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The role of angular momentum conservation in homogeneous turbulence

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Loitsyanky's integral $I = -\int r^2 \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle \mathrm{d}\boldsymbol{r}$ is known to be approximately conserved in certain types of fully developed, isotropic turbulence, and its near conservation controls the rate of decay of kinetic energy. Landau suggested that this integral is related to the angular momentum $\boldsymbol{H} = \int (\boldsymbol{x} \times \boldsymbol{u})\, \mathrm{d}V$ of some large volume V of the turbulence, according to the expression $I = \langle \boldsymbol{H}^2 \rangle / V$. He also suggested that the approximate conservation of I is related to the principle of conservation of angular momentum. However, Landau's analysis can be criticized because, formally, it applies only to inhomogeneous turbulence evolving in a closed domain. So how are we to interpret the near conservation of I? And what is its relationship, if any, to angular momentum conservation? We show that the key to extending Landau's analysis to strictly homogeneous turbulence is to rewrite Loitsyansky's integral in terms of the vector potential of the velocity field, i.e. $I = 6 \int \langle \boldsymbol{A} \cdot \boldsymbol{A}' \rangle \mathrm{d}\boldsymbol{r}$, where $\nabla \times \boldsymbol{A} = \boldsymbol{u}$. This yields $I = 6 \langle [\int_V \boldsymbol{A}\, \mathrm{d}V]^2 \rangle / V$ for any large spherical volume V of radius R. Crucially, $\boldsymbol{J} = 3 \int_V \boldsymbol{A}\, \mathrm{d}V$ can be rewritten as the weighted integral of the angular momentum density throughout all space. This fundamentally changes the way in which we interpret the dynamical behaviour of I. For example, we show that the conservation of $\langle \boldsymbol{J}^2 \rangle / V$, and hence of I, which occurs when the long-range correlations are weak, is a direct consequence of the decorrelation of the flux of angular momentum out through a spherical control surface S and the local angular momentum in the vicinity of S. Thus, within the framework of strictly homogeneous turbulence, we provide the first self-consistent interpretation of Loitsyanky's integral in terms of angular momentum conservation. We also show that essentially the same ideas carry over to certain types of anisotropic turbulence, such as magnetohydrodynamic (MHD), rotating and stratified turbulence. This is important because conservation of angular momentum, which manifests itself in the form of a Loitsyansky-like invariant, places a fundamental restriction on the way in which the integral scales can evolve in such turbulence. This, in turn, controls the rate of decay of energy. We illustrate this by deriving new decay laws for MHD and stratified turbulence. The MHD decay laws are consistent with the available numerical evidence, but further study is required to verify, or otherwise, the predictions for stratified turbulence.

1. Introduction

It is natural to ask if the principles of linear and angular momentum conservation can be used to establish integral invariants for homogenous turbulence. While Saffman

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(1967) used linear momentum conservation to great effect, establishing just such an invariant, our attempts to deploy angular momentum conservation have proved more problematic. This paper explores these difficulties and offers a remedy.

1.1. *The Loitsyansky and Saffman integrals and their relationship to momentum conservation*

We consider isotropic turbulence in which $\langle \boldsymbol{u} \rangle = 0$ and the Reynolds number is high, $Re = u\ell/\nu \gg 1$. (Here u and ℓ are the usual integral scales and ν the viscosity.) The energy spectrum at small wavenumber k then takes the form

$$E\,(k) = Lk^2/4\pi^2 + Ik^4/24\pi^2 + \cdots, \tag{1.1}$$

provided that the two-point velocity correlation $\langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle$ decays sufficiently rapidly with separation $r = |\boldsymbol{r}| = |\boldsymbol{x}' - \boldsymbol{x}|$, i.e. $\langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle \leqslant O(r^{-6})$ (see, for example, Davidson 2004, § 6.3). The scalars L and I are known as the Saffman and Loitsyansky integrals respectively, and can be written as

$$L = \int \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle \mathrm{d}\boldsymbol{r} \tag{1.2}$$

and

$$I = -\int r^2 \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle \mathrm{d}\boldsymbol{r}. \tag{1.3}$$

Equation (1.1) suggests that, as far as the large scales are concerned, there are two canonical cases: $E(k \to 0) \sim Lk^2$, sometimes called a Saffman spectrum, and $E(k \to 0) \sim Ik^4$, the case in which $L = 0$. Of course other possibilities exist, in particular $E(k \to 0) \sim k^n$, $2 < n < 4$. However, these are associated with singularities in the spectral tensor (the transform of $\langle u_i u_j' \rangle$) and so, perhaps, are of less interest.

Note that both $E \sim Lk^2$ and $E \sim Ik^4$ spectra are readily generated in computer simulations (see, for example, Chasnov 1993; Eyink & Thomson 2000; Ossai & Lesieur 2000; Oberlack 2002; Herring *et al.* 2006; Ishida, Davidson, & Kaneda 2006). Which type of turbulence is seen simply depends on the initial conditions. If L is non-zero at $t = 0$, then we obtain a Saffman spectrum for all t, whereas $L = 0$ at $t = 0$ excludes such a spectrum. Opinion is divided, however, as to whether grid turbulence is of the Saffman ($E \sim Lk^2$) or Batchelor ($E \sim Ik^4$) type, since the experimental data is ambiguous on this point.

Let us consider these two classes of turbulence in a little more detail, starting with Saffman spectra. Noting that ensemble and volume averages are equivalent, we may rewrite Saffman's integral in the form

$$L = \left\langle \left[\int_V \boldsymbol{u}\,\mathrm{d}V \right]^2 \right\rangle \bigg/ V \;, \tag{1.4}$$

where V is some large volume embedded within the turbulence. Evidently we obtain a Saffman spectrum whenever the turbulence contains a sufficiently large amount of linear momentum, $\boldsymbol{P} = \int_V \boldsymbol{u}\,\mathrm{d}V$ (Saffman 1967). Note that we cannot make L zero simply through a change of frame of reference. That is we must ensure $\langle \boldsymbol{u} \rangle = 0$, and this implies that there is a preferred frame of reference, in which

$$\lim_{V \to \infty} \frac{\int \boldsymbol{u}\,\mathrm{d}V}{V} = 0. \tag{1.5}$$

However, this is not enough to enforce the stronger condition

$$L = \operatorname*{Lim}_{V\to\infty} \frac{\left(\int \boldsymbol{u}\,\mathrm{d}V\right)^2}{V} = 0. \tag{1.6}$$

Indeed, the central limit theorem suggests that, in general, we would expect $\boldsymbol{P} = \int \boldsymbol{u}\,\mathrm{d}V \sim V^{1/2}$, and hence $L \neq 0$. This can be seen from the following argument: We may consider the turbulence to be composed of a random sea of eddies (blobs of vorticity), each of which has some linear impulse $\boldsymbol{L}_i = (1/2)\int \boldsymbol{x}\times\boldsymbol{\omega}\,\mathrm{d}V$. (Here the subscript i indicates the ith eddy within the volume V.) Moreover, the linear momentum within some large spherical volume V is proportional to the sum of the linear impulses of the individual eddies contained within V, $\int \boldsymbol{u}\,\mathrm{d}V \sim \sum \boldsymbol{L}_i$ (see Appendix A). Now suppose that our control volume is indeed spherical and that there are N eddies within V, each assigned a random value of $\boldsymbol{L}_i$ taken from a probability density function (p.d.f.) of zero mean. Then the central limit theorem suggests that $\int \boldsymbol{u}\,\mathrm{d}V \sim N^{1/2} \sim V^{1/2}$, so that (1.5) is satisfied, but (1.6) is not. In short, a Saffman spectrum will be realized whenever the eddies possess a significant amount of linear impulse.

Note that $\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle$ is related to the longitudinal correlation function $f(r)$ by

$$\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle = \frac{u^2}{r^2}\frac{\partial}{\partial r}(r^3 f), \quad u^2 = \frac{1}{3}\langle \boldsymbol{u}^2\rangle, \tag{1.7}$$

and so Saffman's integral can also be written as $L = 4\pi u^2[r^3 f]_\infty$, where the subscript ∞ indicates $r\to\infty$. Thus a finite value of L implies a slow, algebraic decline in f, i.e. $f_\infty \sim r^{-3}$, though it does not exclude the possibility that $\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle_\infty$ decays more rapidly. Indeed, it is easy to construct kinematically admissible fields of isotropic turbulence in which L is non-zero, yet $\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle_\infty \sim \exp(-r^2/\ell^2)$ (Davidson 2007).

Now Saffman showed that L is an invariant of freely decaying turbulence. This follows from integrating the Kármán–Howarth equation in the form

$$\frac{\partial}{\partial t}\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle = \frac{1}{r^2}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(r^4 u^3 K) + 2\nu\,\nabla^2\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle \tag{1.8}$$

along with the observation that the triple correlation $u^3K(r) = \langle u_x^2(\boldsymbol{x})u_x(\boldsymbol{x}+r\hat{\boldsymbol{e}}_x)\rangle$ falls as $[u^3K]_\infty \sim r^{-4} + O(r^{-5})$ at large r (Batchelor & Proudman 1956). The invariance of L also follows directly from the conservation of linear momentum, in the sense that the flux of linear momentum out through the surface of V turns out to be smaller than $O(V^{1/2})$ and so is too weak to change L in the limit of $V\to\infty$ (see §2.1). One of the practical consequences of the conservation of L is that self-similarity of the large scales demands $u^2\ell^3 = $ constant, which in turn requires that the kinetic energy decays as $u^2 \sim t^{-6/5}$ (Saffman 1967).

Let us now turn to Batchelor spectra, $E \sim Ik^4$, where L is set equal to zero by virtue of our choice of initial condition. For such turbulence Landau suggested that, provided the two-point velocity correlations decay sufficiently rapidly with distance, I can be related to the angular momentum of the fluid, $\boldsymbol{H}$, via an expression similar to (1.4). In particular he proposed

$$I \approx \operatorname*{Lim}_{V\to\infty} \frac{\langle \boldsymbol{H}^2\rangle}{V}, \quad \boldsymbol{H} = \int_V (\boldsymbol{x}\times\boldsymbol{u})\,\mathrm{d}V \tag{1.9}$$

and suggested that the invariance of I follows from the principle of angular momentum conservation (Landau & Lifshitz 1959, p. 142). Thus, at face value, there appears to be certain similarities between I and L, with momentum conservation providing the common theme. However, it turns out that the situation is far more complicated than this. For example, unlike L, the intergal I is not, in general, an invariant. That is the Kármán–Howarth equation integrates to give

$$\frac{\mathrm{d}I}{\mathrm{d}t} = 8\pi\left[u^3 r^4 K\right]_\infty , \tag{1.10}$$

and the work of Batchelor & Proudman (1956) suggests that the long-range pressure forces will, in general, establish long-range triple correlations of the form $K_\infty \sim cr^{-4}$. Certainly, numerical simulations of $E \sim Ik^4$ turbulence usually show a slow rise in I. Curiously, though, recent simulations performed in very large computational domains show $I \approx$ constant once the turbulence has become fully developed (Ishida *et al.* 2006). This, in turn, suggests that $c = (r^4 K)_\infty$ is very small in fully developed turbulence. (The magnitude of c is undetermined in Batchelor & Proudman's 1956 analysis.) One of the consequences of the approximate conservation of I is that Kolmogorov's decay law $u^2 \sim t^{-10/7}$, which rests on the assumption that $I = \text{constant} \sim u^2\ell^5$ (Kolmogorov 1941), is indeed observed in $E \sim Ik^4$ turbulence, at least in the fully developed state (Ishida *et al.* 2006).

Another difference between I and L is that (1.9), unlike (1.4), is not rigorous, since Landau's arguments are formulated for inhomogeneous turbulence (see § 1.2). Moreover, the relationship between the conservation, or otherwise, of I and the principle of angular momentum conservation is not immediately apparent from (1.9). For example, under what conditions, if any, is $\boldsymbol{H}$ conserved in (1.9)? On the other hand, one can establish a very clear link between linear momentum conservation and L = constant. Both of these points are discussed in § 2.

All in all it would seem that the role of momentum conservation in $E \sim Lk^2$ turbulence is well understood, whereas its role in $E \sim Ik^4$ turbulence is not so clear. One of the purposes of this paper is to clarify the link between the behaviour of I and the principle of angular momentum conservation in homogenous turbulence.

1.2. *Landau's analysis for inhomogeneous turbulence*

As a prelude to our discussion of homogenous turbulence it is useful to review briefly Landau's analysis. Our aim is to expose the difficulties associated with his approach.

In order to apply the principle of angular momentum conservation, Landau considered the case of inhomogeneous turbulence evolving in a large, closed domain, V. The fact that the domain is closed turns out to be of crucial importance, as we shall see. The starting point is the identity

$$(\boldsymbol{x} \times \boldsymbol{u}) \cdot (\boldsymbol{x}' \times \boldsymbol{u}') = 2\boldsymbol{x} \cdot \boldsymbol{x}'(\boldsymbol{u} \cdot \boldsymbol{u}') - u_i' x_j' \nabla \cdot [x_i x_j \boldsymbol{u}]$$

which integrates to yield

$$\boldsymbol{H}^2 = \left[\int_V \boldsymbol{x} \times \boldsymbol{u}\, \mathrm{d}V\right]^2 = \int_V \int_V 2\boldsymbol{x} \cdot \boldsymbol{x}'(\boldsymbol{u} \cdot \boldsymbol{u}')\, \mathrm{d}\boldsymbol{x}'\, \mathrm{d}\boldsymbol{x},$$

as $\boldsymbol{u} \cdot \mathrm{d}\boldsymbol{S} = 0$ on the surface of V. Since $\int_V \boldsymbol{u}\, \mathrm{d}V = 0$ for a closed domain, this can be rewritten as

$$\boldsymbol{H}^2 = -\int_V \int_V (\boldsymbol{x}' - \boldsymbol{x})^2 (\boldsymbol{u} \cdot \boldsymbol{u}')\, \mathrm{d}\boldsymbol{x}'\, \mathrm{d}\boldsymbol{x}, \tag{1.11}$$

and on ensemble averaging we obtain

$$\langle \boldsymbol{H}^2 \rangle = \int_V \left[-\int_{V*} r^2 \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle\, \mathrm{d}\boldsymbol{r}\right] \mathrm{d}\boldsymbol{x}. \tag{1.12}$$

Note that the shape of V^* depends on the location of $\boldsymbol{x}$ within V. So far the analysis is rigorous, but its relevance to homogeneous turbulence is unclear. It is now assumed that $\langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle$ falls off rapidly with separation r, say as $\langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle_\infty \sim \exp(-r^2/\ell^2)$, which seems unlikely in the light of the work by Batchelor & Proudman (1956) but cannot be excluded. Then for all points $\boldsymbol{x}$ which are remote from the boundary the inner integral in (1.12) can be replaced by an integral over all $\boldsymbol{r}$. Since $V \gg \ell^3$, this is a good approximation for all points $\boldsymbol{x}$ in V, except those which lie within a distance $O(\ell)$ from the surface. It follows that

$$\langle \boldsymbol{H}^2 \rangle / V = -\int r^2 \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle \, \mathrm{d}\boldsymbol{r} \quad \left[1 + O\left(\ell / V^{1/3}\right)\right]. \tag{1.13}$$

In the limit of $V \gg \ell^3$, we recover (1.9), and so it appears that there is indeed a link between I and angular momentum $\boldsymbol{H}$. Note, however, that two crucial ingredients of Landau's analysis are as follows:

(i) $\int_V \boldsymbol{u} \mathrm{d}V = 0$, since the domain is closed;

(ii) $\langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle$ is assumed to fall off very rapidly with separation r.

