A DEFINITION FOR STRONG RIESZIAN SUMMABILITY AND ITS RELATIONSHIP TO STRONG CESARO SUMMABILITY

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1. *Introduction.* Given a series $\sum_{n=0}^{\infty} a_n$, we define $A_n^{(k)}$, $E_n^{(k)}$, $k > -1$, by the relations

$$A_n^{(k)} = \sum_{\nu=0}^{n} \binom{k+n-\nu}{n-\nu} a_\nu, \quad E_n^{(k)} = \binom{k+n}{n}.$$

The series Σa_n is said to be summable (C, k) to the sum s, if

$$C_n^{(k)} = A_n^{(k)} / E_n^{(k)} \to s$$

as $n \to \infty$, and strongly summable (C, k), $k > 0$, with index p, to the sum s, or summable $[C;\ k, p]$ to the sum s, if

$$\sum_{\nu=0}^{n} | C_\nu^{(k-1)} - s |^p = o(n).$$

It is known * that necessary and sufficient conditions for Σa_n to be summable $[C;\ k, p]$ $k > 0$, $p \geqslant 1$, to the sum s are that Σa_n be summable (C, k) to the sum s and that

$$\sum_{\nu=0}^{n} \nu^p \, | a_\nu^{(k)} |^p = o(n),$$

where $a_\nu^{(k)} = C_\nu^{(k)} - C^{(k)}_{\nu-1}$.

Defining $A_k(\omega)$, $C_k(\omega)$ by the relations

$$A_k(\omega) = \sum_{n<\omega} (\omega - n)^k a_n, \quad C_k(\omega) = \omega^{-k} A_k(\omega),$$

we have the familiar definition that Σa_n is summable (R, k) to the sum s if $C_k(\omega) \to s$ as $\omega \to \infty$ continuously. If, in addition, we have

$$\int_1^{\omega} \left| u \frac{d}{du} C_k(u) \right|^p du = o(\omega),$$

it is then natural to say that Σa_n is strongly summable (R, k), with index p, to the sum s, and write Σa_n is summable $[R;\ k, p]$ to the sum s. In this definition it is assumed that $k > 0$, $p \geqslant 1$.

It should be noted that, for the definition to be valid at all, it is necessary that $kp' > 1$, where $\frac{1}{p} + \frac{1}{p'} = 1$, since, writing $n = [u]$, we have †

$$\left| u \frac{d}{du} C_k(u) \right| = k u^{-k} \left| \sum_{\nu=1}^{n} (u-\nu)^{k-1} \nu a_\nu \right|$$

$$\geqslant k u^{-k} (u-n)^{k-1} n \, | a_n | - k u^{-k} \left| \sum_{\nu=1}^{n-1} (u-\nu)^{k-1} \nu a_\nu \right|,$$

whence

$$k^p u^{-kp} (u-n)^{kp-p} n^p \, | a_n |^p \leqslant 2^p \left| u \frac{d}{du} C_k(u) \right|^p + 2^p k^p u^{-kp} \left| \sum_{\nu=1}^{n-1} (u-\nu)^{k-1} \nu a_\nu \right|^p.$$

* J. M. Hyslop, *Proc.*, Glasgow Math. Assoc., I., p. 16.

† See Lemma 2 below.

Now

$$\int_1^\omega u^{-kp}(u-n)^{kp-p} n^p \mid a_n \mid^p du \geqslant \sum_{n=1}^{[\omega]-1} \int_n^{n+1} u^{-kp}(u-n)^{kp-p} n^p \mid a_n \mid^p du$$
$$\geqslant \sum_{n=1}^{[\omega]-1} (n+1)^{-kp} n^p \mid a_n \mid^p \int_n^{n+1} (u-n)^{kp-p} du$$
$$=\infty,$$

unless $p(k-1) > -1$, that is, unless $k > 1/p'$. Since

$$\int_1^\omega u^{-kp} \left| \sum_{\nu=1}^{n-1} (u-\nu)^{k-1} \nu a_\nu \right|^p du$$

is finite, it follows that

$$\int_1^\omega \left| u \frac{d}{du} C_k(u) \right|^p du$$

is infinite unless $kp' > 1$.

