F_{(u,r)} \leqq \sigma_r F_{(u,v)}$

(b′) $F_{(t,v)} \leqq \tau_t F_{(u,v)}$

and the strong superadditivity condition (1.9) takes the form

(1.9′) $F_{(t,r)} \leqq \tau_t \sigma_r F_{(u,v)}.$

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LEMMA 3.14. *Let* $\{F_{(u,v)}\}_{(u,v)\in C}$ *be a positive strongly superadditive process. Let* $(\alpha, \beta) \in C$, *and define for each* $(u, v) \in C$

$$H^{\alpha\beta}_{(u,v)} = (I - T_u)(I - S_v)\left[\frac{1}{\alpha\beta}\int_0^\alpha\int_0^\beta F_{(s_1,s_2)}ds_2ds_1\right]$$

$$+ (I - T_u)\left[\frac{1}{\alpha\beta}\int_0^\alpha\int_0^v S_{s_2}F_{(s_1,\beta)}ds_2ds_1\right]$$

$$+ (I - S_v)\left[\frac{1}{\alpha\beta}\int_0^u\int_0^\beta T_{s_1}F(\alpha,s_2)ds_2ds_1\right]$$

$$+ \frac{1}{\alpha\beta}\int_0^u\int_0^v S_{s_2}T_{s_1}F_{(\alpha,\beta)}ds_2ds_1.$$

Then $\{H^{\alpha\beta}_{(u,v)}\}_{(u,v)\in C}$ *is a positive additive process and*

$$(3.15)\quad H^{\alpha\beta}_{(u,v)} \geqq \left(1 - \frac{u}{\alpha}\right)\left(1 - \frac{v}{\beta}\right)F_{(u,v)}.$$

Proof. If $\mathbf{O} < (u, v) < (\alpha, \beta)$, then

$$\alpha\beta H^{\alpha\beta}_{(u,v)} = (I - T_u)\left\{\int_0^\alpha\left[(I - S_v)\int_0^\beta F_{(s_1,s_2)}ds_2 + \int_0^v S_{s_2}F_{(s_1,\beta)}ds_2\right]ds_1\right\}$$

$$+ \int_0^u T_{s_1}\left[(I - S_v)\int_0^\beta F_{(\alpha,s_2)}ds_2 + \int^v S_{s_2}F_{(\alpha,\beta)}ds_2\right]ds_1.$$

Let

$$\beta G_v(x) = (I - S_u)\int_0^\beta F_{(x,s_2)}ds_2 + \int_0^v S_{s_2}F_{(x,\beta)}ds_2.$$

Then

$$\alpha\beta H^{\alpha\beta}_{(u,v)} = (I - T_u)\int_0^\alpha \beta G_v(s_1)ds_1 + \int_0^u \beta T_{s_l}G_v(\alpha)ds_1.$$

Now

$$\beta G_v(x) = \int_0^v F_{(x,s_2)}ds_2$$

$$+ \int_v^\beta\left[F_{(x,s_2)} - S_vF_{(x,s_2-v)}\right]ds_2$$

$$+ \int_0^v\left[S_{s_2}F_{(x,\beta)} - S_vF_{(x,\beta+S_2-v)}\right]ds_2$$

$$= \int_0^v F_{(x,s_2)} ds_2 + \int_v^\beta \sigma_v F_{(x,s_2 - v)} ds_2$$
$$+ \int_0^v S_{s_2} \sigma_{v - s_2} F_{(x,\beta + s_2 - v)} ds_2.$$

Also, similarly,

$$\alpha\beta H^{\alpha\beta}_{(u,v)} = \int_0^u \beta G_v(s_1) ds_1 + \int_u^\alpha \beta\tau_u G_v(s_1 - u) ds_1$$
$$+ \int_0^u \beta T_{t_1} \tau_{u - s_1} G_v(\alpha + s_1 - u) ds_1.$$