We shall see that, in general, neither of these conditions holds for a large, open domain embedded within a field of homogenous turbulence.

Note also that, so far, we have not used conservation of angular momentum to explain the invariance of I. (Recall that I can, in principle, be conserved, provided that the long-range correlations are weak enough for $(r^4 K)_\infty = 0$, which may not always be true in practice but is nevertheless the situation considered by Landau.) To do this we need to consider the particular situation in which the closed domain is spherical, with radius R. In such a case $\boldsymbol{H}$ is conserved in each realization, in the sense that the viscous stresses on the surface of V have a negligible influence on $\boldsymbol{H}$ in the limit of $(R/\ell) \to \infty$. Equation (1.13) then tells us that the invariance of I is indeed a consequence of angular momentum conservation, provided, of course, that conditions (i) and (ii) above hold true. However, this kind of logic breaks down when we try to recast the ideas in terms of a large, open domain embedded within a field of homogenous turbulence, since there is a flux of angular momentum out through the open surface of the control volume. It seems that, yet again, any attempt to rework Landau's analysis for strictly homogenous turbulence fails.

1.3. *Structure of the paper*

In this paper we consider homogeneous turbulence, focusing on the $E \sim I k^4$ spectra. In particular, we explore the relationship between I and $\boldsymbol{H}$ and the link between (1.10) and the principle of angular momentum conservation. We shall show that, despite the reservations expressed above, Landau's analysis can indeed be adapted to homogeneous turbulence, although the way in which this is achieved is far from straightforward. Certainly, simply applying angular momentum conservation to a finite-sized control volume embedded within the turbulence does not work. A quite different strategy is required.

The central difficulty is that the angular momentum held in any open control volume of finite size is dominated by the residual linear momentum in V, $\boldsymbol{P} = \int_V \boldsymbol{u} \, \mathrm{d}V$, a problem which Landau avoided by using a closed domain. This residual linear momentum turns out to be important because it changes the scaling $\langle \boldsymbol{H}^2 \rangle \sim V$. Kolmogorov suggested that this difficulty may be circumvented by evaluating a weighted integral of $\boldsymbol{x} \times \boldsymbol{u}$ over an infinite domain, V_∞ . In particular, he proposed evaluating

$$\hat{\boldsymbol{H}} = \int_{V_\infty} \boldsymbol{x} \times \boldsymbol{u} \exp(-|\boldsymbol{x}^2|/R^2)\,\mathrm{d}V \tag{1.14}$$

and then looking at the behaviour of $\langle \hat{\boldsymbol{H}}^2 \rangle / R^3$ in the limit of $R \to \infty$ (see Monin & Yoglam 1975, p. 151, for a discussion of Kolmogorov's analysis). We shall see that such a strategy is indeed fruitful, though the details differ considerably from those proposed by Kolmogorov.

The structure of the paper is as follows: We start, in § 2, by documenting the difficulties which arise when trying to recast Landau's analysis by applying momentum conservation to a large, spherical control volume embedded in a field of homogeneous turbulence. Next, in § 3, we show how these difficulties may be circumvented by rewriting I in terms of the vector potential $\boldsymbol{A}$,

$$I = 6 \int \langle \boldsymbol{A} \cdot \boldsymbol{A}' \rangle \,\mathrm{d}\boldsymbol{r}, \tag{1.15}$$

where $\boldsymbol{u} = \nabla \times \boldsymbol{A}$. This yields

$$I = \lim_{V \to \infty} \frac{6 \left\langle \left[\int_V \boldsymbol{A}\,\mathrm{d}V \right]^2 \right\rangle}{V}, \tag{1.16}$$

where V is our finite-sized control volume of radius R. The crucial point is that $\int_V \boldsymbol{A}\mathrm{d}V$ can be expressed as an integral over the entire space of the angular momentum density:

$$\int_V \boldsymbol{A}\,\mathrm{d}V = \frac{1}{3} \int_{V_\infty} (\boldsymbol{x} \times \boldsymbol{u})\, G\left(|\boldsymbol{x}|/R\right) \mathrm{d}\boldsymbol{x}, \tag{1.17}$$

where

$$G(\chi) = 1 \text{ for } \chi \leqslant 1 \text{ and } G(\chi) = \chi^{-3} \text{ for } \chi > 1.$$

This is reminiscent of (1.14) and shows that I is not proportional to $\langle \boldsymbol{H}^2 \rangle / V$, as suggested by (1.9), but rather to $\langle \boldsymbol{J}^2 \rangle / V$, where $\boldsymbol{J}$ is a weighted integral over the entire space of $\boldsymbol{x} \times \boldsymbol{u}$:

$$\boldsymbol{J} = \int_{V_\infty} (\boldsymbol{x} \times \boldsymbol{u}) G(|\boldsymbol{x}|/R)\,\mathrm{d}\boldsymbol{x}. \tag{1.18}$$

We then show that the conservation of I (in the absence of long-range interactions) can be explained in terms of a lack of correlation between the flux of angular momentum out through a spherical control surface S and the angular momentum in the vicinity of S.

Next, in § 5, we show how these ideas can be extended to include certain types of anisotropic turbulence, such as magnetohydrodynamic (MHD) turbulence, rotating turbulence and stratified turbulence. Indeed, we use the approximate conservation of I in these systems to derive decay laws for MHD and stratified turbulence. The predictions for MHD turbulence are consistent with the available numerical evidence, while those for stratified turbulence remain to be tested.

2. Momentum conservation applied to a spherical control volume embedded in a field of homogeneous turbulence

In § 3 we shall show how a Kolmogorov-like strategy can be used to establish a clear link between I and $\boldsymbol{x} \times \boldsymbol{u}$, thus extending Landau's arguments. First, however, it is instructive to consider what happens when momentum conservation is applied to a large spherical control volume embedded within a field of homogeneous turbulence. We shall examine both $E \sim Lk^2$ turbulence, where linear momentum conservation holds the key, and $E \sim Ik^4$ turbulence, where it is the angular momentum that interests us. We shall see that the procedure works well for $E \sim Lk^2$ turbulence, providing results consistent with Saffman (1967) but that the procedure fails for $E \sim Ik^4$ turbulence, in the sense that we cannot enforce condition (i) above, which in turn leads to a scaling quite different from that of (1.9).

Many of the details in § 2.1 and § 2.2 may be found scattered across various publications, particularly Davidson (2004, 2007), Davidson, Ishida & Kaneda (2006), and Llor (2006). However, we have redeveloped the arguments more or less from first principles because most of the published accounts are incomplete, yet the results are important for the rest of the paper. The purpose of these sections is to set the scene for the central results of § 3 and § 5.

2.1. *The linear momentum balance*

Consider a spherical control volume of radius $R (R \gg \ell)$ embedded in a sea of homogeneous turbulence. We wish to determine the square of the linear momentum $\langle \boldsymbol{P}^2 \rangle$ held within V. Since $u_i = \nabla \cdot (x_i \boldsymbol{u})$, we have

$$\boldsymbol{P}^2 = \int_V u_i' \, \mathrm{d}\boldsymbol{x}' \oint_s x_i \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{S} = \int_V \left[u_i' \oint_s x_i \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{S} \right] \mathrm{d}\boldsymbol{x}', \tag{2.1}$$

where S is the surface enclosing V. Let us take the origin for $\boldsymbol{x}$ and $\boldsymbol{x}'$ to lie at the centre of V. Since all points on S are statistically equivalent, we focus on the particular location $\boldsymbol{x} = R\hat{\boldsymbol{e}}_x$ when evaluating the inner integral. Equation (2.1) then yields

$$\langle \boldsymbol{P}^2 \rangle = 4\pi R^3 \int_V \langle u_x u_x' \rangle \, \mathrm{d}\boldsymbol{r}, \tag{2.2}$$

where $\boldsymbol{x}'$ is an interior point in V, and the displacement vector $\boldsymbol{r} = \boldsymbol{x}' - \boldsymbol{x}$ links $\boldsymbol{x} = R\hat{\boldsymbol{e}}_x$ to the interior point $\boldsymbol{x}'$. Substituting for $\langle u_x u_x' \rangle$ in terms of f yields

$$\langle \boldsymbol{P}^2 \rangle = 2\pi R^3 u^2 \int_V \left[\frac{1}{r} \frac{\partial}{\partial r}(r^2 f) - \frac{r_x^2}{r} \frac{\partial f}{\partial r} \right] \mathrm{d}\boldsymbol{r},$$

which is readily integrated using spherical polar coordinates centred on the point $\boldsymbol{x} = R\hat{\boldsymbol{e}}_x$. In particular, we use the result

$$\int_V g(r) r_x^n \, \mathrm{d}\boldsymbol{r} = (-1)^n \frac{2\pi}{n+1} \int_0^{2R} g(r) r^{n+2} [1 - (r/2R)^{n+1}] \, \mathrm{d}r, \tag{2.3}$$

which holds for any function $g(r)$. Applying (2.3) and then integrating by parts, we find,

$$\langle \boldsymbol{P}^2 \rangle = 4\pi^2 R^2 u^2 \int_0^{2R} r^3 f(r) [1 - (r/2R)^2] \, \mathrm{d}r, \tag{2.4}$$

(Davidson 2004, p. 360). For a Saffman spectrum, in which $f_\infty \sim r^{-3}$, this demands

$$\langle \boldsymbol{P}^2 \rangle / V = 4\pi u^2 [r^3 f]_\infty \tag{2.5}$$

in the limit of $R/\ell \to \infty$. Comparing (2.5) with $L = 4\pi u^2 [r^3 f]_\infty$ we find

$$L = \lim_{V \to \infty} \frac{\langle \boldsymbol{P}^2 \rangle}{V}, \tag{2.6}$$

which is consistent with (1.4). For a Batchelor spectrum, on the other hand, $K_\infty \sim r^{-4}$ combined with the Kármán–Howarth equation requires that f falls off no more slowly than $f_\infty \sim r^{-6}$. Equation (2.4) then reduces to

$$\langle \boldsymbol{P}^2 \rangle = 4\pi^2 R^2 u^2 \int_0^\infty r^3 f(r)\, \mathrm{d}r, \tag{2.7}$$

for $R/\ell \to \infty$. Note that in this case

$$\operatorname*{Lim}_{V\to\infty} \frac{\langle \boldsymbol{P}^2 \rangle}{V} = 0, \tag{2.8}$$

which is consistent with the requirement that $L = 0$. Nevertheless, $\langle \boldsymbol{P}^2 \rangle$ is non-zero and of order R^2, so that we retain some residual linear momentum in V, even though this residual momentum is not large enough to yield a finite value of L. This small but finite linear momentum will turn out to be of crucial importance.

Finally, let us apply the principle of conservation of linear momentum to our control volume V. Ignoring viscous forces we have

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = -\oint_S \boldsymbol{u}\,(\boldsymbol{u}\cdot \mathrm{d}\boldsymbol{S}) - \oint_S p\, \mathrm{d}\boldsymbol{S}, \tag{2.9}$$

from which

$$\frac{\mathrm{d}\boldsymbol{P}^2}{\mathrm{d}t} = -2\int_V \boldsymbol{u}\, \mathrm{d}V \cdot \left[\oint_S \boldsymbol{u}\,(\boldsymbol{u}\cdot\, \mathrm{d}\boldsymbol{S}) + \oint_S p\, \mathrm{d}\boldsymbol{S} \right]. \tag{2.10}$$

(Here p is the pressure, and we have taken the fluid density to be unity.) Since $\langle \boldsymbol{u} p' \rangle = 0$ in isotropic turbulence, the pressure term vanishes on averaging, and we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{P}^2 \rangle = -2\left\langle \int_V \left[u_i' \oint_S u_i\,(\boldsymbol{u}\cdot \mathrm{d}\boldsymbol{S}) \right] \mathrm{d}\boldsymbol{x}' \right\rangle. \tag{2.11}$$

Once again we note that all points on the boundary are statistically equivalent and fixed on the surface point $\boldsymbol{x} = R\hat{\boldsymbol{e}}_x$. Then (2.11) simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{P}^2 \rangle = -8\pi R^2 \int_V \langle u_i u_x u_i' \rangle\, \mathrm{d}\boldsymbol{r}, \tag{2.12}$$

where, as before, $\boldsymbol{x}'$ is an interior point in V, and the displacement vector $\boldsymbol{r} = \boldsymbol{x}' - \boldsymbol{x}$ links $\boldsymbol{x} = R\hat{\boldsymbol{e}}_x$ to the interior point $\boldsymbol{x}'$. We now substitute for $\langle u_i u_x u_i' \rangle$ using the isotropic relationship

$$\langle u_i u_x u_i' \rangle = \frac{u^3 r_x}{2r^4} \frac{\partial}{\partial r}(r^4 K) \tag{2.13}$$

and evaluate the integral in (2.12) using spherical polar coordinates centred on the point $\boldsymbol{x} = R\hat{\boldsymbol{e}}_x$. The calculation is virtually identical to that which led up to (2.4), and so using (2.3) we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{P}^2 \rangle = 4\pi^2 R^2 u^3 \int_0^{2R} [1 - (r/2R)^2] \frac{1}{r} \frac{\partial}{\partial r}(r^4 K)\, \mathrm{d}r \tag{2.14}$$

(Davidson 2004, p. 361). We now consider the limit $R/\ell \to \infty$. Noting that $K_\infty \sim r^{-4}$, (2.14) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{P}^2 \rangle = 4\pi^2 R^2 u^3 \int_0^\infty \frac{1}{r} \frac{\partial}{\partial r}(r^4 K)\, \mathrm{d}r = 4\pi^2 R^2 u^3 \int_0^\infty r^2 K\, \mathrm{d}r. \tag{2.15}$$

The key result here is the scaling $(\mathrm{d}/\mathrm{d}t)\langle \boldsymbol{P}^2\rangle \sim R^2$. Evidently, for large R, the flux of linear momentum out through S is too small to alter $\langle \boldsymbol{P}^2\rangle/V$. Thus, conservation of Saffman's integral can be thought of as a direct consequence of linear momentum conservation, as suggested in § 1.1.