Our object in this paper is to prove the following theorem :

THEOREM. *If $k>0$, $p \geqslant 1$, $\frac{1}{p}+\frac{1}{p'}=1$, $kp'>1$, then summability $[C;\ k,\ p]$ of the series Σa_n implies summability $[R;\ k, p]$ of this series to the same sum, and conversely.*

The proof of the theorem is based on several lemmas, most of which are well known.

2. *Preliminary Lemmas.* LEMMA 1. *If* $k > -1$, $\delta > 0$, then*

$$A_{k+\delta}(\omega) = \frac{\Gamma(k+\delta+1)}{\Gamma(k+1)\Gamma(\delta)} \int_0^\omega (\omega-u)^{\delta-1} A_k(u)\, du.$$

LEMMA 2. *If† $B_k(\omega)$ is the Rieszian sum of order k for the series Σb_n, where $b_n = na_n$, then, for $k>0$,*

$$\omega^{k+1} \frac{d}{d\omega} C_k(\omega) = kB_{k-1}(\omega) = \frac{d}{d\omega} B_k(\omega).$$

LEMMA 3. *We have,‡ for $k > -1$, the formal relations*

$$\sum_{n=0}^\infty A_n^{(k)} x^n = (1-x)^{-k-1} \sum_{n=0}^\infty a_n x^n,$$
$$\sum_{n=0}^\infty nE_n^{(k)} a_n^{(k)} x^n = (1-x)^{-k} \sum_{n=0}^\infty na_n x^n.$$

LEMMA 4. *If § $0 < \theta \leqslant 1$, $k > 0$, q is any positive integer or zero and*

$$\gamma_{n,k}(\theta) = \sum_{\nu=0}^n (n+\theta-\nu)^{k-1} E_\nu^{(-k-1)}$$

then

$$\gamma_{n,k}(\theta) = \delta(\theta) E_n^{(-k-1)} + \beta_{n,k},$$

where

$$\beta_{n,k} = O\{\sum_{\nu=0}^n (\nu+1)^{-k-1}(n-\nu+1)^{k-q-2}\}$$
$$\delta(\theta) = \theta^{k-1} + \sum_{r=0}^q e_r \theta^r,$$

and e_r is independent of n and θ. When $k>1$ the term $\delta(\theta)E_n^{(-k-1)}$ may be incorporated in $\beta_{n,k}$, and θ may take the value zero.

* G. H. Hardy and M. Riesz, *The General Theory of Dirichlet Series* (Cambridge Tract, No. 18), 27.

† See, for example, J. M. Hyslop, *Proc. Edinburgh Math. Soc.*, (2), 5 (1937), 46–54.

‡ E. Kogbetliantz, *Bull. des Sciences Math.*, (2), 49 (1925), 234–56.

§ J. M. Hyslop, *loc. cit.*

LEMMA 5. *In the notation of Lemma 4, by suitable choice of q,*

$$\beta_{n,k}=O\{(n+1)^{-k-1}\}.$$

We have, if $q>2k-1$,

$$\beta_{n,k}=O\{\sum_{0\leqslant\nu\leqslant\frac{1}{2}n}(\nu+1)^{-k-1}(n-\nu+1)^{k-q-2}\}+O\{\sum_{\frac{1}{2}n\leqslant\nu\leqslant n}(\nu+1)^{-k-1}(n-\nu+1)^{k-q-2}\}$$
$$=O\{(\tfrac{1}{2}n+1)^{k-q-2}\sum_{\nu=0}^{\infty}(\nu+1)^{-k-1}\}+O\{(\tfrac{1}{2}n+1)^{-k-1}\sum_{\nu=0}^{\infty}(\nu+1)^{k-q-2}\}$$
$$=O\{(n+1)^{-k-1}\}.$$

LEMMA 6. *If* k is a positive integer or zero, $A_n^{(k)}$ can be expressed in the form*

$$\sum_{\rho=0}^{k}\alpha_\rho A_k\left(n+\frac{\rho}{k}\right),$$

where α_ρ is independent of n.

LEMMA 7. *If $\alpha_\nu\geqslant 0$, $p\geqslant 1$, $\lambda>0$, $\lambda p'>1$,*

$$\sum_{n=0}^{N}\left\{\sum_{\nu=0}^{n}\frac{\alpha_\nu}{(n-\nu+1)^{1+\lambda}}\right\}^p\leqslant K\sum_{n=0}^{N}\alpha_n{}^p$$

where K is independent of N.