Hence, combining the last two equations, we obtain

$$\alpha\beta H^{\alpha\beta}_{(u,v)} = \int_0^u \int_0^v F_{(s_1,s_1)} ds_2 ds_1$$
$$+ \int_0^u \int_v^\beta \sigma_v F_{(s_1,s_2 - v)} ds_2 ds_1$$
$$+ \int_0^u \int_0^v S_{s_2} \sigma_{v - s_2} F_{(s_1,s_2 + \beta - v)} ds_2 ds_1$$
$$+ \int_u^\alpha \int_0^v \tau_u F_{(s_1 - u,s_2} ds_2) ds_1$$
$$+ \int_u^\alpha \int_v^\beta \tau_u \sigma_v F_{(s_1 - u,s_1 - v)} ds_2 ds_1$$
$$+ \int_u^\alpha \int_0^v S_{s_2} \tau_u \sigma_{v - s_2} F_{(s_1 - u,s_2 + \beta - v)} ds_2 ds_1$$
$$+ \int_0^u \int_0^v T_{t_1} \tau_{u - s_1} F_{(\alpha + s_1 - u,s_2)} ds_2 ds_1$$
$$+ \int_0^u \int_v^\beta T_{t_1} \tau_{u - s_1} \sigma_v F_{(s_1 + \alpha - u,s_2 - v)} ds_2 ds_1$$
$$+ \int_0^u \int_0^v S_{s_2} T_{s_1} \tau_{u - s_1} \sigma_{v - s_2} F_{(s_1 + \alpha - u,s_2 + \beta - v)} ds_2 ds_1$$
$$\geqq (\alpha - u)(\beta - v) F_{(u,v)}$$

by (1.8′) and (1.9′) together with the fact that both $\{T_t\}$ and $\{S_r\}$ are positive operators and

$$F_{(u,v)} \geqq 0 \quad \text{for each } (u,v) \in C.$$

Obviously $\{H^{\alpha\beta}_{(u,v)}\}_{(u,v) \in C}$ is an additive process. Since it is positive for small values of $(u, v) \in C$, it is positive for all $(u, v) \in C$, consequently we have (3.15) for each $(u, v) \in C$.

Notice that since $\{H^{\alpha\beta}_{(u,v)}\}_{(u,v)}$ is a positive additive process,

$$h_{\alpha\beta} = q - \lim_{u\to 0^+} \frac{1}{u^2} H_{\mathbf{u}}^{\alpha\beta}$$

exists and is finite a.e. for each $(\alpha, \beta) \in C$ [**3**]. Furthermore, if

$$f = q - \lim_{u\to 0^+} \sup \frac{1}{u^2} F_{\mathbf{u}},$$

then $0 \leqq f \leqq h_{\alpha\beta}$ for each $(\alpha, \beta) \in C$.

LEMMA 3.16. *Let* $\{F_{(u,v)}\}_{(u,v)\in C}$ *be a positive strongly superadditive process and let* $A \in \mathscr{F}$ *be a set. If*

$$\lim_{(u,v)\to\mathbf{O}} \int_A \frac{1}{uv} F_{(u,v)} d\mu = 0,$$

then

$$q - \lim_{u\to 0} \frac{1}{u^2} F_{\mathbf{u}} \text{ exists and is zero a.e. on } A.$$

Proof. Let

$$f = q - \lim_{u\to 0^+} \sup \frac{1}{u^2} F_{\mathbf{u}}.$$

If $f > 0$ on a subset of A with positive measure, then there exists an L_1^+-function h such that

$$\int_A h d\mu > 0$$

and such that

$$0 \leqq h \leqq f \leqq h_{\alpha\beta} \quad \text{for each } (\alpha, \beta) \in C.$$