2.2. *The angular momentum balance*

Let us now apply angular momentum conservation to the same control volume. Ignoring viscous stresses we have

$$\frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}t} = -\oint_S (\boldsymbol{x}\times\boldsymbol{u})\,\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{S}, \tag{2.16}$$

from which

$$\frac{\mathrm{d}\boldsymbol{H}^2}{\mathrm{d}t} = -2\int_V \boldsymbol{x}'\times\boldsymbol{u}'\,\mathrm{d}\boldsymbol{x}'\cdot\oint_S (\boldsymbol{x}\times\boldsymbol{u})\,\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{S}. \tag{2.17}$$

We now deploy the usual trick of focusing on the surface point $\boldsymbol{x}=R\hat{\boldsymbol{e}}_{\boldsymbol{x}}$ while noting that all points on the boundary S are statistically equivalent. Then (2.17) simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{H}^2\rangle = -8\pi R^2\int_V \langle(\boldsymbol{x}'\times\boldsymbol{u}')\cdot(\boldsymbol{x}\times\boldsymbol{u})u_x\rangle\,\mathrm{d}\boldsymbol{r}, \tag{2.18}$$

where, as before, $\boldsymbol{x}'$ is a interior point in V and $\boldsymbol{r}=\boldsymbol{x}'-\boldsymbol{x}$. It is readily confirmed that

$$\left\langle\left(\boldsymbol{x}'\times\boldsymbol{u}'\right)\cdot(\boldsymbol{x}\times\boldsymbol{u})\,u_x\right\rangle = R^2\left[\langle u_iu_xu_i'\rangle - \left\langle u_x^2u_x'\right\rangle\right] + R\left[r_x\left\langle u_iu_xu_i'\right\rangle - r_i\left\langle u_iu_xu_x'\right\rangle\right], \tag{2.19}$$

while isotropy, which allows us to write $\langle u_iu_ju_k'\rangle$ in terms of $u^3K(r)$, demands

$$r_x\langle u_iu_xu_i'\rangle - r_i\langle u_iu_xu_x'\rangle = -\frac{u^3(r^2-3r_x^2)}{4r^2}\frac{\partial}{\partial r}(r^2K) \tag{2.20}$$

and

$$\langle u_x^2u_x'\rangle = \frac{u^3r_x}{2r}\left[\frac{\partial}{\partial r}(rK) - r_x^2\frac{\partial}{\partial r}(K/r)\right]. \tag{2.21}$$

We now substitute for $\langle u_iu_xu_i'\rangle - \langle u_x^2u_x'\rangle$ and $r_x\langle u_iu_xu_i'\rangle - r_i\langle u_iu_xu_x'\rangle$ in (2.19), using (2.13), (2.20) and (2.21). The integral in (2.18) can then be evaluated in the usual way, using spherical polar coordinates centred on the point $\boldsymbol{x}=R\hat{\boldsymbol{e}}_x$. Using (2.3) we find, after a little algebra,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{H}^2\rangle = 4\pi^2R^4u^3\int_0^{2R}[1-3(r/2R)^2+2(r/2R)^4]\frac{1}{r}\frac{\partial}{\partial r}(r^4K)\,\mathrm{d}r, \tag{2.22}$$

which is the angular momentum analogue of (2.14). Finally, noting that the inviscid version of the Kármán–Howarth equation can be written as

$$\frac{\partial}{\partial t}(u^2r^3f(r)) = \frac{u^3}{r}\frac{\partial}{\partial r}(r^4K) \tag{2.23}$$

and substituting for $\partial(r^4K)/\partial r$ in terms of $\partial(u^2f)/\partial t$, we obtain, after integration,

$$\langle \boldsymbol{H}^2\rangle = 4\pi^2R^4u^2\int_0^{2R} r^3f(r)[1-3(r/2R)^2+2(r/2R)^4]\,\mathrm{d}r, \tag{2.24}$$

a result which was obtained by Llor (2006) in a slightly different way. (Note that the constant of integration in (2.24) is set to zero since $\langle \boldsymbol{H}^2\rangle \to 0$ as $u^2\to 0$.) We now consider the limit of $R/\ell \to \infty$. Noting that $K_\infty \sim r^{-4}$, (2.22) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{H}^2\rangle = 4\pi^2 R^4 u^3 \int_0^\infty \frac{1}{r}\frac{\partial}{\partial r}(r^4 K)\,\mathrm{d}r = 4\pi^2 R^4 u^3 \int_0^\infty r^2 K\,\mathrm{d}r \tag{2.25}$$

which, combined with (2.14), yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{H}^2\rangle = R^2 \frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{P}^2\rangle + O((R/\ell)^2). \tag{2.26}$$

Equation (2.24), on the other hand, yields two results:

$$\text{(i)}\quad \langle \boldsymbol{H}^2\rangle = \frac{16}{5}\pi^2 R^5 u^2 [r^3 f(r)]_\infty = \frac{3}{5} R^2 \langle \boldsymbol{P}^2\rangle \tag{2.27}$$

for a Saffman spectrum ($E \sim Lk^2$, $f_\infty \sim r^{-3}$), from which we see that $\langle \boldsymbol{H}^2\rangle / R^5 \sim L$ is an invariant; and

$$\text{(ii)}\quad \langle \boldsymbol{H}^2\rangle = 4\pi^2 R^4 u^2 \int_0^\infty r^3 f(r)\,\mathrm{d}r = R^2\langle \boldsymbol{P}^2\rangle + O((R/\ell)^2) \tag{2.28}$$

for a Batchelor spectrum ($E \sim Ik^4$, $f_\infty \sim r^{-6}$). The most striking feature of (2.28) is that $\langle \boldsymbol{H}^2\rangle$ scales as

$$\langle \boldsymbol{H}^2\rangle = R^2 \langle \boldsymbol{P}^2\rangle \sim R^4, \tag{2.29}$$

instead of the scaling $\langle \boldsymbol{H}^2\rangle \sim R^3$ suggested by (1.13). In short, it seems that $\langle \boldsymbol{H}^2\rangle$ is dominated by the residual linear momentum in V, and this is enough to alter the scaling proposed by Landau, who set $\boldsymbol{P} = 0$ through the choice of a closed domain.

Now Monin & Yaglom (1975) suggest that in an open domain, $\langle \boldsymbol{H}^2\rangle$ will be dominated by contributions that come from a thin layer of fluid adjacent to boundary and that this will influence the scaling of $\langle \boldsymbol{H}^2\rangle$ on R. No proof of this assertion is offered by Monin & Yaglom (1975), but if it is true, it seems likely that this lies behind the difference between the $\langle \boldsymbol{H}^2\rangle \sim R^3$ scaling for a closed domain and the $\langle \boldsymbol{H}^2\rangle \sim R^4$ scaling above.

The kinematic argument given in Appendix B lends support to Monin & Yaglom's (1975) claim and illustrates the nature of the problem. In brief, the argument goes like this: Consider a field of turbulence which consists of a random sea of compact vortex blobs (eddies). Let $\boldsymbol{x}_i$ locate the ith eddy in a large spherical control volume V and $\boldsymbol{r}_i$ be a local coordinate defined by $\boldsymbol{x} = \boldsymbol{x}_i + \boldsymbol{r}_i$. Next we introduce the intrinsic linear and angular impulse of the ith eddy, defined as (Batchelor 1967, p. 519)

$$\boldsymbol{L}_i = \frac{1}{2}\int_{V_i} \boldsymbol{r}_i \times \boldsymbol{\omega}\,\mathrm{d}V, \quad \boldsymbol{M}_i = \frac{1}{3}\int_{V_i} \boldsymbol{r}_i \times (\boldsymbol{r}_i \times \boldsymbol{\omega})\,\mathrm{d}V,$$

where V_i is the volume occupied by the vorticity of the ith eddy. In order to eliminate the possibility of a Saffman spectrum we set $\boldsymbol{L}_i = 0$ for all eddies, including those lying outside V. It is then possible to show that

$$\langle \boldsymbol{H}^2\rangle = \sum_i \langle \boldsymbol{M}_i^2\rangle + \sum_k [\langle [\boldsymbol{x}_k \times \boldsymbol{L}_k]^2\rangle + \text{other surface terms}], \tag{2.30}$$

where the first summation on the right is over all the eddies (blobs of vorticity) which lie entirely within V, and the summation over k corresponds to only those vortex blobs which intersect the surface of our control volume, i.e. those eddies which straddle the surface S of V. The quantity $\boldsymbol{L}_k$ is the impulse-like integral $\boldsymbol{L}_k = (1/2)\int \boldsymbol{r}_i \times \boldsymbol{\omega}\,\mathrm{d}V$ but where the domain of integration is restricted to that part of the volume of the kth eddy which lies within V.

The first term on the right of (2.30) is of order R^3, which is consistent with Landau's analysis, while the second is of order $R^2 S \sim R^4$, which is consistent with (2.29). It would seem, therefore, that there is support for the assertion by Monin & Yaglom (1975) that $\langle \boldsymbol{H}^2 \rangle$ in an open control volume is dominated by contributions that come from a thin layer of fluid adjacent to boundary, and it is this which causes the discrepancy between (1.13) and (2.29). Note, however, that this discrepancy is a result of introducing a control surface which dissects some of the eddies. If, for some (unlikely) reason, none of the eddies were to intersect S, we would recover the $\langle \boldsymbol{H}^2 \rangle \sim R^3$ scaling of Landau (see (B2) in Appendix B).

Clearly, we need an entirely different approach if we are to establish the link between angular momentum and Loitsyansky's integral in homogeneous turbulence (assuming that such a link exists), and it seems appropriate to explore Kolmogorov's suggestion of evaluating a weighted integral of $\boldsymbol{x} \times \boldsymbol{u}$ over the entire space, thus removing the problematic control surface S.

3. A new approach: Loitsyansky's integral and the vector potential

3.1. *Kinematics*

We now turn to the central results of the paper. We restrict ourselves to isotropic turbulence with a Batchelor spectrum ($E \sim I k^4$) in which, according to Batchelor & Proudman (1956), $f(r)$ decays no more slowly than $f_\infty \sim r^{-6}$. Let us introduce the vector potential for $\boldsymbol{u}$, defined in the usual way by $\nabla \times \boldsymbol{A} = \boldsymbol{u}$ and $\nabla \cdot \boldsymbol{A} = 0$. Then $\langle \boldsymbol{A} \cdot \boldsymbol{A}' \rangle$ is related to $\langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle$, $u^2 f(r)$ and I as follows:

$$\langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle = -\nabla^2 \langle \boldsymbol{A} \cdot \boldsymbol{A}' \rangle, \tag{3.1}$$

$$u^2 f(r) = -\frac{1}{r}\frac{\partial}{\partial r} \langle \boldsymbol{A} \cdot \boldsymbol{A}' \rangle, \tag{3.2}$$

$$I = -\int r^2 \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle \, \mathrm{d}\boldsymbol{r} = \int r^2 \nabla^2 \langle \boldsymbol{A} \cdot \boldsymbol{A}' \rangle \, \mathrm{d}\boldsymbol{r}. \tag{3.3}$$

Equation (3.2) tells us that $\langle \boldsymbol{A} \cdot \boldsymbol{A}' \rangle_\infty$ decays no more slowly than $\langle \boldsymbol{A} \cdot \boldsymbol{A}' \rangle_\infty \sim r^{-4}$, so that (3.3) integrates by parts to yield

$$I = 6 \int \langle \boldsymbol{A} \cdot \boldsymbol{A}' \rangle \, \mathrm{d}\boldsymbol{r}. \tag{3.4}$$

It follows from the equivalence of volume and ensemble averages that

$$I = \operatorname*{Lim}_{V \to \infty} \frac{6 \langle [\int_V \boldsymbol{A} \, \mathrm{d}V]^2 \rangle}{V}, \tag{3.5}$$

which is the Loitsyansky analogue of Saffman's relationship (1.4). We note, in passing, that (3.4) integrates to give

$$I = 8\pi \langle \boldsymbol{A}^2 \rangle [r^3 f_A(r)]_\infty,$$

where $f_A(r)$ is the longitudinal correlation function for $\boldsymbol{A}$. (This is the analogue of $L = 4\pi u^2 [r^3 f]_\infty$ for a Saffman spectrum.) Note also that (3.5) can be obtained from (3.4) in a different way, i.e. by repeating the proof of (2.4) and (2.6) but with $\boldsymbol{A}$, $f_A(r)$ and I replacing $\boldsymbol{u}$, $f(r)$ and L, respectively.

Let us now take V to be the usual spherical control volume of radius R. Then we may relate $\int_V \boldsymbol{A} \, \mathrm{d}V$ to $\boldsymbol{x} \times \boldsymbol{u}$ using a well-known result from magnetostatics (Jackson

1975, p. 187), which gives

$$\int_V \boldsymbol{A}\,\mathrm{d}V = \frac{1}{3}\int_{V_\infty} (\boldsymbol{x}\times\boldsymbol{u})G(|\boldsymbol{x}|/R)\,\mathrm{d}\boldsymbol{x}, \tag{3.6}$$

where

$$G(\chi) = 1 \text{ for } \chi \leqslant 1 \text{ and } \ G(\chi) = \chi^{-3} \text{ for } \chi > 1.$$

The relationship between I and angular momentum in homogeneous turbulence is now clear; I is not proportional to $\langle \boldsymbol{H}^2\rangle/V$, as suggested by (1.9), but rather to $\langle \boldsymbol{J}^2\rangle/V$, where $\boldsymbol{J}$ is a weighted integral over the entire space of $\boldsymbol{x}\times\boldsymbol{u}$:

$$I = \lim_{R\to\infty} \frac{2}{3}\frac{\langle \boldsymbol{J}^2\rangle}{V}, \tag{3.7}$$

$$\boldsymbol{J} = \int_{V_\infty} (\boldsymbol{x}\times\boldsymbol{u})G(|\boldsymbol{x}|/R)\,\mathrm{d}\boldsymbol{x}. \tag{3.8}$$

Note the similarity between (3.8) and Kolmogorov's proposal (1.14). In § 5 we shall see how these expressions may be generalized to include certain classes of anisotropic turbulence, such as MHD turbulence and stratified turbulence.