When $p=1$ the result follows immediately on interchanging the order of summation. When $p>1$, we have, by Hölder's inequality,

$$\left\{\sum_{\nu=0}^{n}\frac{\alpha_\nu}{(n-\nu+1)^{1+\lambda}}\right\}^p=\left\{\sum_{\nu=0}^{n}\frac{\alpha_\nu}{n-\nu+1}\frac{1}{(n-\nu+1)^{\lambda}}\right\}^p$$
$$\leqslant\sum_{\nu=0}^{n}\frac{\alpha_\nu{}^p}{(n-\nu+1)^p}\cdot\left\{\sum_{\nu=0}^{n}\frac{1}{(n-\nu+1)^{\lambda p'}}\right\}^{p/p'}$$
$$\leqslant K_1\sum_{\nu=0}^{n}\frac{\alpha_\nu{}^p}{(n-\nu+1)^p},$$

since $\lambda p'>1$. Hence

$$\sum_{n=0}^{N}\left\{\sum_{\nu=0}^{n}\frac{\alpha_\nu}{(n-\nu+1)^{1+\lambda}}\right\}^p\leqslant K_1\sum_{n=0}^{N}\sum_{\nu=0}^{n}\frac{\alpha_\nu{}^p}{(n-\nu+1)^p}$$
$$=K_1\sum_{\nu=0}^{N}\alpha_\nu{}^p\sum_{n=\nu}^{N}\frac{1}{(n-\nu+1)^p}$$
$$\leqslant K\sum_{\nu=0}^{N}\alpha_\nu{}^p,$$

since $p>1$.

3. *Summability* $[C;\ k,p]$ *implies summability* $[R;\ k,p]$. Since summability (C,k) implies summability (R,k) to the same sum, it is sufficient to show that, for $k>0$, $p\geqslant 1$, $kp'>1$,

$$\sum_{\nu=0}^{n}|\nu a_\nu^{(k)}|^p=o(n),$$

implies that

$$\int_1^X\left|\omega\frac{d}{d\omega}C_k(\omega)\right|^p d\omega=o(X).$$

* E. W. Hobson, *The Theory of Functions of a Real Variable*, II (1926), 93.

By Lemmas 2 and 3, if $N=[\omega]$,

$$\begin{aligned}\frac{d}{d\omega}C_k(\omega)&=k\omega^{-k-1}B_{k-1}(\omega)\\&=k\omega^{-k-1}\sum_{n=0}^{N}(\omega-n)^{k-1}na_n\\&=k\omega^{-k-1}\sum_{n=0}^{N}(\omega-n)^{k-1}\sum_{\nu=0}^{n}E_{n-\nu}^{(-k-1)}\nu E_\nu^{(k)}a_\nu^{(k)}.\end{aligned}$$

Write $\omega=N+\theta$, $0<\theta\leqslant 1$, and $n-\nu=\mu$. Then, interchanging the order of summation,

$$\begin{aligned}\omega\frac{d}{d\omega}C_k(\omega)&=k\omega^{-k}\sum_{\nu=0}^{N}\nu E_\nu^{(k)}a_\nu^{(k)}\sum_{\mu=0}^{N-\nu}(N+\theta-\nu-\mu)^{k-1}E_\mu^{(-k-1)}\\&=k\omega^{-k}\sum_{\nu=0}^{N}\nu E_\nu^{(k)}a_\nu^{(k)}\gamma_{N-\nu,\,k}(\theta).\end{aligned}$$

Hence, by Lemmas 4 and 5,

$$\begin{aligned}\left|\omega\frac{d}{d\omega}C_k(\omega)\right|^p&=O\left[\omega^{-kp}\left\{\sum_{\nu=0}^{N}\nu E_\nu^{(k)}|a_\nu^{(k)}|(\omega-N)^{k-1}|E_{N-\nu}^{(-k-1)}|\right\}^p\right]\\&+O\left[\omega^{-k}\left\{\sum_{\nu=0}^{N}\nu E_\nu^{(k)}|a_\nu^{(k)}|\sum_{r=0}^{q}|e_r|\theta^r|E_{N-\nu}^{(-k-1)}|\right\}^p\right]\\&+O\left[\omega^{-kp}\left\{\sum_{\nu=0}^{N}\nu E_\nu^{(k)}|a_\nu^{(k)}|(N-\nu+1)^{-k-1}\right\}^p\right],\end{aligned}$$