Then by (a) of Lemma 2.2 we have

$$I_{(u,v)} h \leqq H_{(u,v)}^{\alpha\beta}.$$

But

$$\begin{aligned} H_{(u,v)}^{\alpha\beta} &\leqq F_{(\alpha,\beta)} + F_{(\alpha,v+\beta)} + F_{(u+\alpha,\beta)} + 2F_{(u+\alpha,v+\beta)} \\ &\leqq 5F_{(u+\alpha,v+\beta)} \end{aligned}$$

since $F_{(u,v)} \geqq 0$ and is increasing as (u, v) increases. Hence, if $\mathbf{O} < (\alpha, \beta) < (u, v)$, then

$$I_{(u,v)} h \leqq 5F_{(2u,2v)},$$

or

$$\int_A \left[\frac{1}{uv} I_{(u,v)} h\right] d\mu \leqq 20 \int_A \left[\frac{1}{4uv} F_{(2u,2v)}\right] d\mu.$$

This is a contradiction since the left hand side converges to $\int_A h d\mu > 0$ as $(u, v) \to \mathbf{O}$, and the right hand side converges to zero.

THEOREM 3.17. *Let* $\{F_{(u,v)}\}_{(u,v)\in C}$ *be a strongly superadditive process such that*

$$\sup_{(u,v)\in C} \frac{1}{uv} \int F_{(u,v)}^{-} d\mu < \infty.$$

Then

$$q - \lim_{u\to 0^+} \frac{1}{u^2} F_{\mathbf{u}}$$

exists and is finite a.e.

Proof. By the remarks of Section 2, without loss of generality we can assume that $\{F_{(u,v)}\}$ is a positive strongly superadditive process that does not dominate any nonzero positive additive process. Hence if we can show that

$$q - \lim_{u\to 0^+} \frac{1}{u^2} F_{\mathbf{u}} = 0 \text{ a.e.},$$

then the proof will be completed. If $E \in \mathscr{F}$ is a bounded set, then

$$\lim_{(u,v)\to\mathbf{O}} \frac{1}{uv} \int_E F_{(u,v)} d\mu = 0$$

by Lemma 3.12. Hence we see that

$$q - \lim_{u\to 0^+} \frac{1}{u^2} F_{\mathbf{u}} = 0 \text{ a.e. on } E$$

by Lemma 3.16. Consequently

$$q - \lim_{u\to 0^+} \frac{1}{u^2} F_{\mathbf{u}} = 0 \text{ a.e. on } X$$

by Lemma 3.9.

REFERENCES

1. M. A. Akçoğlu, *Differentiation of superadditive process,* Preprint.
2. M. A. Akçoğlu and R. V. Chacon, *A local ergodic theorem,* Can. J. Math. *22* (1970), 545-552.
3. M. A. Akçoğlu and A. del Junco, *Differentiation of n-dimensional additive processes,* Can. J. Math. *33* (1981), 749-768.
4. M. A. Akçoğlu and U. Krengel, *A differentiation theorem for additive processes,* Math. Z. *163* (1978), 199-210.
5. ——— *Ergodic theorems for superadditive processes,* J. Reine Angew. Math. *323* (1981), 53-67.

6. M. A. Akçoğlu and L. Sucheston, *A ratio ergodic theorem for superadditive processes*, Z. Wahr. *44* (1978), 269-278.
7. R. Emilion and B. Hachem, *Un theoreme ergodique local sur-additif*, C. R. Acad. Sci. Paris *294* (1982), 337-340.
8. R. Emilion and B. Hachem, *Differentiation des procesus sur-additifs a plusieurs parameters*, Preprint.
9. D. Feyel, *Convergence locale des procesus sur-abeliens et sur-additifs*, to appear.
10. J. F. C. Kingman, *Subadditive ergodic theory*, Ann. Prob. *1* (1973), 883-905.
11. U. Krengel, *A local ergodic theorem*, Invent. Math. *6* (1969), 329-333.
12. M. Lin, *On local ergodic convergence of semigroups and additive processes*, Israel J. Math., to appear.
13. R. T. Smythe, *Multiparameter subadditive processes*, Ann. Prob. *4* (1976), 772-782.
14. T. R. Terrell, *Local ergodic theorems for n-parameter semigroups of operators*, in *Contributions to ergodic theory and probability*, Lecture Notes in Math. *160* (Springer-Verlag, 1970), 262-278.

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