3.2. *Dynamics*

So far we have considered only kinematics. Turning now to dynamics, we note that the Kármán–Howarth equation (1.8) can be combined with (3.1) to give

$$\frac{\partial}{\partial t}\frac{\partial}{\partial r}\langle \boldsymbol{A}\cdot\boldsymbol{A}'\rangle = -\frac{1}{r^3}\frac{\partial}{\partial r}(r^4u^3K) - 2\nu\frac{\partial}{\partial r}\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle. \tag{3.9}$$

This integrates to yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{A}^2\rangle = 3\int_0^\infty u^3K\,\mathrm{d}r - 2\nu\langle \boldsymbol{u}^2\rangle, \tag{3.10}$$

which suggests that $\langle \boldsymbol{A}^2\rangle$ decreases with time, K being negative (except at very large scales), and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[6\int \langle \boldsymbol{A}\cdot\boldsymbol{A}'\rangle\,\mathrm{d}\boldsymbol{r}\right] = 8\pi\left[u^3r^4K\right]_\infty - 12\nu L, \tag{3.11}$$

which is consistent with (1.10) in the form

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[6\int \langle \boldsymbol{A}\cdot\boldsymbol{A}'\rangle\,\mathrm{d}\boldsymbol{r}\right] = 8\pi\left[u^3r^4K\right]_\infty. \tag{3.12}$$

Perhaps the more important question, though, is can we use (3.7) and (3.8) to explain why I is conserved in the absence of long-range correlations? This question may seem somewhat academic in the light of Batchelor & Proudman's (1956) prediction that $[u^3K]_\infty \sim cr^{-4} + O(r^{-5})$. However, it is worth bearing in mind that the magnitude of c is not determined by any rigorous theory, and the simulations of Ishida *et al.* (2006) suggest that c is very small and that $I \approx$ constant in fully developed turbulence. Given the near conservation of I, it seems worth asking what the physical basis for this conservation might be. We consider the idealized case in which all of the two-point correlations fall off exponentially with $|\boldsymbol{r}|$, and our starting point is to determine $\mathrm{d}\boldsymbol{J}/\mathrm{d}t$. From (3.8) we have

$$\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t} = \int_{V_\infty} \nabla\times[pG\boldsymbol{x}]\,\mathrm{d}\boldsymbol{x} - \int_{V_\infty} \nabla\cdot[(\boldsymbol{x}\times\boldsymbol{u})\,G\boldsymbol{u}]\,\mathrm{d}\boldsymbol{x} + \int_{V_\infty} [(\boldsymbol{x}\times\boldsymbol{u})\,\boldsymbol{u}\cdot\nabla G]\,\mathrm{d}\boldsymbol{x}, \tag{3.13}$$

where we have ignored the viscous stresses. The first integral on the right vanishes if we evaluate it over a sphere of radius $\hat{R}$ and then let $\hat{R} \to \infty$. Similarly, we may write the second integral as

$$\int_{V_\infty} \nabla \cdot [(\boldsymbol{x} \times \boldsymbol{u})\, G\boldsymbol{u}]\; \mathrm{d}\boldsymbol{x} = \operatorname*{Lim}_{\hat{R}\to\infty} \frac{R^3}{\hat{R}^3} \oint (\boldsymbol{x} \times \boldsymbol{u})\, \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{S}, \tag{3.14}$$

which also vanishes because the random nature of $\boldsymbol{u}$ means that the integral on the right scales as $\hat{R}^2$. (Strictly, we should show that the second integral on the right of (3.13) makes no contribution to $\mathrm{d}\langle \boldsymbol{J}^2\rangle/\mathrm{d}t$ in the limit of $\hat{R} \to \infty$, and indeed it is not hard to show that this is the case.) We are left with

$$\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t} = -3R^3 \int_R^\infty \mathbf{F}(\rho) \frac{\mathrm{d}\rho}{\rho^4}, \qquad \rho = |\boldsymbol{x}|, \tag{3.15}$$

where $\mathbf{F}(\rho)$ is the angular momentum flux out through the spherical surface S_ρ of radius ρ,

$$\mathbf{F}(\rho) = \oint_{S_\rho} (\boldsymbol{x} \times \boldsymbol{u})\boldsymbol{u} \cdot \mathrm{d}\boldsymbol{S}.$$

From this we conclude

$$2\pi \frac{\mathrm{d}I}{\mathrm{d}t} = \operatorname*{Lim}_{R\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\langle \boldsymbol{J}^2\rangle}{R^3} = -6 \operatorname*{Lim}_{R\to\infty} \int_R^\infty \left\langle \mathbf{F}(\rho) \cdot \int_{V_\infty} (\boldsymbol{x}' \times \boldsymbol{u}')\, G'\, \mathrm{d}\boldsymbol{x}' \right\rangle \frac{\mathrm{d}\rho}{\rho^4}, \tag{3.16}$$

where $G' = G(\rho'/R)$. The key term here is $\langle \mathbf{F}(\rho) \cdot \int (\boldsymbol{x}' \times \boldsymbol{u}')G'\, \mathrm{d}\boldsymbol{x}'\rangle$. Evidently, whether or not I is an invariant depends on the degree to which the angular momentum flux out through S_ρ is statistically correlated to the angular momentum in the vicinity of S_ρ.

Now suppose that there are no long-range interactions, in the sense that all two-point correlations decay rapidly (say exponentially) with separation. Then we may take $|\boldsymbol{r}| = O(\ell)$ in $\langle \mathbf{F}(\rho) \cdot \int (\boldsymbol{x}' \times \boldsymbol{u}')G'\, \mathrm{d}\boldsymbol{x}'\rangle$, since the velocities at $\boldsymbol{x}$ and $\boldsymbol{x}'$ are effectively decorrelated for distances larger than ℓ. Adopting $|\boldsymbol{r}|/\rho \sim \ell/\rho$ as a small parameter, we may make the substitution

$$G(\rho'/R) = G(\rho/R)[1 - 3\boldsymbol{x} \cdot \boldsymbol{r}/\rho^2 + O(\ell^2/\rho^2)]$$

in (3.16), where $\boldsymbol{x}$ locates a point on the surface S_ρ, while $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{r}$ locates a point adjacent to the surface. Taking $\boldsymbol{x} = \rho\hat{\boldsymbol{e}}_x$ as a typical point on S_ρ and noting that all points on the surface are statistically equivalent, (3.16) gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\langle \boldsymbol{J}^2\rangle}{R^3} = -24\pi R^3 \int_R^\infty \int_{V_\infty} \langle u_x (\boldsymbol{x} \times \boldsymbol{u}) \cdot (\boldsymbol{x}' \times \boldsymbol{u}')\rangle \left[1 - 3r_x/\rho + O\left(\frac{\ell^2}{\rho^2}\right)\right] \mathrm{d}\boldsymbol{r}\, \frac{\mathrm{d}\rho}{\rho^5}, \tag{3.17}$$

where, as usual, $\boldsymbol{r} = \boldsymbol{x}' - \boldsymbol{x}$. We shall use isotropy to evaluate all the triple correlations and integrals on the right of (3.17) shortly. First, however, let us examine the implications of (3.17) for $\langle \mathbf{F}(\rho) \cdot \int (\boldsymbol{x}' \times \boldsymbol{u}')G'\, \mathrm{d}\boldsymbol{x}'\rangle$ using arguments of a more general nature. Substituting $\rho\hat{\boldsymbol{e}}_x + \boldsymbol{r}$ for $\boldsymbol{x}'$ in (3.17) and then integrating over ρ yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\langle \boldsymbol{J}^2\rangle}{R^3} = -6 \int_R^\infty \left\langle \mathbf{F}(\rho) \cdot \int_{V_\infty} (\boldsymbol{x}' \times \boldsymbol{u}')G'\, \mathrm{d}\boldsymbol{x}' \right\rangle \frac{\mathrm{d}\rho}{\rho^4} = -4\pi[3RI_1 + (2I_2 - 6I_3) + O(R^{-1})], \tag{3.18}$$

where

$$I_1 = \int_{V_\infty} \langle u_x(\hat{\boldsymbol{e}}_x \times \boldsymbol{u})\cdot(\hat{\boldsymbol{e}}_x \times \boldsymbol{u}')\rangle\,\mathrm{d}\boldsymbol{r} = \int_{V_\infty} \langle u_x(u_y u'_y + u_z u'_z)\rangle\,\mathrm{d}\boldsymbol{r},$$

$$I_2 = \int_{V_\infty} \langle u_x(\hat{\boldsymbol{e}}_x \times \boldsymbol{u})\cdot(\boldsymbol{r} \times \boldsymbol{u}')\rangle\,\mathrm{d}\boldsymbol{r} = 2\int_{V_\infty} [r_x\langle u_x u_y u'_y\rangle - r_y\langle u_x u_y u'_x\rangle]\,\mathrm{d}\boldsymbol{r}$$

and

$$I_3 = \int_{V_\infty} r_x\,\langle u_x(\hat{\boldsymbol{e}}_x \times \boldsymbol{u})\cdot(\hat{\boldsymbol{e}}_x \times \boldsymbol{u}')\rangle\,\mathrm{d}\boldsymbol{r} = \int_{V_\infty} r_x\,\left\langle u_x(u_y u'_y + u_z u'_z)\right\rangle\,\mathrm{d}\boldsymbol{r}.$$

Next we note that in the absence of long-range correlations, $I_3 = 0$, which corresponds to setting $G' = G$ in (3.18). This becomes evident if we use isotropy to rewrite I_3 as

$$I_3 = \langle [u_x u_y]_{\boldsymbol{x}=0} \int (x' u'_y + y' u'_x)\,\mathrm{d}\boldsymbol{x}'\rangle,$$

where the integrand takes the form of a divergence: $(x'u'_y + y'u'_x) = \nabla' \cdot (x'y'\boldsymbol{u}')$. It follows that the integral over $\boldsymbol{x}'$ in I_3 may be expressed as a surface integral evaluated at $|\boldsymbol{x}'| \to \infty$, and hence I_3 is proportional to a two-point correlation evaluated at infinite separation. Since we are assuming that all such long-range interactions are transcendentally small, we have $I_3 = 0$. We may now put $G' = G$ in (3.18), which then simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\langle \boldsymbol{J}^2\rangle}{R^3} = -6R^3 \int_R^\infty \left\langle \mathbf{F}(\rho)\cdot\int_{V_\infty} \left(\boldsymbol{x}' \times \boldsymbol{u}'\right)\mathrm{d}\boldsymbol{x}'\right\rangle \frac{\mathrm{d}\rho}{\rho^7} = -4\pi\,[3RI_1 + 2I_2]\,. \tag{3.19}$$

It seems that $\langle \boldsymbol{J}^2\rangle/V$ is conserved, provided that the contribution of $\langle \mathbf{F}(\rho)\cdot\int(\boldsymbol{x}' \times \boldsymbol{u}')\,\mathrm{d}\boldsymbol{x}'\rangle$ to (3.19) is zero in the limit of $R \to \infty$. However, it is readily confirmed that symmetry demands that both I_1 and I_2 are zero, and hence $\langle \mathbf{F}(\rho)\cdot\int_{V_\infty}(\boldsymbol{x}' \times \boldsymbol{u}')\,\mathrm{d}\boldsymbol{x}'\rangle = 0$. We may show this as follows: Since $\boldsymbol{x}' = \rho\hat{\boldsymbol{e}}_x + \boldsymbol{r}$ we may split $\langle \mathbf{F}(\rho)\cdot\int_{V_\infty}(\boldsymbol{x}' \times \boldsymbol{u}')\,\mathrm{d}\boldsymbol{x}'\rangle$ into two parts,

$$\left\langle \mathbf{F}(\rho)\cdot\int_{V_\infty} \left(\rho\hat{\boldsymbol{e}}_x \times \boldsymbol{u}'\right)\mathrm{d}\boldsymbol{x}'\right\rangle = 4\pi\rho^4 I_1 = 4\pi\rho^4 \int_{V_\infty} \left\langle u_x\left(u_y u'_y + u_z u'_z\right)\right\rangle\mathrm{d}\boldsymbol{r},$$

$$\left\langle \mathbf{F}(\rho)\cdot\int_{V_\infty} \left(\boldsymbol{r} \times \boldsymbol{u}'\right)\mathrm{d}\boldsymbol{x}'\right\rangle = 4\pi\rho^3 I_2 = 8\pi\rho^3 \int_{V_\infty} \left[r_x\left\langle u_x u_y u'_y\right\rangle - r_y\left\langle u_x u_y u'_x\right\rangle\right]\mathrm{d}\boldsymbol{r}.$$

The first of these is zero by virtue of reflectional symmetry. That is $\langle u_x(u_y u'_y + u_z u'_z)\rangle$ changes sign under a reversal of the x-axis and so is odd in r_x. The second is zero by virtue of rotational symmetry. For example, consider a new coordinate system obtained by rotating about the z-axis by $\pi/2$. Then $y \to -x$ and $x \to y$, and the integrand in I_2 changes sign, which in turn requires $I_2 = 0$. We conclude, therefore, that symmetry demands that $\langle \mathbf{F}(\rho)\cdot\int_{V_\infty}(\boldsymbol{x}' \times \boldsymbol{u}')\,\mathrm{d}\boldsymbol{x}'\rangle = 0$, and so there is no statistical correlation between the flux of angular momentum out through the surface S_ρ and the angular momentum in the vicinity of S_ρ. Expression (3.19) then tells us that in the absence of long-range interactions, $\langle \boldsymbol{J}^2\rangle/R^3$, and hence I, is an invariant. This provides a simple physical interpretation for the conservation of I. Finally, it is worth noting that the requirement for I to be conserved is that the flux $\mathbf{F}(\rho)$ is decorrelated with $\int_{V_\infty}(\boldsymbol{x}' \times \boldsymbol{u}')\,\mathrm{d}\boldsymbol{x}'$ and not with $\int_{V_\rho}(\boldsymbol{x}' \times \boldsymbol{u}')\,\mathrm{d}\boldsymbol{x}'$. Indeed, we know from (2.22) that $\mathbf{F}(\rho)$ has a finite correlation with the latter integral.