and, since $E_{N-\nu}^{(-k-1)}=O\{(N-\nu+1)^{-k-1}\}$, $0<\theta\leqslant 1$, the result will follow if we show that the two integrals *

$$\int_1^X(\omega-N)^{p(k-1)}\left\{\omega^{-k}\sum_{\nu=1}^{N}\frac{\nu^k\nu|a_\nu^{(k)}|}{(N-\nu+1)^{1+k}}\right\}^p d\omega,$$

$$\int_1^X\left\{\omega^{-k}\sum_{\nu=1}^{N}\frac{\nu^k.\nu|a_\nu^{(k)}|}{(N-\nu+1)^{1+k}}\right\}^p d\omega$$

are each $o(X)$.

The second integral is not greater than

$$\begin{aligned}\int_1^X\left\{\sum_{\nu=1}^{N}\frac{\nu|a_\nu^{(k)}|}{(N-\nu+1)^{1+k}}\right\}^p d\omega&\leqslant\sum_{N=1}^{[X]}\int_N^{N+1}\left\{\sum_{\nu=1}^{N}\frac{\nu|a_\nu^{(k)}|}{(N-\nu+1)^{1+k}}\right\}^p d\omega\\&=\sum_{N=1}^{[X]}\left\{\sum_{\nu=1}^{N}\frac{\nu|a_\nu^{(k)}|}{(N-\nu+1)^{1+k}}\right\}^p\\&=O\left\{\sum_{N=1}^{[X]}|\nu a_\nu^{(k)}|^p\right\}=o(X)\end{aligned}$$

by hypothesis and Lemma 7. The first integral is not greater than

$$\begin{aligned}\int_1^X(\omega-N)^{pk-p}\left\{\sum_{\nu=1}^{N}\frac{\nu|a_\nu^{(k)}|}{(N-\nu+1)^{1+k}}\right\}^p d\omega&\leqslant\sum_{N=1}^{[X]}\left\{\sum_{\nu=1}^{N}\frac{\nu|a_\nu^{(k)}|}{(N-\nu+1)^{1+k}}\right\}^p\int_N^{N+1}(\omega-N)^{kp-p}d\omega\\&\leqslant K\sum_{N=1}^{[X]}\left\{\sum_{\nu=1}^{N}\frac{\nu|a_\nu^{(k)}|}{(N-\nu+1)^{1+k}}\right\}^p,\end{aligned}$$

since $kp'>1$. It now follows as above that this integral is also $o(X)$.

* The term $\nu=0$ in each of the preceding expressions is, of course, zero, and may therefore be omitted.

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4. *Summability* $[R;\ k, p]$ *implies summability* $[C;\ k, p]$.

Since summability (R, k) implies summability (C, k) to the same sum, it is sufficient to show that, for $k>0$, $p\geqslant 1$, $kp'>1$,

$$\int_1^X \left| \omega \frac{d}{d\omega} C_k(\omega) \right| d\omega = o(X),$$

implies that

$$\sum_{n=1}^{N} | na_n^{(k)} |^p = o(N).$$

We have, from Lemma 3,

$$nE_n^{(k)} a_n^{(k)} = \sum_{\nu=0}^{n} E_{n-\nu}^{(k-1)} b_\nu$$
$$= \sum_{\nu=0}^{n} E_{n-\nu}^{(k-1)} \sum_{\mu=0}^{\nu} E_{\nu-\mu}^{(-i-2)} B_\mu^{(i)},$$

where i is the integer next greater than k. From Lemmas 1, 2 and 6, it then follows that

$$nE_n^{(k)} a_n^{(k)} = \sum_{\rho=0}^{i} d_\rho \sum_{\nu=0}^{n} E_{n-\nu}^{(k-1)} \sum_{\mu=0}^{\nu} E_{\nu-\mu}^{(-i-2)} B_i(\mu+\phi)$$
$$= D_k \sum_{\rho=0}^{i} d_\rho \sum_{\nu=0}^{n} E_{n-\nu}^{(k-1)} \sum_{\mu=0}^{\nu} E_{\nu-\mu}^{(-i-2)} \int_0^{\mu+\phi} (\mu+\phi-u)^{i-k} \frac{d}{du} B_k(u)\, du,$$