The same conclusions may be reached through a detailed evaluation of the triple correlations in (3.17). The starting point is to note that (2.13) and (2.19)–(2.21) give us

$$\left\langle u_x\,(\boldsymbol{x}\times\boldsymbol{u})\cdot\left(\boldsymbol{x}'\times\boldsymbol{u}'\right)\right\rangle = \rho^2\,[g_1(r_x,r)-g_3(r_x,r)]+\rho g_2(r_x,r),$$

where

$$g_1(r_x,r)=\langle u_i u_x u_i'\rangle=\frac{u^3 r_x}{2r^4}\frac{\partial}{\partial r}\left(r^4K\right), \qquad (3.20)$$

$$g_2(r_x,r)=r_x\,\langle u_i u_x u_i'\rangle - r_i\,\langle u_i u_x u_x'\rangle = -\frac{u^3(r^2-3r_x^2)}{4r^2}\frac{\partial}{\partial r}\left(r^2K\right), \qquad (3.21)$$

$$g_3(r_x,r)=\left\langle u_x^2 u_x'\right\rangle=\frac{u^3 r_x}{2r}\left[\frac{\partial}{\partial r}(rK)-r_x^2\frac{\partial}{\partial r}(K/r)\right], \qquad (3.22)$$

and hence (3.17) can be rewritten as

$$\operatorname*{Lim}_{R\to\infty}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\langle \boldsymbol{J}^2\rangle}{R^3} = -24\pi \operatorname*{Lim}_{R\to\infty}\left[R^3\int_R^\infty\int_{V_\infty}[\rho^2(g_1-g_3)+\rho g_2]\left[1-3r_x/\rho+O\left(\frac{\ell^2}{\rho^2}\right)\right]\mathrm{d}\boldsymbol{r}\frac{\mathrm{d}\rho}{\rho^5}\right]$$

Now g_1 and g_3 are both odd in r_x and so integrate to zero, as does g_2 because of the factor $(r^2-3r_x^2)$ in (3.21). This corresponds to the vanishing of I_1 and I_2 in (3.19) and confirms that $\langle \mathbf{F}(\rho)\cdot\int(\boldsymbol{x}'\times\boldsymbol{u}')\,\mathrm{d}\boldsymbol{x}'\rangle=0$ by virtue of symmetry. Next, noting that all the contributions arising from the term $O(\ell^2/\rho^2)$ vanish in the limit of $R\to\infty$, as does the contribution from $r_x g_2$, the expression above simplifies to

$$\operatorname*{Lim}_{R\to\infty}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\langle \boldsymbol{J}^2\rangle}{R^3} = 2\pi\frac{\mathrm{d}I}{\mathrm{d}t} = 24\pi\int_{V_\infty} r_x\,[g_1-g_3]\,\mathrm{d}\boldsymbol{r}. \qquad (3.23)$$

Finally, substituting for g_1 and g_3 in (3.23) and integrating by parts yields

$$\operatorname*{Lim}_{R\to\infty}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\langle \boldsymbol{J}^2\rangle}{R^3} = 2\pi\frac{\mathrm{d}I}{\mathrm{d}t}\sim u^3\left[r^4K\right]_\infty = 0. \qquad (3.24)$$

Note that this final step relies on there being no long-range interactions and corresponds to the vanishing of I_3. We are thus led to the same conclusion as before. Either way, we find that the flux $\mathbf{F}(\rho)$ is not correlated to the local angular momentum adjacent to S_ρ, and so $\langle \boldsymbol{J}^2\rangle/V$, and hence I, is conserved in the absence of long-range interactions. (In practice, a complete absence of long-range interactions in unlikely, with K falling off as a power law, so it would be interesting to redo this calculation for the weaker assumption of $K<O(r^{-4})$.)

4. A comparison with Kolmogorov's analysis

The similarity between (1.14) and (3.8) is striking, and it is natural to ask if there are significant links between the analysis of § 3.1 and that of Kolmogorov. Let us first recall what Kolmogorov did, following the somewhat sketchy outline given in Monin & Yaglom (1975, p. 151), adapted here to our notation. We introduce $\hat{\boldsymbol{H}}$ defined by (1.14) and note that

$$\langle \hat{\boldsymbol{H}}^2\rangle = \int_{V_\infty}\int_{V_\infty}\left[\langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle\,\boldsymbol{x}\cdot\boldsymbol{x}'-\left\langle u_i u_j'\right\rangle x_j x_i'\right]\exp[-(\boldsymbol{x}^2+\boldsymbol{x}'^2)/R^2]\,\mathrm{d}\boldsymbol{x}'\,\mathrm{d}\boldsymbol{x}. \qquad (4.1)$$

Introducing $\boldsymbol{r}=\boldsymbol{x}'-\boldsymbol{x}$ and $\mathbf{s}=(\boldsymbol{x}+\boldsymbol{x}')/2$, substituting for $\boldsymbol{x}$ and $\boldsymbol{x}'$ and throwing out terms in the integrand which are odd in $\mathbf{s}$ yields

$$\langle \hat{\boldsymbol{H}}^2 \rangle = \int_{V_\infty} \int_{V_\infty} \left[\langle \boldsymbol{u}\cdot\boldsymbol{u}' \rangle \left(\frac{2}{3}s^2 - \frac{1}{4}r^2 \right) + \frac{1}{4} \langle u_i u_j' \rangle r_j r_i \right] \exp[-2s^2/R^2] \exp[-r^2/2R^2]\, \mathrm{d}\boldsymbol{r}\, \mathrm{d}\mathbf{s}. \tag{4.2}$$

Finally, integrating over $\mathbf{s}$ and using isotropy to relate $\langle u_i u_j' \rangle$ to $u^2 f(r)$, we find, after some algebra,

$$\frac{\langle \hat{\boldsymbol{H}}^2 \rangle}{R^3} = -\frac{\pi^{\frac{3}{2}}}{8\sqrt{2}} \int_{V_\infty} \exp\left[-r^2/2R^2\right]\, r^3 \frac{\partial}{\partial r}(u^2 f)\, \mathrm{d}\boldsymbol{r}, \tag{4.3}$$

from which

$$\operatorname*{Lim}_{R\to\infty} \frac{\langle \hat{\boldsymbol{H}}^2 \rangle}{R^3} = -\frac{\pi^{\frac{5}{2}}}{2\sqrt{2}} \int_{V_\infty} r^5 \frac{\partial}{\partial r}(u^2 f)\, \mathrm{d}r = \frac{5\pi^{\frac{3}{2}}}{16\sqrt{2}} I. \tag{4.4}$$

(Note that there is an error in Monin & Yaglom (1975), who quote $5\pi^{3/2}I/4\sqrt{2}$, rather than $5\pi^{3/2}I/16\sqrt{2}$, for the term on the right of (4.4).) In short, we have

$$I \sim \operatorname*{Lim}_{R\to\infty} \frac{\langle \hat{\boldsymbol{H}}^2 \rangle}{R^3}, \tag{4.5}$$

which is analogous to (3.7).

Evidently, there are similarities between the two analyses. However, there are also crucial differences. For example, (3.8) is motivated by the kinematic expression (3.4), whereas (1.14) is introduced as a matter of mathematical expediency. Moreover, Kolmogorov's analysis is purely kinematic, whereas ours embraces dynamics, showing that the conservation of I (in the absence of long-range interactions) can be explained in terms of a decorrelation between the flux of angular momentum out through the control surface S_ρ and the local angular momentum density surrounding S_ρ.

5. Extension to other types of homogeneous turbulence

5.1. *Why Landau's theory is important*

One might legitimately question whether Landau's analysis, or its various refinements, are of any consequence. After all, (1.10) tells us that I is not, in general, an invariant. Moreover, isotropic turbulence is itself an idealized state, rarely achieved. However, the central value of Landau's theory lies in the fact that $c=[r^4K]_\infty$ is observed to be very small in fully developed, isotropic turbulence (Ishida *et al.* 2006). Thus the long-range correlations predicted by Batchelor & Proudman (1956) are weak, at least in mature turbulence, and so for most practical purposes I can be treated as a constant. Thus, for example, the decay exponent in the power law $u^2 \sim t^{-n}$ is close to Kolmogorov's prediction of $n=10/7$ for $E \sim Ik^4$ turbulence (Ishida *et al.* 2006). So the underlying assumption in Landau's theory, that long-range interactions may be neglected, turns out to be a reasonable approximation in isotropic turbulence.

Moreover, if the long-range interactions are weak in other forms of homogeneous turbulence, such as MHD, rotating and stratified turbulence, then it is possible to generalize Landau's analysis to embrace these anisotropic systems. As we shall see, the only requirement is that there is no net torque associated with the body force (the buoyancy, Coriolis or Lorentz force) in at least one direction. In such a case one can repeat the steps of § 1.2 but focusing on the conserved component of angular momentum only. Now it turns out that the Lorentz, Coriolis and buoyancy forces do indeed satisfy this constraint, and so the inhomogeneous theory of § 1.2 is readily adapted to these anisotropic flows. In this sense, then, Landau's analysis constitutes an important, general result, provided, of course, that the long-range interactions are weak in these more complex systems. One problem, however, is that these various anisotropic extensions of § 1.2 all suffer from the same problem as Landau's original argument: they are formulated for inhomogeneous turbulence evolving in a closed domain. In § 5.3 we shall show how this shortcoming may be remedied following the logic of § 3, thus providing a self-consistent theory within a homogeneous framework. First, however, it is useful to review briefly the way in which Landau's inhomogeneous analysis may be extended to these more complex systems.

5.2. *Landau's inhomogeneous analysis extended to MHD, rotating and stratified turbulence*

The details of how to generalize the inhomogeneous analysis of § 1.2 to stratified and MHD turbulence are spelt out in Davidson (1997, 2004, pp. 514, 541). In the case of rotating or stratified turbulence, that is turbulence evolving in the presence of a background rotation or stratification, the component of angular momentum parallel to the rotation axis or gravitational acceleration is clearly conserved. In MHD turbulence, on the other hand, it turns out to be the component of $\boldsymbol{H}$ parallel to the applied magnetic field which is conserved. This latter claim is not particularly obvious but can be seen as follows: Consider a turbulent, conducting fluid evolving in a large electrically insulated spherical domain and subject to an imposed magnetic field $\boldsymbol{B}_0$. Let $\boldsymbol{j}$ be the current density induced in the fluid, which satisfies the boundary condition $\boldsymbol{j} \cdot \mathrm{d}\boldsymbol{S} = 0$, and $\boldsymbol{b}$ be the magnetic field associated with $\boldsymbol{j}$ by virtue of the Biot–Savart law or, equivalently, Ampere's law $\nabla \times \boldsymbol{b} = \boldsymbol{j}$. (For simplicity, we shall take the permeability of free space to be unity.) Then the net torque exerted on the fluid by the Lorentz force is

$$\mathbf{T} = \int_V \boldsymbol{x} \times (\boldsymbol{j} \times \boldsymbol{B}_0)\, \mathrm{d}V + \int_V \boldsymbol{x} \times (\boldsymbol{j} \times \boldsymbol{b})\, \mathrm{d}V,$$

where V is the closed spherical domain. However, a closed system of currents produces zero net torque when interacting with its self-field $\boldsymbol{b}$, and it follows that the second integral on the right is zero. The first integral, on the other hand, can be transformed using the identity

$$2\boldsymbol{x} \times [\boldsymbol{j} \times \boldsymbol{B}_0] = (\boldsymbol{x} \times \boldsymbol{j}) \times \boldsymbol{B}_0 + \nabla \cdot [\boldsymbol{x} \times (\boldsymbol{x} \times \boldsymbol{B}_0)\, \boldsymbol{j}]$$

and the boundary condition $\boldsymbol{j} \cdot \mathrm{d}\boldsymbol{S} = 0$. This yields

$$\mathbf{T} = \frac{1}{2} \int_{\mathrm{v}} (\boldsymbol{x} \times \boldsymbol{j})\, \mathrm{d}V \times \boldsymbol{B}_0 = \boldsymbol{m} \times \boldsymbol{B}_0,$$

where $\boldsymbol{m}$ is the dipole moment associated with $\boldsymbol{j}$. Ignoring the viscous stresses on the surface of V and noting that we are taking the fluid density to be unity, the angular momentum of the fluid evolves according to $\mathrm{d}\boldsymbol{H}/\mathrm{d}t = \boldsymbol{m} \times \boldsymbol{B}_0$. Evidently, the component of $\boldsymbol{H}$ parallel to $\boldsymbol{B}_0$ is conserved, as claimed above. Note that this true for both poorly conducting and highly conducting fluids.

Let us now denote the conserved component of $\boldsymbol{H}$ (for MHD, rotating or stratified turbulence) by $H_{//}$ and introduce the Loitsyansky-like integral

$$I_{//} = -\int r_\perp^2 \langle \boldsymbol{u}_\perp \cdot \boldsymbol{u}'_\perp \rangle \,\mathrm{d}\boldsymbol{r}.$$

Then, repeating the steps in § 1.2, but focusing on $H_{//}$ only, leads to

$$I_{//} = \lim_{V\to\infty} \frac{\langle H_{//}^2 \rangle}{V} = -\int r_\perp^2 \langle \boldsymbol{u}_\perp \cdot \boldsymbol{u}'_\perp \rangle \,\mathrm{d}\boldsymbol{r} = \text{constant}, \tag{5.1}$$

where $\boldsymbol{r}_\perp$ and $\boldsymbol{u}_\perp$ are the components of $\boldsymbol{r}$ and $\boldsymbol{u}$ in the plane normal to $\boldsymbol{H}_{//}$. This is the anisotropic analogue of (1.13). Of course, as in Landau's original analysis, (5.1) relies on the assumption that all two-point correlations decay rapidly with separation and on the fact that $\boldsymbol{P} = \int \boldsymbol{u}\,\mathrm{d}V = 0$ in a closed domain.

An independent check on the validity of at least part of (5.1), i.e.

$$I_{//} = -\int r_\perp^2 \langle \boldsymbol{u}_\perp \cdot \boldsymbol{u}'_\perp \rangle \,\mathrm{d}\boldsymbol{r} = \text{constant}, \tag{5.2}$$

can be made by integrating the appropriate form of the generalized, anisotropic Kármán–Howarth equation, incorporating the appropriate body forces (the buoyancy, Coriolis or Lorentz force). It may be confirmed that, provided the two-point correlations decay sufficiently rapidly with separation, (5.2) does indeed hold for homogeneous turbulence (Davidson 2004, pp. 514, 542). However, the attraction of Landau's approach is that it yields (5.1) with relative ease, whereas a derivation of (5.2) via the generalized Kármán–Howarth equation is typically a long and tedious calculation. In any event, whichever derivation we follow, (5.2) holds whenever the long-range interactions are sufficiently weak.

Unfortunately, as noted above, these variants of § 1.2 all have a weakness: they are formulated for inhomogeneous turbulence evolving in a closed domain. Thus, for example, the link between (5.2) and angular momentum conservation becomes unclear as we move from inhomogeneous to homogeneous turbulence and from closed to open control volumes. Indeed, since $I \sim \langle \boldsymbol{J}^2 \rangle / V$, rather than $I \sim \langle \boldsymbol{H}^2 \rangle / V$, in isotropic turbulence, we might anticipate that $I_{//} \sim \langle J_{//}^2 \rangle / V$ in these anisotropic, homogeneous flows, rather than the expression $I_{//} \sim \langle H_{//}^2 \rangle / V$ suggested by (5.1). It is natural to ask, therefore, if our reformulation of Landau's theory, outlined in § 3, can be adapted to these anisotropic systems. It turns out that it can.

5.3. *Reformulating our analysis for anisotropic, homogeneous turbulence*

Our starting point is to note that in homogeneous turbulence,

$$\langle u_i u'_j \rangle = -\delta_{ij} \nabla^2 \langle \boldsymbol{A} \cdot \boldsymbol{A}' \rangle + \frac{\partial^2}{\partial r_i \partial r_j} \langle \boldsymbol{A} \cdot \boldsymbol{A}' \rangle + \nabla^2 \langle A_j A'_i \rangle,$$

from which

$$\langle \boldsymbol{u}_\perp \cdot \boldsymbol{u}'_\perp \rangle = -\nabla^2 \langle A_{//} A'_{//} \rangle - \frac{\partial^2}{\partial r_{//}^2} \langle \boldsymbol{A} \cdot \boldsymbol{A}' \rangle. \tag{5.3}$$

It follows that, if the two-point correlations decay sufficiently rapidly with r, then

$$I_{//} = 4 \int \langle A_{//} A'_{//} \rangle \,\mathrm{d}\boldsymbol{r} = \lim_{V\to\infty} \frac{4\langle [\int_V A_{//}\,\mathrm{d}V]^2 \rangle}{V}, \tag{5.4}$$

which when combined with (3.6) and (5.2) yields

$$I_{//} = \operatorname*{Lim}_{R\to\infty} \frac{4\left\langle J_{//}^2\right\rangle}{9V} = \text{constant}, \tag{5.5}$$

$$J_{//} = \int_{V_\infty} (\boldsymbol{x}\times\boldsymbol{u})_{//}\, G\left(|\boldsymbol{x}|/R\right) \mathrm{d}\boldsymbol{x}. \tag{5.6}$$

Thus, as anticipated above, we have $I_{//} \sim \langle J_{//}^2\rangle/V$ rather than $I_{//} \sim \langle H_{//}^2\rangle/V$. All of the arguments of § 3.2 can now be adapted to MHD, rotating and stratified turbulence but with attention focused on $(\boldsymbol{x}\times\boldsymbol{u})_{//}$ rather than $\boldsymbol{x}\times\boldsymbol{u}$. In particular, since there is no net torque $\mathbf{T}_{//}$ associated with the body forces (the Lorentz, Coriolis or buoyancy force), one can explain the conservation of $I_{//}$ in terms of a decorrelation between the flux of $(\boldsymbol{x}\times\boldsymbol{u})_{//}$ out through the surface S_ρ and the angular momentum $(\boldsymbol{x}\times\boldsymbol{u})_{//}$ adjacent to S_ρ.

The importance of (5.2) and (5.5) cannot be underestimated. If the large scales are self-similar, then (5.2) demands

$$u_\perp^2 \ell_\perp^4 \ell_{//} = \text{constant}, \tag{5.7}$$

where $\ell_\perp$ and $\ell_{//}$ are integral scales, defined, say, by

$$\ell_\perp = \frac{1}{\left\langle \boldsymbol{u}_\perp^2\right\rangle}\int \langle \boldsymbol{u}_\perp(\boldsymbol{x})\cdot\boldsymbol{u}_\perp(\boldsymbol{x}+r\hat{\boldsymbol{e}}_x)\rangle\, \mathrm{d}r, \quad \ell_{//} = \frac{1}{\left\langle \boldsymbol{u}_\perp^2\right\rangle}\int \langle \boldsymbol{u}_\perp(\boldsymbol{x})\cdot\boldsymbol{u}_\perp(\boldsymbol{x}+r\hat{\boldsymbol{e}}_z)\rangle\, \mathrm{d}r.$$

This imposes a powerful constraint on the evolution of the large scales. For example, (5.7) can be used in the spirit of Kolmogorov (1941) to estimate the rate of decay of energy in MHD turbulence (Davidson 2004, p. 543; Okamoto, Davidson & Kaneda 2008), and the resulting predictions are close to the numerical and experimental evidence (see below).

A key question, of course, is whether or not the long-range interactions are indeed weak in these anisotropic systems, just as they are in mature, isotropic turbulence. If they are not weak, then (5.5) fails in the sense that $I_{//}$ is not constant. It is reassuring in this respect to note that the Coriolis and buoyancy forces are less efficient at transmitting information over large distances than the pressure force. That is the Coriolis and buoyancy forces can generate internal waves (inertial waves and gravitational waves), but these waves are less efficient at propagating information than the pressure force, which acts instantaneously over large distances in the form of infinitely fast acoustic waves. The situation is a little more complicated in MHD turbulence, where the Lorentz force is a non-local function of $\boldsymbol{u}$, but it is shown in Davidson (1997) that the resulting long-range interactions are no stronger than in conventional isotropic turbulence.

5.4. *Decay laws in MHD turbulence*

We close § 5 by showing how (5.7) can play a crucial role in freely decaying, homogeneous turbulence. To make the point we focus on three particular cases: (i) MHD turbulence in which the magnetic Reynolds number is small, $R_m = u\ell/\lambda \ll 1$, λ being the magnetic diffusivity; (ii) MHD turbulence at high R_m; and (iii) turbulence evolving in the presence of a background stratification. In each case we derive an energy decay law for the system and predict the rate of change of the integral scales. We wish to emphasis, however, that (5.7) is likely to play an equally important role in other homogenous systems, such as rotating turbulence, rotating–MHD turbulence and rotating–stratified turbulence. Our analysis of stratified and high-R_m MHD turbulence is new, whereas our discussion of low-R_m MHD turbulence, which is included for completeness, is based on that given in Davidson (2004) and Okamoto *et al.* (2008).

5.4.1. *Low-R_m MHD turbulence*

Let us start with MHD turbulence at low R_m. The low-R_m regime, which typifies almost all terrestrial MHD, is characterized by a low electrical conductivity σ and weak induced currents, and hence $\boldsymbol{b} \ll \boldsymbol{B}_0$ (Moffatt 1978). The statistically averaged kinetic energy equation is readily shown to be

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\langle \boldsymbol{u}^2\rangle = -\nu\langle \boldsymbol{\omega}^2\rangle - \langle \boldsymbol{j}^2\rangle/\sigma, \tag{5.8}$$

where the second term on the right is the Joule dissipation. (As in § 2, we take the fluid density to be unity.) The Joule dissipation in (5.8) may be estimated using the curl of Ohm's law $\nabla \times \boldsymbol{j} = \sigma \boldsymbol{B}_0 \cdot \nabla \boldsymbol{u}$ from which we obtain the estimate

$$\frac{\langle \boldsymbol{j}^2\rangle}{\sigma} \sim \left(\frac{\ell_{min}}{\ell_{//}}\right)^2 \frac{\langle \boldsymbol{u}^2\rangle}{\tau}, \quad \tau = (\sigma B_0^2)^{-1}. \tag{5.9}$$

Here ℓ_{min} and $\ell_{//}$ are suitably defined integral scales, and τ in known as the Joule dissipation time. Now we know that the effect of $\boldsymbol{B}_0$ is to introduce anisotropy into the turbulence, with $\ell_{//} > \ell_{\perp}$. Thus we have

$$\frac{\langle \boldsymbol{j}^2\rangle}{\sigma} = \frac{\beta}{2}\left(\frac{\ell_{\perp}}{\ell_{//}}\right)^2 \frac{\langle \boldsymbol{u}^2\rangle}{\tau}, \tag{5.10}$$

where β is a coefficient of order unity – in fact it can be shown that $\beta = 2/3$ when the turbulence is isotropic (Davidson 2004). Using this to estimate the Joule dissipation in (5.8) we obtain

$$\frac{\mathrm{d}u^2}{\mathrm{d}t} = -\alpha\frac{u^3}{\ell_{\perp}} - \beta\left(\frac{\ell_{\perp}}{\ell_{//}}\right)^2 \frac{u^2}{\tau}, \tag{5.11}$$

where we have made the usual high-*Re* estimate of the viscous dissipation term, with α a coefficient of the order of unity. Now in low-R_m turbulence it is conventional to categorize the flow according to the value of the so-called interaction parameter $N = \ell_{\perp}/u\tau$. When N is small (negligible magnetic effects), (5.7) and (5.11) reduce to

$$\frac{\mathrm{d}u^2}{\mathrm{d}t} = -\alpha\frac{u^3}{\ell}, \quad u^2\ell^5 = \text{constant}, \tag{5.12}$$

which yields the familiar Kolmogorov decay law $u^2 \sim t^{-10/7}$. When N is large, on the other hand, inertia is unimportant, and we have $\ell_{\perp} = \text{constant}$, since diffusive Alfven waves increase $\ell_{//}$ but leave $\ell_{\perp}$ unchanged on times of order τ. Thus the high-N case is governed by a combination of (5.7) and

$$\frac{\mathrm{d}u^2}{\mathrm{d}t} = -\beta\left(\frac{\ell_{\perp}}{\ell_{//}}\right)^2 \frac{u^2}{\tau}, \ \ \ell_{\perp} = \text{constant}, \tag{5.13}$$

from which we obtain the familiar results

$$u^2 = u_0^2\left[1 + 2\beta t/\tau\right]^{\frac{-1}{2}}, \quad \ell_{//} = \ell_0\left[1 + 2\beta t/\tau\right]^{\frac{1}{2}}. \tag{5.14}$$

(Here the subscript 0 indicates a value at $t = 0$.) For intermediate values of N, however, we have a problem. Equations (5.7) and (5.11) between them contain three unknowns,

u^2, $\ell_\perp$ and $\ell_{//}$. To close the system we follow Davidson (2004) and introduce the heuristic equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\ell_{//}/\ell_\perp\right)^2 = 2\beta/\tau, \tag{5.15}$$

which has the merit of being exact for $N \to 0$ and $N \to \infty$ but cannot be formally justified for intermediate N. However, the direct numerical simulations (DNSs) of Okamoto *et al.* (2008) suggest that (5.15) is indeed a reasonable approximation for the intermediate values of $N \sim 1$, lending some confidence to this estimate. Finally, integrating (5.7), (5.11) and (5.15) yields the decay laws

$$\begin{aligned} u^2/u_0^2 &= \hat{t}^{-1/2}[1 + (7\alpha/15\beta)(\hat{t}^{3/4} - 1)N_0^{-1}]^{\frac{-10}{7}}, \\ \ell_\perp/\ell_0 &= [1 + (7\alpha/15\beta)(\hat{t}^{3/4} - 1)N_0^{-1}]^{\frac{2}{7}}, \\ \ell_{//}/\ell_0 &= \hat{t}^{1/2}[1 + (7\alpha/15\beta)(\hat{t}^{3/4} - 1)N_0^{-1}]^{\frac{2}{7}}, \end{aligned} \tag{5.16}$$

where N_0 is the initial value of N and $\hat{t} = 1 + 2\beta(t/\tau)$. These expressions reduce to Kolmogorov's decay law for small N_0 and to (5.14) at large N_0, as they must. Moreover, for the case of $N_0 = 7\alpha/15\beta$ we obtain the power laws

$$u^2/u_0^2 \sim \hat{t}^{\frac{-11}{7}}, \quad \ell_{//}/\ell_0 \sim \hat{t}^{5/7}, \quad \ell_\perp/\ell_0 \sim \hat{t}^{\frac{3}{14}}, \tag{5.17}$$

and indeed these power laws are reasonable approximations to (5.16) for all values of N_0 around unity. Interestingly, experiments of low-R_m, homogeneous turbulence suggest $u^2 \sim t^{-1.6}$ for $N_0 \sim 1$ (Davidson 2004), which compares favourably with the prediction of (5.17), i.e. $u^2 \sim t^{-1.57}$. Moreover, the DNSs of Okamoto *et al.* (2008), which were performed at $N_0 \sim 1$, show that (5.17) provides good estimates of u^2, $\ell_\perp$ and $\ell_{//}$.

However, perhaps the particular details of decay laws (5.16) are unimportant for the present purposes. The main point is that the Loitsyansky-like constraint (5.7), which is related to angular momentum conservation, plays a critical role in determining the rate of decay of energy in low-R_m MHD turbulence, just as it does in Kolmogorov's decay law for conventional hydrodynamic turbulence. It seems likely, therefore, that (5.7) is equally important in determining the rate of energy decay in high-R_m MHD turbulence and in rotating or stratified turbulence.