where $\phi=\phi(\rho)=\rho/i$ and

$$D_k = \frac{\Gamma(i+1)}{\Gamma(k+1)\Gamma(1+i-k)}.$$

We now obtain, from Lemma 2,

$$nE_n^{(k)} a_n^{(k)} = D_k \sum_{\rho=0}^{i} d_\rho \sum_{\mu=0}^{n} \sum_{\nu=\mu}^{n} E_{n-\nu}^{(k-1)} E_{\nu-\mu}^{(-i-2)} \int_0^{\mu+\phi} \{u^{k+1} \frac{d}{du} C_k(u)\} (\mu+\phi-u)^{i-k}\, du$$
$$= D_k \sum_{\rho=0}^{i} d_\rho \sum_{\mu=0}^{n} \int_0^{\mu+\phi} (\mu+\phi-u)^{i-k} u^{k+1} \frac{d}{du} C_k(u)\, du \sum_{\nu=\mu}^{n} E_{n-\nu}^{(k-1)} E_{\nu-\mu}^{(-i-2)}.$$

But

$$\sum_{\nu=\mu}^{n} E_{n-\nu}^{(k-1)} E_{\nu-\mu}^{(-i-2)} = \sum_{s=0}^{n-\mu} E_{n-\mu-s}^{(k-1)} E_s^{(-i-2)},$$

which is the coefficient of $x^{n-\mu}$ in the expansion of $(1-x)^{-k}(1-x)^{i+1}$, and is therefore equal to $E_{n-\mu}^{(k-i-2)}$ Hence, since $0\leqslant\phi\leqslant 1$, we have

$$n\left| a_n^{(k)} \right| = O\left\{ n^{-k} \sum_{\rho=0}^{i} \int_0^{n+\phi} u^{k+1} \left| \frac{d}{du} C_k(u) \right| du \left| \sum_{\mu=[u-\phi]}^{n} (\mu+\phi-s)^{i-k} E_{n-\mu}^{(k-i-2)} \right| \right\}$$
$$= O\left\{ n^{-k} \sum_{\rho=0}^{i} \sum_{s=0}^{n} \int_s^{s+1} u^{k+1} \left| \frac{d}{du} C_k(u) \right| du \left| \sum_{\mu=s}^{n} (\mu+\phi-s)^{i-k} E_{n-\mu}^{(k-i-2)} \right| \right\}.$$

The innermost sum in the expression is

$$\sum_{\lambda=0}^{n-s} (n-s-\lambda+\phi)^{i-k} E_\lambda^{(k-i-2)} = \gamma_{n-s,\, i+1-k}(\phi),$$

which, by Lemmas 4 and 5, is

$$O\{(n-s+1)^{k-i-2}\}, \quad (\rho=0, 1, 2, \ldots, i).$$

Writing

$$\alpha_s = \int_s^{s+1} \left| u \frac{d}{du} C_k(u) \right| du,$$

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it follows that

$$n\left|a_n^{(k)}\right| = O\left\{n^{-k}\sum_{s=0}^{n}(n-s+1)^{k-i-2}\int_s^{s+1}u^k\left|u\frac{d}{du}C_k(u)\right|du\right\}$$

$$= O\left\{\sum_{s=0}^{n}\alpha_s(n-s+1)^{k-i-2}\right\},$$

and, by Lemma 7, that, for $p \geqslant 1$,

$$\sum_{n=1}^{N}\left|na_n^{(k)}\right|^p = O\left\{\sum_{s=1}^{N}\alpha_s^p\right\}$$

$$= O\left\{\sum_{s=1}^{N}\left(\int_s^{s+1}u\left|\frac{d}{du}C_k(u)\right|du\right)^p\right\}$$

$$= O\left\{\sum_{s=1}^{N}\left(\int_s^{s+1}\left|u\frac{d}{du}C_k(u)\right|^p du\right)\left(\int_s^{s+1}(1^{p'}\,du\right)^{p/p'}\right\}$$

$$= O\left\{\int_1^{N+1}\left|u\frac{d}{du}C_k(u)\right|^p du\right\} = o(N).$$

The theorem is therefore completely proved.

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