5.4.2. *High-R_m MHD turbulence*

Consider now the case of high-R_m turbulence evolving in the presence of a uniform, imposed field. Here the energy associated with the induced field $\boldsymbol{b}$ cannot be neglected, and a term $\boldsymbol{b}^2$ should be added to the right of (5.8). However, Alfven waves travelling along the mean field tend to promote an equipartition of energy between $\boldsymbol{u}_\perp^2$ and $\boldsymbol{b}^2$, and so (5.8) remains qualitatively correct. Indeed, the dissipation of energy in such turbulence is observed to be of the order of $u_\perp^3/\ell_\perp$, as in hydrodynamic turbulence (Cho & Vishniac 2000). Thus we have

$$\frac{\mathrm{d}u_\perp^2}{\mathrm{d}t} = -\alpha\frac{u_\perp^3}{\ell_\perp}, \quad \alpha \sim 1, \tag{5.18}$$

where α is a constant. Moreover, the ratio of integral length scales $\ell_{//}/\ell_\perp$ is set by the so-called critical-balance condition (again, see Cho & Vishniac 2000), which requires

$$\ell_{//}/\ell_\perp \sim V_A/u_\perp, \tag{5.19}$$

where V_A is the Alfven wave speed associated with the mean field, $V_A \sim B_0$. Integrating (5.18) subject to constraints (5.7) and (5.19) yields

$$\frac{u_\perp^2}{u_0^2} = \left[1 + \frac{3\alpha}{5}\frac{u_0 t}{\ell_0}\right]^{\frac{-5}{3}}, \tag{5.20}$$

$$\frac{\ell_\perp}{\ell_0} = \left[1 + \frac{3\alpha}{5}\frac{u_0 t}{\ell_0}\right]^{\frac{1}{6}}, \tag{5.21}$$

$$\frac{\ell_{//}}{\ell_0} \sim \frac{V_A}{u_0}\left[1 + \frac{3\alpha}{5}\frac{u_0 t}{\ell_0}\right], \tag{5.22}$$

where u_0 and ℓ_0 are the initial values of $u_\perp$ and $\ell_\perp$. Yet again, our angular momentum constraint yields simple decay laws. Note that at large times, (5.22) simplifies to $\ell_{//} \sim V_A t$. To date, the available numerical evidence is insufficient to put (5.20)–(5.22) to the test.

5.5. *The decay of stratified turbulence: speculative decay laws*

We conclude by considering turbulence evolving in the presence of a uniform, stable stratification. Let N be the Väisälä–Brunt frequency, defined in the usual way with N^2 proportional to the normalized density gradient, and

$$Fr = u_\perp / N\ell_\perp \tag{5.23}$$

be the corresponding Froude number. For low Fr and high Reynolds number it has been suggested by several researchers, partly on the basis of empirical observation, that the horizontal kinetic energy decays according to

$$\frac{\mathrm{d}u_\perp^2}{\mathrm{d}t} = -\alpha\frac{u_\perp^3}{\ell_\perp}, \quad \alpha \sim 1, \tag{5.24}$$

while $\ell_{//}$ is fixed by the condition

$$\frac{u_\perp}{N\ell_{//}} = C \sim 1. \tag{5.25}$$

(Here $u_\perp^2 = (1/2)\langle \boldsymbol{u}_\perp^2 \rangle$ and α and C are constants of order unity.) Expressions (5.24) and (5.25), or their equivalent, are proposed in, for example, Godeferd & Staquet (2003), Riley & deBruynKops (2003), Waite & Bartello (2004), Lindborg (2006) and Brethouwer *et al.* (2007). Combining these with (5.7) and integrating yields

$$\frac{u_\perp^2}{u_0^2} = \left[1 + \frac{7\alpha}{8}\frac{u_0 t}{\ell_0}\right]^{\frac{-8}{7}}, \tag{5.26}$$

$$\frac{\ell_\perp}{\ell_0} = \left[1 + \frac{7\alpha}{8}\frac{u_0 t}{\ell_0}\right]^{\frac{3}{7}}, \tag{5.27}$$

$$\frac{\ell_{//}}{\ell_0} = \frac{1}{C}\frac{u_0}{N\ell_0}\left[1 + \frac{7\alpha}{8}\frac{u_0 t}{\ell_0}\right]^{\frac{-4}{7}}, \tag{5.28}$$

where u_0 and ℓ_0 are the initial values of $u_\perp$ and $\ell_\perp$. Once again, (5.2) has determined the rate of energy decay. Note, however, that unlike MHD turbulence, where our predicted decay laws have been verified by numerical simulations, predictions (5.26)–(5.28) have yet to be put to the test. This is important, as the physical basis of (5.24) and (5.25) is still a mater of debate. Moreover, there is the possibility that, depending on how the turbulence is generated, the large scales may be of the Saffman type, in which case (5.26)–(5.28) will not apply.

6. A problem: the elusive nature of the long-range interactions.

Although we have successfully reformulated Landau's analysis for homogeneous turbulence, there is still the mystery as to why the long-range correlations, as measured by $c=[r^4K]_\infty$, are so weak in fully developed, isotropic turbulence. After all, $K_\infty \sim cr^{-4}$ seems to be an inevitable consequence of the long-range pressure forces or equivalently the Biot–Savart law (Davidson 2004). Moreover, simple closure models, such as the quasi-normal closure, have c as an order one quantity (Proudman & Reid 1954), while more complex Markovianized closure models, such as the so-called eddy-damped quasi-normal Markovian (EDQNM) closure, predict a small but finite time dependence for $I(t)$. For example, in Appendix C we show that EDQNM predicts $I \sim (u^2)^{-m}$, where

$$m = \frac{7\pi}{20a_1A}\frac{\sqrt{3}}{\sqrt{2}}\int_0^\infty \frac{F(x)}{x}\,\mathrm{d}x\left[\int_0^\infty F(x)\,\mathrm{d}x\right]^{\frac{-3}{2}}\int_1^\infty \frac{F^2(x)/x^2}{\sqrt{\int_0^x y^2F(y)\,\mathrm{d}y}}\,\mathrm{d}x. \tag{6.1}$$

Here $F(k/k_i)$ is the normalized energy spectrum $F \sim k_iE(k)/u^2$; k_i is the wavenumber at which E is a maximum; a_1 is a model constant of order unity; and A is a dimensionless measure of dissipation, $A \approx 1/3$. It turns out (see Appendix C) that the standard version of EDQNM has $m \approx 0.12$, leading to $I \sim t^{0.16}$ (for a discussion of EDQNM see, for example, Lesieur 1990; Sagaut & Cambon 2008).

In summary, then, conventional wisdom and quasi-normal-like closure models would have $c=[r^4K]_\infty$ finite, and hence I time dependant. If that were true, Landau's theory and its various extensions would be of limited interest. However, high-resolution DNSs suggest that after a transient, $I \approx$ constant.

So why is c so small in practice, leading to $I \approx$ constant in fully developed turbulence? This is an issue which has yet to be adequately addressed in the literature, though there have been some tentative suggestions. For example, Ruelle (1990) speculated that the long-range correlations might vanish by analogy with Debye–Huckel screening in plasmas, whereby the long-range Coulomb forces are suppressed through a clustering of oppositely signed charges, leaving the plasma electrically neutral at each point, at least in a coarse-grained sense. Ruelle (1990) noted that, in a similar way, current loops interacting via their induced magnetic fields can, in certain situations, exhibit a form of partial screening, in the sense that the magnetic energy associated with far-field interactions is reduced by a reorientation of the dipoles. Of course, there is a kinematic analogy between such current loops and vortex tubes in a turbulent flow, and it was this analogy that motivated Ruelle's (1990) speculation about screening in turbulence. It turns out, however, that the analogy is imperfect. This becomes apparent if one looks at dynamics rather than kinematics, as illustrated by the following example: Consider two distinct vortex loops interacting remotely in an inviscid fluid via their induced velocity fields. Let loop 1 have a vorticity field $\boldsymbol{\omega}_1$, which is confined to the region V_1, and loop 2 have a vorticity field $\boldsymbol{\omega}_2$, confined to the region V_2. We take V_1 and V_2 to be non-overlapping, though of course the induced velocity fields $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ pervade the entire space. We now consider the influence of loop 2 on the angular impulse of loop 1,

$$\boldsymbol{M}_1 = \frac{1}{3}\int_{V_1} \boldsymbol{x} \times (\boldsymbol{x} \times \boldsymbol{\omega}_1)\,\mathrm{d}V,$$

and the corresponding influence of loop 1 on $\boldsymbol{M}_2$. We start by noting that the inviscid vorticity equation yields, after some algebra,

$$\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{x} \times (\boldsymbol{x} \times \boldsymbol{\omega})) = 3\boldsymbol{x} \times (\boldsymbol{u} \times \omega) + \boldsymbol{\omega} \cdot \nabla (\boldsymbol{x} \times (\boldsymbol{x} \times \boldsymbol{u})). \tag{6.2}$$

Moreover, using $\boldsymbol{u} \times \boldsymbol{\omega} = \nabla(\boldsymbol{u}^2/2) - \boldsymbol{u} \cdot \nabla \boldsymbol{u}$ to rewrite $\int \boldsymbol{x} \times (\boldsymbol{u} \times \omega)\,\mathrm{d}V$ as a surface integral, we have

$$\int_{V_\infty} \boldsymbol{x} \times (\boldsymbol{u}_1 \times \boldsymbol{\omega}_1)\,\mathrm{d}V = \int_{V_\infty} \boldsymbol{x} \times (\boldsymbol{u}_2 \times \boldsymbol{\omega}_2)\,\mathrm{d}V = \int_{V_\infty} \boldsymbol{x} \times (\boldsymbol{u} \times \boldsymbol{\omega})\,\mathrm{d}V = 0, \tag{6.3}$$

where $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2$ and $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$. Combining (6.2) and (6.3) yields

$$\frac{\mathrm{d}\boldsymbol{M}_1}{\mathrm{d}t} = -\int_{V_1} \boldsymbol{x} \times (\boldsymbol{\omega}_1 \times \boldsymbol{u}_2)\,\mathrm{d}V, \quad \frac{\mathrm{d}\boldsymbol{M}_2}{\mathrm{d}t} = -\int_{V_2} \boldsymbol{x} \times (\boldsymbol{\omega}_2 \times \boldsymbol{u}_1)\,\mathrm{d}V, \tag{6.4}$$

and $\boldsymbol{M}_1 + \boldsymbol{M}_2 = \text{constant}$. (An expression similar to (6.4) appears in Saffman 1992, p. 59.)

Let us now compare this with the equivalent problem in magnetostatics. Here we have two current loops with current densities $\boldsymbol{j}_1$ and $\boldsymbol{j}_2$, exerting forces and torques on each other through their induced magnetic fields, $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$, with $\nabla \times \boldsymbol{B} = \boldsymbol{j}$. Clearly, there is a kinematic analogy, with $\boldsymbol{j} \leftrightarrow \boldsymbol{\omega}$ and $\boldsymbol{B} \leftrightarrow \boldsymbol{u}$. When we turn to dynamics, however, the analogy breaks down, since the counterpart of (6.4) is

$$\boldsymbol{T}_1 = \int_{V_1} \boldsymbol{x} \times (\boldsymbol{j}_1 \times \boldsymbol{B}_2)\,\mathrm{d}V, \quad \boldsymbol{T}_2 = \int_{V_2} \boldsymbol{x} \times (\boldsymbol{j}_2 \times \boldsymbol{B}_1)\,\mathrm{d}V,$$

where $\boldsymbol{j} \times \boldsymbol{B}$ is the Lorentz force and $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ are the torques exerted on the two current loops. Evidently, the torques in the two problems are in opposite directions. The implication is that the behaviour of interacting current loops will be quite different from that of interacting vortex tubes. It seems unlikely, therefore, that there is screening in turbulence of the type seen in magnetostatics.

All in all, our understanding of the nature of the long-range correlations remains poor, and we have no convincing explanation for the weakness of these interactions. This is important because, if we cannot explain why the long-range interactions are often so weak, then we cannot predict when they will be significant. Perhaps this is the main drawback of Landau's theory and its various extensions discussed here.

7. Conclusions

Landau's analysis, linking Loitsyansky's integral to the angular momentum of the turbulence, can be criticized because, formally, it applies to inhomogeneous turbulence only. Any attempt to recast this analysis for homogenous turbulence, by applying momentum conservation to an open control volume, is doomed to failure, as the angular momentum in such a control volume is dominated by the residual linear momentum. As noted in Monin & Yaglom (1975), the problem lies in the fact that the fluid lying adjacent to the control surface dominates $\langle \boldsymbol{H}^2 \rangle$. Clearly, an entirely different approach is needed. We have shown that the key is to rewrite Loitsyansky's integral in terms of the vector potential $\boldsymbol{A}$,

$$I = 6\int \langle \boldsymbol{A}\cdot\boldsymbol{A}'\rangle \,\mathrm{d}\boldsymbol{r} = 6\left\langle \left[\int_V \boldsymbol{A}\,\mathrm{d}V\right]^2 \right\rangle \Big/ V.$$

It follows immediately that I is not equal to $\langle \boldsymbol{H}^2\rangle/V$, as suggested by Landau's analysis, but rather proportional to $\langle \boldsymbol{J}^2\rangle/V$, where $\boldsymbol{J}$ is the weighted integral of the angular momentum density throughout the space. We have also shown that the conservation of I, in the absence of long-range correlations, can be understood in terms of the decorrelation of the flux of angular momentum out through a spherical control surface and the local angular momentum itself. Our analysis bears some similarities to that of Kolmogorov (1.14) but is essentially distinct. It is also readily generalized to certain types of anisotropic turbulence, such as MHD, rotating and stratified turbulence, yielding the important constraint $u_\perp^2 \ell_\perp^4 \ell_{//} = \text{constant}$. We have shown that this constraint yields decay laws for both MHD and stratified turbulence and that the former are consistent with the available numerical data.

The author would like to thank Jim Riley whose discussions of stratified turbulence helped establish (5.26)–(5.28).

Appendix A. The physical interpretation of Saffman's scaling $\langle \boldsymbol{P}^2\rangle \sim V$

We suggested in § 1.1 that the scaling $\langle \boldsymbol{P}^2\rangle \sim V$ could be thought of as a consequence of the central limit theorem, provided that individual eddies (blobs of vorticity) retain some finite linear impulse. This view can be confirmed as follows: As usual, we let $\boldsymbol{P} = \int_V \boldsymbol{u}\,\mathrm{d}V$, where V is a large spherical control volume of radius R. Moreover, let us suppose that the turbulence consists of a random distribution of discrete eddies, each occupying a volume V_i and with linear impulse $\boldsymbol{L}_i = (1/2)\int_{V_i} \boldsymbol{x}\times\boldsymbol{\omega}\,\mathrm{d}V$. The contribution that each eddy makes to $\boldsymbol{P}$ depends on whether or not it lies inside V. In particular, it may be shown that (Jackson 1998, p. 187)

$$\boldsymbol{P} = \int_V \boldsymbol{u}\,\mathrm{d}V = \frac{2}{3}\sum_i \boldsymbol{L}_i + V\sum_j \boldsymbol{u}_{0j}, \tag{A 1}$$

where the subscripts i and j refer to eddies inside and outside V respectively and $\boldsymbol{u}_{0j}$ is the velocity induced at the centre of V by the jth external eddy. If these eddies are taken to be statistically independent, which is somewhat simplistic, then we have

$$\langle \boldsymbol{P}^2\rangle = \frac{4}{9}\sum_i \langle \boldsymbol{L}_i^2\rangle + V^2\sum_j \langle (\boldsymbol{u}_{0j})^2\rangle$$

or

$$\langle \boldsymbol{P}^2\rangle = \frac{4}{9} nV \left\langle \boldsymbol{L}_i^2\right\rangle + V^2\sum_j \langle (\boldsymbol{u}_{0j})^2\rangle, \tag{A 2}$$

where n is the number density of eddies, defined by $nV\langle \boldsymbol{L}_i^2\rangle = \sum \langle \boldsymbol{L}_i^2\rangle$. Next we note that for $R \gg \ell$, the Biot–Savart law yields

$$4\pi \boldsymbol{u}_{0j} = \boldsymbol{L}_j \cdot \nabla\left(\nabla\left(r_j^{-1}\right)\right) + O\left(r_j^{-4}\right) \quad \text{(no summation over } j\text{)},$$

where r_j is the distance of the jth eddy from the centre of V. For an isotropic distribution of eddies this yields

$$V^2\langle(\boldsymbol{u}_{0j})^2\rangle = \frac{2}{9}\left\langle \boldsymbol{L}_j^2\right\rangle \frac{R^6}{r_j^6} + O\left(r_j^{-7}\right),$$

and if these eddies are uniformly distributed in space, with $\langle \boldsymbol{L}_j^2\rangle$ independent of position, we find

$$V^2\sum_j \langle(\boldsymbol{u}_{0j})^2\rangle = \frac{2}{9}\langle \boldsymbol{L}_i^2\rangle nV + O(R^2).$$

Substituting back into (A 2) yields the desired result:

$$\langle \boldsymbol{P}^2\rangle = \frac{2}{3}nV\langle \boldsymbol{L}_i^2\rangle + O(R^2) \sim V. \tag{A 3}$$

Of course, this analysis is somewhat naïve, as the eddies within V will not be statistically independent. Nevertheless, this does provide a kinematically admissible field of isotropic turbulence and so does lend support to the scaling $\langle \boldsymbol{P}^2\rangle \sim V$ in Saffman turbulence.

Note that, if $\boldsymbol{L}_i = 0$, which would be typical of Batchelor spectra, then (A 3) tells us that $\langle \boldsymbol{P}^2\rangle \sim R^2$, which is consistent with (2.7).

Appendix B. The physical reason for the $\langle \mathrm{H}^2\rangle \sim R^4$ scaling in an open control volume

Consider an artificial field of turbulence which consists of a random sea of compact vortex blobs (eddies), each occupying a volume V_i within which the vortex lines are closed. Let $\boldsymbol{x}_i$ locate the ith eddy (i.e. $\boldsymbol{x}_i$ lies within V_i) and $\boldsymbol{r}_i$ be a local coordinate defined by $\boldsymbol{x} = \boldsymbol{x}_i + \mathbf{r}_i$. The intrinsic linear and angular impulse of the ith eddy are (Batchelor 1967, p. 519)

$$\boldsymbol{L}_i = \frac{1}{2}\int_{V_i} \boldsymbol{r}_i \times \boldsymbol{\omega}\,\mathrm{d}V, \quad \boldsymbol{M}_i = \frac{1}{3}\int_{V_i} \boldsymbol{r}_i \times (\boldsymbol{r}_i \times \boldsymbol{\omega})\,\mathrm{d}V.$$

In addition, $\int_{V_i} \boldsymbol{\omega}\,\mathrm{d}V = 0$, since the vortex lines are closed in V_i. Now the total angular momentum held in a spherical control volume V can be written in terms of the global angular impulse as follows:

$$\boldsymbol{H} = \int_V \boldsymbol{x} \times \boldsymbol{u}\,\mathrm{d}V = \frac{1}{3}\int_V \boldsymbol{x} \times (\boldsymbol{x} \times \boldsymbol{\omega})\,\mathrm{d}V + \frac{1}{3}R^2\int_V \boldsymbol{\omega}\,\mathrm{d}V, \tag{B 1}$$

where, as usual, R is the radius of the control volume. (This comes from integrating the identity $6\boldsymbol{x} \times \boldsymbol{u} = 2\boldsymbol{x} \times (\boldsymbol{x} \times \boldsymbol{\omega}) + 3\nabla \times (\boldsymbol{x}^2\boldsymbol{u}) - \boldsymbol{\omega}\cdot\nabla(\boldsymbol{x}^2\boldsymbol{x})$.) Moreover, if none of the eddies straddles the surface S of our control volume but rather lies inside or outside V, then the second integral on the right of (B1) is zero. Making the substitution $\boldsymbol{x} = \boldsymbol{x}_i + \boldsymbol{r}_i$ and assuming that none of the eddies straddles the surface S, we find

$$\boldsymbol{H} = \frac{1}{3}\int_V \boldsymbol{x} \times (\boldsymbol{x} \times \boldsymbol{\omega})\,\mathrm{d}V = \sum_i [\boldsymbol{M}_i + \boldsymbol{x}_i \times \boldsymbol{L}_i], \tag{B 2}$$

where the summation is over all the eddies in the interior of V (Davidson 2004, p. 363). (Note that in going from (B1) to (B2) we have made use of a second identity, i.e. $2\boldsymbol{x} \times (\boldsymbol{x}_0 \times \boldsymbol{\omega}) = \boldsymbol{x}_0 \times (\boldsymbol{x} \times \boldsymbol{\omega}) + \boldsymbol{\omega}\cdot\nabla(\boldsymbol{x} \times (\boldsymbol{x}_0 \times \boldsymbol{x}))$.)

However, it is inevitable that the control surface will intersect some of the eddies V_i, in which case the second integral on the right of (B1) is non-zero. Repeating the analysis of Davidson (2004) but allowing some of the eddies to straddle the surface S, we find after a little algebra

$$\boldsymbol{H} = \sum_i \left[\boldsymbol{M}_i + \boldsymbol{x}_i \times \boldsymbol{L}_i + \frac{1}{3}\left[\boldsymbol{x}_i \cdot \int_{\delta S_i} \boldsymbol{r}_i\,(\boldsymbol{\omega}\cdot \mathrm{d}\boldsymbol{S})\right]\boldsymbol{x}_i + \frac{1}{6}\int_{\delta S_i} \boldsymbol{r}_i \times (\boldsymbol{x}_i \times \boldsymbol{r}_i)(\boldsymbol{\omega}\cdot \mathrm{d}\boldsymbol{S})\right], \tag{B 3}$$

where $\boldsymbol{L}_i$ is now evaluated over that part of V_i that lies in V. The two extra terms on the right clearly arise from those eddies which straddle S and δS_i is that part of S which intersects such an eddy. Now for the eddies which intersect S we are free to take $\boldsymbol{x}_i$ to lie on S. It follows that $\boldsymbol{r}_i \cdot \boldsymbol{x}_i = 0$ on δS_i, at least for $R/\ell \to \infty$. Thus, to leading order in R/ℓ, (B3) simplifies to

$$\boldsymbol{H} = \sum_i \left[\boldsymbol{M}_i + \boldsymbol{x}_i \times \boldsymbol{L}_i + \frac{1}{6}\boldsymbol{x}_i \int_{\delta S_i} \boldsymbol{r}_i^2\,(\boldsymbol{\omega}\cdot \mathrm{d}\boldsymbol{S})\right]. \tag{B 4}$$

Next we note that the easiest way to generate a kinematically admissible field of turbulence with a Batchelor spectrum is to create a random sea of eddies (vortex blobs), each of which possesses some angular impulse but no linear impulse (Davidson 2004, p. 370). (If the eddies possess some linear impulse, we get a Saffman spectrum, as discussed in § 2.1.) In such a case $\boldsymbol{L}_i = 0$, except for the eddies which intersect the boundary, and so we have

$$\boldsymbol{H} = \sum_i \boldsymbol{M}_i + \sum_k \left[\boldsymbol{x}_k \times \boldsymbol{L}_k + \frac{1}{3}\boldsymbol{x}_k \int_{\delta V_k} (\boldsymbol{r}_k \cdot \boldsymbol{\omega})\,\mathrm{d}V\right], \tag{B 5}$$

where the summation over k corresponds to the eddies which intersect S and δV_k is that part of V_k which lies in V. Evidently we have two contributions to $\boldsymbol{H}$, one from the intrinsic angular impulse of the interior eddies and another from the eddies which intersect the boundary. The final step is to assume that the eddies are statistically independent, in which case

$$\langle \boldsymbol{H}^2 \rangle = \sum_i \langle \boldsymbol{M}_i^2 \rangle + \sum_k \left\langle \left[\boldsymbol{x}_k \times \boldsymbol{L}_k + \frac{1}{3}\boldsymbol{x}_k \int_{\delta V_k} (\boldsymbol{r}_k \cdot \boldsymbol{\omega})\,\mathrm{d}V\right]^2 \right\rangle. \tag{B 6}$$

The first term on the right is of order R^3, which is consistent with Landau's analysis, while the second is of order $R^2 S \sim R^4$, which is consistent with (2.29).

Appendix C. The EDQNM closure prediction for the evolution of $\boldsymbol{I}(t)$.

The EDQNM closure model is discussed in detail in Lesieur (1990) and Sagaut & Cambon (2008). If $E(k \to 0) = C(t)k^4$, then the model predicts that the large to intermediate scales evolve in a self-similar manner,

$$E(k) = Ck_i^4 F\left(k/k_i\right), \tag{C 1}$$

where $C = I/24\pi^2$ and $k_i(t)$ is the wavenumber at which the spectrum peaks. Moreover, it predicts that a good approximation to $F(x)$, except in the vicinity of $k_i(t)$, is

$$F(x) = x^4,\ \ x < 1,\ \ \text{and}\ \ F(x) = x^{\frac{-5}{3}},\ \ x > 1, \tag{C 2}$$

corresponding to the model spectrum $E(k) = Ck^4, k < k_i$, and $E(k) = \alpha\varepsilon^{2/3}k^{-5/3}, k > k_i$. (Here $\alpha \approx 1.52$ is the Kolmogorov constant and ε the energy dissipation rate.) Note that the characteristic velocity u, defined via $u^2=(1/3)\langle \boldsymbol{u}^2\rangle$, and the integral scale ℓ, defined in the usual way as the integral of the longitudinal correlation function, are related to $F(x)$ by

$$u^2 = \frac{2}{3}Ck_i^5\int_0^\infty F(x)\,\mathrm{d}x, \tag{C 3}$$

$$\ell = \frac{\pi}{2u^2}\int_0^\infty \frac{E(k)}{k}\,\mathrm{d}k = \frac{\pi}{2u^2}Ck_i^4\int_0^\infty \frac{F(x)}{x}\,\mathrm{d}x. \tag{C 4}$$

The EDQNM model replaces the quasi-normal estimate

$$\frac{\mathrm{d}^2C}{\mathrm{d}t^2} = \frac{14}{15}\int_0^\infty \frac{E^2(p)}{p^2}\,\mathrm{d}p \tag{C 5}$$

with

$$\frac{\mathrm{d}C}{\mathrm{d}t} = \frac{14}{15}\int_{k_i}^\infty \theta_{0pp}\frac{E^2(p)}{p^2}\,\mathrm{d}p, \tag{C 6}$$

where θ_{0pp} is a model parameter whose precise prescription varies somewhat from one version of EDQNM to another. For large times and large Reynolds number Re, a commonly used prescription for θ_{0pp} is (Lesieur 1990)

$$\theta_{0pp} = \left[4a_1^2\int_0^p q^2E(q)\,\mathrm{d}q\right]^{\frac{-1}{2}}, \quad a_1\approx 0.22\alpha^{\frac{3}{2}}. \tag{C 7}$$

Substituting for $E(k)$ in (C6) using (C1) and making use of (C3), (C4) and (C7), we obtain the EDQNM prediction

$$\mathrm{d}C/C = Bu\,\mathrm{d}t/\ell, \tag{C 8}$$

where B is the constant,

$$B = \frac{7}{15a_1}\frac{\sqrt{3}}{\sqrt{2}}\frac{3\pi}{4}\int_0^\infty \frac{F(x)}{x}\,\mathrm{d}x\left[\int_0^\infty F(x)\,\mathrm{d}x\right]^{\frac{-3}{2}}\int_1^\infty \frac{F^2(x)/x^2}{\sqrt{\int_0^x y^2F(y)\,\mathrm{d}y}}\,\mathrm{d}x. \tag{C 9}$$

Next we note that

$$\frac{\mathrm{d}u^2}{\mathrm{d}t} = -A\frac{u^3}{\ell}, \tag{C 10}$$

where measurements at large Re show $A\approx 1/3$ (Davidson 2004). Combining (C8) and (C10) yields $I\sim C\sim (u^2)^{-m}$, where $m=B/A$. Note that estimate (C9) of B, and hence that of the exponent m, assumes only self-similarity of the spectrum, which is consistent with the predictions of EDQNM. If we now adopt the model spectrum (C2) to estimate A, then A is fixed by the requirement that $Ck_i^4=\alpha\varepsilon^{2/3}k_i^{-5/3}$, which, combined with (C3) and (C10), demands

$$A = \left(\frac{3}{2\alpha}\right)^{\frac{3}{2}}\frac{\pi}{2}\int_0^\infty \frac{F(x)}{x}\,\mathrm{d}x\left[\int_0^\infty F(x)\,\mathrm{d}x\right]^{\frac{-5}{2}} = 0.347. \tag{C 11}$$

This estimate of A, which follows directly from (C2) and has nothing to do with EDQNM, is remarkably close to the observed value of $A\approx 1/3$. Full numerical simulations of the EDQNM model, which do not prejudge the form of $F(x)$, are reported in Lesieur (1990), and these suggest that (C9) gives the modest value of $m\approx 0.12$. Now (C10) requires $\ell\sim ut$, while self-similarity of the large scales demands

$C = cu^2\ell^5$, where c is a constant. It follows that m is related to the exponents γ and n in the expressions $I \sim C \sim t^\gamma$ and $u^2 \sim t^{-n}$ by $\gamma = 10/(7/m + 2)$ and $n = \gamma/m$. The value of $m \approx 0.12$ leads to $\gamma \approx 0.16$ and $n = 1.38$.

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